IMRN International Mathematics Research Notices
2004, No. 30

Minimality and Nondegeneracy of Degree-One Ginzburg-Landau Vortex as a Hardy's Type Inequality

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1 Introduction

We consider the complex-valued Ginzburg-Landau equation in the plane

$$\Delta w + \left(1 - |w|^2\right)w = 0, \quad \text{in } \mathbb{R}^2. \tag{1.1}$$

The standard one-vortex solution of degree one in the plane is the solution $w(x)$ of (1.1) of the form

$$w(x) = U(r)e^{i\theta}, \tag{1.2}$$

where (r, θ) designate usual polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, and $U(r)$ is the unique solution of the problem

$$\begin{aligned} &U'' + \frac{U'}{r} - \frac{U}{r^2} + \left(1 - U^2\right)U = 0, \quad \text{in } (0, \infty), \\ &U(0) = 0, \qquad U(+\infty) = 1. \end{aligned} \tag{1.3}$$

It is known that $U'(0) > 0$ and that

$$U(r) \sim 1 - \frac{1}{r^2}, \quad U'(r) \sim \frac{2}{r^3}, \quad \text{as } r \to +\infty, \tag{1.4}$$

Received 25 November 2003. Revision received 19 January 2004.

see, for instance, [3]. An important feature of this solution is its *locally minimizing character*. The energy functional associated to equation (1.1) is given by

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^2} \left(1-|v|^2\right)^2. \tag{1.5}$$

While $E(w) = +\infty$, it turns out that for any ϕ smooth and compactly supported, $E(w) - E(w+\phi) \le 0$, which implies $B(\phi,\phi) \ge 0$ for all such ϕ, where B is the bilinear form given by the (formal) second variation of E around w,

$$B(\phi,\phi) = \int_{\mathbb{R}^2} |\nabla\phi|^2 - \int_{\mathbb{R}^2} \left(1-U^2\right)|\phi|^2 + 2\int_{\mathbb{R}^2} \left|\operatorname{Re}(\bar{w}\phi)\right|^2. \tag{1.6}$$

We observe that $B(\phi,\phi) = \langle L(\phi), \phi\rangle$, where here and in what follows

$$\langle u, v\rangle = \operatorname{Re}\int_{\mathbb{R}^2} u\bar{v}, \tag{1.7}$$

and L is the linearization of (1.1) around w,

$$L(\phi) = \Delta\phi + \left(1-U^2\right)\phi - 2\operatorname{Re}(\bar{w}\phi)w, \quad \text{in } \mathbb{R}^2. \tag{1.8}$$

Direct substitution shows that

$$L\left(\frac{\partial w}{\partial x_1}\right) = L\left(\frac{\partial w}{\partial x_2}\right) = L(iw) = 0, \tag{1.9}$$

which accounts for the invariance of equation (1.1) under space translations of the solution and under multiplication by complex scalars of absolute value one, which introduces degeneracy of this minimizer.

The locally minimizing character of w, $B(\psi,\psi) \ge 0$, follows by combining known results in the literature as pointed out to us by Mironescu. A local minimizer v with $v(0) = 0$ can be found by considering global minimizers of the Ginzburg-Landau energy in a ball with large radius and boundary condition $e^{i\theta}$. The analysis in [1] shows that (taking as the origin one of their zeros) these functions converge, up to subsequences, locally over compacts of $\mathbb{R}^2$ to a solution of (1.1) with $v(0) = 0$, clearly a local minimizer which, besides, satisfies $\int_{\mathbb{R}^2}(1-|u|^2)^2 < +\infty$. From [2, 12], it follows that v has a degree at infinity which is equal to 1 or -1. From [9], v is necessarily radial so that it must be equal to w or to its conjugate. Stability of radial solutions in a ball was previously studied in [7, 8].

A different proof of this result was obtained by Ovchinnikov and Sigal [10] who analyzed the spectrum of the operator L in L^2 and found that it had 0 as its lower limit.

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More specifically, see also [4, 6] for related results and methods, $B(\phi, \phi)$ is positive if ϕ lies in an L^2-orthogonal to the space

$$\mathcal{Z} = \text{span}\left\{ \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, iw \right\}. \tag{1.10}$$

Spectral theory for Schrödinger operators and a form of Perron-Frobenius method applied for spectral analysis of each element of a block decomposition of the operator L are the ingredients used in [4, 6, 10].

An observation we should make however is that, in principle, L^2 is not an ideal environment space for L since $\mathcal{Z}$ is not contained in L^2. In particular, a decaying solution of $L(\phi) = h$ for, say h, compactly supported, should not decay in general at a faster rate than that of ∇w. Thus, it is not obvious how to produce a satisfactory solvability theory for this problem from the L^2-information.

The purpose of this paper is to present an elementary direct proof of the strict minimizing character of w for perturbations ϕ in the "natural" Hilbert space H for the bilinear form B of all locally H^1 functions for which

$$\|\phi\|_H^2 = \int_{\mathbb{R}^2} \left[|\nabla\phi|^2 + \left(1 - U^2\right)|\phi|^2 + \left| \operatorname{Re}(\bar{w}\phi) \right|^2 \right] < +\infty. \tag{1.11}$$

Theorem 1.1. The following inequality holds:

$$B(\phi, \phi) \geq 0 \quad \forall \phi \in H. \tag{1.12}$$

Besides, if $\phi \in H$ is such that $B(\phi, \phi) = 0$, then

$$\phi = c_1 \frac{\partial w}{\partial x_1} + c_2 \frac{\partial w}{\partial x_2}, \tag{1.13}$$

for certain real constants c_1, c_2. □

The proof of this result uses the standard observation that B can be decomposed in an additive way among different Fourier modes in θ. We decompose ϕ into the form

$$\phi = \phi^0 + \sum_{j=1}^{\infty} \phi_j^1 + \sum_{j=1}^{\infty} \phi_j^2, \tag{1.14}$$

where

$$\begin{aligned} \phi^0 &= e^{i\theta}\left[\phi_1^0(r) + i\phi_2^0(r)\right], \\ \phi_j^1 &= e^{i\theta}\left[\phi_{j1}^1(r)\sin j\theta + i\phi_{j2}^1(r)\cos j\theta\right], \\ \phi_j^2 &= e^{i\theta}\left[\phi_{j1}^2(r)\cos j\theta + i\phi_{j2}^2(r)\sin j\theta\right]. \end{aligned} \tag{1.15}$$

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Then, we get that

$$B(\phi,\phi) = B\big(\phi^0,\phi^0\big) + \sum_{j=1}^{\infty} B\big(\phi_j^1,\phi_j^1\big) + \sum_{j=1}^{\infty} B\big(\phi_j^2,\phi_j^2\big). \tag{1.16}$$

This decomposition is naturally associated to the elements of $\mathcal{Z}$ since

$$\begin{aligned} iw &= e^{i\theta}\big[0 + iU(r)\big],\\ \frac{\partial w}{\partial x_1} &= e^{i\theta}\left[U'(r)\cos\theta - i\frac{U(r)}{r}\sin\theta\right],\\ \frac{\partial w}{\partial x_2} &= e^{i\theta}\left[U'(r)\sin\theta + i\frac{U(r)}{r}\cos\theta\right]. \end{aligned} \tag{1.17}$$

We need to establish nonnegativity of each of the individual terms in (1.16). The most delicate step is to establish that $B(\phi_1^\ell,\phi_1^\ell) \geq 0$, $\ell = 1,2$, and that equality holds only if respectively $\phi_1^1 = \partial w/\partial x_2$, $\phi_1^2 = \partial w/\partial x_1$. We present a novel proof of this fact which amounts to a Hardy's type inequality for vector-valued functions (Proposition 2.1 below), which is interesting in its own right and has an elementary proof.

As a corollary, we will show that the following Fredholm alternative for the operator L holds.

Theorem 1.2. Consider the equation

$$L(\phi) = h, \quad \text{in } \mathbb{R}^2, \tag{1.18}$$

where it is assumed that for certain $\sigma > 0$, $\int_{\mathbb{R}^2} |h|^2(1 + r^{2+\sigma}) < +\infty$. If additionally

$$\left\langle h, \frac{\partial w}{\partial x_1}\right\rangle = \left\langle h, \frac{\partial w}{\partial x_2}\right\rangle = \langle h, iw\rangle = 0, \tag{1.19}$$

then (1.18) has a solution $\phi_0 \in H$ which satisfies

$$\|\phi_0\|_H^2 \leq C\int_{\mathbb{R}^2} |h|^2\big(1 + r^{2+\sigma}\big). \tag{1.20}$$

Moreover, all solutions $\phi \in H$ have the form

$$\phi = \phi_0 + c_1\frac{\partial w}{\partial x_1} + c_2\frac{\partial w}{\partial x_2}, \tag{1.21}$$

with $c_1, c_2 \in \mathbb{R}$. □

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A solvability theory for the linearized operator is of crucial importance in the use of singular perturbation methods for the construction of vortex solutions of problems where the rescaled vortex w provides a canonical profile. This is a subject broadly developed in [11], where an entirely different approach is used with respect to the issue.

The present paper was originally motivated by the authors' study of the existence of vortex lines in three-dimensional domains. In this case, Fredholm alternative for the three-dimensional profiles which are built up from two-dimensional vortices depends on the application of Theorem 1.2. Another area where our results could potentially be used is the question of orbital stability of vortex solutions in $\mathbb{R}^2$. In this context, the positivity of the bilinear form corresponding to Ginzburg-Landau energy E appears to play the key role as shown in [5], where orbital stability of magnetic vortices is analyzed.

We devote the rest of this paper to the proof of Theorems 1.1 and 1.2.

2 Nondegeneracy of w

We first consider a smooth function ϕ compactly supported, whose support does not contain the origin. It is convenient to define ψ by the relation

$$\phi = iw\psi, \tag{2.1}$$

and introduce the bilinear form

$$\mathbb{B}(\psi,\psi) = B(iw\psi, iw\psi). \tag{2.2}$$

Then we have, writing $\psi = \psi_1 + i\psi_2$,

$$\mathbb{B}(\psi,\psi) = \int_{\mathbb{R}^2} U^2|\nabla\psi|^2 - 2\,\mathrm{Re}\int_{\mathbb{R}^2} \frac{iU^2}{r^2}\frac{\partial\psi}{\partial\theta}\bar{\psi} + 2\int_{\mathbb{R}^2} U^4\left|\psi_2\right|^2. \tag{2.3}$$

On the other hand, defining $\phi^0 = iw\psi^0$ and $\phi_j^\ell = iw\psi_j^\ell$ for $j \in \mathbb{N}$ and $\ell = 1, 2$ and using (1.16), we find

$$\mathbb{B}(\psi,\psi) = \mathbb{B}\left(\psi^0,\psi^0\right) + \sum_{j=1}^{\infty} \mathbb{B}\left(\psi_j^1,\psi_j^1\right) + \mathbb{B}\left(\psi_j^2,\psi_j^2\right). \tag{2.4}$$

We set

$$\begin{aligned}
\psi^0 &= \psi_1^0(r) + i\psi_2^0(r),\\
\psi_j^1 &= \psi_{j1}^1(r)\cos j\theta + i\psi_{j2}^1(r)\sin j\theta,\\
\psi_j^2 &= \psi_{j1}^2(r)\sin j\theta + i\psi_{j2}^2(r)\cos j\theta.
\end{aligned} \tag{2.5}$$

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We consider functions $\varphi : [0,\infty) \to \mathbb{R}^2$ and the bilinear forms

$$\mathcal{B}_j^\ell(\varphi,\varphi) = \int_0^\infty rU^2|\varphi'|^2 + \int_0^\infty rU^2 B_j^\ell \varphi\cdot\varphi, \tag{2.6}$$

where B_j^ℓ is the matrix

$$B_j^\ell = \frac{1}{r^2}\begin{pmatrix} j^2 & (-1)^\ell 2j \\ (-1)^\ell 2j & j^2+2U^2r^2 \end{pmatrix}. \tag{2.7}$$

With these definitions, it is direct to check that the following fact holds:

$$\mathbb{B}\big(\psi_j^\ell,\psi_j^\ell\big) = \pi\mathcal{B}_j^\ell\big(\varphi_j^\ell,\varphi_j^\ell\big), \quad \ell = 1,2, \tag{2.8}$$

where

$$\varphi_j^\ell(r) = \big(\psi_{j1}^\ell(r),\psi_{j2}^\ell(r)\big), \quad \ell = 1,2. \tag{2.9}$$

At the core of the proof of Theorem 1.1 is the positivity of the bilinear forms $\mathcal{B}_1^1$, which can be written as

$$\mathcal{B}_1^1(\varphi,\varphi) = \int_0^\infty rU^2\left[|\varphi'|^2 + \frac{1}{r^2}|\varphi|^2 - \frac{4}{r^2}\varphi_1\varphi_2 + 2U^2|\varphi_2|^2\right] \geq 0. \tag{2.10}$$

This inequality is a vector-valued form of Hardy's inequality. In fact, Hardy's inequality for radially symmetric functions in $\mathbb{R}^N$, $N \geq 3$, asserts that

$$\int_0^\infty |u'|^2 r^{N-1}\,dr - \left(\frac{N-2}{2}\right)^2 \int_0^\infty \frac{|u|^2}{r^2} r^{N-1}\,dr \geq 0. \tag{2.11}$$

We observe that $\mathcal{B}_1^1(\varphi,\varphi) \geq 0$ for $\varphi = (\nu,-\nu)$ means that

$$\int_0^\infty |\nu'|^2 U^2 r\,dr - \int_0^\infty \frac{|\nu|^2}{r^2} U^2 r\,dr + \int_0^\infty \nu^2 U^4 r\,dr \geq 0. \tag{2.12}$$

Near $r = 0$, $U^2(r)r \sim r^3$. Replacing ν by $u(r/\delta)$ with u compactly supported and taking limit as $\delta \to 0$, we obtain

$$\int_0^\infty |u'|^2 r^3\,dr - \int_0^\infty \frac{|u|^2}{r^2} r^3\,dr \geq 0, \tag{2.13}$$

which is precisely the optimal Hardy's inequality in dimension $N = 4$. In precise terms, the result we obtain is the following proposition.

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Proposition 2.1. For any $\mathbb{R}^2$-valued smooth function φ with compact support away from the origin,

$$\mathcal{B}_1^1(\varphi,\varphi) = \int_0^\infty U^2 r\left|\varphi' - A_1(r)\varphi\right|^2 dr, \tag{2.14}$$

where $A_1(r)$ is a 2×2 matrix of smooth functions in $(0,\infty)$ with the property that the only solutions of the system $\varphi' = A_1(r)\varphi$ such that $\int_0^1 |\varphi|^2 U^2 r\, dr < +\infty$ are given by constant multiples of $\varphi_0(r) = (1/r, U'/U)$. □

Proof of Proposition 2.1. We will write $\mathcal{B}_1^1(\varphi,\varphi)$ in the form

$$\mathcal{B}_1^1(\varphi,\varphi) = \int_0^\infty U^2 r\left|\varphi' - A_1(r)\varphi\right|^2 dr, \tag{2.15}$$

where $A_1(r)$ is a 2×2 symmetric matrix of functions which we will determine next. First we expand

$$\int_0^\infty U^2 r\left|\varphi' - A_1(r)\varphi\right|^2 = \int_0^\infty rU^2|\varphi'|^2 - 2\int_0^\infty rU^2 A_1\varphi'\cdot\varphi + \int_0^\infty rU^2 A_1^2\varphi\cdot\varphi. \tag{2.16}$$

Now,

$$2\int_0^\infty rU^2 A_1\varphi'\cdot\varphi = \int_0^\infty \frac{d}{dr}(rU^2 A_1\varphi\cdot\varphi) - \int_0^\infty \left(rU^2 A_1\right)'\varphi\cdot\varphi. \tag{2.17}$$

Since φ is compactly supported in $(0,\infty)$, we get

$$\int_0^\infty U^2 r\left|\varphi' - A_1\varphi\right|^2 = \int_0^\infty rU^2|\varphi'|^2 + \int_0^\infty \left(rU^2 A_1\right)'\varphi\cdot\varphi + \int_0^\infty rU^2 A_1^2\varphi\cdot\varphi. \tag{2.18}$$

Thus, the requirement (2.15) is equivalent to

$$\left(rU^2 A_1\right)' + rU^2 A_1^2 = B_1^1, \tag{2.19}$$

where B_1^1 is the matrix defined in (2.7) with $j = 1$. We write

$$A_1 = \begin{pmatrix} a & c \\ c & b \end{pmatrix}. \tag{2.20}$$

We write $\varphi_0 = (\rho_1, \rho_2)$. Since $\mathcal{B}_1^1(\varphi_0,\varphi_0) = 0$, the matrix A_1 should satisfy

$$\varphi_0' = A_1\varphi_0. \tag{2.21}$$

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This yields the relations

$$a = \frac{\rho_1' - c\rho_2}{\rho_1}, \qquad b = \frac{\rho_2' - c\rho_1}{\rho_2}. \tag{2.22}$$

Now, substituting these relations into (2.19), direct inspection leads to the fact that (2.19) holds if and only if

$$\left(U^2 rc\right)' = U^2 rc^2\left(\frac{\rho_2}{\rho_1} + \frac{\rho_1}{\rho_2}\right) - U^2 rc\left(\frac{\rho_1'}{\rho_1} + \frac{\rho_2'}{\rho_2}\right) - \frac{2U^2}{r}, \tag{2.23}$$

where, we recall $\rho_1 = 1/r$, $\rho_2 = U'/U$. Expanding the above equation, we find

$$c' = -c\left(\frac{U''}{U} + \frac{U'}{U}\right) + c^2\left(\frac{\rho_2}{\rho_1} + \frac{\rho_1}{\rho_2}\right) - \frac{2}{r^2}. \tag{2.24}$$

This is a Ricatti equation. We observe that if u satisfies the equation

$$\left(p(r)u'\right)' + q(r)u + p(r)z(r)u' = 0, \tag{2.25}$$

then $c = -pu'/u$ satisfies

$$c' = -\frac{(pu')'}{u} + p\frac{u'^2}{u^2} = q(r) + \frac{c^2}{p} - cz. \tag{2.26}$$

We set

$$\begin{aligned} p &= \frac{\rho_1\rho_2}{\rho_2^2 + \rho_1^2}, \qquad q = -\frac{2}{r^2}, \\ z &= \left(\frac{U'}{U} + \frac{U''}{U'}\right). \end{aligned} \tag{2.27}$$

Then $c = -pu'/u$ satisfies equation (2.24) on $(0, \infty)$ if u is a positive solution of

$$\left(\frac{\rho_1\rho_2 UU'}{\rho_2^2 + \rho_1^2}u'\right)' - \frac{2UU'}{r^2}u = 0, \quad r \in (0, \infty). \tag{2.28}$$

Observe that

$$\frac{\rho_1\rho_2 UU'}{\rho_2^2 + \rho_1^2} = \frac{U'^2 r}{1 + \dfrac{rU'}{U}}. \tag{2.29}$$

It is easy to check that equation (2.28) has a solution $u(r)$ which decays to zero and it is positive as $r \to +\infty$. In fact, as $r \to +\infty$, this equation resembles

$$\left(r^{-5}u'\right)' - 2r^{-5}u = 0, \tag{2.30}$$

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or

$$u'' - \frac{5}{r}u' - 2u = 0, \tag{2.31}$$

which is a Bessel-type equation with a decaying solution $u(r) \sim r^{5/2}e^{-\sqrt{2}r}$. We can find with the use of barriers a solution $u(r)$ of (2.28) with this property. We observe that then, $u(r)$ is actually positive all over $r \in (0,\infty)$, since equation (2.28) satisfies the maximum principle. As $r \to 0$, the equation gets similar to $(ru')' - (4/r)u = 0$, or

$$r^2u'' + ru' - 4u = 0. \tag{2.32}$$

Hence, the behavior of u is like $u(r) \sim r^{-2}$ as $r \to 0$. In fact an application of Frobenius method for equation (2.28) provides this fact. Summarizing, we conclude that equation (2.24) has a globally defined positive solution $c(r)$, $r \in (0,\infty)$, where $c = -pu'/u$. Since from its definition, $p(0^+) = 1/2$, we get $c(r) = 1/r + o(r^{-1})$ as $r \to 0$. The matrix A_1 as desired has thus been built. Moreover, the following property for $A_1(r)$ is automatically checked:

$$A_1(r) = \begin{pmatrix} -\frac{2}{r} & \frac{1}{r} \\ \frac{1}{r} & -\frac{2}{r} \end{pmatrix} + o\left(r^{-1}\right), \tag{2.33}$$

as $r \to 0$. The system $\varphi' = A_1(r)\varphi$ has as a solution $\varphi_0(r) = (1/r, U'/U)$. Let $\varphi_1(r)$ be a second linearly independent solution. Then Liouville's formula for the Wronskian gives

$$W(\varphi_0,\varphi_1) = Ce^{-\int_r^{r_0} \mathrm{tr}[A_1(s)]ds} \sim \frac{C}{r^4}. \tag{2.34}$$

It follows that $|\varphi_1(r)| \geq C/r^3$ for all small $r > 0$. The conclusion is that the unique solutions of $\varphi' = A_1(r)\varphi$ for which $\int_0^1 |\varphi(r)|^2U^2r\,dr < +\infty$ are scalar multiples of φ_0, and the proof of the proposition is complete. ■

Corollary 2.2. For any $\mathbb{R}^2$-valued smooth function φ with compact support away from the origin,

$$\mathcal{B}_1^2(\varphi,\varphi) = \int_0^\infty U^2r\left|\varphi' - A_2(r)\varphi\right|^2 dr, \tag{2.35}$$

where $A_2(r)$ is a 2×2 matrix of smooth functions in $(0,\infty)$ with the property that the only solutions of the system $\varphi' = A_2(r)\varphi$ such that $\int_0^1 |\varphi|^2U^2r\,dr < +\infty$ are given by constant multiples of $\overline{\varphi}_0(r) = (1/r, -U'/U)$. □

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Proof. It is enough to apply Proposition 2.1 with $\overline{\varphi} = (\varphi_1, -\varphi_2)$ and consider

$$A_2 = \begin{pmatrix} a & -c \\ -c & b \end{pmatrix}, \tag{2.36}$$

where a, b, and c were defined in the proof of the proposition. ■

Proof of Theorem 1.1. We have to estimate from below the quantities $\mathcal{B}_j^\ell(\varphi_j^\ell, \varphi_j^\ell)$, $j \geq 1$, $\ell = 1, 2$, with $\mathcal{B}_j$ the bilinear form given by (2.6) and φ_j^ℓ defined by (2.9). We assume that $j \geq 2$. Then

$$\mathcal{B}_j^\ell(\varphi, \varphi) = \int_0^\infty rU^2|\varphi'|^2 + \int_0^\infty rU^2 B_j^\ell \varphi \cdot \varphi, \tag{2.37}$$

where B_j^ℓ is given by (2.7). We observe that

$$(B_j^\ell - B_1^\ell)\varphi \cdot \varphi \geq \frac{j-1}{r^2} \begin{pmatrix} j+1 & (-1)^\ell 2 \\ (-1)^\ell 2 & j+1 \end{pmatrix} \varphi \cdot \varphi \geq \frac{(j-1)^2}{r^2}|\varphi|^2. \tag{2.38}$$

Hence, we find

$$\mathcal{B}_j^\ell(\varphi, \varphi) \geq (j-1)^2 \int_0^\infty \frac{U^2}{r^2}|\varphi|^2 r\, dr. \tag{2.39}$$

Gathering the above estimates, we find the following inequality:

$$\sum_{j=1}^\infty \mathbb{B}(\psi_j^\ell, \psi_j^\ell) \geq \int_0^\infty \left|\varphi_1^{\ell\,\prime} - A_\ell(r)\varphi_1^\ell\right|^2 U^2 r\, dr + \sum_{j=2}^\infty (j-1)^2 \int_{\mathbb{R}^2} \frac{U^2}{r^2}|\psi_j^\ell|^2, \tag{2.40}$$

$\ell = 1, 2$. On the other hand, we observe that

$$\mathbb{B}(\psi^0, \psi^0) = \int_{\mathbb{R}^2} U^2 \left|\nabla\psi^0\right|^2 + 2\int_{\mathbb{R}^2} U^4 \left|\psi_2^0\right|^2. \tag{2.41}$$

These facts give the following inequality. Whenever ϕ is smooth and compactly supported away from $r = 0$ with $\phi = iw\psi$, we have

$$\begin{aligned} B(\phi, \phi) \geq & \int_{\mathbb{R}^2} U^2 \left|\nabla\psi^0\right|^2 + 2\int_{\mathbb{R}^2} U^4 \left|\psi_2^0\right|^2 \\ & + \sum_{\ell=1}^2 \int_0^\infty \left|\varphi_1^{\ell\,\prime} - A_\ell(r)\varphi_1^\ell\right|^2 U^2 r\, dr \\ & + \sum_{\ell=1}^2 \sum_{j=2}^\infty (j-1)^2 \int_{\mathbb{R}^2} \left|\psi_j^\ell\right|^2 \frac{U^2}{r^2}. \end{aligned} \tag{2.42}$$

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We claim that inequality (2.42) remains true for smooth ϕ and with compact support now containing the origin. Let $\eta(s)$ be a smooth cutoff with $\eta = 0$ for $s < 1$ and $\eta = 1$ for $s > 2$, and set $\eta_\sigma(r) = \eta(r/\delta)$, then (2.42) is valid for ϕ replaced by $\eta_\delta\phi$. Now,

$$\int_{\mathbb{R}^2} \left|\nabla(\eta_\delta\phi)\right|^2 = \int_{\mathbb{R}^2} \eta_\delta^2|\nabla\phi|^2 + 2\int_{\mathbb{R}^2} \psi\eta_\delta\nabla\eta_\delta\nabla\phi + \int_{\mathbb{R}^2} \psi^2\left|\nabla\eta_\delta\right|^2. \tag{2.43}$$

Integrating by parts,

$$2\int_{\mathbb{R}^2} \phi\eta_\delta\nabla\eta_\delta\nabla\phi = -\int_{\mathbb{R}^2} \phi^2\left|\nabla\eta_\delta\right|^2 - \int_{\mathbb{R}^2} \phi^2\Delta\eta_\delta. \tag{2.44}$$

Now,

$$\int_{\mathbb{R}^2} \phi^2\Delta\eta_\delta = \int_{\mathbb{R}^2} \phi^2(\delta x)\Delta\eta\, dx. \tag{2.45}$$

Thus,

$$\lim_{\delta\to 0}\int_{\mathbb{R}^2} \psi^2\Delta\eta_\delta = \phi^2(0)\int_{\mathbb{R}^2} \Delta\eta = 0. \tag{2.46}$$

Combining the above computations, we find

$$\lim_{\delta\to 0}\int_{\mathbb{R}^2} \left|\nabla(\eta_\delta\phi)\right|^2 = \int_{\mathbb{R}^2} |\nabla\phi|^2. \tag{2.47}$$

Using this, a density argument, and Fatou's lemma, inequality (2.42) is readily obtained not only for ϕ compactly supported but actually for any $\phi \in H$. In particular, $B(\phi,\phi) \geq 0$ for any $\phi \in H$.

Finally, we assume that $\phi \in H$ is such that $B(\phi,\phi) = 0$. Inequality (2.42) then clearly implies that $\psi^0 = 0$, $\psi^j_\ell = 0$, for all $j \geq 2$, $\ell = 1, 2$. Besides, we also have for $r > 0$, the validity of the differential equations

$$\varphi_1^{\ell\,\prime}(r) = A_\ell(r)\varphi_1^\ell(r). \tag{2.48}$$

On the other hand, $\phi \in H$ implies that $\int_0^1 |\varphi_1^\ell|^2 U^2 r\, dr < +\infty$. We then get $\varphi_1^\ell = c_\ell(1/r, (-1)^\ell(U'/U))$. This translates exactly into the desired form for ϕ, thus concluding the proof. ■

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3 Fredholm alternative: proof of Theorem 1.2

Finding a solution of $L(\phi) = h$ in H corresponds to finding a critical point in H of the functional

$$J(\phi) = \frac{1}{2}B(\phi,\phi) - \langle \phi, h\rangle. \tag{3.1}$$

Assume that h has the form described in Theorem 1.2. We decompose h in the following way:

$$h = h^0 + \sum_{j=1}^{\infty} h_j^1 + \sum_{j=1}^{\infty} h_j^2, \tag{3.2}$$

where now

$$\begin{aligned} h^0 &= e^{i\theta}\left[h_0(r) + ih_1(r)\right],\\ h_j^1 &= e^{i\theta}\left[h_{j1}^1(r)\sin j\theta + ih_{j2}^1(r)\cos j\theta\right],\\ h_j^2 &= e^{i\theta}\left[h_{j1}^2(r)\cos j\theta + ih_{j2}^2(r)\sin j\theta\right]. \end{aligned} \tag{3.3}$$

We decompose ϕ as in (1.14). Then the equation $L(\phi) = h$ is equivalent to solving each of the individual equations

$$L\left(\phi^0\right) = h^0, \tag{3.4}$$

$$L\left(\phi_j^\ell\right) = h_j^\ell, \quad j \in \mathbb{N},\ \ell = 1, 2, \tag{3.5}$$

where ϕ^0, ϕ_j^ℓ have the form in (1.15). We begin by solving problem (3.5) for $\ell = 1, j = 1$. Let H_* be the space of functions $\tilde\phi(r) = (\tilde\phi_1(r), \tilde\phi_2(r))$, $r \in (0,\infty)$, such that $\phi = e^{i\theta}[\tilde\phi_1(r)\sin\theta + i\tilde\phi_2(r)\cos\theta] \in H$ endowed with the norm $\|\tilde\phi\|_{H_*} = \|\phi\|_H$. Explicitly, we choose the (equivalent) norm

$$\|\tilde\phi\|_{H_*}^2 = \int_0^\infty \left[\left|\tilde\phi'\right|^2 + \frac{1}{r^2}\left(\tilde\phi_1 - \tilde\phi_2\right)^2 + \left(1 - U^2\right)\left|\tilde\phi\right|^2 + U^2\left|\tilde\phi_2\right|^2\right] r\,dr. \tag{3.6}$$

We also set

$$\tilde h(r) = \left(\tilde h_1(r), \tilde h_2(r)\right) = \left(h_{11}^1(r), h_{12}^1(r)\right). \tag{3.7}$$

Solving (3.5) for $\ell = 1$, $j = 1$ in H corresponds exactly to finding a critical point of the functional J_1 in H_* defined as

$$J_1\left(\tilde\phi\right) = \frac{1}{2}\mathbf{B}\left(\tilde\phi, \tilde\phi\right) - \int_0^\infty \tilde h(r)\cdot\tilde\phi(r) r\,dr, \tag{3.8}$$

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where

$$\mathbf{B}(\tilde\varphi,\tilde\varphi) = \mathcal{B}_1^1(\varphi_1^1,\varphi_1^1), \tag{3.9}$$

and where $\mathcal{B}_1^1$ is given by (2.10) and $\varphi_1^1 = U^{-1}(\tilde\varphi_2, -\tilde\varphi_1)$. Theorem 1.1, in terms of the bilinear form $\mathbf{B}$, reads like this: for any $\tilde\varphi \in H_*$, we have the inequality

$$\mathbf{B}(\tilde\varphi,\tilde\varphi) \geq 0. \tag{3.10}$$

Equality holds if and only if $\tilde\varphi = C(-U', U/r)$ for some $C \in \mathbb{R}$. Set $Z_0(r) = (-U', U/r)$ and observe that from the assumptions on h we get

$$\int_0^\infty \tilde h(r)\cdot Z_0(r) r\,dr = 0. \tag{3.11}$$

We will find a minimizer of $J_1(\tilde\varphi)$ in a suitable subspace of H_*. To do so we need to establish the following lemma.

Lemma 3.1. There exists a constant $C > 0$ such that for any $\tilde\varphi \in H_*$ with

$$\int_0^\infty (1-U^2)\tilde\varphi\cdot Z_0 r\,dr = 0, \tag{3.12}$$

it holds that

$$C\|\tilde\varphi\|_{H_*}^2 \leq \mathbf{B}(\tilde\varphi,\tilde\varphi). \tag{3.13}$$

□

Proof of Lemma 3.1. We start by observing the following:

$$\mathbf{B}(\tilde\varphi,\tilde\varphi) = \int_0^\infty \left[|\tilde\varphi'|^2 + \frac{2}{r^2}(\tilde\varphi_1-\tilde\varphi_2)^2 - (1-U^2)|\tilde\varphi|^2 + 2U^2|\tilde\varphi_2|^2\right] r\,dr. \tag{3.14}$$

Since $1-U^2(r) \sim 1/r^2$ for large r, we see that given $\delta > 0$, there exists $R > 0$ such that for all $r > R$, we have

$$\begin{aligned}&\frac{2-\delta}{r^2}(\tilde\varphi_1-\tilde\varphi_2)^2 - (1-\delta)(1-U^2)|\tilde\varphi|^2 + (2-\delta)U^2|\tilde\varphi_2|^2\\ &\quad\geq \frac{1-2\delta}{r^2}|\tilde\varphi|^2 + (2-2\delta)|\tilde\varphi_2|^2 - \frac{2}{r^2}|\tilde\varphi_1||\tilde\varphi_2| \geq \frac{1}{2r^2}|\tilde\varphi|^2.\end{aligned} \tag{3.15}$$

It then follows that for certain positive numbers C_1, C_2, we have

$$\mathbf{B}(\tilde\varphi,\tilde\varphi) \geq C_1\|\tilde\varphi\|_{H_*}^2 - C_2\int_0^R |\tilde\varphi|^2 r\,dr. \tag{3.16}$$

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Now in order to establish the lemma, we assume the opposite, namely, existence of a sequence $\tilde{\phi}_n$ with $\|\tilde{\phi}_n\|_{H_*} = 1$ such that $\int_0^\infty (1-U^2)\tilde{\phi}_n \cdot Z_0 r\,dr = 0$ and $\mathbf{B}(\tilde{\phi}_n, \tilde{\phi}_n) \to 0$. Let $\widehat{\phi}$ be a weak limit of $\tilde{\phi}_n$ in the sense of $\|\cdot\|_{H_*}$. We claim that $\widehat{\phi} \neq 0$. Indeed, $\tilde{\phi}_n \to \widehat{\phi}$ locally strongly in L^2-sense. Hence, if $\widehat{\phi} = 0$, we would have $\int_0^R |\tilde{\phi}_n|^2 r\,dr \to 0$ and estimate (3.16) would yield $\|\tilde{\phi}_n\|_{H_*} \to 0$, which is impossible. Strong L^2-convergence over compacts and weak lower semicontinuity of L^2-norms give $\mathbf{B}(\widehat{\phi}, \widehat{\phi}) = 0$. But then, we must have that $\widehat{\phi} = CZ_0$. Weak convergence in $\|\cdot\|_{H_*}$ norm finally gives $\int_0^\infty (1-U^2)\widehat{\phi} \cdot Z_0 r\,dr = 0$ so $C = 0$, a contradiction that proves the lemma. ■

We consider the problem of minimizing the functional J_1 in the closed subspace of H_*,

$$H_0 = \left\{ \tilde{\phi} \in H_* / \int_0^\infty (1-U^2)\tilde{\phi} \cdot Z_0 r\,dr = 0 \right\}. \tag{3.17}$$

We observe that by assumption on h,

$$\int_0^\infty |\tilde{h}|^2 (1 + r^{2+\sigma}) r\,dr \le \int_{\mathbb{R}^2} |h|^2 (1 + r^{2+\sigma}) \tag{3.18}$$

and additionally that $\int_0^\infty \tilde{h} \cdot Z_0 r\,dr = 0$. From Lemma 3.1, it easily follows that the functional J_1 is continuous, coercive, and strictly convex in H_0. Hence, there is a unique minimizer $\tilde{\phi}$ for this functional. Obviously, $\tilde{\phi}$ satisfies

$$\mathbf{B}(\tilde{\phi}, \eta) + \int_0^\infty \eta \cdot h r\,dr = 0, \tag{3.19}$$

for all $\eta \in H_0$. Now, any $\eta \in H_*$ can be decomposed as $\eta = \eta_1 + C_\eta Z_0$ with $\eta_1 \in H_0$, therefore, the above equation is actually satisfied for all $\eta \in H_*$. This by definition means that $\tilde{\phi}$ is a critical point of J_1 in the whole H_*. Moreover, by this construction, we easily see that

$$\int_0^\infty \left[|\tilde{\phi}'|^2 + (1-U^2)|\tilde{\phi}|^2 \right] r\,dr \le C \int_0^\infty |\tilde{h}|^2 (1 + r^2) r\,dr. \tag{3.20}$$

It is straightforward to check that the inherited solution ϕ_1^1 of (3.5), $\ell = 1$, $j = 1$, indeed satisfies

$$\|\phi_1^1\|_H^2 \le C \int_{\mathbb{R}^2} |h_1^1|^2 (1 + r^{2+\sigma}). \tag{3.21}$$

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In an exactly symmetric way, we find a solution ϕ_1^2 of $L(\phi_1^2) = h_1^2$ with analogous estimate. We will solve the remaining equations (3.5) for $j \geq 2$, all at once. We set

$$h^\perp = \sum_{j\geq 2} h_j^1 + h_j^2, \qquad \phi^\perp = \sum_{j\geq 2} \phi_j^1 + \phi_j^2. \tag{3.22}$$

We then consider the equation

$$L(\phi^\perp) = h^\perp \tag{3.23}$$

in the closed subspace $H^\perp$ of all functions $\phi \in H$ that can be written in the form $\phi^\perp$. A minimizer of the functional

$$J(\phi^\perp) = \frac{1}{2}B(\phi^\perp, \phi^\perp) - \langle h^\perp, \phi^\perp \rangle \tag{3.24}$$

in $H^\perp$ automatically gives a solution. Existence of such a minimizer is in this case a direct matter since we have from (2.42) the inequality

$$B(\phi^\perp, \phi^\perp) \geq c \int_{\mathbb{R}^2} \frac{1}{r^2} |\phi^\perp|^2, \tag{3.25}$$

with $c > 0$, from where it is straightforward to deduce

$$B(\phi^\perp, \phi^\perp) \geq c \|\phi^\perp\|_H^2, \tag{3.26}$$

with $c > 0$. Finally, we consider the mode-zero case, (3.4). Then, in terms of $\psi = -iw^{-1}\phi^0 = \psi_1(r) + i\psi_2(r)$, we get two uncoupled equations:

$$\psi_1'' + \left(\frac{1}{r} + \frac{2U'}{U}\right)\psi_1' = g_1, \tag{3.27}$$

$$\psi_2'' + \left(\frac{1}{r} + \frac{2U'}{U}\right)\psi_2' - 2U^2\psi_2 = g_2, \quad \text{for } 0 < r < \infty, \tag{3.28}$$

where $-iw^{-1}h^0 = g_1(r) + ig_2(r)$. By the assumption made on h, we have the properties

$$\int_0^\infty rU^2 g_1 = 0, \qquad \int_0^\infty rU^2 g_1^2 (1 + r^{2+\sigma})\, dr \leq C \int_{\mathbb{R}^2} |h|^2 (1 + r^{2+\sigma}). \tag{3.29}$$

The following explicit formula is then directly checked to represent a solution of (3.27):

$$\psi_1(r) = -\int_r^\infty \frac{ds}{sU^2(s)} \int_0^s g_1(t) t U^2(t)\, dt. \tag{3.30}$$

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Moreover, ψ_1 satisfies

$$\int_0^\infty \left[|\psi_1'(r)|^2 + (1-U^2)\psi_1^2 \right] rU^2 dr \le C \int_{\mathbb{R}^2} |h|^2 (1 + r^{2+\sigma}). \tag{3.31}$$

On the other hand, equation (3.28) has a solution which simply minimizes the functional

$$\frac{1}{2}\int_0^\infty \left[|\psi_1'(r)|^2 + 2U^2|\psi_2|^2 \right] rU^2 dr + \int_0^\infty g_2\psi_2 rU^2 dr \tag{3.32}$$

in its natural H^1-weighted space. With these definitions, we inherit for ϕ^0 the estimate

$$\|\phi^0\|_H^2 \le C \int_{\mathbb{R}^2} |h|^2 (1 + r^{2+\sigma}). \tag{3.33}$$

Adding up the above-constructed solutions, we have found a solution ϕ_0 of $L\phi = h$ with the required property. The fact that all solutions in H can be written as the sum of ϕ_0 and a linear combination of the partial derivatives of w follows immediately from Theorem 1.1. This concludes the proof.

Acknowledgments

The first and second authors have been partly supported by FONDAP Matemáticas Aplicadas and, respectively, by FONDECYT Grants 1030840 and 1030929.

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