Communications in Contemporary Mathematics
Vol. 9, No. 5 (2007) 639–680

PERTURBING SINGULAR SOLUTIONS OF THE GELFAND PROBLEM

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Received 8 February 2006
Revised 25 April 2006

The equation $-\Delta u = \lambda e^u$ posed in the unit ball $B \subseteq \mathbb{R}^N$, with homogeneous Dirichlet condition $u|_{\partial B} = 0$, has the singular solution $U = \log \frac{1}{|x|^2}$ when $\lambda = 2(N-2)$. If $N \geq 4$ we show that under small deformations of the ball there is a singular solution (u, λ) close to $(U, 2(N-2))$. In dimension $N \geq 11$ it corresponds to the extremal solution — the one associated to the largest λ for which existence holds. In contrast, we prove that if the deformation is sufficiently large then even when $N \geq 10$, the extremal solution remains bounded in many cases.

Keywords: Singular solutions; nonlinear elliptic equations; Hardy potential; Gelfand problem; stability; supercritical problems; weighed Hölder spaces; isolated singularity.

Mathematics Subject Classification 2000: 35J65, 35B05, 35P30, 35Q55

1. Introduction

We consider the Gelfand problem [17], namely the equation

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary and $\lambda \geq 0$ is a parameter.

Equation (1) and many variants have been widely considered in the literature, see for instance [6, 8, 12, 13, 18, 19], from which the following general properties are known:

Proposition 1.1. *There exists* $\lambda^* \in (0, \infty)$ *such that*

- (1) *has a smooth solution for* $0 \le \lambda < \lambda^*$,
- (1) *has a unique weak solution for* $\lambda = \lambda^*$,
- (1) *has no solution for* $\lambda > \lambda^*$ (*even in the weak sense*).

Above we have used the following definition: a function $u \in L^1(\Omega)$ is a weak solution to (1) if $e^u \mathrm{dist}(x, \partial\Omega) \in L^1(\Omega)$ and

$$-\int_\Omega u\Delta\zeta = \lambda \int_\Omega e^u \zeta, \quad \forall \zeta \in C^2(\bar{\Omega}), \quad \zeta = 0 \text{ on } \partial\Omega.$$

It is also known that for $0 \le \lambda < \lambda^*$, there exists a minimal solution u_λ which is smooth. u_λ depends smoothly on λ and is monotone increasing with respect to this parameter. Also, u_λ is stable in the sense that the linearized operator at u_λ is positive, i.e.

$$\inf_{\varphi \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 - \lambda \int_\Omega e^{u_\lambda}\varphi^2}{\int_\Omega \varphi^2} > 0. \tag{2}$$

The monotone limit $u^* := \lim_{\lambda \nearrow \lambda^*} u_\lambda$ is the weak solution for $\lambda = \lambda^*$ and satisfies

$$\lambda^* \int_\Omega e^{u^*}\varphi^2 \le \int_\Omega |\nabla\varphi|^2, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{3}$$

It is then natural to ask the following question: given a smooth bounded domain, is u^* a smooth solution?

Joseph and Lundgren [18] studied the case where Ω is a ball and completely determined the structure of the radial solutions of (1). In particular, they showed that if $\Omega = B_1$ then u^* is bounded if and only if $N < 10$, and in the case $N \ge 10$ then $u^* = \log \frac{1}{|x|^2}$ and $\lambda^* = 2(N-2)$. It was shown later in [13, 22] that if $N < 10$ then for *any* smooth and bounded domain Ω, the extremal solution u^* is bounded. Brezis and Vázquez [8] gave an interesting alternative proof of $u^* = \log \frac{1}{|x|^2}$ when $\Omega = B_1$ in the case $N \ge 10$, making use of Hardy's inequality, which we recall below: if $N \ge 3$ then

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \le \int_{\mathbb{R}^N} |\nabla\varphi|^2, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

Thus far, the ball in dimension $N \ge 10$ is the only domain where it is known that u^* is singular.

In this work we consider (1) in a domain that is sufficiently close to a ball in the following sense. Let $\psi : \bar{B}_1 \to \mathbb{R}^N$ be a C^2 map, $t > 0$ and define

$$\Omega_t = \{x + t\psi(x) : x \in B_1\}.$$

We choose henceforth t so small that Ω_t is a smooth bounded domain diffeomorphic to B_1 and we consider the Gelfand problem in Ω_t:

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega_t \\ u = 0 & \text{on } \partial\Omega_t. \end{cases} \tag{4}$$

Our main result is the following:

Theorem 1.2. *Let $N \geq 11$. Given $t > 0$ small, let $u^*(t)$ denote the extremal solution to* (4) (*defined by Proposition* 1.1).

Then there exists $t_0 = t_0(N, \psi) > 0$ such that if $t < t_0$, $u^*(t)$ is singular. In addition, there exists $\xi(t) \in B_1$ such that $\|u(x,t) - \log \frac{1}{|x-\xi(t)|^2}\|_{L^\infty(\Omega_t)} \to 0$ as $t \to 0$.

In fact, one can construct a singular solution of Problem (4) in any dimension $N \geq 4$:

Theorem 1.3. *Let $N \geq 4$. Then there exists $t_0 = t_0(N, \psi) > 0$ and a curve $t \mapsto (\lambda(t),\ u(t))$, defined for $t \in [0, t_0)$, such that $(\lambda(t),\ u(t))$ is a solution to* (4) *and $\lambda(0) = 2(N-2)$, $u(0) = \log \frac{1}{|x|^2}$. Moreover $u(t)$ is singular and there exists $\xi(t) \in B_1$ such that $\|u(x,t) - \log \frac{1}{|x-\xi(t)|^2}\|_{L^\infty(\Omega_t)} \to 0$ as $t \to 0$.*

The behavior of the singular solution at the origin is characterized as follows:

Corollary 1.4. *Fix $t < t_0$ and let $(\lambda(t), u(t), \xi(t))$ denote the solution of* (4) *given by Theorem* 1.3. *Then,*

$$u(x,t) = \ln \frac{1}{|x-\xi(t)|^2} + \ln\left(\frac{\lambda(0)}{\lambda(t)}\right) + \epsilon(|x-\xi(t)|), \tag{5}$$

where $\lim_{s\to 0} \epsilon(s) = 0$.

Remark 1.5. If $N \geq 5$, the curve $t \mapsto (\lambda(t), \xi(t), u(t))$ given by Theorem 1.3 is differentiable in the following sense: for any $x \notin \xi([0, t_0))$, the limit

$$\lim_{\tau\to 0} \frac{u(x, t+\tau) - u(x,t)}{\tau}$$

exists.

Theorem 1.2 is a consequence of this more general result and is obtained thanks to a lemma of Brezis and Vázquez [8] which asserts that a singular solution in H^1 which is stable must be the extremal solution.

Remark 1.6. The natural restriction on the dimension in Theorem 1.2 should perhaps be $N \geq 10$. We do not know whether Theorem 1.2 holds in dimension $N = 10$.

A similar result (which proof we omit) can be obtained for power-type nonlinearities: given $p > 1$, consider the problem

$$\begin{cases} -\Delta u = \lambda(1+u)^p & \text{in } \Omega_t \\ u = 0 & \text{on } \partial\Omega_t. \end{cases} \tag{6}$$

When $t = 0$, i.e. when the domain is the unit ball, it is known (see e.g. [8]) that the extremal solution is unbounded and given by $u^* = |x|^{-2/(p-1)} - 1$ if and only if $N \geq 11$ and

$$N \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}.$$

We have:

Theorem 1.7. *Let $N \geq 11$ and $p > 1$ such that*

$$N > 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}.$$

Given $t > 0$ small, let $u^*(t)$ denote the extremal solution to (6)*. Then there exists $t_0 = t_0(N, \psi, p) > 0$ such that if $t < t_0$, $u^*(t)$ is singular. In addition, there exists $\xi(t) \in B_1$ such that $\|u(x,t) - (|x - \xi(t)|^{-2/(p-1)} - 1)\|_{L^\infty(\Omega_t)} \to 0$ as $t \to 0$.*

Concerning Theorem 1.3, we point out the work of Rébaï [25], who produced singular solutions of (1) in the ball, having a prescribed singularity at a point $\xi \neq 0$ sufficiently close to the origin, whenever $N = 3$. According to the author, this result was also proved by Matano.

When the boundary condition is not prescribed (i.e. $u = 0$ may not hold on $\partial\Omega$), Pacard [23] proved that for $N > 10$, there exist a (dumbbell shaped) domain Ω and a positive solution u of $-\Delta u = e^u$ in Ω having prescribed singularities at finitely many points. Rebaï [26] extended this result to the case $N = 3$. Bidaut-Véron and Véron [9] studied the behavior of solutions to the Gelfand problem around an isolated singularity and at infinity in dimension 3.

When the exponential nonlinearity is replaced by $f(u) = u^\alpha$, Mazzeo and Pacard proved that for any exponent α lying in a certain range and for *any* bounded domain Ω, there exist solutions of $-\Delta u = u^\alpha$ in Ω with $u = 0$ on $\partial\Omega$, with a non-removable singularity on a finite union of smooth manifolds without boundary. Further results in this direction are provided in [27, 24] and their bibliography.

Returning to (1), one may be tempted to conjecture that if Ω is any smooth bounded domain and $N \geq 10$, u^* is singular. But if Ω is an annulus it is easily seen that with no restriction on N the extremal solution u^* is smooth. This lead Brezis and Vázquez [8] to stating the following question: is it true that if $N \geq 10$ and Ω is a convex smooth, bounded domain then u^* is singular?

Added in proof: after completing this work, we have been informed that this question had already been answered by E. N. Dancer (*see* [14] *pp.* 54–56).

As in [14], we provide a negative answer to the question of Brezis and Vázquez by considering some thin domains. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary. We assume furthermore that Ω is convex and $\partial\Omega$ is *uniformly convex*, i.e. its principal curvatures are bounded away from zero. Write $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $x = (x_1, x_2) \in \mathbb{R}^N$ with $x_1 \in \mathbb{R}^{N_1}$, $x_2 \in \mathbb{R}^{N_2}$. For $\varepsilon > 0$ set

$$\Omega_\varepsilon = \{x = (y_1, \varepsilon y_2) : (y_1, y_2) \in \Omega\} \tag{7}$$

and consider the Gelfand problem in Ω_ε:

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega_\varepsilon \\ u = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{8}$$

Theorem 1.8. *Let $N = N_1 + N_2 \geq 10$. Given $\varepsilon > 0$, let u_ε^* be the extremal solution to* (8).

If $N_2 \leq 9$ then there exists $\varepsilon_0 = \varepsilon_0(N, \Omega) > 0$ such that if $\varepsilon < \varepsilon_0$, u_ε^* is smooth.

The proof of Theorem 1.8 is given in Sec. 4.

Remark 1.9. Let $\Omega = B_1$ in dimension $N \geq 11$ and let Ω_ε be as in (7) with $N_2 = 1$. Combining Theorems 1.2 and 1.8 we can say that for ε close to 1, u^* is singular while for ε close to 0, u^* is regular.

The proof of Theorem 1.2 is based on the study of the following model equation

$$\begin{cases} -\Delta u = \lambda e^u + f(x,t) & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \tag{9}$$

where $B = B_1(0) \subset \mathbb{R}^N$ with $N \geq 11$. Here $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a smooth function such that $f(\cdot, 0) \equiv 0$.

For each $t \geq 0$, there exists an extremal parameter $\lambda^*(t)$ and an extremal solution $u^*(t)$.

Theorem 1.10. *Let $u^*(t)$ denote the extremal solution to* (9). *There exists $t_0 = t_0(N, f) > 0$ such that if $t < t_0$ and $N \geq 11$ then $u^*(t)$ is singular.*

Let us sketch the main idea of the proofs of Theorems 1.2 and 1.10. For simplicity we do this for (9) assuming that for all t the function $x \to f(x,t)$ is radially symmetric so that $u^*(t)$ may only be singular at the origin.

We know that $u^*(0)(x) = \log \frac{1}{|x|^2}$ and $\lambda^*(0) = 2(N-2)$. Assume that $u^*(t)$ and $\lambda^*(t)$ are differentiable functions of t, differentiate (9) with respect to t and evaluate at $t = 0$. Writing for convenience $v = \frac{du^*}{dt}(0)$, $\lambda' = \frac{d\lambda^*}{dt}(0)$ and $c^* = 2(N-2)$ we find

$$\begin{cases} -\Delta v - \dfrac{c^*}{|x|^2} v = \lambda' \dfrac{1}{|x|^2} + \dfrac{\partial f}{\partial t}(x, 0) & \text{in } B \\ v = 0 & \text{on } \partial B. \end{cases} \tag{10}$$

Since in dimension $N \geq 11$ we have $c^* < \frac{(N-2)^2}{4}$, by Hardy's inequality the operator $-\Delta - \frac{c^*}{|x|^2}$ is invertible in $H_0^1(\Omega)$. This suggests that the extremal solution of (9) can be constructed (for small t) by means of the implicit function theorem and we shall indeed use a similar scheme.

As observed by Brezis [5] one must be careful in this situation. To illustrate the difficulty, let $\Omega = B$ in dimension $N \geq 10$ and $F(\lambda, u) = -\Delta u - \lambda e^u$. Then, informally, $D_u F(\lambda^*, u^*) = -\Delta - \frac{c^*}{|x|^2}$. As mentioned earlier this operator is invertible from $H_0^1(B)$ to $H^{-1}(B)$ if $N \geq 11$ (and one may use another space if $N = 10$). Nonetheless, by Proposition 1.1, there are no solutions to $F(\lambda, u) = 0$ for $\lambda > \lambda^*$. As observed in [8], this phenomenon can be thought of as a lack of appropriate functional spaces to set up the implicit function theorem: good spaces for the linear operator seem to be $H_0^1(B)$ and $H^{-1}(\Omega)$ but $u \mapsto e^u$ is not well defined from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ (recall that $N \geq 10$). See [7] for similar situations in other nonlinear problems.

Going back to (10) we observe that besides the difficulty mentioned above, this equation apparently does not give any information on λ'. Thinking of λ' as a given parameter we will examine closer equation (10) in Sec. 2 and we will show that there exists a unique value of λ' for which the solution v is bounded. This is the good value of $\lambda' = \frac{d\lambda^*}{dt}(0)$. Then for small t we look for a solution to (9) of the form

$$u(x) = \log \frac{1}{|x|^2} + \phi.$$

Writing $\lambda = \lambda^*(0) + \mu$ Eq. (9) is equivalent to

$$\begin{cases} -\Delta\phi - \dfrac{c^*}{|x|^2}\phi = \dfrac{c^*}{|x|^2}(e^\phi - 1 - \phi) + \mu \dfrac{1}{|x|^2} e^\phi + f(x,t) & \text{in } B \\ \phi = 0 & \text{on } \partial B \end{cases}$$

where the unknowns are ϕ, μ. The objective is to find for t small a solution with $\|\phi\|_{L^\infty(B)}$ and $|\mu|$ small. This can be done using a fixed point theorem, where at each iteration we select the good value of μ, i.e. the one for which the solution is bounded. We explain this and prove Theorems 1.2 and 1.10 in Sec. 3.

2. A Linear Equation with the Inverse Square Potential

We study the linear equation

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x-\xi|^2}\phi = g & \text{in } B \\ \phi = h & \text{on } \partial B, \end{cases} \tag{11}$$

where $B = B_1(0)$, $\xi \in B$ and c is any real number. Later on, we shall state results in more general domains, which we are able to prove only for values of c in a restricted range.

As mentioned in the introduction, we would like to obtain bounded solutions of equations of the form (11). In general, this cannot be achieved without assumptions on the data. For example, if $c > 0$ and g, h are nonnegative functions, ϕ, if it exists (and is nontrivial), is *always* singular (this was first observed by Baras and

Goldstein [3]. See also [16]). We will establish a result saying that if the functions g and h satisfy orthogonality conditions with respect to appropriate functions then (11) is uniquely solvable in a suitable space.

Such conditions will not come as a surprise to the reader, taking into account that the operator $L = -\Delta - \frac{c}{|x|^2}$ is symmetric and that it has a nontrivial kernel, as the following paragraph shows:

2.1. *The kernel of* $L = -\Delta - \frac{c}{|x|^2}$

Recall the following properties of the Laplace–Beltrami operator $-\Delta$ on the sphere S^{N-1}. The eigenvalues of $-\Delta$ on S^{N-1} are given by

$$\lambda_k = k(N+k-2), \quad k \geq 0.$$

See [4]. Let m_k denote the multiplicity of λ_k and $\varphi_{k,l}$, $l = 1, \ldots, m_k$ the eigenfunctions associated to λ_k. We normalize these eigenfunctions so that $\{\varphi_{k,l} : k \geq 0,\, l = 1, \ldots, m_k\}$ is an orthonormal system in $L^2(S^{N-1})$. We choose the first functions to be

$$\varphi_{0,1} = \frac{1}{|S^{N-1}|^{1/2}}, \quad \varphi_{1,l} = \frac{x_l}{\left(\int_{S^{N-1}} x_l^2\right)^{1/2}} = \left(\frac{N}{|S^{N-1}|}\right)^{1/2} x_l, \quad l = 1, \ldots, N.$$

We seek solutions of

$$-\Delta w - \frac{c}{|x|^2} w = 0 \quad \text{in } \mathbb{R}^N \backslash \{0\} \tag{12}$$

of the form $w(x) = f(r)\varphi_{k,l}(\sigma)$, where $r = |x|$ and $\sigma = x/r$ for $x \in \mathbb{R}^N \backslash \{0\}$. This is equivalent to asking that f solves the following ordinary differential equation:

$$f'' + \frac{N-1}{r} f' + \frac{c - \lambda_k}{r^2} f = 0, \quad \text{for } r > 0. \tag{13}$$

Equation (13) is of Euler type and it admits a basis of solutions of the form $f(r) = r^{-\alpha_k^{\pm}}$, where $\alpha_k^{\pm}$ are the roots of the associated characteristic equation, i.e.

$$\alpha_k^{\pm} = \frac{N-2}{2} \pm \sqrt{\left(\frac{N-2}{2}\right)^2 - c + \lambda_k}.$$

Note that $\alpha_k^{\pm}$ may have a nonzero imaginary part only for finitely many k's. If k_0 is the first integer k such that $\alpha_k^{\pm} \in \mathbb{R}$ then

$$\cdots < \alpha_{k_0+1}^- < \alpha_{k_0}^- \leq \frac{N-2}{2} \leq \alpha_{k_0}^+ < \alpha_{k_0+1}^+ < \cdots,$$

whereas, if $k < k_0$, we denote the imaginary part of α_k^+ by

$$b_k = \sqrt{c - \left(\frac{N-2}{2}\right)^2 - \lambda_k}.$$

For $k \geq 0$, $l = 1, \ldots, m_k$, we have just found a family of real-valued solutions of (12), denoted by $w^1 = w^1_{k,l}$, $w_2 = w^2_{k,l}$ and defined on $\mathbb{R}^N \setminus \{0\}$ by

$$\begin{cases} \text{if } \left(\frac{N-2}{2}\right)^2 - c + \lambda_k > 0: & w^1(x) = |x|^{-\alpha_k^+} \varphi_{k,l}\left(\frac{x}{|x|}\right), \\ & w^2(x) = |x|^{-\alpha_k^-} \varphi_{k,l}\left(\frac{x}{|x|}\right), \\ \text{if } \left(\frac{N-2}{2}\right)^2 - c + \lambda_k = 0: & w^1(x) = |x|^{-\frac{N-2}{2}} \log|x|\, \varphi_{k,l}\left(\frac{x}{|x|}\right), \\ & w^2(x) = |x|^{-\frac{N-2}{2}} \varphi_{k,l}\left(\frac{x}{|x|}\right), \\ \text{if } \left(\frac{N-2}{2}\right)^2 - c + \lambda_k < 0: & w^1(x) = |x|^{-\frac{N-2}{2}} \sin(b_k \log|x|) \varphi_{k,l}\left(\frac{x}{|x|}\right), \\ & w^2(x) = |x|^{-\frac{N-2}{2}} \cos(b_k \log|x|) \varphi_{k,l}\left(\frac{x}{|x|}\right). \end{cases} \tag{14}$$

Each of the functions $W_{k,l}$ defined by

$$\begin{cases} \text{if } \left(\frac{N-2}{2}\right)^2 - c + \lambda_k > 0: & W_{k,l}(x) = w^1(x) - w^2(x), \\ \text{if } \left(\frac{N-2}{2}\right)^2 - c + \lambda_k \leq 0: & W_{k,l}(x) = w^1(x), \end{cases} \tag{15}$$

then solves (12) and

$$W_{k,l}|_{\partial B} = 0.$$

2.2. *Functional setting*

Our results are stated for functions behaving like a power of $|x - \xi|$. More precisely, we shall work in the following functional setting (see [25, 2, 11]).

Given Ω a smooth domain, $\xi \in \Omega$, $k \geq 0$, $0 < \alpha < 1$, $0 < r \leq \operatorname{dist}(x, \partial\Omega)/2$ and $u \in C^{k,\alpha}_{loc}(\bar{B} \setminus \{\xi\})$ we define:

$$|u|_{k,\alpha,r,\xi} = \sup_{r \leq |x-\xi| \leq 2r} \sum_{j=0}^{k} r^j |\nabla^j u(x)| + r^{k+\alpha} \left[\sup_{r \leq |x-\xi|, |y-\xi| \leq 2r} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha} \right].$$

Let $d = \operatorname{dist}(\xi, \partial\Omega)$ and for any $\nu \in \mathbb{R}$ let

$$\|u\|_{k,\alpha,\nu,\xi;\Omega} = \|u\|_{C^{k,\alpha}(\bar{\Omega} \setminus B_{d/2}(\xi))} + \sup_{0 < r \leq \frac{d}{2}} r^{-\nu} |u|_{k,\alpha,r,\xi}.$$

Define the space

$$C^{k,\alpha}_{\nu,\xi}(\Omega) = \{u \in C^{k,\alpha}_{loc}(\bar{\Omega} \setminus \{\xi\}) : \|u\|_{k,\alpha,\nu,\xi;\Omega} < \infty\}.$$

One can easily check that $C^{k,\alpha}_{\nu,\xi}(\Omega)$ is a Banach space. It embeds continuously in the space of bounded functions whenever $\nu \geq 0$.

From now on, given $h \in C(\partial B)$ and $g \in C(B\backslash\{\xi\})$, we shall say that a function $\phi \in C^{k,\alpha}_{\nu,\xi}(B)$ $(k \geq 2)$ solves (11) whenever the boundary condition $\phi|_{\partial B} = h$ holds and $-\Delta\phi(x) - \frac{c}{|x-\xi|^2}\phi(x) = g(x)$ for all $x \in B\backslash\{\xi\}$.

2.3. *The case* $\boldsymbol{\xi = 0}$

In the case $\Omega = B$ and $\xi = 0$, we have the following

Lemma 2.1. *Let $c, \nu \in \mathbb{R}$ and assume*

$$\exists k_1 \text{ such that } \alpha^-_{k_1} \in \mathbb{R} \quad \text{and} \quad -\alpha^-_{k_1} < \nu < -\alpha^-_{k_1+1}. \tag{16}$$

Let $g \in C^{0,\alpha}_{\nu-2,0}(B)$ and $h \in C^{2,\alpha}(\partial B)$ and consider

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x|^2}\phi = g & \text{in } B \\ \phi = h & \text{on } \partial B. \end{cases} \tag{17}$$

Then (17) *has a solution in $C^{2,\alpha}_{\nu,0}(B)$ if and only if*

$$\int_B gW_{k,l} = \int_{\partial B} h\frac{\partial W_{k,l}}{\partial n}, \quad \forall k = 0,\ldots,k_1, \quad \forall l = 1,\ldots,m_k. \tag{18}$$

Under this condition the solution $\phi \in C^{2,\alpha}_{\nu,0}(B)$ to (17) *is unique and it satisfies*

$$\|\phi\|_{2,\alpha,\nu,0;B} \leq C(\,\|g\|_{0,\alpha,\nu-2,0;B} + \|h\|_{C^{2,\alpha}(\partial B)}) \tag{19}$$

where C is independent of g and h.

Remark 2.2. Under the hypotheses of Lemma 2.1 we have

$$\nu > -\alpha^-_{k_1} \geq -\frac{N-2}{2}, \tag{20}$$

where the last inequality follows from the discussion in Sec. 2.1. This implies that the integrals in the left-hand side of (18) are finite.

Remark 2.3. By taking k_1 sufficiently large, one can choose $\nu \geq 0$ in the above lemma. In particular, the corresponding solution ϕ is bounded.

Corollary 2.4. *Assume* (16)–(18) *hold. Assume in addition that $\nu \geq 0$.*

If $|x|^2 g$ is continuous at the origin, then so is ϕ.

Proof of Lemma 2.1. Write ϕ as

$$\phi(x) = \sum_{k=0}^{\infty}\sum_{l=1}^{m_k} \phi_{k,l}(r)\varphi_{k,l}(\sigma), \quad x = r\sigma, \quad 0 < r < 1, \quad \sigma \in S^{N-1}$$

Then ϕ solves $-\Delta\phi - \frac{c}{|x|^2}\phi = g$ in $B\setminus\{0\}$ if and only if $\phi_{k,l}$ satisfies the ODE

$$\phi''_{k,l} + \frac{N-1}{r}\phi'_{k,l} + \frac{c-\lambda_k}{r^2}\phi_{k,l} = -g_{k,l} \quad 0 < r < 1, \tag{21}$$

for all $k \geq 0$ and $l = 1, \ldots, m_k$, where

$$g_{k,l}(r) = \int_{S^{N-1}} g(r\sigma)\varphi_{k,l}(\sigma)\,d\sigma, \quad 0 < r < 1, \quad \sigma \in S^{N-1}.$$

Note that if $\phi \in C^{2,\alpha}_{\nu,0}(B)$ then there exists a constant $C > 0$ independent of r such that

$$|\phi_{k,l}(r)| \leq Cr^{\nu}. \tag{22}$$

Furthermore, $\phi = h$ on ∂B if and only if $\phi_{k,l}(1) = h_{k,l}$ for all k, l, where

$$h_{k,l} = \int_{S^{N-1}} h(\sigma)\varphi_{k,l}(\sigma)\,d\sigma.$$

Step 1. Clearly, $\sup_{0\leq t\leq 1} t^{2-\nu}|g_{k,l}(t)| < \infty$ and observe that (18) still holds when g is replaced by $g_{k,l}\varphi_{k,l}$ and h by $h_{k,l}\varphi_{k,l}$. We claim that there is a unique $\phi_{k,l}$ that satisfies (21), (22) and

$$\phi_{k,l}(1) = h_{k,l}. \tag{23}$$

We also have

$$|\phi_{k,l}(r)| \leq C_k r^{\nu}\left(\sup_{0\leq t\leq 1} t^{2-\nu}|g_{k,l}(t)| + |h_{k,l}|\right), \quad 0 < r < 1. \tag{24}$$

Case $\boldsymbol{k = 0, \ldots, k_1}$. A solution to (21) is given by:

- if $\alpha^{\pm}_{k,l} \notin \mathbb{R}$

$$\phi_{k,l}(r) = \frac{1}{b}\int_0^r s\left(\frac{s}{r}\right)^{\frac{N-2}{2}} \sin\left(b_k \log\frac{s}{r}\right) g_{k,l}(s)\,ds, \tag{25}$$

- if $\alpha^{+}_{k,l} = \alpha^{-}_{k,l} = \frac{N-2}{2}$:

$$\phi_{k,l}(r) = \int_0^r s\left(\frac{s}{r}\right)^{\frac{N-2}{2}} \log\left(\frac{s}{r}\right) g_{k,l}(s)\,ds, \tag{26}$$

- if $\alpha^{\pm}_{k,l} \in \mathbb{R}$, $\alpha^{\pm}_{k,l} \neq \frac{N-2}{2}$:

$$\phi_{k,l}(r) = \frac{1}{\alpha^+_k - \alpha^-_k}\int_0^r s\left(\left(\frac{s}{r}\right)^{\alpha^+_k} - \left(\frac{s}{r}\right)^{\alpha^-_k}\right) g_{k,l}(s)\,ds. \tag{27}$$

In each case, (24) holds and (23) follows from (18).

Concerning uniqueness, suppose that $\phi_{k,l}$ satisfies (21) with $g_{k,l} = 0$ and (23) with $h_{k,l} = 0$. Then $\phi_{k,l}$ is a linear combination of the functions w^1, w^2 defined in (14). By (16), (20) and (24), $\phi_{k,l}$ has to be zero.

Case $k \geq k_1 + 1$. Observe that (21) is equivalent to

$$-\Delta\tilde{\phi}_{k,l} + \frac{\lambda_k - c}{|x|^2}\tilde{\phi}_{k,l} = \tilde{g}_{k,l} \quad \text{in } B\backslash\{0\},$$

where $\tilde{\phi}_{k,l}(x) = \phi_{k,l}(|x|)$ and $\tilde{g}_{k,l}(x) = g_{k,l}(|x|)$. Since $\alpha_k^{\pm} \in \mathbb{R}$ we must have $\lambda_k - c \geq -(\frac{N-2}{2})^2$ and hence the equation

$$\begin{cases} -\Delta\tilde{\phi}_{k,l} + \dfrac{\lambda_k - c}{|x|^2}\tilde{\phi}_{k,l} = \tilde{g}_{k,l} & \text{in } B \\ \tilde{\phi}_{k,l} = h_{k,l} & \text{on } \partial B, \end{cases} \tag{28}$$

has a unique solution $\tilde{\phi}_{k,l} \in H$, where H is the completion of $C_0^\infty(B)$ with the norm

$$\|\varphi\|_H^2 = \int_B |\nabla\varphi|^2 + \frac{\lambda_k - c}{|x|^2}\varphi^2,$$

see [28].

To show (24), observe that for some constant C depending only on N, λ_k and ν,

$$A_{k,l}(r) = r^\nu C\left(\sup_{0<t\leq 1} t^{2-\nu}|g_{k,l}(t)| + |h_{k,l}|\right)$$

is a supersolution to (28) and $-A_{k,l}$ is a subsolution. To see this, we emphasize that the condition $-\alpha_k^- > \nu > -(N-2)/2$ implies $\nu^2 + (N-2)\nu + c - \lambda_k < 0$. It follows that $|\tilde{\phi}_{k,l}(x)| \leq A_{k,l}(|x|)$ for $0 < |x| \leq 1$.

To show that $\tilde{\phi}_{k,l}$ is uniquely determined, we simply observe that any solution w of (28) such that $|w(x)| \leq C|x|^\nu$ must belong to H (where uniqueness holds). Indeed, by scaling, it can be checked that $|\nabla w(x)| \leq C|x|^{\nu-1}$ (see Claim 1 below) and this together with (20) implies $w \in H^1(B)$, which is contained in H.

The computations above also yield the necessity of condition (18). Indeed, assuming a solution $\phi \in C^{2,\alpha}_{\nu,0}(B)$ exists, since $\phi_{k,l}$ satisfies the ODE (21) we see that for $k = 0, \ldots, k_1$ the difference between $\phi_{k,l}$ and one of the particular solutions (25), (26) or (27) can be written in the form $c_{k,l}r^{-\alpha_k^+} + d_{k,l}r^{-\alpha_k^-}$. Since $|\phi_{k,l}(r)| \leq Cr^\nu$ and $\nu > -\alpha_{k_1}^-$ we have $c_{k,l} = d_{k,l} = 0$ and this implies (18).

Step 2. Define for $m \geq 1$

$$\mathcal{G}_m = \left\{ g = \sum_{k=0}^m \sum_l g_{k,l}(r)\varphi_{k,l}(\sigma) : |x|^{2-\nu}g(x) \in L^\infty(B) \right\}$$

and

$$\mathcal{H}_m = \left\{ h = \sum_{k=0}^m \sum_l h_{k,l}\varphi_{k,l}(\sigma) : h_{k,l} \in \mathbb{R} \right\}.$$

Let $g_m \in \mathcal{G}_m$, $h_m \in \mathcal{H}_m$ be such that (18) holds. Write $g_m(x) = \sum_{k=0}^m \sum_l g_{k,l}(r)\varphi_{k,l}(\sigma)$ and $h_m(\sigma) = \sum_{k=0}^m h_{k,l}\varphi_{k,l}(\sigma)$. Let $\phi_{k,l}$ be the unique solution to (21)–(23) associated to $g_{k,l}$, $h_{k,l}$ and define $\phi_m(x) = \sum_{k=0}^m \sum_l \phi_{k,l}(r)\varphi_{k,l}(\sigma)$.

We claim that there exists C independent of m such that

$$|\phi_m(x)| \le C|x|^\nu \left(\sup_B |y|^{2-\nu}|g_m(y)| + \sup_{\partial B} |h_m| \right), \quad 0 < |x| < 1. \tag{29}$$

By the previous step, (29) holds for some constant C which may depend on m. In particular, choosing $m = k_1$, we obtain a bound on the first components $\phi_{k,l}$, $k = 0, \dots, k_1$. Hence, it suffices to prove (29) in the case $g_{k,l} \equiv 0$ and $h_{k,l} = 0$, $k = 0, \dots, k_1$. Working as in [25] (the argument already appeared in unpublished notes of Pacard), we argue by contradiction assuming that

$$\|\phi_m\,|x|^{-\nu}\|_{L^\infty(B)} \ge C_m(\|g_m\,|x|^{2-\nu}\|_{L^\infty(B)} + \|h_m\|_{L^\infty(\partial B)}),$$

where $C_m \to \infty$ as $m \to \infty$. Replacing ϕ_m by $\phi_m/\|\phi_m|x|^{-\nu}\|_{L^\infty(B)}$ if necessary, we may assume

$$\begin{gathered} \|\phi_m|x|^{-\nu}\|_{L^\infty(B)} = 1, \\ \|g_m|x|^{2-\nu}\|_{L^\infty(B)} + \|h_m\|_{L^\infty(\partial B)} \to 0 \quad \text{as } m \to \infty. \end{gathered} \tag{30}$$

Let $x_m \in B\backslash\{0\}$ be such that $|\phi_m(x_m)||x_m|^{-\nu} \in [\frac{1}{2}, 1]$. Let us show that $x_m \to 0$ as $m \to \infty$. Otherwise, up to a subsequence $x_m \to x_0 \neq 0$. By standard elliptic regularity, up to another subsequence, $\phi_m \to \phi$ uniformly on compact sets of $\bar{B}\backslash\{0\}$ and hence

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x|^2}\phi = 0 & \text{in } B\backslash\{0\} \\ \phi = 0 & \text{on } \partial B. \end{cases}$$

Moreover ϕ satisfies $|\phi(x_0)||x_0|^{-\nu} \in [\frac{1}{2}, 1]$ and $|\phi(x)| \le |x|^\nu$ in B. Writing

$$\phi(x) = \sum_{k \ge k_1+1} \sum_l \phi_{k,l}(r)\varphi_{k,l}(\sigma),$$

we see that $\phi_{k,l}$ solves (13). The growth restriction $|\phi_{k,l}(r)| \le Cr^\nu$ and the explicit functions w^1, w^2 given by (14) rule out the cases $\alpha_k^\pm \notin \mathbb{R}$, $\alpha_k^- = \alpha_k^+$ and force $\phi_{k,l} = a_{k,l}r^{-\alpha_k^-}$. But $\phi_{k,l}(1) = 0$ so we deduce $\phi_{k,l} \equiv 0$ and hence $\phi \equiv 0$, contradicting $|\phi(x_0)||x_0|^{-\nu} \neq 0$.

The above argument shows that $x_m \to 0$. Define $r_m = |x_m|$ and

$$v_m(x) = r_m^{-\nu}\phi_m(r_m x), \quad x \in B_{1/r_m}.$$

Then $|v_m(x)| \le |x|^\nu$ in B_{1/r_m}, $|v_m(\frac{x_m}{r_m})| \in [\frac{1}{2}, 1]$ and

$$-\Delta v_m(x) - \frac{c}{|x|^2}v_m(x) = r_m^{2-\nu}g(r_m x) \quad \text{in } B_{1/r_m}\backslash\{0\}.$$

But

$$r_m^{2-\nu}\,|g(r_m x)| \le \|g_m(y)|y|^{2-\nu}\|_{L^\infty(B)}|x|^{\nu-2} \to 0, \quad \text{as } m \to \infty$$

by (30). Passing to a subsequence, we have that $\frac{x_m}{r_m} \to x_0$ with $|x_0| = 1$, $v_m \to v$ uniformly on compact sets of $\mathbb{R}^N \setminus \{0\}$ and v satisfies

$$-\Delta v - \frac{c}{|x|^2} v = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Furthermore, $|v(x)| \le |x|^\nu$ in $\mathbb{R}^N \setminus \{0\}$ and $|v(x_0)| \ne 0$. Write

$$v(x) = \sum_{k=0}^{\infty} \sum_{l} v_{k,l}(r) \varphi_{k,l}(\sigma).$$

Then $|v_{k,l}(r)| \le C_k r^\nu$ for $r > 0$. But $v_{k,l}$ has to be a linear combination of the functions w^1, w^2 given in (14), and none of these is bounded by Cr^ν for all $r > 0$. Thus $v \equiv 0$ yielding a contradiction and (29) is proved.

Step 3. Fix an integer $d \ge 3(N-2)/2 + 1$. Suppose now that $g \in C^\infty(\bar{B} \setminus \{0\})$ and $|\nabla^i g(x)| \le C|x|^{\nu-2-i}$ for $0 < |x| < 1$ and for $i = 0, \ldots, d$. Let $h \in C^\infty(\partial B)$ such that (18) holds. We will show that there exists $\phi \in C^{2,\alpha}_{\nu,0}(B)$ solution to (17), satisfying the estimate

$$\|\phi |x|^{-\nu}\|_{L^\infty(B)} \le C(\|g|x|^{2-\nu}\|_{L^\infty(B)} + \|h\|_{L^\infty(\partial B)}). \tag{31}$$

To prove this, define for $m \in \mathbb{N}$

$$g_m(x) = \sum_{k=0}^{m} \sum_{l} g_{k,l}(r) \varphi_{k,l}(\sigma) \quad \text{and} \quad h_m(\sigma) = \sum_{k=0}^{m} \sum_{l} h_{k,l} \varphi_{k,l}(\sigma).$$

We have

$$\begin{aligned}
\sum_l |g_{k,l}(r)| &= \sum_l \left| \int_{S^{N-1}} g(r\sigma) \varphi_{k,l}(\sigma)\, d\sigma \right| = \sum_l \frac{1}{\lambda_k} \left| \int_{S^{N-1}} g(r\sigma) \Delta \varphi_{k,l}(\sigma)\, d\sigma \right| \\
&\le \frac{C m_k r^{2d}}{\lambda_k^d} \sup_{|x|=r} |\nabla^{2d} g(x)|\, \|\varphi_{k,l}\|_{L^\infty(S^{N-1})} \\
&\le C r^{\nu-2} k^{-2d+2(N-2)},
\end{aligned}$$

where we used integration by parts d times to obtain the inequality and the facts: $\lambda_k \sim k^2$ as $k \to \infty$, $|\varphi_{k,l}| \le C k^{N-2}$ in S^{N-1} and $m_k \le C k^{N-2}$, where m_k is the multiplicity of λ_k, see [1]. It follows that $g_m(x)|x|^{2-\nu}$ converges uniformly in B to $g(x)|x|^{2-\nu}$ and hence $\|g_m |x|^{2-\nu}\|_{L^\infty(B)} \to \|g|x|^{2-\nu}\|_{L^\infty(B)}$ as $m \to \infty$. Similarly h_m converges uniformly to h on ∂B and thus $\lim_{m\to\infty} \|h_m\|_{L^\infty(\partial B)} = \|h\|_{L^\infty(\partial B)}$.

Now $g_m \in \mathcal{G}_m$ and $h_m \in \mathcal{H}_m$ verify the orthogonality conditions (18). By the previous step, the associated solution ϕ_m satisfies

$$\|\phi_m |x|^{-\nu}\|_{L^\infty(B)} \le C(\|g_m |x|^{2-\nu}\|_{L^\infty(B)} + \|h_m\|_{L^\infty(\partial B)}).$$

Using elliptic regularity, up to a subsequence, $\phi_m \to \phi$ uniformly in $B \setminus \{0\}$, for some ϕ satisfying the equations $-\Delta \phi - \frac{c}{|x|^2} \phi = g$ in $B \setminus \{0\}$, $\phi = h$ on ∂B and the estimate (31).

Claim 1. ϕ is a solution to the equation in the whole ball B.

To see this, it suffices to prove that

$$|\nabla\phi(x)| \le C|x|^{\nu-1} \quad \text{for } x \in B_{1/2}. \tag{32}$$

Recall that $\nu - 1 > -\frac{N}{2}$. This implies that $\phi \in H^1(B)$ and thus solves the equation in B (since $\mathrm{cap}(\{0\}) = 0$ whenever $N \ge 3$).

Let $x_0 \in B_{1/2}$, $d = |x_0|$ and for $x \in B_{3/4}$, $v(x) = \phi(x_0 + dx)$. Then,

$$-\Delta v - \frac{cd^2}{|x_0 + dx|^2} v = d^2 g(x_0 + dx) \quad \text{in } B_{3/4}.$$

Observing that $0 \le \frac{c\,d^2}{|x_0+dx|^2} \le 16c$ for $x \in B_{3/4}$, it follows by elliptic regularity that for some constants C independent of d,

$$\begin{aligned} |\nabla v(0)| &\le C(\|d^2 g(x_0 + dx)\|_{L^\infty(B_{3/4})} + \|v\|_{L^\infty(B_{3/4})}) \\ &\le C d^\nu(\|g|x|^{2-\nu}\|_{L^\infty(B)} + \|\phi|x|^{-\nu}\|_{L^\infty(B)}) \\ &\le C|x_0|^\nu(\|g|x|^{2-\nu}\|_{L^\infty(B)} + \|h\|_{L^\infty(\partial B)}), \end{aligned}$$

where we used (31) in the last inequality. Hence, $|\nabla\phi(x_0)| \le C'|x_0|^{\nu-1}$, which is the desired result.

Step 4. We assume now that $g \in C^{0,\alpha}_{\nu-2,0}(B)$ and $h \in C^{2,\alpha}(\partial B)$ satisfy (18). For $\varepsilon > 0$ let h_ε be the convolution product of h with a standard mollifier on the sphere ∂B. Let ρ_ε be a standard mollifier in $\mathbb{R}^N$ and define $g_\varepsilon(x) = |x|^{\nu-2}\rho_\varepsilon(x) * (g|x|^{2-\nu})$, where g is first extended by zero outside B. Since $g(x)|x|^{2-\nu} \in L^\infty(B)$, we have $g_\varepsilon \in C^\infty(\bar{B}\setminus\{0\})$ and

$$|\nabla^i g_\varepsilon(x)| \le C(i,\varepsilon)|x|^{\nu-2-i}.$$

Moreover, $g_\varepsilon \to g$ a.e. in B, $h_\varepsilon \to h$ a.e. on ∂B as $\varepsilon \to 0$ and

$$\|g_\varepsilon|x|^{2-\nu}\|_{L^\infty(B)} \le \|g|x|^{2-\nu}\|_{L^\infty(B)} \quad \text{and} \quad \|h_\varepsilon\|_{L^\infty(\partial B)} \le \|h\|_{L^\infty(\partial B)}.$$

From this and (18), we deduce that for all $k = 0, \dots, k_1$ and $l = 1, \dots, m_k$,

$$\int_B g_\varepsilon W_{k,l} - \int_{\partial B} h_\varepsilon \frac{\partial W_{k,l}}{\partial n} \to 0 \quad \text{as } \varepsilon \to 0.$$

Let

$$a^{(\varepsilon)}_{k,l} = \frac{1}{\displaystyle\int_{\partial B} W_{k,l}\frac{\partial W_{k,l}}{\partial n}} \left(\int_B g_\varepsilon W_{k,l} - \int_{\partial B} h_\varepsilon \frac{\partial W_{k,l}}{\partial n}\right)$$

and

$$\tilde{h}_\varepsilon = h_\varepsilon + \sum_{k=0}^{k_1}\sum_{l=1}^{m_k} a^{(\varepsilon)}_{k,l} W_{k,l}.$$

Then g_ε, $\tilde{h}_\varepsilon$ satisfy the orthogonality conditions (18). Let $\phi_\varepsilon \in C^{2,\alpha}_{\nu,0}(B)$ denote the solution to (17) with data g_ε, $\tilde{h}_\varepsilon$. We have

$$\begin{aligned}\|\phi_\varepsilon|x|^{-\nu}\|_{L^\infty(B)} &\leq C(\|g_\varepsilon|x|^{2-\nu}\|_{L^\infty(B)} + \|\tilde{h}_\varepsilon\|_{L^\infty(\partial B)})\\ &\leq C(\|g|x|^{2-\nu}\|_{L^\infty(B)} + \|h\|_{L^\infty(\partial B)}).\end{aligned}$$

As in the previous step, from here we deduce that $\phi = \lim_{\varepsilon\to 0}\phi_\varepsilon$ is a solution to (17) with data g, h. In addition, (31) holds.

Finally, the estimate (19) is obtained by scaling, working as in Claim 1. $\square$

Proof of Corollary 2.4. Let (α_n) denote an arbitrary sequence of real numbers converging to zero, $\tilde{g}(x) = |x|^2 g(x)$ and $\phi_n(x) = \phi(\alpha_n x)$ for $x \in B_{1/\alpha_n}(0)$. Then ϕ_n solves

$$-\Delta\phi_n - \frac{c}{|x|^2}\phi_n = \frac{\tilde{g}(\alpha_n x)}{|x|^2} \quad \text{in } B_{1/\alpha_n}(0).$$

Also, (ϕ_n) is uniformly bounded so that up to a subsequence, it converges in the topology of $C^{1,\alpha}(\mathbb{R}^N\setminus\{0\})$ to a bounded solution Φ of

$$-\Delta\Phi - \frac{c}{|x|^2}\Phi = \frac{\tilde{g}(0)}{|x|^2} \quad \text{in } \mathbb{R}^N\setminus\{0\}.$$

Now $\Phi + \tilde{g}(0)/c$ is bounded and solves (12), so it must be identically zero. It follows that the whole sequence (ϕ_n) converges to $-\tilde{g}(0)/c$. Let now (x_n) denote an arbitrary sequence of points in $\mathbb{R}^N$ converging to 0 and $\alpha_n = |x_n|$. Then, $\phi(x_n) = \phi_n(\frac{x_n}{|x_n|})$ and up to a subsequence, $\phi(x_n) \to -\tilde{g}(0)/c$. Again, since the limit of such a subsequence is unique, the whole sequence converges. $\square$

2.4. *The case $\xi \neq 0$*

As we observed earlier, one cannot expect to obtain bounded solutions of (11) for general data g and h. But Lemma 2.1 suggests that one can *modify* the data so that the necessary orthogonality conditions hold. This is what we prove below, in the more general case where ξ may be chosen different from the origin.

Let indeed $\epsilon_0 > 0$ and $\eta \in C^\infty(\mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta \not\equiv 0$ and $\operatorname{supp}(\eta) \subset [\frac{1}{4}, \frac{1}{2}]$. For $\xi \in B_{1/2}$ we construct functions $V_{k,l,\xi}$ $(k \geq 1,\ l = 1, \ldots, m_k)$ as

$$V_{k,l,\xi}(x) = \eta(|x-\xi|)W_{k,l}\left(\frac{x-\xi}{1-2\epsilon_0}\right). \tag{33}$$

We prove:

Proposition 2.5. *Assume*

$$\exists k_1 \textit{ such that } \alpha^-_{k_1} \in \mathbb{R} \quad \textit{and} \quad -\alpha^-_{k_1} < \nu < -\alpha^-_{k_1+1}. \tag{34}$$

Then there exists $\epsilon_0 > 0$ such that if $|\xi| < \epsilon_0$ and $g_0 \in C^{0,\alpha}_{\nu,\xi}(B)$ satisfies

$$\|g_0 - 1\|_{L^\infty(B)} < \epsilon_0,$$

then given any $g \in C^{0,\alpha}_{\nu,\xi}(B)$ *and* $h \in C^{2,\alpha}(\partial B)$, *there exist unique* $\phi \in C^{2,\alpha}_{\nu,\xi}(B)$ *and* $\mu_0, \mu_{k,l} \in \mathbb{R}$ $(k = 1, \ldots, k_1,\ l = 1, \ldots, m_k)$ *solution to*

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x-\xi|^2}\phi = \dfrac{g}{|x-\xi|^2} + \mu_0 \dfrac{g_0}{|x-\xi|^2} + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \mu_{k,l} V_{k,l,\xi} & \text{in } B \\ \phi = h & \text{on } \partial B. \end{cases} \tag{35}$$

Moreover we have for some constant $C > 0$ *independent of* g *and* h

$$\|\phi\|_{2,\alpha,\nu,\xi;B} + |\mu_0| + \sum_{k=1}^{k_1}\sum_{l=1}^{m_k} |\mu_{k,l}| \le C(\|g\|_{0,\alpha,\nu,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)}). \tag{36}$$

Proof. We work with $0 < |\xi| < \epsilon_0$ where $\epsilon_0 \in (0, 1/2)$ is going to be fixed later on, small enough. Let $R = 1 - 2\epsilon_0$. This implies in particular that $B_R(\xi) \subset B$.

We define an operator $T_1 : C^{2,\alpha}(\partial B_R(\xi)) \to C^{1,\alpha}(\partial B_R(\xi)) \times \mathbb{R}$ as follows: given $\phi_0 \in C^{2,\alpha}(\partial B_R(\xi))$, find $\phi \in C^{2,\alpha}_{\nu,\xi}(B_R(\xi))$ and γ_0, $\gamma_{k,l}$ the unique solution to

$$\begin{cases} -\Delta\phi_1 - \dfrac{c}{|x-\xi|^2}\phi_1 = \gamma_0 \dfrac{g_0}{|x-\xi|^2} + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \gamma_{k,l} V_{k,l,\xi} & \text{in } B_R(\xi) \\ \phi_1 = \phi_0 & \text{on } \partial B_R(\xi). \end{cases} \tag{37}$$

and set $T_1(\phi_0) = (\frac{\partial\phi_1}{\partial n}, \gamma_0)$. This can be done (see Step 1 below) by adjusting the constants γ_0 and $\gamma_{k,l}$ in such a way that the orthogonality relations (18) in Lemma 2.1 are satisfied. Similarly, there is a unique $\tilde\phi_1 \in C^{2,\alpha}_{\nu,\xi}(B_R(\xi))$ and $\tilde\gamma_0$, $\tilde\gamma_{k,l}$ such that

$$\begin{cases} -\Delta\tilde\phi_1 - \dfrac{c}{|x-\xi|^2}\tilde\phi_1 = \dfrac{g}{|x-\xi|^2} + \tilde\gamma_0 \dfrac{g_0}{|x-\xi|^2} + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \tilde\gamma_{k,l} V_{k,l,\xi} & \text{in } B_R(\xi) \\ \tilde\phi_1 = 0 & \text{on } \partial B_R(\xi). \end{cases} \tag{38}$$

Given $\tilde\phi_1, \tilde\gamma_0$ as in (38), we define $\tilde\phi_2$ by

$$\begin{cases} -\Delta\tilde\phi_2 - \dfrac{c}{|x-\xi|^2}\tilde\phi_2 = \dfrac{g}{|x-\xi|^2} + \tilde\gamma_0 \dfrac{g_0}{|x-\xi|^2} & \text{in } B \backslash B_R(\xi) \\ \dfrac{\partial\tilde\phi_2}{\partial n} = \dfrac{\partial\tilde\phi_1}{\partial n} & \text{on } \partial B_R(\xi) \\ \tilde\phi_2 = h & \text{on } \partial B. \end{cases} \tag{39}$$

We also define an operator $T_2 : C^{1,\alpha}(\partial B_R(\xi)) \times \mathbb{R} \to C^{2,\alpha}(\partial B_R(\xi))$ by

$$T_2(\Psi, \gamma_0) = \phi_2|_{\partial B_R(\xi)}$$

where ϕ_2 is the solution to

$$\begin{cases} -\Delta\phi_2 - \dfrac{c}{|x-\xi|^2}\phi_2 = \gamma_0 \dfrac{g_0}{|x-\xi|^2} & \text{in } B\backslash B_R(\xi) \\ \dfrac{\partial\phi_2}{\partial n} = \Psi & \text{on } \partial B_R(\xi) \\ \phi_2 = 0 & \text{on } \partial B. \end{cases} \tag{40}$$

As we shall see later (see Step 2), Eqs. (39) and (40) possess indeed a unique solution if ξ is sufficiently small, because the domain $B\backslash B_R(\xi)$ is small.

We construct a solution ϕ of (35) as follows: choose $\phi_0 \in C^{2,\alpha}(\partial B_R(\xi))$, let ϕ_1 be the solution to (37) and let ϕ_2 be the solution to (40) with $\Psi = \frac{\partial\phi_1}{\partial n}$ and γ_0 from problem (37) . Then set

$$\phi = \begin{cases} \phi_1 + \tilde{\phi}_1 & \text{in } B_R(\xi) \\ \phi_2 + \tilde{\phi}_2 & \text{in } B\backslash B_R(\xi), \end{cases}$$

and $\mu_0 = \gamma_0 + \tilde{\gamma}_0$, $\mu_{k,l} = \gamma_{k,l} + \tilde{\gamma}_{k,l}$. If we have in addition

$$\phi_1 + \tilde{\phi}_1 = \phi_2 + \tilde{\phi}_2 \quad \text{on } \partial B_R(\xi), \tag{41}$$

then ϕ, μ_0 and $\mu_{k,l}$ form a solution to (35).

With this notation, solving Eq. (35) thus reduces to finding $\phi_0 \in C^{2,\alpha}(\partial B_R(\xi))$ such that (41) holds i.e.

$$T_2 \circ T_1(\phi_0) + \widetilde{\phi}_2 = \phi_0 \quad \text{in } \partial B_R(\xi).$$

The fact that this equation is uniquely solvable (when ξ is small) will follow once we show that $\|T_2\| \to 0$ as $\epsilon_0 \to 0$, while $\|T_1\|$ remains bounded.

Step 1. Given $\phi_0 \in C^{2,\alpha}(\partial B_R(\xi))$ there exist γ_0 and $\gamma_{k,l}$ such that (37) has a unique solution ϕ_1 in $C^{2,\alpha}_{\nu,\xi}(\partial B_R(\xi))$.

In this step we change variables $y = x - \xi$ and work in $B_R(0)$. Solving for γ_0 in the orthogonality relations (18) yields

$$\gamma_0 = \frac{\dfrac{1}{R}\displaystyle\int_{\partial B_R} \phi_0 \frac{\partial W_{0,0}}{\partial n}\left(\frac{y}{R}\right)}{\displaystyle\int_{B_R} g_0(y+\xi)|y|^{-2} W_{0,0}\left(\frac{y}{R}\right)} \tag{42}$$

and a computation, using $\|g_0 - 1\|_{L^\infty(B_R)} < \epsilon_0$ shows that

$$\int_{B_R} g_0(y+\xi)|y|^{-2} W_{0,0}\left(\frac{y}{R}\right) = R^{\nu+N-2} C(N,c) + O(\epsilon_0),$$

where $C(N,c) \neq 0$. In particular this integral remains bounded away from zero as $R \to 1$ ($R = 1 - 2\epsilon_0$ and $\epsilon_0 \to 0$) and hence γ_0 stays bounded.

Regarding $\gamma_{k,l}$ we have

$$\gamma_{k,l} = \frac{\frac{1}{R}\int_{\partial B_R(0)} \phi_0 \frac{\partial W_{k,l}}{\partial n}\left(\frac{y}{R}\right) - \gamma_0 \int_{B_R} g_0(y+\xi)|y|^{-2} W_{k,l}\left(\frac{y}{R}\right)}{\int_{B_R} \eta(|y|) W_{k,l}\left(\frac{y}{R}\right)^2}. \tag{43}$$

and we observe that $\int_{B_R} \eta(|y|)W_{k,l}(\frac{y}{R})^2$ is a positive constant depending on k, l and R (which stays bounded away from zero as $R \to 1$). Using Lemma 2.1, it follows that $\|T_1\|$ remains bounded as $R \to 1$ i.e. when $\epsilon_0 \to 0$.

Step 2. For ξ small enough Eq. (40) is uniquely solvable and $\|T_2\| \le C|\xi|$. Let $z_0 = 1 - |x|^2$. Then $z_0(|\gamma_0| \sup_{B\setminus B_R(\xi)} \frac{|g_0|}{|x-\xi|^2} + \sup_{\partial B_R(\xi)} |\Psi|)$ is a positive supersolution of (40). This shows that this equation is solvable and that for its solution ϕ_2 we have the estimate $|\phi_2| \le C|\xi|(|\gamma_0| + \sup_{\partial B_R(\xi)} |\Psi|)$. This and Schauder estimates yield $\|\phi_2\|_{C^{2,\alpha}(\partial B_R(\xi))} \le C|\xi|(|\gamma_0| + \|\Psi\|_{C^{2,\alpha}(\partial B_R(\xi))})$, which is the desired estimate.

Finally, estimate (36) follows from (19) and formulas (42), (43). □

2.5. *Differentiability*

Suppose now that for each $\xi \in B_{\epsilon_0}$ we have functions $g_0(\cdot,\xi)$, $g(\cdot,\xi) \in C^{0,\alpha}_{\nu,\xi}(B)$ and $h(\cdot,\xi) \in C^{2,\alpha}(\partial B)$. By Proposition 2.5, there is a unique $\phi(\cdot,\xi) \in C^{2,\alpha}_{\nu,\xi}(B)$ solution to (35). We want to investigate the differentiability properties of the map $\xi \mapsto \phi(\cdot,\xi)$.

Proposition 2.6. *Assume the following conditions*:

$$\exists k_1 \text{ such that } \alpha^-_{k_1} \in \mathbb{R} \quad \text{and} \quad -\alpha^-_{k_1} < \nu < -\alpha^-_{k_1+1},$$

$$\nu > -\frac{N}{2} + 2 \tag{44}$$

and

$$\nu - 1 \ne -\alpha^-_{k_1}.$$

Let $\varepsilon_0 > 0$ *and for* $\xi \in B_{\varepsilon_0}$, *let* $g_0(\cdot,\xi), g(\cdot,\xi) \in C^{1,\alpha}_{\nu,\xi}(B)$ *be such that*

$$A_0 := \sup_{\xi\in B_{\varepsilon_0}} (\|g_0(\cdot,\xi)\|_{1,\alpha,\nu,\xi;B} + \|D_\xi g_0(\cdot,\xi)\|_{0,\alpha,\nu-1,\xi;B}) < \infty \tag{45}$$

and

$$A := \sup_{\xi\in B_{\varepsilon_0}} (\|g(\cdot,\xi)\|_{1,\alpha,\nu,\xi;B} + \|D_\xi g(\cdot,\xi)\|_{0,\alpha,\nu-1,\xi;B}) < \infty.$$

Let $h(\cdot,\xi) \in C^{3,\alpha}(\partial B)$ *with*

$$\sup_{\xi\in B_{\varepsilon_0}} (\|h(\cdot,\xi)\|_{C3(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}) < \infty.$$

Let $\phi(\cdot,\xi)$ denote the solution to (35). *Then there exists $\bar{\epsilon}_0 > 0$ and a constant C such that if $\varepsilon_0 < \bar{\varepsilon}_0$ and if $\|g_0(\cdot,\xi) - 1\|_{L^\infty(B)} < \epsilon_0$, $|t| < \epsilon_0$ and $\xi_1, \xi_2 \in B_{\epsilon_0}$ then*

$$\|\phi(\cdot + \xi_2, \xi_2) - \phi(\cdot + \xi_1, \xi_1)\|_{2,\alpha,\nu-1,0;B_{1/2}} \le C|\xi_2 - \xi_1|. \tag{46}$$

Moreover the map $\xi \in B_{\epsilon_0} \mapsto \phi(\cdot;\xi)$ is differentiable in the sense that

$$D_\xi \phi(x,\xi)\eta = \lim_{\tau \to 0} \frac{1}{\tau}(\phi(x, \xi + \tau\eta) - \phi(x,\xi)) \quad \textit{exists for all } x \in B \setminus \{\xi\} \tag{47}$$

and $\eta \in \mathbb{R}^N$. Furthermore $D_\xi \phi(\cdot,\xi) \in C^{2,\alpha}_{\nu-1,\xi}(B)$, the maps $\xi \in B_{\epsilon_0} \mapsto \mu_0, \mu_{k,l} \in \mathbb{R}$ are differentiable and

$$\begin{aligned} \|D_\xi \phi(\cdot,\xi)\|_{2,\alpha,\nu-1,\xi;B} &+ |D_\xi \mu_0| + \sum_{k=1}^{k_1} \sum_{l=1}^{m_k} |D_\xi \mu_{k,l}| \\ &\le C(\|g(\cdot,\xi)\|_{0,\alpha,\nu,\xi;B} + \|D_\xi g(\cdot,\xi)\|_{0,\alpha,\nu-1,\xi;B} \\ &\quad + \|h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}). \end{aligned} \tag{48}$$

Remark 2.7. For simplicity we have stated Proposition 2.6 under the assumption $\nu - 1 \neq -\alpha^-_{k_1}$. A similar result also holds if $\nu - 1 = -\alpha^-_{k_1}$, but estimate (46) has to be replaced by:

$$\|\phi(\cdot + \xi_2, \xi_2) - \phi(\cdot + \xi_1, \xi_1)\|_{2,\alpha,\bar{\nu}-1,0;B_{1/2}} \le C|\xi_2 - \xi_1|,$$

where $\nu - \delta < \bar{\nu} < \nu$ for some $\delta > 0$ and with the constant C now depending on $\bar{\nu}$. Similarly, (48) is replaced by

$$\begin{aligned} \|D_\xi \phi(\cdot,\xi)\|_{2,\alpha,\bar{\nu}-1,\xi;B} &+ |D_\xi \mu_0| + \sum_{k=1}^{k_1} \sum_{l=1}^{m_k} |D_\xi \mu_{k,l}| \\ &\le C(\|g(\cdot,\xi)\|_{0,\alpha,\bar{\nu},\xi;B} + \|D_\xi g(\cdot,\xi)\|_{0,\alpha,\bar{\nu}-1,\xi;B} \\ &\quad + \|h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}). \end{aligned}$$

Proof. We change coordinates $y = x - \xi \in B - \xi$. Then (35) is equivalent to finding $\phi \in C^{2,\alpha}_{\nu,0}(B - \xi)$ such that

$$\begin{cases} -\Delta\phi - \dfrac{c}{|y|^2}\phi = \dfrac{g(y+\xi,\xi)}{|y|^2} + \mu_0 \dfrac{g_0(y+\xi,\xi)}{|y|^2} + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \mu_{k,l} V_{k,l,0} & \text{in } B - \xi \\ \phi = h(y+\xi,\xi) & \text{on } \partial B - \xi. \end{cases} \tag{49}$$

This equation can also be seen as the fixed point problem:

$$T_2(T_1(\phi_0,\xi),\xi) + \tilde{\phi}_2(y;\xi) = \phi_0, \quad \phi_0 \in C^{2,\alpha}(\partial B_R) \tag{50}$$

where $R = 1 - 2\epsilon_0$ and

- the operator $T_1 : C^{2,\alpha}(\partial B_R) \times B_{\epsilon_0} \to C^{1,\alpha}(\partial B_R) \times \mathbb{R}$ is defined by $T_1(\phi_0, \xi) = (\phi_1, \gamma_0)$ and ϕ_1, γ_0, $\gamma_{k,l}$ is the unique solution in $C^{2,\alpha}_{\nu,0}(B - \xi)$ to

$$\begin{cases} -\Delta\phi_1 - \dfrac{c}{|y|^2}\phi_1 = \gamma_0 \dfrac{g_0(y+\xi,\xi)}{|y|^2} + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \gamma_{k,l} V_{k,l,0} & \text{in } B_R \\ \phi_1 = \phi_0 & \text{on } \partial B_R. \end{cases} \tag{51}$$

- $T_2 : C^{1,\alpha}(\partial B_R(\xi)) \times \mathbb{R} \times B_{\epsilon_0} \to C^{2,\alpha}(\partial B_R(\xi))$ is defined by

$$T_2(\Psi, \gamma_0, \xi) = \phi_2|_{\partial B_R}$$

where ϕ_2 is the solution to

$$\begin{cases} -\Delta\phi_2 - \dfrac{c}{|y|^2}\phi_2 = \gamma_0 \dfrac{g_0(y+\xi,\xi)}{|y|^2} & \text{in } (B-\xi)\backslash B_R \\ \dfrac{\partial \phi_2}{\partial n} = \Psi & \text{on } \partial B_R \\ \phi_2 = 0 & \text{on } \partial B - \xi. \end{cases} \tag{52}$$

- $\tilde\phi_2(x;\xi)$ is the solution defined in (39) and can be computed by solving for $\tilde\phi_1 \in C^{2,\alpha}_{\nu,\xi}(B_R(\xi))$ and $\tilde\gamma_0$, $\tilde\gamma_{k,l}$ such that

$$\begin{cases} -\Delta\tilde\phi_1 - \dfrac{c}{|y|^2}\tilde\phi_1 = \dfrac{g(y+\xi,\xi)}{|y|^2} + \tilde\gamma_0 \dfrac{g_0(y+\xi,\xi)}{|y|^2} + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \tilde\gamma_{k,l} V_{k,l,0} & \text{in } B_R \\ \tilde\phi_1 = 0 & \text{on } \partial B_R, \end{cases} \tag{53}$$

and then $\tilde\phi_2$ is given by

$$\begin{cases} -\Delta\tilde\phi_2 - \dfrac{c}{|y|^2}\tilde\phi_2 = \dfrac{g(y+\xi,\xi)}{|y|^2} + \tilde\gamma_0 \dfrac{g_0(y+\xi,\xi)}{|y|^2} & \text{in } (B-\xi)\backslash B_R \\ \dfrac{\partial \tilde\phi_2}{\partial n} = \dfrac{\partial \tilde\phi_1}{\partial n} & \text{on } \partial B_R \\ \tilde\phi_2 = h(y+\xi,\xi) & \text{on } \partial B - \xi. \end{cases} \tag{54}$$

We shall derive the following Lipschitz estimate for T_1, where we write $(\phi_1(\cdot,\xi_i), \gamma_0^{(i)}) = T_1(\phi_0, \xi_i)$ $(i = 1, 2)$:

$$\begin{aligned} &\|\phi_1(\cdot,\xi_2) - \phi_1(\cdot,\xi_1)\|_{2,\alpha,\nu-1,0;B_R} + |\gamma_0^{(2)} - \gamma_0^{(1)}| + \sum_{k=1}^{k_1}\sum_{l=1}^{m_k} |\gamma_{kl}^{(2)} - \gamma_{kl}^{(1)}| \\ &\quad \le C|\xi_2 - \xi_1|. \end{aligned} \tag{55}$$

Indeed, by formulas (42) and (43) and condition (44) we deduce

$$|\gamma_0^{(2)} - \gamma_0^{(1)}| + \sum_{k=1}^{k_1}\sum_{l=1}^{m_k} |\gamma_{kl}^{(2)} - \gamma_{kl}^{(1)}| \le C|\xi_2 - \xi_1|. \tag{56}$$

Now write $\tau = |\xi_2 - \xi_1|$, $\phi \equiv \frac{\phi_1(\cdot,\xi_2) - \phi_1(\cdot,\xi_1)}{\tau}$. Then

$$\begin{cases} -\Delta\phi - \dfrac{c}{|y|^2}\phi = \dfrac{1}{\tau}\left[\dfrac{\gamma_0^{(2)}}{|y|^2} g_0(y+\xi_2,\xi_2) - \dfrac{\gamma_0^{(1)}}{|y|^2} g_0(y+\xi_1,\xi_1)\right] & \\ \qquad\qquad + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \frac{\gamma_{kl}^{(2)} - \gamma_{kl}^{(1)}}{\tau} V_{k,l,0} & \text{in } B_R \\ \phi = 0 & \text{on } \partial B_R. \end{cases}$$

By (36) we have

$$\begin{aligned} \|\phi\|_{2,\alpha,\nu-1,0;B_R} &\le C\left\| \frac{\gamma_0^{(2)} g_0(y+\xi_2,\xi_2) - \gamma_0^{(1)} g_0(y+\xi_1,\xi_1)}{\tau} \right\|_{0,\alpha,\nu-1,0;B_R} \\ &\le C + C\left\| \frac{g_0(y+\xi_2,\xi_2) - g_0(y+\xi_1,\xi_1)}{\tau} \right\|_{0,\alpha,\nu-1,0;B_R} \end{aligned}$$

where we have used (56). Using (45), we obtain that

$$\left\| \frac{g_0(y+\xi_2,\xi_2) - g_0(y+\xi_1,\xi_1)}{\tau} \right\|_{0,\alpha,\nu-1,0;B_R} \le C.$$

This implies (55).

Similarly we have

$$\begin{aligned} &\|\tilde\phi_1(\cdot,\xi_2) - \tilde\phi_1(\cdot,\xi_1)\|_{2,\alpha,\nu-1,0;B_R} + |\tilde\gamma_0^{(2)} - \tilde\gamma_0^{(1)}| + \sum_{k=1}^{k_1}\sum_{l=1}^{m_k} |\tilde\gamma_{kl}^{(2)} - \tilde\gamma_{kl}^{(1)}| \\ &\quad \le C|\xi_2 - \xi_1|. \end{aligned} \tag{57}$$

Using standard local elliptic regularity arguments, applied to $u \equiv \tilde\phi_2(\cdot,\xi_2) - \tilde\phi_2(\cdot,\xi_2)$ in $B_1(-\xi_2) \cap B_1(-\xi_1) \setminus B_R$, we have

$$\|\tilde\phi_2(\cdot,\xi_2) - \tilde\phi_2(\cdot,\xi_1)\|_{C^{2,\alpha}(\partial B_R)} \le C|\xi_2 - \xi_1| \tag{58}$$

and similarly

$$\|T_2(\Psi,\gamma_0^{(2)},\xi_2) - T_2(\Psi,\gamma_0^{(1)},\xi_1)\|_{C^{2,\alpha}(\partial B_R)} \le C(|\xi_2 - \xi_1| + |\gamma_0^{(2)} - \gamma_0^{(1)}|). \tag{59}$$

Using the fixed point characterization (50) of ϕ_0 and estimates (55), (57)–(59) we deduce

$$\|\phi_0(\cdot,\xi_2) - \phi_0(\cdot,\xi_1)\|_{C^{2,\alpha}(\partial B_R)} \le C|\xi_2 - \xi_1|.$$

The solution ϕ to (49) is then given by

$$\phi = \begin{cases} \phi_1 + \tilde\phi_1 & \text{in } B_R \\ \phi_2 + \tilde\phi_2 & \text{in } (B-\xi)\setminus B_R \end{cases}$$

and thanks to (55), (57) we obtain (46).

Let us show now that (47) holds. We return to the problem (35), without translating, and let $\phi(\cdot,\xi)$ denote the solution to (35). Let $x \neq \xi$ and write

$$\begin{aligned}\phi(x,\xi+\tau\eta)-\phi(x,\xi) &= \phi(x,\xi+\tau\eta)-\phi(x+\tau\eta,\xi+\tau\eta)\\ &\quad+\phi(x+\tau\eta,\xi+\tau\eta)-\phi(x,\xi).\end{aligned}$$

Since $\phi(\cdot,\xi+\tau\eta)\in C^{2,\alpha}_{\nu,\xi+\tau\eta}(B)$ by the mean value theorem

$$\frac{\phi(x,\xi+\tau\eta)-\phi(x+\tau\eta,\xi+\tau\eta)}{\tau} = -\nabla\phi(x+s\eta,\xi+\tau\eta)\eta$$

for some $|s|<|\tau|$ and letting $\tau\to 0$ we see that $\lim_{\tau\to 0}\frac{\phi(x,\xi+\tau\eta)-\phi(x+\tau\eta,\xi+\tau\eta)}{\tau} = -\nabla\phi(x,\xi)\eta$. For the other term, changing variables $y=x-\xi$ we have

$$\begin{aligned}\frac{\phi(x+\tau\eta,\xi+\tau\eta)-\phi(x,\xi)}{\tau} &= \frac{\phi(y+\xi+\tau\eta,\xi+\tau\eta)-\phi(y+\xi,\xi)}{\tau}\\ &= \frac{\phi^{tr}(y,\xi+\tau\eta)-\phi^{tr}(y,\xi)}{\tau},\end{aligned}\tag{60}$$

where now $\phi^{tr}(\cdot,\xi)$ denotes the solution to the shifted problem (49). From estimate (46) we deduce that

$$\left\|\frac{\phi^{tr}(\cdot,\xi+\tau\eta)-\phi^{tr}(\cdot,\xi)}{\tau}\right\|_{2,\alpha,\nu-1,0;B_{1/2}} \le C \tag{61}$$

with C independent of τ.

Observe now that the quotient $\frac{\phi(x,\xi+\tau\eta)-\phi(x,\xi)}{\tau}$ is uniformly bounded in $B\backslash B_{1/4}$ which can be seen from estimates (55), (57)–(59). It follows from standard local elliptic regularity arguments that this quotient is uniformly bounded in $C^{2,\alpha}(B\backslash B_{1/4})$.

Fix $0<\beta<\alpha$. Then for any sequence $\tau_n\to 0$ we can extract a subsequence (denoted the same) such that $\frac{\phi(x,\xi+\tau_n\eta)-\phi(x,\xi)}{\tau_n}$ converges in $C^{2,\beta}(B\backslash B_{1/4})$. Set

$$\psi_1 = \lim_{n\to\infty}\frac{\phi(x,\xi+\tau_n\eta)-\phi(x,\xi)}{\tau_n} - \nabla_x\phi(x,\xi)\eta \quad \text{in } B\backslash B_{1/4}$$

so that $\psi_1\in C^{2,\alpha}(B\backslash B_{1/4})$. Note that

$$\begin{aligned}\psi_1(x) &= \lim_{n\to\infty}\frac{\phi(x+\tau_n\eta,\xi+\tau_n\eta)-\phi(x,\xi)}{\tau_n}\\ &= \lim_{n\to\infty}\frac{\phi^{tr}(x-\xi,\xi+\tau_n\eta)-\phi^{tr}(x-\xi,\xi)}{\tau_n} \quad \forall x\in B\backslash B_{1/4}.\end{aligned}\tag{62}$$

In addition, from (61) we find $\psi_2\in C^{2,\alpha}_{\nu-1,0}(B_{1/2})$ such that

$$\left\|\frac{\phi^{tr}(\cdot,\xi+\tau_n\eta)-\phi^{tr}(\cdot,\xi)}{\tau_n} - \psi_2\right\|_{2,\beta,\nu-1,0;B_{1/2}} \to 0 \quad \text{as } \tau_n\to 0.$$

Set

$$\psi(x) = \begin{cases} \psi_1(y+\xi) & y \in (B \backslash B_{1/4}) - \xi \\ \psi_2(y) & y \in B_{1/2}. \end{cases}$$

Clearly ψ belongs to $C^{2,\beta}_{\nu-1,0}(B-\xi)$. Moreover by (42), (43) and similar formulas for $\tilde{\gamma}_0$, $\tilde{\gamma}_{k,l}$ we have that the functions $\mu_0(\xi)$, $\mu_{k,l}(\xi)$ are differentiable and hence

$$\begin{cases} -\Delta\psi - \dfrac{c}{|y|^2}\psi = \dfrac{D_x g(y+\xi,\xi)\eta + D_\xi g(y+\xi,\xi)\eta}{|y|^2} + \dfrac{\partial\mu_0}{\partial\eta}\dfrac{g_0(y+\xi,\xi)}{|y|^2} \\ \qquad\qquad + \mu_0 \dfrac{D_x g(y+\xi,\xi)\eta + D_\xi g(y+\xi,\xi)\eta}{|y|^2} + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \dfrac{\partial\mu_{k,l}}{\partial\eta} V_{k,l,0} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } B-\xi \\ \psi = D_y\phi^{tr}(y,\xi)\eta + D_\xi h(y+\xi,\xi)\eta \qquad\qquad \text{for } y \in \partial B - \xi. \end{cases}$$

The boundary condition is obtained by observing that for fixed $x \in \partial B$, we have $\phi^{tr}(x-\xi,\xi) = h(x,\xi)$ and differentiating with respect to ξ.

To show the convergence of (60) (as $\tau \to 0$), it suffices to verify that ψ is uniquely determined. Let $\mu_0(\xi)$, $\mu_{k,l}(\xi)$ be the constants associated to $\phi(\cdot,\xi)$ in (35). This equation possesses at most one solution $\psi \in C^{2,\alpha}_{\nu-1,0}(B)$ by Proposition 2.5.

Estimate (48) now follows from the formulas (42), (43) and the equation satisfied by $D_\xi\phi$. □

2.6. *Perturbations of the operator* $-\Delta - \frac{c}{|x-\xi|^2}$

We wish to extend Proposition 2.5 to an operator of the form $-\Delta - L_t - \frac{c}{|x-\xi|^2}$ where L_t is a suitably small second order differential operator. We will take L_t of the form

$$L_t w = a_{ij}(x,t)D_{ij}w + b_i(x,t)D_i w + c(x,t)w. \tag{63}$$

Lemma 2.8. *Suppose that the coefficients of L_t satisfy: $a_{ij}(\cdot,t)$, $b_i(\cdot,t)$, $c_i(\cdot,t)$ are $C^\alpha(\bar{B})$ and for some C it holds*

$$\|a_{ij}(\cdot,t)\|_{C^\alpha(\bar{B})} + \|b_i(\cdot,t)\|_{C^\alpha(\bar{B})} + \|c(\cdot,t)\|_{C^\alpha(\bar{B})} \le C|t|.$$

Assume

$$\exists k_1 \text{ such that } \alpha^-_{k_1} \in \mathbb{R} \quad \text{and} \quad -\alpha^-_{k_1} < \nu < -\alpha^-_{k_1+1}.$$

Then there exists $\epsilon_0 > 0$ such that if $|\xi| < \epsilon_0$, $|t| < \epsilon_0$ and $g_0 \in C^{0,\alpha}_{\nu,\xi}(B)$ satisfies $\|g_0 - 1\|_{L^\infty(B)} < \epsilon_0$, then given any $g \in C^{0,\alpha}_{\nu,\xi}(B)$ and $h \in C^{2,\alpha}(\partial B)$, there exist *unique* $\phi \in C^{2,\alpha}_{\nu,\xi}(B)$ *and* $\mu_0, \mu_{k,l} \in \mathbb{R}$ $(k = 1, \ldots, k_1,\ l = 1, \ldots, m_k)$ *solution to*

$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c}{|x-\xi|^2}\phi \\ \qquad = \dfrac{g}{|x-\xi|^2} + \mu_0 \dfrac{g_0}{|x-\xi|^2} + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \mu_{k,l} V_{k,l,\xi} & \text{in } B \\ \phi = h & \text{on } \partial B. \end{cases} \tag{64}$$

Moreover

$$\|\phi\|_{2,\alpha,\nu,\xi;B} + |\mu_0| + \sum_{k=1}^{k_1}\sum_{l=1}^{m_k} |\mu_{k,l}| \le C(\|g\|_{0,\alpha,\nu,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)}). \tag{65}$$

Proof. Fix $h \in C^{2,\alpha}(\partial B)$ and $|\xi| < \epsilon_0$, where ϵ_0 is the constant appearing in Proposition 2.5. For $g \in C^{0,\alpha}_{\nu,\xi}(B)$ let $\phi = T(g/|x-\xi|^2)$ be the solution to (35) as defined in Proposition 2.5. Then (64) is equivalent to $\phi = T(g/|x-\xi|^2 + L_t\phi)$. Define

$$\widetilde{T}(\phi) = T(g/|x-\xi|^2 + L_t\phi).$$

We apply the Picard Fixed Point Theorem to the operator $\tilde{T}$ in a closed ball $\mathcal{B}_R$ of the Banach space $C^{2,\alpha}_{\nu,\xi}(B)$ equipped with the norm $\|\cdot\|_{2,\alpha,\nu,\xi;B}$.

Note that by Proposition 2.5 we have $\|T(g/|x-\xi|^2)\|_{2,\alpha,\nu,\xi;B} \le C(\|g\|_{0,\alpha,\nu,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)})$. Using this inequality, for $\|\phi\|_{2,\alpha,\nu,\xi;B} \le R$ we have

$$\begin{aligned} \|\widetilde{T}(\phi)\|_{2,\alpha,\nu,\xi;B} &\le C(\|g\|_{0,\alpha,\nu,\xi;B} + \|L_t\phi\|_{0,\alpha,\nu-2,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)}) \\ &\le C(\|g\|_{0,\alpha,\nu,\xi} + |t|R + \|h\|_{C^{2,\alpha}(\partial B)}) \le R, \end{aligned}$$

where the last inequality holds if we first take t so small that $C|t| \le \frac{1}{2}$, and then choose R so large that $C(\|g\|_{0,\alpha,\nu,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)}) \le \frac{R}{2}$.

For $\|\phi_1\|_{2,\alpha,\nu,\xi;B} \le R$, $\|\phi_2\|_{2,\alpha,\nu,\xi;B} \le R$ we have

$$\begin{aligned} \|\widetilde{T}(\phi_1) - \widetilde{T}(\phi_2)\|_{2,\alpha,\nu,\xi;B} &\le C\|L_t(\phi_1 - \phi_2)\|_{0,\alpha,\nu-2,\xi;B} \\ &\le C|t|\,\|\phi_1 - \phi_2\|_{2,\alpha,\nu,\xi;B}, \end{aligned}$$

and we see that $\widetilde{T}$ is a contraction on the ball $\mathcal{B}_R$ of $C^{2,\alpha}_{\nu,\xi}(B)$ if t is chosen small enough. □

We now extend the results of the previous section on differentiability to perturbed operators of the form $-\Delta - L_t - \frac{c}{|x-\xi|^2}$.

Proposition 2.9. *Assume the following conditions*:

$$\exists k_1 \text{ such that } \alpha^-_{k_1} \in \mathbb{R} \quad \text{and} \quad -\alpha^-_{k_1} < \nu < -\alpha^-_{k_1+1}$$

$$\nu > -\frac{N}{2} + 2,$$

and

$$\nu - 1 \neq -\alpha_{k_1}^{-}. \tag{66}$$

Let $\varepsilon_0 > 0$ *and for* $\xi \in B_{\varepsilon_0}$ *let* $g_0(\cdot,\xi), g(\cdot,\xi) \in C^{1,\alpha}_{\nu,\xi}(B)$ *be such that*

$$A_0 \equiv \sup_{\xi \in B_{\epsilon_0}} (\|g_0(\cdot,\xi)\|_{1,\alpha,\nu,\xi;B} + \|D_\xi g_0(\cdot,\xi)\|_{0,\alpha,\nu-1,\xi;B}) < \infty$$

and

$$A \equiv \sup_{\xi \in B_{\epsilon_0}} (\|g(\cdot,\xi)\|_{1,\alpha,\nu,\xi;B} + \|D_\xi g(\cdot,\xi)\|_{0,\alpha,\nu-1,\xi;B}) < \infty.$$

On the operator L_t *we assume*

$$\|a_{ij}(\cdot,t)\|_{C^{1,\alpha}(\bar{B})} + \|b_i(\cdot,t)\|_{C^{1,\alpha}(\bar{B})} + \|c(\cdot,t)\|_{C^{1,\alpha}(\bar{B})} \leq C|t|.$$

Let $h(\cdot,\xi) \in C^{3,\alpha}(\partial B)$ *with*

$$\sup_{\xi \in B_{\epsilon_0}} (\|h(\cdot,\xi)\|_{C^3(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}) < \infty$$

and let $\phi(\cdot,\xi)$ *denote the solution to* (64). *Then there exist* $\bar{\epsilon}_0 > 0$, $C > 0$ *such that if* $\varepsilon_0 < \bar{\varepsilon}_0$, $\|g_0(\cdot,\xi) - 1\|_{L^\infty(B)} < \epsilon_0$, $|t| < \epsilon_0$ *and* $\xi_1, \xi_2 \in B_{\epsilon_0}$, *we have*

$$\|\phi(\cdot + \xi_2, \xi_2) - \phi(\cdot + \xi_1, \xi_1)\|_{2,\alpha,\nu-1,0;B_{1/2}} \leq C|\xi_2 - \xi_1|. \tag{67}$$

Furthermore,

$$D_\xi \phi(x;\xi)\eta = \lim_{t \to 0} \frac{1}{t}(\phi(x;\xi + t\eta) - \phi(x;\xi)) \quad \textit{exists } \forall x \in B\backslash\{\xi\}, \forall \eta \in \mathbb{R}^N,$$

the maps $\xi \in B_{\epsilon_0}(0) \mapsto \mu_0, \mu_{k,l} \in \mathbb{R}$ *are differentiable and*

$$\begin{aligned}\|D_\xi \phi(x;\xi)\|_{2,\alpha,\nu-1,\xi;B} \leq C(&\|g(\cdot,\xi)\|_{0,\alpha,\nu,\xi;B} + \|D_\xi g(\cdot,\xi)\|_{0,\alpha,\nu-1,\xi;B} \\ &+ \|h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}).\end{aligned} \tag{68}$$

Proof. To prove this result we use again a fixed point argument. Consider the Banach space X of functions $\phi(x,\xi)$ defined for $x \in B$, $\xi \in B_{\epsilon_0}$, which are twice continuously differentiable with respect to x and once with respect to ξ for $x \neq \xi$, for which the following norm is finite

$$\|\phi\|_X = \sup_{\xi \in B_{\epsilon_0}} (\|\phi(\cdot,\xi)\|_{2,\alpha,\nu,\xi;B} + \|D_\xi \phi(\cdot,\xi)\|_{2,\alpha,\nu-1,\xi;B}).$$

Let $\mathcal{B}_R$ denote the closed ball of radius R in X where $R > 0$ is to be chosen. For $\xi \in B_{\epsilon_0}$, $g(\cdot,\xi) \in C^{0,\alpha}_{\nu,\xi}(B)$, $h(\cdot,\xi) \in C^{3,\alpha}(\partial B)$ let $\phi = T(g(\cdot,\xi)/|x-\xi|^2, \xi)$ be the solution to (35) as defined in Proposition 2.5 with $g_0(\cdot,\xi)$ in place of g_0. Let $\widetilde{T} : \mathcal{B}_R \to X$ be defined by $\widetilde{T}(\phi) = T(g(\cdot,\xi)/|x-\xi|^2 + L_t\phi(\cdot,\xi), \xi)$. Then

$$\begin{aligned}\|\widetilde{T}(\phi)\|_{2,\alpha,\nu,\xi;B} &\leq C(\|g\|_{0,\alpha,\nu,\xi;B} + \|L_t\phi\|_{0,\alpha,\nu-2,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)}) \\ &\leq C(\|g\|_{0,\alpha,\nu,\xi} + |t|R + \|h\|_{C^{2,\alpha}(\partial B)}) \leq R,\end{aligned}$$

if $|t|$ is small and R large. Moreover from (48) we have

$$\begin{aligned}
\|D_\xi \widetilde{T}(\phi)\|_{2,\alpha,\nu-1,\xi;B} &\le C(\|g\|_{0,\alpha,\nu,\xi;B} + \|D_\xi g\|_{0,\alpha,\nu-1,\xi;B} \\
&\quad + \|L_t\phi\|_{0,\alpha,\nu-2,\xi;B} + \|L_t D_\xi \phi\|_{0,\alpha,\nu-3,\xi;B} \\
&\quad + \|h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}) \\
&\le C(A + |t|\|\phi\|_X + \|h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}) \\
&\le R \qquad (69)
\end{aligned}$$

if we take $|t|$ small enough and then R large. Taking the supremum over $\xi \in B_{\epsilon_0}$ then yields the estimate

$$\|\widetilde{T}(\phi)\|_X \le R \quad \forall \|\phi\|_X \le R.$$

That $\widetilde{T}$ is a contraction is proved in a similar manner as in the previous lemma. For $\|\phi_1\|_X \le R$, $\|\phi_2\|_X \le R$ we have

$$\begin{aligned}
\|\widetilde{T}(\phi_1) - \widetilde{T}(\phi_2)\|_{2,\alpha,\nu,\xi;B} &\le C\|L_t(\phi_1 - \phi_2)\|_{0,\alpha,\nu-2,\xi;B} \\
&\le C|t|\|\phi_1 - \phi_2\|_{2,\alpha,\nu,\xi;B},
\end{aligned}$$

and using (48)

$$\begin{aligned}
\|D_\xi(\widetilde{T}(\phi_1) - \widetilde{T}(\phi_2))\|_{2,\alpha,\nu-1,\xi;B} &\le C(\|L_t(\phi_1 - \phi_2)\|_{0,\alpha,\nu-2,\xi;B} \\
&\quad + \|L_t D_\xi(\phi_1 - \phi_2)\|_{0,\alpha,\nu-3,\xi;B}) \\
&\le C|t|\|\phi_1 - \phi_2\|_X,
\end{aligned}$$

and it follows that

$$\|\widetilde{T}(\phi_1) - \widetilde{T}(\phi_2)\|_X \le C|t|\|\phi_1 - \phi_2\|_X.$$

Thus $\widetilde{T}$ is a contraction on the ball $\mathcal{B}_R$ of X if $|t|$ is chosen small enough.

To prove (68) we observe that, if $\phi = \widetilde{T}(\phi)$ then as in (69) we have

$$\begin{aligned}
\|D_\xi \phi\|_{2,\alpha,\nu-1,\xi;B} &= \|D_\xi \widetilde{T}(\phi)\|_{2,\alpha,\nu-1,\xi;B} \\
&\le C(\|g\|_{0,\alpha,\nu,\xi;B} + \|D_\xi g\|_{0,\alpha,\nu-1,\xi;B} \\
&\quad + |t|(\|\phi\|_{0,\alpha,\nu,\xi;B} + \|D_\xi\phi\|_{0,\alpha,\nu,\xi;B}) \\
&\quad + \|h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}).
\end{aligned}$$

Combining this and (65) we deduce (68). □

2.7. *Additional results when* $0 < c < \frac{(N-2)^2}{4}$

The purpose of this section is to extend Lemma 2.1 to a general bounded, smooth domain Ω of $\mathbb{R}^N$, $N \ge 3$ and general $\xi \in \Omega$, by redefining the functions $W_{k,l}$ which appear in (18). For this we restrict ourselves to values of c in the range

$$0 < c < \frac{(N-2)^2}{4}$$

which guarantees $\alpha_0^- < \frac{N-2}{2} < \alpha_0^+$.

Take $g \in C^{0,\alpha}_{\nu-2,0}(\Omega) \cap H^{-1}(\Omega)$ and $h \in C^{2,\alpha}(\partial\Omega)$.

Hardy's inequality and the condition $c < \frac{(N-2)^2}{4}$ ensure that equation

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x-\xi|^2}\phi = g & \text{in } \Omega \\ \phi = h & \text{on } \partial\Omega, \end{cases} \tag{70}$$

has a unique solution $\phi \in H^1(\Omega)$. If we do not impose a restriction on ϕ of the form $\phi \in H^1(\Omega)$ then uniqueness in (70) is lost, see for instance [15, 16].

We define $W_{k,l,\xi}$, which will play the same role as in (18), to be smooth functions in $\Omega\backslash\{\xi\}$ satisfying

$$\begin{cases} -\Delta W_{k,l,\xi} - \dfrac{c}{|x-\xi|^2} W_{k,l,\xi} = 0 & \text{in } \Omega\backslash\{\xi\} \\ W_{k,l,\xi} = 0 & \text{on } \partial\Omega \\ W_{k,l,\xi}(x) \sim |x-\xi|^{-\alpha_k^+}\varphi_{k,l}\left(\frac{x-\xi}{|x-\xi|}\right) & x \sim \xi. \end{cases} \tag{71}$$

Let indeed

$$W_{k,l,\xi}(x) = |x-\xi|^{-\alpha_k^+}\varphi_{k,l}\left(\frac{x-\xi}{|x-\xi|}\right) - \psi_{k,l,\xi}(x), \tag{72}$$

where $\psi_{k,l,\xi} \in H^1(\Omega)$ is the unique solution to

$$\begin{cases} -\Delta\psi_{k,l,\xi} - \dfrac{c}{|x-\xi|^2}\psi_{k,l,\xi} = 0 & \text{in } \Omega \\ \psi_{k,l,\xi} = |x-\xi|^{-\alpha_k^+}\varphi_{k,l}\left(\dfrac{x-\xi}{|x-\xi|}\right) & \text{on } \partial\Omega. \end{cases}$$

Observe that for $C > 0$ large enough, $C|x-\xi|^{-\alpha_0^-}$ and $-C|x-\xi|^{-\alpha_0^-}$ are respectively a super and a subsolution of the above equation, whence by the maximum principle (which is valid in virtue of Hardy's inequality and the restriction $c < \frac{(N-2)^2}{4}$), $|\psi_{k,1,\xi}| \le C|x-\xi|^{-\alpha_0^-}$ and $W_{k,l,\xi}$ satisfies (71).

Remark 2.10. If $\Omega = B_1(0)$ and $\xi = 0$, our definition is consistent with (15), since

$$\psi_{k,l,\xi} = |x|^{-\alpha_k^-}\varphi_{k,l}\left(\frac{x}{|x|}\right) \quad \text{and} \quad W_{k,l,\xi} = (|x|^{-\alpha_k^+} - |x|^{-\alpha_k^-})\varphi_{k,l}\left(\frac{x}{|x|}\right). \tag{73}$$

Theorem 2.11. *Let $c \in \mathbb{R}$ and assume*

$$\exists k_1 \ge k_0 \quad -\alpha_{k_1+1}^- > \nu > -\alpha_{k_1}^-. \tag{74}$$

Let Ω a smooth bounded domain of $\mathbb{R}^N$, $N \ge 3$, $\xi \in \Omega$, $g \in C^{0,\alpha}_{\nu-2,\xi}(\Omega) \cap H^{-1}(\Omega)$ and $h \in C^{2,\alpha}(\partial\Omega)$. If

$$\int_\Omega g W_{k,l,\xi} = \int_{\partial\Omega} h \frac{\partial W_{k,l,\xi}}{\partial n}, \quad \forall k = 0,\dots,k_1, \quad \forall l = 1,\dots,m_k \tag{75}$$

then there exists a unique $\phi \in C^{2,\alpha}_{\nu,\xi}(\Omega) \cap H^1(\Omega)$ solution to

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x-\xi|^2}\phi = g & \textit{in } \Omega \\ \phi = h & \textit{on } \partial\Omega, \end{cases} \tag{76}$$

and it satisfies

$$\|\phi\|_{2,\alpha,\nu,0} \le C(\,\|g\|_{0,\alpha,\nu-2,0} + \|h\|_{C^{2,\alpha}(\partial B)}) \tag{77}$$

where C is independent of g and h.

By translating the domain we consider from now on $\xi = 0$. By Lemma 2.1, a straightforward scaling argument implies that Theorem 2.11 holds when $\Omega = B_R(0)$ and $\xi = 0$. In this case $W_{k,l,0}$ takes the form

$$\widetilde{W}_{k,l}(x) = \left(\left(\frac{|x|}{R}\right)^{-\alpha_k^+} - \left(\frac{|x|}{R}\right)^{-\alpha_k^-}\right)\varphi_{k,l}\left(\frac{x}{|x|}\right). \tag{78}$$

This is obtained by scaling the functions in (73) and is the same as in definition (72) except for a multiplicative constant.

Proof of Theorem 2.11. As mentioned earlier, we shall give the proof in the case $\xi = 0$. Take $R > 0$ small such that $B_R(0) \subset \Omega$. Then the unique solution $\phi \in H^1(\Omega)$ of (76) satisfies (77) if

$$\int_{B_R} g\widetilde{W}_{k,l} = \int_{\partial B_R} \phi \frac{\partial \widetilde{W}_{k,l}}{\partial n}, \quad \forall k = 0, \dots, k_1, \quad \forall l = 1, \dots, m_k, \tag{79}$$

where $\widetilde{W}_{k,l}$ is defined in (78). Since $\widetilde{W}_{k,l}$ satisfies

$$-\Delta\widetilde{W}_{k,l} - \frac{c}{|x|^2}\widetilde{W}_{k,l} = 0 \quad \text{in } \mathbb{R}^N\backslash\{0\},$$

multiplying this equation by ϕ and integrating in $\Omega\backslash B_R$ we obtain

$$\int_{\partial\Omega}\left(\frac{\partial\widetilde{W}_{k,l}}{\partial n}\phi - \widetilde{W}_{k,l}\frac{\partial\phi}{\partial n}\right) - \int_{\partial B_R}\frac{\partial\widetilde{W}_{k,l}}{\partial n}\phi = \int_{\Omega\backslash B_R} g\widetilde{W}_{k,l} \tag{80}$$

where n denotes the exterior normal vector to $\partial\Omega$ and ∂B_R. Adding (79) and (80) we see that (79) is equivalent to

$$\int_{\partial\Omega}\left(\frac{\partial\widetilde{W}_{k,l}}{\partial n}\phi - \widetilde{W}_{k,l}\frac{\partial\phi}{\partial n}\right) = \int_{\Omega} g\widetilde{W}_{k,l}. \tag{81}$$

Let $\tilde\psi_{k,l} \in H^1(\Omega)$ be the solution to

$$\begin{cases} -\Delta\tilde\psi_{k,l} - \dfrac{c}{|x|^2}\tilde\psi_{k,l} = 0 & \text{in } \Omega \\ \tilde\psi_{k,l} = \widetilde{W}_{k,l} & \text{on } \partial\Omega. \end{cases}$$

Multiplying this equation by ϕ and integrating by parts yields

$$\int_{\partial\Omega}\left(\frac{\partial\tilde\psi_{k,l}}{\partial n}\phi - \frac{\partial\phi}{\partial n}\tilde\psi_{k,l}\right) = \int_{\Omega} g\tilde\psi_{k,l}.$$

Subtracting this equation from (81) we obtain that (79) is equivalent to

$$\int_{\partial\Omega} \frac{\partial(\widetilde{W}_{k,l} - \tilde{\psi}_{k,l})}{\partial n}\phi = \int_{\Omega} g(\widetilde{W}_{k,l} - \tilde{\psi}_{k,l}).$$

Up to multiplicative constant $\widetilde{W}_{k,l} - \tilde{\psi}_{k,l}$ is the same as $W_{k,l,0}$ as defined in (72). □

3. Solution to the Nonlinear Equation

We first study (4). Recall

$$\Omega_t = \{x + t\psi(x) : x \in B_1\},$$

where t is small and $\psi : \bar{B}_1 \to \mathbb{R}^N$ a C^2 map.

We change variables to replace (4) with a problem in the unit ball. The map $id + t\psi$ is invertible for t small and we write the inverse of $y = x + t\psi(x)$ as $x = y + t\tilde{\psi}(y,t)$. Define v by

$$u(y) = v(y + t\tilde{\psi}(y,t)).$$

Then

$$\Delta_y u = \Delta_x v + L_t v$$

where L_t is a second order operator given by

$$L_t v = 2t \sum_{i,k} v_{x_i x_k} \frac{\partial \tilde{\psi}_k}{\partial y_i} + t \sum_{i,k} v_{x_k} \frac{\partial^2 \tilde{\psi}_k}{\partial y_i^2} + t^2 \sum_{i,j,k} v_{x_j x_k} \frac{\partial \tilde{\psi}_j}{\partial y_i} \frac{\partial \tilde{\psi}_k}{\partial y_i}.$$

We look for a solution of the form

$$v(x) = \log \frac{1}{|x-\xi|^2} + \phi, \quad \lambda = c^* + \mu,$$

where $c^* = 2(N-2)$. Then (4) is equivalent to

$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c^*}{|x-\xi|^2}\phi = \dfrac{c^*}{|x-\xi|^2}(e^\phi - 1 - \phi) + \dfrac{\mu}{|x-\xi|^2}e^\phi & \\ \qquad\qquad + L_t\left(\log \dfrac{1}{|x-\xi|^2}\right) & \text{in } B \\ \phi = -\log \dfrac{1}{|x-\xi|^2} & \text{on } \partial B. \end{cases} \tag{82}$$

We observe that if $N \geq 4$ then $N - 1 < c^* < 2N$ and hence $\alpha_1^- > 0$, $\alpha_2^- < 0$.

We fix from now on $\nu = 0$ and $k_1 = 1$. We may thus apply Proposition 2.5 and Lemma 2.8, since (34) is satisfied. In dimension $N \geq 5$, since (44) and (66) hold, we may also apply Propositions 2.6 and 2.9.

We simplify our notation and write

$$V_{\ell,\xi} := V_{1,\ell,\xi} \quad \ell = 1, \ldots, N,$$

where $V_{1,\ell,\xi}$ is defined in (33).

Define

$$\tilde{f}(x,t) = L_t\left(\log\frac{1}{|x-\xi|^2}\right)$$

and note that

$$\|\tilde{f}(x,t)|x-\xi|^2\|_{0,\alpha,-2,\xi} \le C|t|. \tag{83}$$

Concerning (82) we prove:

Lemma 3.1. *Write $c = c^* = 2(N-2)$. Then there exists $\epsilon_0 > 0$ such that if $|\xi| < \epsilon_0$, $|t| < \epsilon_0$, there exist $\phi \in C^{2,\alpha}_{0,\xi}(B)$ and $\mu_0, \ldots, \mu_N \in \mathbb{R}$ such that*

$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c}{|x-\xi|^2}\phi = \dfrac{c}{|x-\xi|^2}(e^\phi - 1 - \phi) + \mu_0\dfrac{1}{|x-\xi|^2}e^\phi \\ \qquad\qquad + \tilde{f}(x,t) + \displaystyle\sum_{i=1}^N \mu_i V_{i,\xi} \qquad \text{in } B \\ \phi = -\log\dfrac{1}{|x-\xi|^2} \qquad \text{on } \partial B. \end{cases} \tag{84}$$

If $N \ge 5$, we have in addition that:

- *the map $\xi \in B_{\epsilon_0} \mapsto \phi(\cdot,\xi)$ is differentiable in the sense that*

$$D_\xi\phi(x,\xi)\eta = \lim_{\tau\to 0}\frac{1}{\tau}(\phi(x,\xi+\tau\eta) - \phi(x,\xi)) \quad \text{exists for all } x \in B\backslash\{\xi\}$$

  and $\eta \in \mathbb{R}^N$.
- *for $\bar{\nu} < 0$ small, $D_\xi\phi(\cdot,\xi) \in C^{2,\alpha}_{\bar{\nu}-1,\xi}(B)$, the maps $\xi \in B_{\epsilon_0} \mapsto \mu_0, \mu_i \in \mathbb{R}$ are differentiable and there exists a constant C independent of ξ such that*

$$\|D_\xi\phi(\cdot,\xi)\|_{2,\alpha,\bar{\nu}-1,\xi;B} + |D_\xi\mu_0| + \sum_{k=1}^{k_1}\sum_{l=1}^{m_k}|D_\xi\mu_{k,l}| \le C. \tag{85}$$

Proof. Case $N \ge 5$. Let ϵ_0 be as in Lemma 2.8. Consider the Banach space X of functions $\phi(x,\xi)$ defined for $x \in B$, $\xi \in B_{\epsilon_0}$, which are twice continuously differentiable with respect to x and once with respect to ξ for $x \neq \xi$ for which the following norm is finite

$$\|\phi\|_X = \sup_{\xi\in B_{\epsilon_0}} \|\phi(\cdot,\xi)\|_{2,\alpha,0,\xi;B} + \lambda\|D_\xi\phi(\cdot,\xi)\|_{2,\alpha,\bar{\nu}-1,\xi;B},$$

where $\lambda > 0$ is a parameter to be fixed later on and $\bar{\nu} < 0$ is close to zero.

Let $\mathcal{B}_R = \{\phi \in X \mid \|\phi\|_X \le R\}$. Using Lemma 2.8 we may define a nonlinear map $F : \mathcal{B}_R \to X$ by $F(\psi) = \phi$, where $\phi(\cdot,\xi)$ is the solution to (64) with

$$g = c(e^\psi - 1 - \psi) + |x-\xi|^2\tilde{f}(x,t), \quad g_0 = e^\psi, \quad h = -\log\frac{1}{|x-\xi|^2}. \tag{86}$$

We shall choose later on $R > 0$ small. Observe that in Lemma 2.8 the constants C in (65) and ϵ_0 associated to $g_0 = e^\psi$, stay bounded and bounded away from zero respectively as we make R smaller, since $e^{-R} \le e^\psi \le e^R$ for $\psi \in \mathcal{B}_R$.

Let us show that if t is small then one can choose R small and $\lambda > 0$ small so that $F : \mathcal{B}_R \to \mathcal{B}_R$. Indeed, let $\psi \in \mathcal{B}_R$ and $\phi = F(\psi)$. Then by (65), (83) we have

$$\begin{aligned}\|\phi\|_{2,\alpha,0,\xi;B} &\le C(\|c(e^\psi - 1 - \psi) + |x-\xi|^2 \tilde f(x,t)\|_{0,\alpha,0,\xi;B} + |\xi|)\\ &\le C(R^2 + |t| + |\xi|) < \frac{R}{2},\end{aligned} \tag{87}$$

provided R is first taken small enough and then $|t|$ and $|\xi| < \epsilon_0$ are chosen small. Similarly, recalling Remark 2.7,

$$\begin{aligned}\|D_\xi\phi\|_{2,\alpha,\bar\nu-1,\xi;B} &\le C(\|c(e^\psi - 1 - \psi) + |x-\xi|^2 \tilde f(x,t)\|_{0,\alpha,0,\xi;B}\\ &\quad + \|cD_\xi(e^\psi - 1 - \psi) + D_\xi(|x-\xi|^2 \tilde f(x,t))\|_{0,\alpha,\bar\nu-1,\xi;B} + 1)\\ &\le C\left(R^2 + t + \frac{R^2}{\lambda} + 1\right) \le \frac{R}{2\lambda},\end{aligned}$$

if we choose now λ small enough.

Next we show that F is a contraction on $\mathcal{B}_R$. Let ψ_1, $\psi_2 \in \mathcal{B}_R$ and $\phi_\ell = F(\psi_\ell)$, $\ell = 1, 2$. Let $\mu_i^{(\ell)}$, $i = 0, \ldots, N$ be the constants in (64) associated with ψ_ℓ. By (65) and repeating the calculation in (87)

$$\sum_{i=0}^N |\mu_i^{(\ell)}| \le R. \tag{88}$$

Let $\phi = \phi_1 - \phi_2$. Then ϕ satisfies

$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c}{|x-\xi|^2}\phi = c\left(\dfrac{e^{\psi_1} - 1 - \psi_1}{|x-\xi|^2} - \dfrac{e^{\psi_2} - 1 - \psi_2}{|x-\xi|^2}\right) & \\ \qquad + \mu_0^{(2)} \dfrac{e^{\psi_1} - e^{\psi_2}}{|x-\xi|^2} + (\mu_0^{(1)} - \mu_0^{(2)}) \dfrac{e^{\psi_1}}{|x-\xi|^2} & \\ \qquad + \displaystyle\sum_{i=1}^N (\mu_i^{(1)} - \mu_i^{(2)}) V_{i,\xi} & \text{in } B \\ \phi = 0 & \text{on } \partial B. \end{cases} \tag{89}$$

Apply (65) with $g_0 = \frac{e^{\psi_1}}{|x-\xi|^2}$, $h = 0$ and

$$g := c\left(\frac{e^{\psi_1} - 1 - \psi_1}{|x-\xi|^2} - \frac{e^{\psi_2} - 1 - \psi_2}{|x-\xi|^2}\right) + \mu_0^{(2)} \frac{e^{\psi_1} - e^{\psi_2}}{|x-\xi|^2}, \tag{90}$$

to conclude that

$$\|\phi\|_{2,\alpha,0,\xi} + \sum_{i=0}^N |\mu_i^{(1)} - \mu_i^{(2)}| \le C\|g\|_{0,\alpha,0,\xi}. \tag{91}$$

Using (88), we have in particular that $|\mu_0^{(2)}| \le R$ and it follows from (90) and (91) that

$$\|\phi_1 - \phi_2\|_{2,\alpha,0,\xi} \le CR\|\psi_1 - \psi_2\|_{2,\alpha,0,\xi}. \tag{92}$$

Thanks to (68) we also have the bound

$$\begin{aligned} \|D_\xi(\phi_1 - \phi_2)\|_{1,\alpha,\bar\nu-1,\xi;B} &\le C(\|e^{\psi_1} - \psi_1 - (e^{\psi_1} - \psi_2)\|_{0,\alpha,0,\xi;B} \\ &\quad + \|D_\xi(e^{\psi_1} - \psi_1 - (e^{\psi_1} - \psi_2))\|_{0,\alpha,\bar\nu-1,\xi;B}) \\ &\le CR\|\psi_1 - \psi_2\|_{2,\alpha,0,\nu;B} + CR\|D_\xi(\psi_1 - \psi_2)\|_{0,\alpha,\bar\nu-1,\xi;B} \end{aligned} \tag{93}$$

Combining (92), (93) we obtain

$$\|F(\psi_1) - F(\psi_2)\|_X \le CR\|\psi_1 - \psi_2\|_X.$$

This shows that F is a contraction if R is taken small enough.

Case $N = 4$. In this case (44) fails for $\nu = 0$ and estimates like (67) or (68) may not hold. So we work with the Banach space X of functions $\phi(x,\xi)$ which are twice continuously differentiable with respect to x and continuous with respect to ξ for $x \ne \xi$, for which the norm

$$\|\phi\|_X = \sup_{\xi \in B_{\epsilon_0}} \|\phi(\cdot,\xi)\|_{2,\alpha,0,\xi;B}$$

is finite. Working as in the previous case, we easily obtain that F is a contraction on some ball $\mathcal{B}_R$ of X. □

Proof of Theorem 1.3. We define the map $(\xi,t) \mapsto \phi(\xi,t)$ as the small solution to (84) constructed in Lemma 3.1 for t, ξ small. We need to show that for t small enough there is a choice of ξ such that $\mu_i = 0$ for $i = 1,\dots,N$. Let

$$\widehat{V}_j(x;\xi) = W_{1,j}(x-\xi)\eta_1(|x-\xi|), \quad j = 0,\dots,N, \tag{94}$$

where $\eta_1 \in C^\infty(\mathbb{R})$ is a cut-off function such that $0 \le \eta_1 \le 1$,

$$\begin{cases} \eta_1(r) = 0 & \text{for } r \le \dfrac{1}{8}, \\ \eta_1(r) = 1 & \text{for } r \ge \dfrac{1}{4}. \end{cases} \tag{95}$$

Multiplication of (84) by $\widehat{V}_j(x;\xi)$ and integration in B gives

$$\begin{aligned} &\int_B \left(-\Delta\widehat{V}_j(x;\xi) - L_t\widehat{V}_j(x;\xi) - \frac{c}{|x-\xi|^2}\widehat{V}_j(x;\xi)\right)\phi \\ &\quad + \int_{\partial B} \log\frac{1}{|x-\xi|^2}\frac{\partial\widehat{V}_j(x;\xi)}{\partial n} - \int_{\partial B}\frac{\partial\phi}{\partial n}\widehat{V}_j(x;\xi) \\ &= \int_B \frac{c}{|x-\xi|^2}(e^\phi - 1 - \phi)\widehat{V}_j(x;\xi) + \mu_0\int_B \frac{e^\phi}{|x-\xi|^2}\widehat{V}_j(x;\xi) \\ &\quad + \int_B \tilde f(x,t)\widehat{V}_j(x;\xi) + \sum_{i=1}^N \mu_i \int_B V_{i,\xi}\widehat{V}_j(x;\xi). \end{aligned}$$

When $\xi = 0$ the matrix $A = A(\xi)$ defined by

$$A_{i,j}(\xi) = \int_B V_{i,\xi} \widehat{V}_j(x;\xi) \quad \text{for } i,j = 1,\dots,N$$

is diagonal and invertible and by continuity it is still invertible for small ξ. Thus, we see that $\mu_i = 0$ for $i = 1,\dots,N$ if and only if

$$H_j(\xi,t) = 0, \quad \forall j = 1,\dots,N, \tag{96}$$

where, given $j = 1,\dots,N$,

$$\begin{aligned} H_j(\xi,t) = &\int_B \frac{c}{|x-\xi|^2}(e^\phi - 1 - \phi)\widehat{V}_j(x;\xi) + \mu_0 \int_B \frac{e^\phi}{|x-\xi|^2}\widehat{V}_j(x;\xi) \\ &+ \int_B \tilde{f}(x,t)\widehat{V}_j(x;\xi) - \int_{\partial B} \log\frac{1}{|x-\xi|^2}\frac{\partial \widehat{V}_j(x;\xi)}{\partial n} + \int_{\partial B} \frac{\partial\phi}{\partial n}\widehat{V}_j(x;\xi) \\ &- \int_B \left(-\Delta\widehat{V}_j(x;\xi) - L_t\widehat{V}_j(x;\xi) - \frac{c}{|x-\xi|^2}\widehat{V}_j(x;\xi)\right)\phi. \end{aligned}$$

If this holds, then $\mu_1(\xi,t) = \cdots = \mu_N(\xi,t) = 0$ and $\phi(\xi,t)$ is the desired solution to (82) (with μ in (82) equal to $\mu_0(\xi,t)$).

Observe that

$$\begin{aligned} &\frac{\partial}{\partial\xi_k}\left[\int_{\partial B} \log\frac{1}{|x-\xi|^2}\,\frac{\partial\widehat{V}_j(x;\xi)}{\partial n}\right]_{\xi=0} \\ &\quad = 2\int_{\partial B} x_k \frac{\partial\widehat{V}_j(x;0)}{\partial n} + \int_{\partial B} \log\frac{1}{|x-\xi|^2}\frac{\partial}{\partial\xi_k}\frac{\partial\widehat{V}_j(x;\xi)}{\partial n}\bigg|_{\xi=0} \\ &\quad = 2\int_{\partial B} x_k \frac{\partial\widehat{V}_j(x;0)}{\partial n}. \end{aligned} \tag{97}$$

For $j = 1,\dots,N$ we have $W_{1,j}(x) = (|x|^{-\alpha_j^+} - |x|^{-\alpha_j^-})\varphi_j(\frac{x}{|x|})$ for $x \in \partial B$, and hence $\frac{\partial W_{1,j}}{\partial n}(x) = (\alpha_j^- - \alpha_j^+)\varphi_j(x) = \frac{\alpha_j^- - \alpha_j^+}{(\int_{S^{N-1}} x_j^2)^{1/2}}\, x_j$.

Case $N \geq 5$. By Lemma 3.1, $\phi(\cdot,\xi)$ is differentiable with respect to ξ. We may then compute the derivatives of the other terms of H_j. For instance

$$\frac{\partial}{\partial\xi_k}\int_B \frac{c}{|x-\xi|^2}(e^\phi - 1 - \phi)\widehat{V}_j(x;\xi)\bigg|_{\xi=0,t=0} = 0$$

because the expression above is quadratic in ϕ and the computation can be justified using estimate (85).

Similarly

$$\frac{\partial}{\partial\xi_k}\left[\mu_0\int_B \frac{e^\phi}{|x-\xi|^2}\widehat{V}_j(x;\xi)\right]_{\xi=0} = 0.$$

Finally, using that $\phi|_{\xi=0} \equiv 0$ and integration by parts, we find

$$\frac{\partial}{\partial \xi_k}\left[\int_{\partial B_1} \frac{\partial \phi}{\partial n}\widehat{V}_j - \int_B \left(-\Delta \widehat{V}_j - L_t \widehat{V}_j - \frac{c}{|x|^2}\widehat{V}_j\right)\phi\right]_{\xi=0,t=0}$$
$$= \int_{\partial B} \frac{\partial \widehat{V}_j}{\partial n}\frac{\partial \phi}{\partial \xi_k} - \int_B \left(-\Delta \frac{\partial \phi}{\partial \xi_k} - \frac{c}{|x|^2}\frac{\partial \phi}{\partial \xi_k}\right)\widehat{V}_j. \tag{98}$$

But when $\xi = 0$, $\frac{\partial \phi}{\partial \xi_k}$ satisfies

$$\begin{cases} -\Delta \dfrac{\partial \phi}{\partial \xi_k} - \dfrac{c}{|x|^2}\dfrac{\partial \phi}{\partial \xi_k} = \dfrac{\partial \mu_0}{\partial \xi_k}\dfrac{1}{|x|^2} + \displaystyle\sum_{i=1}^N \dfrac{\partial \mu_i}{\partial \xi_k} V_{i,0} & \text{in } B \\ \dfrac{\partial \phi}{\partial \xi_k} = 2x_k & \text{on } \partial B \end{cases} \tag{99}$$

since at $\xi = 0$, $\phi = 0$ and $\mu_i = 0$ for $0 \le i \le N$. By the conditions (18) we find $\frac{\partial \mu_0}{\partial \xi_k} = 0$ and

$$\frac{\partial \mu_i}{\partial \xi_k} = 2\frac{\displaystyle\int_{\partial B} x_k \frac{\partial W_{1,i}}{\partial \nu}}{\displaystyle\int_B V_{i,0} W_{1,i}}, \quad 1 \le i \le N. \tag{100}$$

The integral above is zero whenever $i \ne k$ and thus, using (99), (100) in (98) we obtain

$$\frac{\partial}{\partial \xi_k}\left[\int_{\partial B_1} \frac{\partial \phi}{\partial n}\widehat{V}_j - \int_B \left(-\Delta \widehat{V}_j - L_t \widehat{V}_j - \frac{c}{|x|^2}\widehat{V}_j\right)\phi\right]_{\xi=0,t=0}$$
$$= 2\int_{\partial B} x_k \frac{\partial \widehat{V}_j}{\partial \nu} - 2\frac{\displaystyle\int_{\partial B} x_k \frac{\partial W_{1,k}}{\partial \nu}}{\displaystyle\int_B V_{k,0} W_{1,k}} \int_B V_{k,0}\widehat{V}_j = 0$$

thanks to (95). This and (97) imply that the matrix $\left(\frac{\partial H_j}{\partial \xi_k}(0,0)\right)_{ij}$ is invertible.

We may then apply the Implicit Function Theorem, to conclude that there exists a differentiable curve $t \to \xi(t)$ defined for $|t|$ small, such that (96) holds for $\xi = \xi(t)$. Letting $v(x) = \log \frac{1}{|x-\xi(t)|^2} + \phi(x, \xi(t))$ for $x \in B$ and $u(y) = v(y + t\tilde{\psi}(y))$ for $y \in \Omega_t$, we conclude that u is the desired solution of (4).

Case $N = 4$. Lemma 3.1 yields no information on the differentiability of ϕ and μ_i with respect to ξ. In particular, we may not apply the Implicit Function Theorem as above. We use instead the Brouwer Fixed Point Theorem as follows. Define $H = (H_1, \ldots, H_N)$ and

$$B(\xi) = (B_1, \ldots, B_N) \quad \text{with} \quad B_j(\xi) = \int_{\partial B} \log \frac{1}{|x-\xi|^2}\frac{\partial W_{j,\xi}}{\partial n}.$$

By (97), B is differentiable and $DB(0)$ is invertible. (96) is then equivalent to

$$\xi = G(\xi),$$

where

$$G(\xi) = DB(0)^{-1}\left(DB(0)\xi - H(\xi,t)\right).$$

To apply the Brouwer Fixed Point Theorem it suffices to prove that for t, ρ small, G is a continuous function of ξ and $G : \bar{B}_\rho \to \bar{B}_\rho$. This is the object of the next two lemmas.

Lemma 3.2. *G is continuous for t, ξ small.*

Proof. Observe first that for t, ξ small such that $\|\phi\|_{L^\infty(B)} \le R$ we have

$$\begin{aligned}\|\phi\|_{L^\infty(B)} &\le C(\|c(e^\phi - 1 - \phi) + |x-\xi|^2 \tilde{f}(x,t)\|_{L^\infty(B)} + |\xi|)\\ &\le C(R\|\phi\|_{L^\infty(B)} + |t| + |\xi|),\end{aligned}$$

and we deduce (taking R smaller if necessary)

$$\|\phi\|_{L^\infty(B)} \le C(|t| + |\xi|). \tag{101}$$

Similarly

$$|\mu_i| \le C(|t| + |\xi|), \quad \forall i = 0,\ldots,N. \tag{102}$$

Now let $\xi_k \to \xi$, $\phi_k = \phi(\xi_k,t)\mu_i^{(k)}$ be the solutions and parameters associated to (84). By (101) and Eq. (84) and using elliptic estimates we see that (ϕ_k) is bounded in $C^{1,\alpha}$ on compact sets of $\bar{B}\setminus\{\xi\}$. By passing to a subsequence we may assume that $\phi_k \to \phi$ uniformly on compact sets of $\bar{B}\setminus\{\xi\}$ and by (102) that $\mu_i^{(k)} \to \mu_i$. Then ϕ is a solution of (84) with $\|\phi\|_{L^\infty(B)} \le R$ and with parameters ξ and μ_i. This solution is unique by Lemma 3.1 and this shows that in fact, the complete sequence converges. Then all terms in the definition of $H(\xi,t)$ converge. In fact

$$\int_B c\frac{e^{\phi_k} - 1 - \phi_k}{|x-\xi|^2}\widehat{V}_j(x,\xi) \to \int_B c\frac{e^{\phi} - 1 - \phi}{|x-\xi|^2}\widehat{V}_j(x,\xi) \quad \text{as } k \to \infty,$$

by dominated convergence, because

$$\left|\frac{e^{\phi_k} - 1 - \phi_k}{|x-\xi|^2}\widehat{V}_j(x,\xi)\right| \le \frac{C}{|x-\xi|^2}.$$

Similarly

$$\mu_0^{(k)}\int_B \frac{e^{\phi_k}}{|x-\xi|^2}\widehat{V}_j(x,\xi) \to \mu_0\int_B \frac{e^{\phi}}{|x-\xi|^2}\widehat{V}_j(x,\xi) \quad \text{as } k \to \infty. \tag{103}$$

□

Lemma 3.3. *If $\rho > 0$ and $|t|$ are small enough then $G : \bar{B}_\rho \to \bar{B}_\rho$.*

Proof. By (101)

$$\left| \int_B c \frac{e^{\phi(\xi)} - 1 - \phi(\xi)}{|x-\xi|^2} \widehat{V}_j(x,\xi) \right| \le C\|\phi\|^2_{L^\infty(B)} \le C(|\xi| + |t|)^2.$$

Let $\sigma > 0$ to be fixed later. From (103) we have

$$\left| \int_B \frac{e^\phi}{|x-\xi|^2} \widehat{V}_j(x,\xi) \right| \le \sigma$$

if t and ξ are small enough. Also

$$|DB(0)\xi - B(\xi)| \le C|\xi|^2,$$

and

$$\left| \int_B \tilde{f}(x,t) \widehat{V}_j(x,\xi) \right| \le C|t|$$

for some constant C. Thus if $|\xi| < \rho$ and ρ is small we have

$$|G(\xi)| \le C(\rho^2 + |t| + \sigma\rho).$$

First fix σ such that $C\sigma < \frac{1}{4}$. We can then fix $\rho > 0$ so small that $C(\rho^2 + \sigma\rho) < \frac{\rho}{2}$. Then, for $|t|$ small, $|G(\xi)| \le \rho$. □

Proof of Corollary 1.4. We have just constructed a solution $\phi \in C^{2,\alpha}_{0,\xi}(B)$ of (82), when $\xi = \xi(t)$. Change variables and let $\tilde{\phi}(y) = \phi(x)$, where $x = y + t\tilde{\psi}(y,t)$ for $y \in \Omega_t$. Then,

$$-\Delta_y \tilde{\phi} = \frac{\lambda(t)}{|x-\xi|^2} e^{\tilde{\phi}} + \Delta_y \ln \frac{1}{|x-\xi|^2} \quad \text{in } \Omega_t.$$

Letting $\tilde{\xi} = \xi + t\psi(\xi)$ and $\Psi(y) = \tilde{\phi}(y) + \ln \frac{|y-\tilde{\xi}|^2}{|x-\xi|^2}$, the above equation can be rewritten as

$$-\Delta_y \Psi = \frac{\lambda(t)}{|y-\tilde{\xi}|^2} e^{\Psi} - \frac{\lambda(0)}{|y-\tilde{\xi}|^2} \quad \text{in } \Omega_t,$$

where we used the fact that $\Delta_y \ln \frac{1}{|y-\tilde{\xi}|^2} = -\frac{\lambda(0)}{|y-\tilde{\xi}|^2}$. Since Ψ is bounded, it follows by Corollary 2.4 and the fixed point characterization of Ψ that Ψ is continuous at $y = \tilde{\xi}$. Define the sequence (Ψ_n) by

$$\Psi_n(y) = \Psi\left(\frac{1}{n}(y-\tilde{\xi}) + \tilde{\xi}\right), \quad \text{for } y \in \Omega^n := n(\Omega_t - \tilde{\xi}) + \tilde{\xi}.$$

Clearly, (Ψ_n) converges pointwise to the constant $\Psi(\tilde{\xi})$. Also, Ψ_n solves

$$-\Delta_y \Psi_n = \frac{\lambda(t)}{|y-\tilde{\xi}|^2} e^{\Psi_n} - \frac{\lambda(0)}{|y-\tilde{\xi}|^2} \quad \text{in } \Omega^n. \tag{104}$$

Away from $y = \tilde{\xi}$, the right-hand side in the above equality remains bounded. It follows by elliptic regularity that up to a subsequence, (Ψ_n) converges to $\Psi(\tilde{\xi})$ in the topology of $C^\infty(\mathbb{R}^N \backslash \{\tilde{\xi}\})$. In particular, passing to the limit for $y \neq \tilde{\xi}$ in (104), we obtain

$$0 = \frac{\lambda(t)}{|y-\tilde{\xi}|^2} e^{\Psi(\tilde{\xi})} - \frac{\lambda(0)}{|y-\tilde{\xi}|^2},$$

whence $\Psi(\tilde{\xi}) = \ln \frac{\lambda(0)}{\lambda(t)}$. Since the solution $u(t)$ of (4) we constructed is given by $u(t) = \ln \frac{1}{|y-\tilde{\xi}|^2} + \Psi$, we just have proved Corollary 1.4. □

Proof of Theorem 1.2. We recall that if $u \in H^1(\Omega)$ is an unbounded solution of (1) such that

$$\int_\Omega |\nabla\varphi|^2 \geq \lambda \int_\Omega e^u \varphi^2 \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

then $\lambda = \lambda^*$ and $u = u^*$. This result is due to Brezis and Vázquez, see [8].

Given $t > 0$ small, let $u = u(t)$ denote the solution of (4) obtained in Theorem 1.3. Since $N \geq 11$, $2(N-2) < \frac{(N-2)^2}{4}$ and it follows from Theorem 1.3 that if t is chosen small enough,

$$\lambda(t) e^{\left\| u - \ln \frac{1}{|x-\xi(t)|^2} \right\|_{L^\infty(\Omega_t)}} < \frac{(N-2)^2}{4}.$$

Hence for $\varphi \in C_0^\infty(\Omega_t)$,

$$\lambda(t) \int_{\Omega_t} e^u \varphi^2 \leq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x-\xi(t)|^2} \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2,$$

in virtue of Hardy's inequality. Hence, $u(t)$ is the extremal solution of (4). □

4. u^* is Bounded for Some Thin Domains

Proof of Theorem 1.8. We assume by contradiction that for a sequence $\varepsilon_j \searrow 0$, we have $u^*_{\varepsilon_j} \notin L^\infty(\Omega_{\varepsilon_j})$. Let $M > 0$ be a constant to be fixed later. By continuity, we can select a number λ_j with $0 < \lambda_j < \lambda^*_{\varepsilon_j}$ such that the minimal solution u_j of (8) with parameter λ_j satisfies

$$\max_{\bar{\Omega}_{\varepsilon_j}} u_j = M. \tag{105}$$

Define

$$v_j(y_1, y_2) = u_j(y_1, \varepsilon_j y_2).$$

Then v_j is defined in $\bar{\Omega}$ and satisfies

$$\begin{cases} -(\varepsilon_j^2 \Delta_{y_1} + \Delta_{y_2}) v_j = \varepsilon_j^2 \lambda_j e^{v_j} & \text{in } \Omega \\ v_j = 0 & \text{on } \partial\Omega, \end{cases} \tag{106}$$

where Δ_{y_i} denotes the Laplacian with respect to the variables y_i, $i = 1, 2$.

For some constant C_0 we have

$$\lambda^*_\varepsilon \leq \frac{C_0}{\varepsilon^2}. \tag{107}$$

Indeed, let μ_ε denote the first eigenvalue for $-\Delta$ in Ω_ε with Dirichlet boundary condition and $\varphi_\varepsilon > 0$ the associated eigenfunction, that is

$$\begin{cases} -\Delta\varphi_\varepsilon = \mu_\varepsilon\varphi_\varepsilon & \text{in } \Omega_\varepsilon \\ \varphi_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

We normalize φ_ε so that $\|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$. Multiplying (8) by φ_ε and integrating by parts we find

$$\mu_\varepsilon \int_{\Omega_\varepsilon} u_\varepsilon^* \varphi_\varepsilon = \lambda_\varepsilon^* \int_{\Omega_\varepsilon} e^{u_\varepsilon^*}\varphi_\varepsilon.$$

Since $e^u \geq u$ for all $u \in \mathbb{R}$, it follows that $\lambda_\varepsilon^* \leq \mu_\varepsilon$. But changing variables $(x_1, x_2) = (y_1, \varepsilon y_2)$ we find

$$\mu_\varepsilon = \inf_{\substack{\varphi \in C_0^\infty(\Omega_\varepsilon) \\ \varphi \neq 0}} \frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla\varphi|^2}{\displaystyle\int_{\Omega_\varepsilon} \varphi^2} = \inf_{\substack{\psi \in C_0^\infty(\Omega) \\ \psi \neq 0}} \frac{\displaystyle\int_\Omega |\nabla_{y_1}\psi|^2 + \frac{1}{\varepsilon^2}|\nabla_{y_2}\psi|^2}{\displaystyle\int_\Omega \psi^2}.$$

Fixing $\psi \in C_0^\infty(\Omega)$, $\psi \neq 0$ we deduce $\mu_\varepsilon \leq \frac{C_0}{\varepsilon^2}$. Note that $C_0 = C_0(\Omega, N)$ does not depend on M. We have just proved (107).

Next we show that for some constant C independent of j

$$\|\nabla v_j\|_{L^\infty(\Omega)} \leq C. \tag{108}$$

For this, using the uniform convexity of Ω, find $R > 0$ large enough so that for any $y_0 \in \partial\Omega$ there exists $z_0 \in \mathbb{R}^N$ such that the ball $B_R(z_0)$ satisfies $\Omega \subset B_R(z_0)$ and $y_0 \in \partial B_R(z_0)$. For convenience write for $\varepsilon > 0$

$$L_\varepsilon = \varepsilon^2\Delta_{y_1} + \Delta_{y_2}.$$

Define $\zeta(y) = R^2 - |y - z_0|^2$ so that $\zeta \geq 0$ in Ω and $-L_\varepsilon\zeta = 2\varepsilon N_1 + 2N_2$ (this can be computed easily by shifting so that z_0 is at the origin and writing $|(y_1, y_2)|^2 = |y_1|^2 + |y_2|^2$). From (107) we have the uniform bound $\varepsilon_j^2\lambda_j \leq C$. It follows from (106) and the Maximum Principle that $v_j \leq C\zeta$ with C independent of j and y_0. Since $v_j(y_0) = \zeta(y_0) = 0$, this in turn implies that

$$|\nabla v_j(y_0)| \leq C \quad \forall j,\, y_0 \in \partial\Omega. \tag{109}$$

Recall that the minimal solution u_j is strictly stable in the sense that the linearized operator $w \mapsto -\Delta w - \lambda_j e^{u_j} w$ has a positive first eigenvalue (i.e. (2) holds). By changing variables, the same holds true for the linearization of (106) at v_j, i.e. the operator $w \mapsto -L_{\varepsilon_j} w - \varepsilon_j^2\lambda_j e^{v_j} w$ has a positive first eigenvalue. This implies that we have the Maximum Principle in the form: if $w \in C^2(\bar{\Omega})$ satisfies $-L_{\varepsilon_j} w - \varepsilon_j^2\lambda_j e^{v_j} w = 0$ in Ω then

$$\max_{\bar{\Omega}} |w| \leq \max_{\partial\Omega} |w|.$$

Applying this to the partial derivatives of v_j and using (109), we deduce (108). By (105), (108) and (107) we can find subsequences, denoted for simplicity (v_j), (ε_j)

and (λ_j), such that $v_j \to v$ uniformly in $\bar{\Omega}$ and $\varepsilon_j^2 \lambda_j \to \lambda_0 \geq 0$. Multiplying (106) by $\varphi \in C_0^\infty(\Omega)$ and integrating by parts twice we find

$$-\int_\Omega v_j(\varepsilon_j^2 \Delta_{y_1} \varphi + \Delta_{y_2} \varphi) = \varepsilon_j^2 \lambda_j \int_\Omega e^{v_j} \varphi.$$

Letting $j \to \infty$ we obtain

$$-\int_\Omega v \Delta_{y_2} \varphi = \lambda_0 \int_\Omega e^v \varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

Writing $v_{y_1}(y_2) := v(y_1, y_2)$ for $(y_1, y_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \cap \Omega$, we see that for each non-empty slice

$$\Omega_{y_1} = \{y_2 \in \mathbb{R}^{N_2} : (y_1, y_2) \in \Omega\},$$

we have

$$\begin{cases} -\Delta_{y_2} v_{y_1} = \lambda_0 e^{v_{y_1}} & \text{in } \Omega_{y_1} \\ v_{y_1} = 0 & \text{on } \partial\Omega_{y_1}. \end{cases} \tag{110}$$

Let $y_j \in \Omega$ denote the point of maximum of v_j, that is, $v_j(y_j) = \max_{\bar{\Omega}} v_j = M$. For a subsequence, $y_j \to y_0 \in \bar{\Omega}$ as $j \to \infty$ and since v_j converges uniformly to v, we have $M = v_j(y_j) \to v(y_0)$. Since $v|_{\partial\Omega} = 0$, we must have $y_0 \in \Omega$.

Let $y_0 = (a, b)$ and observe that Ω_a is non-empty since $y_0 \in \Omega$. Then $v_a(y_2) = v(a, y_2)$ solves (110) in Ω_a. Moreover $\max_{\bar{\Omega}_a} v_a = M$ and v_a is weakly stable in the sense that

$$\lambda_0 \int_{\Omega_a} e^{v_a} \varphi^2 \leq \int_{\Omega_a} |\nabla \varphi|^2, \quad \forall \varphi \in C_0^\infty(\Omega_a). \tag{111}$$

To see this, let $\varphi \in C_0^\infty(\Omega_a)$ and $\chi \in C_0^\infty(\mathbb{R}^{N_1})$ be such that $\chi \equiv 1$ in a neighborhood of a and $\operatorname{supp}(\chi(y_1)\varphi(y_2)) \subset \Omega$. By stability of u_j and changing variables we have

$$\varepsilon_j^2 \lambda_j \int_\Omega e^{v_j} \chi(y_1)^2 \varphi(y_2)^2 \leq \int_\Omega \varepsilon_j^2 \varphi(y_2)^2 |\nabla \chi(y_1)|^2 + \chi(y_1)^2 |\nabla \varphi(y_2)|^2.$$

Letting $j \to \infty$ yields

$$\lambda_0 \int_\Omega e^v \chi(y_1)^2 \varphi(y_2)^2 \leq \int_\Omega \chi(y_1)^2 |\nabla \varphi(y_2)|^2.$$

Choosing a sequence $\chi_k \in C_0^\infty(\mathbb{R}^{N_1})$ such that $\chi_k \equiv 1$ in a neighborhood of a and $\operatorname{supp}(\chi_k) \subset B_{1/k}(a)$ we obtain (111).

Let $y_{\min}^{(1)} = \min\{y_1 : \Omega_{y_1} \neq \emptyset\}$, $y_{\max}^{(1)} = \max\{y_1 : \Omega_{y_1} \neq \emptyset\}$. For any $y_{\min}^{(1)} < y_1 < y_{\max}^{(1)}$ the slice Ω_{y_1} is a smooth open non-empty set and hence for the problem

$$\begin{cases} -\Delta_{y_2} v = \lambda e^v & \text{in } \Omega_{y_1} \\ v = 0 & \text{on } \partial\Omega_{y_1} \end{cases} \tag{112}$$

there exists a number $0 < \lambda_{y_1}^* < \infty$ such that (see [6, 8, 19, 22])

- if $0 \le \lambda < \lambda^*_{y_1}$ then (112) has a unique minimal solution $v_{y_1,\lambda}$. Moreover $v_{y_1,\lambda}$ is smooth and characterized as the unique semi-stable solution to (112), i.e. the unique solution satisfying

$$\lambda \int_{\Omega_{y_1}} e^{v_{y_1,\lambda}} \varphi^2 \le \int_{\Omega_{y_1}} |\nabla \varphi|^2, \quad \forall \varphi \in C_0^\infty(\Omega_{y_1}).$$

- If $\lambda > \lambda^*_{y_1}$ then (112) has no weak solution.
- If $\lambda = \lambda^*_{y_1}$ then (112) has a unique weak solution $v^*_{y_1}$ and $v^*_{y_1} = \lim_{\lambda \nearrow \lambda^*_{y_1}} v_{y_1,\lambda}$.
- If $N_2 \le 9$ (recall that $\Omega_{y_1} \subset \mathbb{R}^{N_2}$) then $v^*_{y_1}$ is bounded.

We claim that for any $\lambda > 0$ there exists $M_\lambda > 0$ depending only on Ω and λ such that such for any $y^{(1)}_{\min} < y_1 < y^{(1)}_{\max}$ we have

$$\max_{\bar{\Omega}_{y_1}} v_{y_1,\lambda} \le M_\lambda. \tag{113}$$

That is, we assert that if we have some *a priori* control on λ, the boundedness of $v_{y_1,\lambda}$ is uniform when $y^{(1)}_{\min} < y_1 < y^{(1)}_{\max}$.

Using (107) we have the bound $\lambda_0 \le C_0$. Hence, choosing $M = M_{\lambda_0} + 1$ at the beginning of the proof, (113) contradicts (105).

Proof of (113). The argument is the same as in [13, 22] but we shall emphasize that the bound does not depend on $y^{(1)}_{\min} < y_1 < y^{(1)}_{\max}$.

For simplicity we write $v = v_{y_1,\lambda}$. Let $0 < \alpha < 2$ and multiply Eq. (112) by $e^{2\alpha v} - 1$. Integrating in Ω_{y_1} we find

$$2\alpha \int_{\Omega_{y_1}} e^{2\alpha v} |\nabla v|^2 = \lambda \int_{\Omega_{y_1}} (e^{(2\alpha+1)v} - e^v). \tag{114}$$

Using (111) with $e^{\alpha v} - 1$ yields

$$\lambda \int_{\Omega_{y_1}} e^v (e^{\alpha v} - 1)^2 \le \alpha^2 \int_{\Omega_{y_1}} e^{2\alpha v} |\nabla v|^2. \tag{115}$$

Combining (114) and (115) gives

$$\left(1 - \frac{\alpha}{2}\right) \int_{\Omega_{y_1}} e^{(2\alpha+1)v} \le 2 \int_{\Omega_{y_1}} e^{(\alpha+1)v} \le 2 \left[\int_{\Omega_{y_1}} e^{(2\alpha+1)v}\right]^{\frac{\alpha+1}{2\alpha+1}} |\Omega_{y_1}|^{1 - \frac{\alpha+1}{2\alpha+1}}.$$

For $0 < p < 5$ we deduce the bound

$$\|e^v\|_{L^p(\Omega_{y_1})} \le C$$

with C independent of $y^{(1)}_{\min} < y_1 < y^{(1)}_{\max}$.

In dimension $N_2 \le 9$, we thus have $\|e^v\|_{L^p(\Omega_{y_1})} \le C$ for some $p > N_2/2$. Recalling (112), this shows that $\|v\|_{L^\infty(\Omega_{y_1})} \le C$ and the constant is independent of $y^{(1)}_{\min} < y_1 < y^{(1)}_{\max}$, as can be seen using Moser's iteration technique and working on a large ball U such that $\Omega_{y_1} \subset U$ for all $y^{(1)}_{\min} < y_1 < y^{(1)}_{\max}$, considering all functions on Ω_{y_1} to be extended by zero in $U \backslash \Omega_{y_1}$. $\square$

Acknowledgments

J. Dávila was partially supported by Fondecyt Grant 1050725, Fondap Matemáticas Aplicadas and an invitation of Université Picardie Jules Verne. L. Dupaigne was partially supported by Fondecyt Grants 7040153 and 7050147.

References

[1] S. Axler, P. Bourdon and W. Ramey, *Harmonic Function Theory*, Graduate Texts in Mathematics, Vol. 137, 2nd edn. (Springer-Verlag, New York, 2001).

[2] S. Baraket and F. Pacard, Construction of singular limits for a semilinear elliptic equation in dimension 2, *Calc. Var. Partial Differential Equations* **6**(1) (1998) 1–38.

[3] P. Baras and J. A. Goldstein, The heat equation with a singular potential, *Trans. Amer. Math. Soc.* **284** (1984) 121–139.

[4] M. Berger, P. Gauduchon and E. Mazet, *Le spectre d'une variété Riemannienne* (Springer Verlag, Berlin, 1971).

[5] H. Brezis, *Is the failure of the inverse function theorem?* in *Morse Theory, Minimax Theory and Their Applications in Nonlinear Differential Equations*, Proc. Workshop Chinese Acad. of Sciences, Beijing, 1999, eds. H. Brezis *et al.* (International Press, 2003), pp. 23–33.

[6] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, Blow up for $u_t - \Delta u = g(u)$ revisited, *Adv. Differential Equ.* **1**(1) (1996) 73–90.

[7] H. Brezis and X. Cabré, Some simple nonlinear PDEs without solutions, *Boll. Unione Mat. Ital.* **1** (1998) 223–262.

[8] H. Brezis and J. L. Vázquez, Blow-up solutions of some nonlinear elliptic problems, *Rev. Mat. Univ. Complut. Madrid* **10** (1997) 443–469.

[9] M. F. Bidaut-Véron and L. Véron, Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations, *Invent. Math.* **106**(3) (1991) 489–539.

[10] X. Cabré and A. Capella, Regularity of radial minimizers and extremal solutions of semilinear elliptic equations, *J. Funct. Anal.* **238** (2006) 709–733.

[11] L. Caffarelli, R. Hardt and L. Simon, Minimal surface with isolated singularities, *Manuscripta Math.* **48** (1984) 1–18.

[12] D. S. Cohen and H. B. Keller, Some positone problems suggested by nonlinear heat generation, *J. Math. Mech.* **16** (1967) 1361–1376.

[13] M. G. Crandall and P. Rabinowitz, Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, *Arch. Rational Mech. Anal.* **58**(3) (1975) 207–218.

[14] E. N. Dancer, Weakly nonlinear Dirichlet problems on long or thin domains, *Mem. Amer. Math. Soc.* **501** (1993) 1–66.

[15] J. Dávila and L. Dupaigne, Comparison results for PDEs with a singular potential, *Proc. Roy. Soc. Edinburgh Sect. A* **133** (2003) 61–83.

[16] L. Dupaigne, A nonlinear elliptic PDE with the inverse square potential, *J. Anal. Math.* **86** (2002) 359–398.

[17] I. M. Gelfand, Some problems in the theory of quasilinear equations, Section 15, due to G. I. Barenblatt, *Amer. Math. Soc. Transl. II. Ser.* **29** (1963) 295–381.

[18] D. D. Joseph and T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, *Arch. Rational Mech. Anal.* **49** (1972/73) 241–269.

[19] Y. Martel, Uniqueness of weak extremal solutions of nonlinear elliptic problems, *Houston J. Math.* **23**(1) (1997) 161–168.

[20] R. Mazzeo and N. Smale, Conformally flat metrics of constant positive scalar curvature on subdomains of the sphere, *J. Differential Geom.* **34** (1991) 581–621.

[21] R. Mazzeo and F. Pacard, A construction of singular solutions for a semilinear elliptic equation using asymptotic analysis, *J. Differential Geom.* **44** (1996) 331–370.

[22] F. Mignot and J. P. Puel, Sur une classe de problèmes non linéaires avec nonlinéarité positive, croissante, convexe, *Comm. Partial Differential Equations* **5**(8) (1980) 791–836.

[23] F. Pacard, Solutions de $\Delta u = -\lambda e^u$ ayant des singularités ponctuelles prescrites, *C. R. Acad. Sci. Paris Sér. I Math.* **311**(6) (1990) 317–320.

[24] F. Pacard, Existence de solutions faibles positives de $-\Delta u = u^\alpha$ dans des ouverts bornés de R^n, $n \geq 3$, *C. R. Acad. Sci. Paris Sér. I Math.* **315**(7) (1992) 793–798.

[25] Y. Rébaï, Solutions of semilinear elliptic equations with one isolated singularity, *Differential Integral Equations* **12**(4) (1999) 563–581.

[26] Y. Rébaï, Solutions of semilinear elliptic equations with many isolated singularities: The unstable case, *Nonlinear Anal.* **38**(2) (1999) 173–191.

[27] Y. Rébaï, Weak solutions of nonlinear elliptic equations with prescribed singular set, *J. Differential Equations* **127**(2) (1996) 439–453.

[28] J. L. Vázquez and E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, *J. Funct. Anal.* **173**(1) (2000) 103–153.