Communications in Contemporary Mathematics
Vol. 6, No. 1 (2004) 165–193

GLOBAL REGULARITY FOR A SINGULAR EQUATION AND LOCAL H^1 MINIMIZERS OF A NONDIFFERENTIABLE FUNCTIONAL

JUAN DÁVILA
Departamento de Ingeniería Matemática, CMM,
Universidad de Chile, Casilla 170/3 – Correo 3, Santiago, Chile
jdavila@dim.uchile.cl

Received 10 December 2002

We prove optimal Hölder estimates up to the boundary for the maximal solution of a singular elliptic equation. The techniques used in this argument are applied to show that in some situations the maximal solution is a local minimizer of the corresponding functional in the topology of H^1.

Keywords: Singular elliptic equation; gradient estimates.

Mathematics Subject Classification 2000: 35J65, 35R05, 35R35

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. We are interested in nonnegative solutions to the equation

$$\begin{cases} -\Delta u + u^{-\beta} = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $0 < \beta < 1$, $\lambda > 0$ and $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is a nonnegative function, measurable in x, and increasing and concave in u for a.e. $x \in \Omega$. We assume also that $f_u(x,\cdot)$ is continuous on $(0,\infty)$ for a.e. $x \in \Omega$ and that f is sublinear in u uniformly in x, that is,

$$\lim_{u\to\infty} \frac{f(x,u)}{u} = 0 \quad \text{uniformly for } x \in \Omega\,. \tag{1.2}$$

For a function $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ and $u > 0$ in Ω, it is clear what it means to be a solution of (1.1). If a function $u \ge 0$ vanishes in parts of the domain, we replace (1.1) by

$$\begin{cases} -\Delta u = \chi_{\{u>0\}}(-u^{-\beta} + \lambda f(x,u)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.3}$$

where $\chi_{\{u>0\}}$ stands for the characteristic function of the set $\{u > 0\}$.

Definition 1.1. We say that a function $u \in H_0^1(\Omega)$ is a solution of (1.3) if $u \geq 0$,

$$-u^{-\beta} + \lambda f(x,u) \in L^1(\{u > 0\}),$$

and

$$\int_\Omega \nabla u \nabla \varphi = \int_{\{u>0\}} (-u^{-\beta} + \lambda f(x,u))\varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

Let us define the distance function to the boundary as

$$\delta(x) = \operatorname{dist}(x, \partial\Omega).$$

The following result was proved in [2].

Theorem 1.2. *For any $\lambda > 0$ there is a unique maximal solution $\bar{u}_\lambda$ to* (1.3)*. Moreover there exists $\lambda^* \in (0,\infty)$ such that for $\lambda > \lambda^*$ the maximal solution $\bar{u}_\lambda$ is positive in Ω, belongs to $C(\bar{\Omega}) \cap C^{1,\mu}_{\text{loc}}(\Omega) \forall 0 < \mu < 1$ and satisfies*

$$a\delta \leq \bar{u}_\lambda \leq b\delta \quad \text{in } \Omega, \tag{1.4}$$

where a, b are positive constants depending on Ω, λ and f.

For $0 < \lambda \leq \lambda^*$ the maximal solution $\bar{u}_\lambda$ has regularity $C(\bar{\Omega}) \cap C^{1,\gamma}_{\text{loc}}(\Omega)$ with $\gamma = \frac{1-\beta}{1+\beta}$, and for $0 < \lambda < \lambda^*$ the set $\{\bar{u}_\lambda = 0\}$ has positive measure.

The first result in this work asserts that $\bar{u}_\lambda$ is $C^{1,\gamma}$ up to the boundary.

Theorem 1.3. *The maximal solution $\bar{u}_\lambda$ of* (1.3) *belongs to $C^{1,\gamma}(\bar{\Omega})$ with $\gamma = \frac{1-\beta}{1+\beta}$. Moreover, if $\lambda > \lambda^*$ then $\bar{u}_\lambda \in C^{1,1-\beta}(\bar{\Omega})$ and $\bar{u}_\lambda \in C^{1,\mu}_{\text{loc}}(\Omega) \forall \mu \in (0,1)$.*

Remark 1.4. Let us mention that the exponent $\gamma = \frac{1-\beta}{1+\beta}$ is the best possible for the case $\lambda \leq \lambda^*$. In the case $\lambda = \lambda^*$ there are examples where the behavior of the maximal solution near the boundary is $\delta^{\frac{2}{1+\beta}}$, see [2, Example 2.5]. When $\lambda < \lambda^*$ the maximal solution vanishes somewhere in the domain, and its behavior near the free boundary $FB = \Omega \cap \partial\{\bar{u}_\lambda > 0\}$ is of the form $\operatorname{dist}(x, FB)^{\frac{2}{1+\beta}}$ (see [8]).

The case $\lambda > \lambda^*$ is simpler from the point of view of the regularity of the maximal solution. In this case, as a consequence of (1.4) we have $|\Delta \bar{u}_\lambda| \leq C\delta^{-\beta}$. We can then immediately apply a result of Gui and Lin [7] to conclude that $\bar{u}_\lambda \in C^{1,1-\beta}(\bar{\Omega})$ (see Lemma 2.1) and the exponent $1-\beta$ is the best possible in this situation.

The difficulty in proving Theorem 1.3 stems from the fact that in general the maximal solution has a free boundary when $\lambda < \lambda^*$, which can touch the boundary of the domain. This actually happens in some cases, and in Sec. 5 we construct different examples where the following situations occur: the support of the maximal solution is compact; the support of the maximal solution "touches" $\partial\Omega$ but is not the entire domain; and the set where the maximal solution vanishes is compact.

In these examples f depends on x, but when $f = f(u)$ we can say something about the support of $\bar{u}_\lambda$. For example, it can not be compact (see Sec. 5 for details).

The proof of Theorem 1.3 that we present here relies on the approach first developed by Phillips [8], and then applied to obtain the interior regularity for

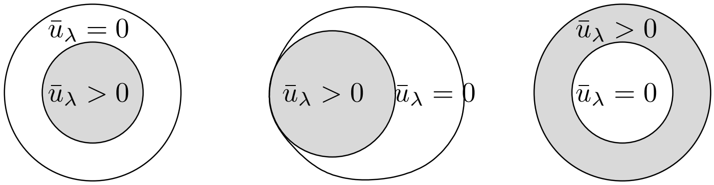

Fig. 1. Possible situations for the support of $\bar{u}_\lambda$.

(1.3) in [2], as well as on some estimates of Gui and Lin [7]. Using other techniques Giaquinta and Giusti [5, 6] (see also [4]) proved interior gradient estimates for local minimizers of general nondifferentiable functionals, which include the functional Φ defined in (1.5) below. It is not clear though that those results can be applied to our situation when $\lambda \leq \lambda^*$, which is in some sense the interesting case, because it is not known whether or not $\bar{u}_\lambda$ is a local minimum of Φ in this range of λ. The second result is related to this variational property of $\bar{u}_\lambda$ in the range $\lambda > \lambda^*$.

Consider the cone K of nonnegative functions in $H_0^1(\Omega)$

$$K = \{u \in H_0^1(\Omega) | u \geq 0 \text{ a.e. in } \Omega\}$$

and for $u \in K$ let

$$\Phi(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{u^{1-\beta}}{1-\beta} - \lambda F(x, u(x))dx\,, \tag{1.5}$$

where $F(x,u) = \int_0^u f(x,t)dt$.

Our second result is the following:

Theorem 1.5. *For $\lambda > \lambda^* \bar{u}_\lambda$ is a strict local minimum of Φ on K in the H^1 topology, that is, there exists $\rho > 0$ such that for $u \in K$ with $0 < \|u - \bar{u}_\lambda\|_{H^1} < \rho$, we have*

$$\Phi(\bar{u}_\lambda) < \Phi(u)\,.$$

The strategy in the proof of Theorem 1.5 consists of the two following steps:

(1) first we show that $\bar{u}_\lambda$ is a strict local minimum of Φ in the C^1 topology, which makes sense because of Theorem 1.3.
(2) Then we prove that a local minimum of Φ in the C^1 topology is also a local minimum in the H^1 topology.

The reason for the first claim is that the first eigenvalue for the linearization of (1.3) at $\bar{u}_\lambda$ is positive for $\lambda > \lambda^*$, that is

$$\Lambda(\bar{u}_\lambda) > 0 \quad \forall \lambda > \lambda^*\,, \tag{1.6}$$

where $\Lambda(u)$ is given, for a function $u > 0$ a.e. in Ω, by

$$\Lambda(u) = \inf_{\|\varphi\|_{L^2}=1} \int_\Omega |\nabla\varphi|^2 - (\beta u^{-\beta-1} + \lambda f_u(x,u))\varphi^2\,,$$

(see [2, Theorem 2.3]). Using (1.4) and (1.6) we prove in Lemma 4.1, Sec. 4, that for $\lambda > \lambda^*$ $\bar{u}_\lambda$ is a strict local minimum of Φ in the $C^1(\bar{\Omega})$ topology.

The second step is inspired by the work of Brezis and Nirenberg [1] where they proved that for a class of functionals on H_0^1, a local minimum u_0 in the C^1 topology is also a local minimum in the H^1 topology. The basic point in their proof, is to obtain estimates in $C^{1,\alpha}(\bar{\Omega})$ for the minimizer of their functional in a ball $\{u \| \|u - u_0\|_{H^1(\Omega)} \le \varepsilon\}$ that are independent of ε. The class of functionals in their work does not include Φ, as defined in (1.5).

In our case, instead of minimizing Φ in a ball $\{u \| \|u - \bar{u}_\lambda\|_{H^1(\Omega)} \le \varepsilon\}$ we consider a penalized functional:

$$\Psi_\varepsilon(u) = \Phi(u) + P_\varepsilon(u)\,,$$

where P_ε is the penalization and is given by

$$P_\varepsilon(u) = \frac{1}{\varepsilon^2}\left(\int_\Omega (u - \bar{u}_\lambda)^2 - \varepsilon\right)^{+^2}\,.$$

This functional depends on λ but for convenience we will omit this dependence from the notation. The infimum of Φ_ε over K is always attained. If $\bar{u}_\lambda$ is not a strict local minimum of Φ, then for any $\varepsilon > 0$ there exists a minimizer $u_\varepsilon \in K$ of Ψ_ε with $u_\varepsilon \ne \bar{u}_\lambda$ such that

$$\Psi_\varepsilon(u_\varepsilon) \le \Phi(\bar{u})\,.$$

(see Sec. 4 for details). The key result we will derive in Sec. 3 is

Theorem 1.6. *Let $\lambda > 0$ be fixed and for $\varepsilon > 0$ let u_ε be a minimizer of Ψ_ε. Then there exists $C > 0$ independent of ε such that*

$$\|u_\varepsilon\|_{C^{1,\gamma}(\bar{\Omega})} \le C\,, \tag{1.7}$$

where $\gamma = \frac{1-\beta}{1+\beta}$.

Remark 1.7. We note that this theorem holds for any $\lambda > 0$ fixed (actually, one can let λ to vary as long as $0 \le \lambda \le \lambda_0$ with $\lambda_0 < \infty$ fixed, and then the constant in (1.7) depends on λ_0). As a consequence, if $\lambda > 0$ and the maximal solution $\bar{u}_\lambda$ is a local minimizer of Φ in the topology of C^1, then it is also a minimizer in the topology of H^1. We don't know in general, whether for $\lambda \le \lambda^*$ the maximal solution $\bar{u}_\lambda$ is a local minimizer of Φ in the C^1 topology.

In summary, in Sec. 2 we prove Theorem 1.3. Section 3 is devoted to the estimates for the minimizers of Ψ_ε and establishes Theorem 1.6. We give the necessary arguments to complete the proof of Theorem 1.5 in Sec. 4. Finally in Sec. 5 we give some constructions of maximal solutions.

2. Estimates up to the Boundary for the Maximal Solution

This section is devoted to the proof of Theorem 1.3. Throughout this section $u := \bar{u}_\lambda$ denotes the maximal solution of (1.3). We also use the following notation

$$\alpha = \frac{2}{1+\beta},$$

$$\gamma = \alpha - 1 = \frac{1-\beta}{1+\beta},$$

so that $1 < \alpha < 2$, $0 < \gamma < 1$ (recall that $0 < \beta < 1$).

We will always use the notation $\delta(x) = \operatorname{dist}(x, \partial\Omega)$, whereas the distance from x to any set A will be denoted by $\operatorname{dist}(x, A)$.

Since Ω is smooth, there is $r_0 > 0$ (possibly small) so that for $p \in \Omega$ and $r \in (0, r_0)$ one can construct an open connected set $D_{p,r}$ with the following properties:

(a) $B_{3r/4}(p) \cap \Omega \subset D_{p,r} \subset B_r(p) \cap \Omega$,
(b) the scaled domain

$$\tilde{D}_{p,r} = \frac{1}{r}(D_{p,r} - p)$$

has smooth boundary, with smoothness independent of p and r.

We will write $\tilde{D} = \tilde{D}_{p,r}$ when there is no confusion about p and r. We use also the notation

$$\partial_1 \tilde{D} = \partial\tilde{D} \cap \left(\frac{1}{r}(\partial\Omega - p)\right),$$

$$\partial_2 \tilde{D} = \partial\tilde{D} \setminus \partial_1 \tilde{D}.$$

Consider $p \in \Omega$, $r \in (0, r_0)$ and translate so that p is at the origin. Given u a solution of (1.3),we will work with the rescaled function

$$\tilde{u}(y) = r^{-\alpha} u(ry) \quad \forall y \in \tilde{D}.$$

Then $\tilde{u}$ satisfies

$$\begin{cases} -\Delta\tilde{u} = \chi_{\{\tilde{u}>0\}}(-\tilde{u}^{-\beta} + r^{2-\alpha} f(ry, r^\alpha \tilde{u}(y))) & \text{in } \tilde{D} \\ \quad \tilde{u} = 0 & \text{on } \partial_1 \tilde{D}. \end{cases} \tag{2.1}$$

The next lemma is essentially proved in [7] (see the proof of their Theorem 1.1).

Lemma 2.1. *Let U be a bounded open set with smooth boundary. Consider $k : \Omega \to \mathbb{R}$ a measurable function such that*

$$\sup_{x \in U} |k(x)| \operatorname{dist}(x, \partial U)^\beta < \infty,$$

where $\beta \in (0, 1)$. Let v solve

$$\begin{cases} \Delta v = k & \text{in } U, \\ \quad v = 0 & \text{on } \partial U. \end{cases}$$

Then

$$\|v\|_{C^{1,1-\beta}(\bar{U})} \le C \sup_{x\in U} |k(x)| \operatorname{dist}(x,\partial U)^{\beta} . \tag{2.2}$$

Remark 2.2. When $U = \tilde{D}_{p,r}$ the constant C appearing in (2.2) can be chosen independently of $p \in \Omega$ and $r \in (0, r_0)$.

The result that follows is an adaptation of [8, Theorem II]; for completeness we present its proof below.

Lemma 2.3. *There exist constants c_0, $c_1 > 0$ depending only on Ω and β with the following property. Let $p \in \Omega$, $r \in (0, r_0)$ and $\tilde{D} = \frac{1}{r}(D_{p,r} - p)$. Let $u_0 \in H^1(\tilde{D})$, $u_0 \ge 0$ and assume that*

$$\fint_{\partial\tilde{D}} u_0 \ge c_0 .$$

Then there exists $w_0 \in H^1(\tilde{D})$ satisfying

$$\begin{cases} \Delta w_0 \ge w_0^{-\beta} & \text{in } \tilde{D}, \\ w_0 = u_0 & \text{on } \partial\tilde{D}, \end{cases} \tag{2.3}$$

and

$$w_0(y) \ge c_1 \left(\fint_{\partial\tilde{D}} u_0 \right) \operatorname{dist}(y, \partial\tilde{D}), \quad \forall y \in \tilde{D}. \tag{2.4}$$

Proof. Let

$$\tilde{\delta}(y) = \operatorname{dist}(y, \partial\tilde{D}),$$

and let h be the solution to

$$\begin{cases} \Delta h = 0 & \text{in } \tilde{D}, \\ h = u_0 & \text{on } \partial\tilde{D}. \end{cases}$$

By Hopf's lemma and the strong maximum principle there is a constant $\bar{c} > 0$ (which depends on the smoothness of $\tilde{D}$, but that can be chosen independent of p, r) such that

$$h \ge \bar{c} \left(\fint_{\partial\tilde{D}} u_0 \right) \tilde{\delta} \quad \text{in } \tilde{D}. \tag{2.5}$$

Now let v solve

$$\begin{cases} -\Delta v = \tilde{\delta}^{-\beta} & \text{in } \tilde{D}, \\ v = 0 & \text{in } \partial\tilde{D}. \end{cases}$$

By Lemma 2.1 $v \in C^{1,1-\beta}(\overline{\tilde{D}})$, and therefore there exists $M > 0$ (independent of p, r) such that

$$v \le M\tilde{\delta} \quad \text{in } \tilde{D}. \tag{2.6}$$

Let $m = \fint_{\partial\tilde{D}} u_0$, set $\varepsilon = \frac{\bar{c}m}{2M}$ and define

$$w_0 = h - \varepsilon v\,.$$

Then w_0 satisfies

$$w_0 \geq c_1 m\tilde{\delta}$$

with $c_1 = \bar{c}/2$. Indeed, by (2.5) and (2.6)

$$\begin{aligned} w_0 &\geq \bar{c}m\tilde{\delta} - \varepsilon M\tilde{\delta} \\ &= \frac{1}{2}\bar{c}m\tilde{\delta}\,. \end{aligned}$$

We now check that if m is suitable large, then $\Delta w_0 \geq w_0^{-\beta}$, which is equivalent to

$$\tilde{\delta} + \left(\frac{\bar{c}m}{2M}\right)^{1+1/\beta} v \leq \left(\frac{\bar{c}m}{2M}\right)^{1/\beta} h\,.$$

In fact, on one hand

$$\tilde{\delta} + \left(\frac{\bar{c}m}{2M}\right)^{1+1/\beta} v \leq \tilde{\delta}\left(1 + \left(\frac{\bar{c}m}{2M}\right)^{1+1/\beta} M\right), \tag{2.7}$$

and on the other

$$\left(\frac{\bar{c}m}{2M}\right)^{1/\beta} h \geq \left(\frac{\bar{c}m}{2M}\right)^{1/\beta} \bar{c}m\tilde{\delta}\,. \tag{2.8}$$

By (2.7) and (2.8) it is enough to show that

$$1 + \frac{(\bar{c}m)^{1+1/\beta}}{2^{1+1/\beta}M^{1/\beta}} \leq \frac{(\bar{c}m)^{1+1/\beta}}{2^{1/\beta}M^{1/\beta}}\,,$$

which is the same as

$$1 \leq \frac{(\bar{c}m)^{1+1/\beta}}{2^{1+1/\beta}M^{1/\beta}}\,.$$

This in turn holds if $m \geq c_0$ where

$$c_0 = \frac{2}{\bar{c}}M^{1/(\beta+1)}\,.$$

□

Before proceeding we make an important observation.

Remark 2.4. The maximal solution to (1.3) is also characterized as the maximal (pointwisely) function in $H^1(\Omega)$ satisfying

$$\begin{cases} -\Delta u + \chi_{\{u>0\}}u^{-\beta} \leq \lambda f(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \Omega\,. \end{cases}$$

Now we can use a scaling argument and the previous lemma to obtain:

Lemma 2.5. *Let u denote the maximal solution to* (1.3). *Let $p \in \Omega$, $r \in (0, r_0)$ and $D = D_{p,r}$. If*

$$\fint_{\partial D} u \geq c_0 r^\alpha \,, \tag{2.9}$$

then

$$u(x) \geq c_1 \left(\fint_{\partial D} u \right) \operatorname{dist}(x, \partial D)/r \,, \quad \forall x \in D \,. \tag{2.10}$$

Proof. By translation we can assume that $p = 0$. Consider $\tilde{D} = \frac{1}{r} D$ and the rescaled function

$$\tilde{u}(y) = r^{-\alpha} u(ry) \,, \quad y \in \tilde{D} \,.$$

Then $\tilde{u}$ is the maximal solution of the rescaled problem

$$\begin{cases} -\Delta w = \chi_{\{w>0\}}(-w^{-\beta} + r^{2-\alpha} f(ry, r^\alpha w(y))) & \text{in } \tilde{D} \,, \\ \quad w = \tilde{u} & \text{on } \partial \tilde{D} \,. \end{cases} \tag{2.11}$$

We can apply Lemma 2.3 (with $u_0 = \tilde{u}$) provided $\fint_{\partial \tilde{D}} \tilde{u} \geq c_0$ which is equivalent to (2.9). Thus, if (2.9) holds we conclude that there exists w_0 satisfying (2.3) and (2.4). Since $\tilde{u}$ is the maximal solution of (2.11) we deduce that

$$\tilde{u}(y) \geq w_0(y) \geq c_1 \left(\fint_{\partial \tilde{D}} \tilde{u} \right) \operatorname{dist}(y, \partial \tilde{D}) \,, \quad \forall y \in \tilde{D} \,.$$

Rescaling back we obtain (2.10). □

We state without proof a basic elliptic estimate that will be used in the sequel.

Lemma 2.6. *Let $p \in \Omega$, $r \in (0, r_0)$ and consider $\tilde{D} = \tilde{D}_{p,r}$. Suppose that $\operatorname{dist}(0, \partial_1 \tilde{D}) < 1/4$ and suppose that $u_1 \in H^1(\tilde{D})$ satisfies*

$$\begin{cases} -\Delta u_1 \leq h & \text{in } \tilde{D} \,, \\ \quad u_1 = 0 & \text{on } \partial_1 \tilde{D} \,. \end{cases}$$

Then

$$u_1(y) \leq \bar{C} \operatorname{dist}(y, \partial_1 \tilde{D}) \left(\|h\|_{L^\infty(\tilde{D})} + \fint_{\partial \tilde{D}} |u_1| \right) \,, \quad \forall y \in B_{1/2} \,.$$

The constant $\bar{C}$ can be chosen independently of p and $r \in (0, r_0)$.

The next two lemmas provide the essential steps toward the Hölder estimates for the gradient of u. Roughly speaking, the behavior of the solution u near the boundary can be of two types: either $u \sim \delta$ or $u \sim \delta^\alpha$. The first lemma deals with the case $u \sim \delta$ near $\partial\Omega$, which is expressed concretely as condition (2.12) below.

Lemma 2.7. *There exist positive constants* θ_1, C_1 *such that if* $p \in \Omega$ *and*

$$\delta(p) \le \theta_1 u(p)^{1/\alpha} \tag{2.12}$$

then

$$|Du(p)| \le C_1 \frac{u(p)}{\delta(p)}\,.$$

Moreover, if p, $q \in \Omega$ *and in addition to* (2.12) *we have*

$$|p-q| \le \theta_1 \left(\frac{u(p)}{\delta(p)}\right)^{1/(\alpha-1)}\,, \tag{2.13}$$

then

$$|Du(p) - Du(q)| \le C_1 |p-q|^{\gamma}\,,$$

θ_1 *and* C_1 *depend only on* Ω, β *and* $\lambda\|f(x,u(x))\|_\infty$.

Proof. Define

$$L = \lambda\|f(x,u(x))\|_\infty\,. \tag{2.14}$$

Let $\bar{C}$ be the constant from Lemma 2.6, and choose

$$r = \left(\frac{u(p)}{\bar{C}(c_0+L)\delta(p)}\right)^{1/(\alpha-1)}\,.$$

Using (2.12) we see that

$$\delta(p) \le r(\theta_1^{\alpha}\bar{C}(c_0+L))^{1/(\alpha-1)}\,.$$

By choosing θ_1 small one gets

$$\delta(p) < \frac{r}{4}\,. \tag{2.15}$$

Translating we can assume that p is at the origin. Let

$$\tilde{u}(y) = r^{-\alpha}u(ry)\,, \quad y \in \tilde{D}\,,$$

and note that $\tilde{u}$ satisfies (2.1). Using Lemma 2.6 (note that $\operatorname{dist}(0,\partial\tilde{D}) < 1/4$ by (2.15)), we conclude that

$$\tilde{u}(y) \le \bar{C}\operatorname{dist}(y,\partial_1\tilde{D})\left(r^{2-\alpha}L + \fint_{\partial\tilde{D}} \tilde{u}\right) \quad \forall y \in B_{1/2}\,.$$

In particular, at $y=0$

$$\frac{\tilde{u}(0)}{\operatorname{dist}(0,\partial_1\tilde{D})} \le \bar{C}\left(r^{2-\alpha}L + \fint_{\partial\tilde{D}} \tilde{u}\right)\,. \tag{2.16}$$

But

$$\frac{\tilde{u}(0)}{\operatorname{dist}(0,\partial_1\tilde{D})} = \frac{u(p)}{r^{\alpha-1}\delta(p)} = \bar{C}(c_0+L)\,. \tag{2.17}$$

Combining (2.16) and (2.17) we see that

$$\fint_{\partial\tilde{D}} \tilde{u} \geq c_0 \,, \tag{2.18}$$

(we can assume that $r_0 < 1$, hence $r < 1$). By Lemma 2.3 we thus find that

$$\tilde{u}(y) \geq c_1 \left(\fint_{\partial\tilde{D}} \tilde{u} \right) \operatorname{dist}(y, \partial\tilde{D}) \,, \quad \forall y \in \tilde{D} \,. \tag{2.19}$$

This in combination with (2.18) implies that

$$\tilde{u}(y) \geq c_1 c_0 \operatorname{dist}(y, \partial\tilde{D}) \,, \quad \forall y \in \tilde{D} \,. \tag{2.20}$$

Write $\tilde{u} = h + v$ where h is harmonic in $\tilde{D}$ and $h = \tilde{u}$ on $\partial\tilde{D}$. Then

$$\begin{cases} -\Delta v = \chi_{\{\tilde{u}>0\}}(-\tilde{u}^{-\beta} + \lambda r^{2-\alpha} f(ry, r^\alpha \tilde{u}(y))) & \text{in } \tilde{D} \,, \\ \quad v = 0 & \text{on } \partial\tilde{D} \,. \end{cases}$$

Using (2.20) we can apply Lemma 2.1 to conclude that

$$\|v\|_{C^{1,1-\beta}(\overline{\tilde{D}})} \leq C \,.$$

To estimate h, observe that when we take $y = 0$ in (2.19) we obtain

$$\fint_{\partial\tilde{D}} \tilde{u} \leq \frac{\tilde{u}(0)}{c_1 \operatorname{dist}(0, \partial\tilde{D})} = \frac{\bar{C}(c_0 + L)}{c_1} \,.$$

Hence by standard estimates for harmonic functions

$$\|h\|_{C^2(\overline{B_{1/2} \cap \tilde{\Omega}})} \leq C \,, \quad \tilde{\Omega} = \frac{1}{r}\Omega \,,$$

and thus

$$\|\tilde{u}\|_{C^{1,1-\beta}(\overline{B_{1/2} \cap \tilde{\Omega}})} \leq C \,.$$

The definition of $\tilde{u}$ immediately yields

$$|Du(0)| = r^{\alpha-1}|D\tilde{u}(0)| \leq C r^{\alpha-1} = C_1 \frac{u(p)}{\delta(p)} \,.$$

If $q \in \Omega$ and $q = ry$ with $|y| < 1/2$, which is the same as

$$|p - q| < r/2 = \frac{1}{2} \left(\frac{u(p)}{\bar{C}(c_0 + L)\delta(p)} \right)^{1/(\alpha-1)} \,, \tag{2.21}$$

we have

$$|D\tilde{u}(0) - D\tilde{u}(y)| \leq C|y|^{1-\beta} \,.$$

Hence

$$|Du(p) - Du(q)| \leq C r^{\alpha-1} \left(\frac{|p - q|}{r} \right)^{1-\beta} \leq C|p - q|^{\alpha-1} \,.$$

This finishes the proof of the lemma (by taking θ_1 smaller if necessary, so that (2.13) implies (2.21)). □

The next lemma deals with the situation $u \sim \delta^\alpha$ near $\partial\Omega$.

Lemma 2.8. *There exists a constant $C_2 > 0$ depending only on $\lambda\|f(x,u(x))\|_\infty$, Ω and β, such that if $p \in \Omega$ and*

$$\delta(p) \geq \theta_1 u(p)^{1/\alpha} > 0\,, \tag{2.22}$$

then

$$|Du(p)| \leq C_2 u(p)^{(1-\beta)/2}\,. \tag{2.23}$$

Moreover, there is $\theta_2 > 0$ ($\theta_2 = \theta_2(\lambda\|f(x,u(x))\|_\infty, \Omega, \beta)$) such that if $q \in \Omega$ and in addition to (2.22) *one has*

$$|p - q| \leq \theta_2 u(p)^{1/\alpha}\,,$$

then

$$|Du(p) - Du(q)| \leq C_2|p-q|^\gamma\,. \tag{2.24}$$

Proof. Let L be as in (2.14) and

$$r = \left(\frac{u(p)}{\bar{C}(c_0 + L)}\right)^{1/\alpha}.$$

Translating so that $p = 0$, let $\tilde{u}(y) = r^{-\alpha}u(ry)$. Note that (2.22) and the choice of r implies that

$$\delta(p) \geq r\theta_1(\bar{C}(c_0 + L))^{1/\alpha}\,.$$

Let

$$\rho = \theta_1(\bar{C}(c_0 + L))^{1/\alpha} > 0\,.$$

Then $B_{r\rho} \subset \Omega$. By taking θ_1 smaller, we can assume that $\rho < 1$.

Elliptic estimates imply that

$$\tilde{u}(y) \leq \bar{C}\left(r^{2-\alpha}L + \fint_{\partial B_\rho} \tilde{u}\right), \quad \forall y \in B_{\rho/2}\,.$$

In particular, at $y = 0$, we find

$$\bar{C}\left(r^{2-\alpha}L + \fint_{\partial B_\rho} \tilde{u}\right) \geq \tilde{u}(0) = r^{-\alpha}u(p) = \bar{C}(c_0 + L)\,.$$

Hence

$$\fint_{\partial B_\rho} \tilde{u} \geq c_0 \geq c_0 \rho^\alpha \,. \tag{2.25}$$

Using Lemma 2.5 (applied to $\tilde{u}$ and $D = B_\rho$), we find that

$$\tilde{u}(y) \geq c_1 \left(\fint_{\partial B_\rho} \tilde{u} \right) \operatorname{dist}(y, \partial B_\rho)/\rho \geq c_1 c_0 \operatorname{dist}(y, \partial B_\rho)/\rho \quad \forall y \in B_\rho \,. \tag{2.26}$$

As in the previous lemma we write $\tilde{u} = h + v$ where h is harmonic in B_ρ and $h = \tilde{u}$ on ∂B_ρ. Using the lower bound (2.26) on $\tilde{u}$ and Lemma 2.1, we again find that

$$\|v\|_{C^{1,1-\beta}(\bar{B}_\rho)} \leq C \,.$$

To estimate h we only need an upper bound for $\fint_{\partial B_\rho} \tilde{u}$, which we get from (2.26) by setting $y = 0$

$$c_1 \fint_{\partial B_\rho} \tilde{u} \leq \tilde{u}(0) = \bar{C}(c_0 + L) \,.$$

Thus we establish

$$\|\tilde{u}\|_{C^{1,1-\beta}(\bar{B}_\rho)} \leq C \,.$$

As before, (2.23) and (2.24) follow immediately observing that $y = q/r$ satisfies $|y| < \rho$ if

$$|p - q| < \rho r = \theta_2 u(p)^{1/\alpha} \,. \qquad \square$$

Proof of Theorem 1.3. We first show that $u \in C^{1,\gamma}(\bar{\Omega})$. Let $p, q \in \Omega$, with $p \neq q$ and $u(p)$, $u(q) > 0$. We need to consider several cases.

Case 1. Suppose $\delta(p) < \theta_1 u(p)^{1/\alpha}$ and $\delta(q) < \theta_1 u(q)^{1/\alpha}$. If

$$|p - q| \leq \theta_1 \max \left(\frac{u(p)}{\delta(p)}, \frac{u(q)}{\delta(q)} \right)^{1/(\alpha-1)} ,$$

by Lemma 2.7 we immediately deduce $|Du(p) - Du(q)| \leq C|p - q|^\gamma$. Otherwise, again using Lemma 2.7

$$\begin{aligned} |Du(p) - Du(q)| &\leq |Du(p)| + |Du(q)| \\ &\leq C_1 \left(\frac{u(p)}{\delta(p)} + \frac{u(q)}{\delta(q)} \right) \\ &\leq \frac{C_1}{\theta_1^{\alpha-1}} |p - q|^{\alpha-1} \\ &= C|p - q|^\gamma \,. \end{aligned}$$

Case 2. Suppose $\delta(p) \geq \theta_1 u(p)^{1/\alpha}$ and $\delta(q) \geq \theta_1 u(q)^{1/\alpha}$. This case is analogous to the previous one, but one uses Lemma 2.8 instead of Lemma 2.7.

Case 3. Suppose $\delta(p) < \theta_1 u(p)^{1/\alpha}$ and $\delta(q) \ge \theta_1 u(q)^{1/\alpha}$. If either

$$|p - q| \le \theta_1 (u(p)/\delta(p))^{1/(\alpha-1)} \tag{2.27}$$

or

$$|p - q| \le \theta_2 u(q)^{1/\alpha}\,, \tag{2.28}$$

hold, then Lemma 2.7 or Lemma 2.8 can be used to deduce that $|Du(p) - Du(q)| \le C|p - q|^\gamma$. If neither (2.27), (2.28) hold, then

$$\begin{aligned} |Du(p) - Du(q)| &\le |Du(p)| + |Du(q)| \\ &\le C_1 \frac{u(p)}{\delta(p)} + C_2 u(q)^{(1-\beta)/2} \\ &\le \left[\frac{C_1}{\theta_1^{\alpha-1}} + \frac{C_2}{\theta_2^{\alpha}}\right] |p - q|^\gamma\,. \end{aligned}$$

Finally observe that for $\lambda > \lambda^* u = \bar{u}_\lambda$ satisfies (1.4). Therefore applying Lemma 2.1 we conclude that $u \in C^{1,1-\beta}(\bar{\Omega})$ and since $\Delta u \in L^\infty_{\rm loc}(\Omega)$ we also have $u \in C^{1,\mu}_{\rm loc}(\Omega)$ for all $\mu \in (0,1)$.

This completes the proof of Theorem 1.3. □

3. Global Estimates for the Minimizers of Ψ_ε

In this section we let u_ε denote a minimizer of Ψ_ε and we let $\bar{u} = \bar{u}_\lambda$.

We will prove Theorem 1.6 by showing that u_ε satisfies the same property derived for $\bar{u}$ in Lemma 2.5, with constants independent of ε. This will be done in Lemma 3.4 below. Then the same arguments as in Lemmas 2.7 and 2.8 and Theorem 1.3 apply to u_ε and this will establish Theorem 1.6.

We start with some observations.

Lemma 3.1. *For all $\varphi \in K$*

$$\int_\Omega \nabla u_\varepsilon \nabla \varphi + u_\varepsilon^{-\beta}\varphi \ge \int_\Omega f(x, u_\varepsilon)\varphi - M_\varepsilon \int_\Omega (u_\varepsilon - \bar{u})\varphi\,, \tag{3.1}$$

where

$$M_\varepsilon = \frac{4}{\varepsilon^2}\left(\int_\Omega |u_\varepsilon - \bar{u}|^2 - \varepsilon\right)^+\,.$$

In (3.1) $u_\varepsilon^{-\beta}$ is regarded as ∞ if $u_\varepsilon = 0$.

If $\varphi \in K$ and $\varphi \le Cu_\varepsilon$ for some $C > 0$, then we also have the opposite inequality:

$$\int_\Omega \nabla u_\varepsilon \nabla \varphi + u_\varepsilon^{-\beta}\varphi \le \int_\Omega f(x, u_\varepsilon)\varphi - M_\varepsilon \int_\Omega (u_\varepsilon - \bar{u})\varphi\,. \tag{3.2}$$

Note that since $\varphi \le Cu_\varepsilon$, the term $u_\varepsilon^{-\beta}\varphi$ is integrable in Ω.

Remark 3.2. Since in formula (3.1) $u_\varepsilon(x)^{-\beta}$ is ∞ if $u_\varepsilon(x) = 0$, the left hand side of that inequality can be infinite. To prove (3.1), we use $\Psi_\varepsilon(u_\varepsilon) \le \Psi_\varepsilon(u_\varepsilon + t\varphi)$ for any $t > 0$. The proof of (3.2) exploits $\Psi_\varepsilon(u_\varepsilon) \le \Psi_\varepsilon(u_\varepsilon - t\varphi)$ for any $t > 0$ small, noting that $u_\varepsilon - t\varphi \in K$ for t small if $\varphi \le C u_\varepsilon$.

Lemma 3.3. *$u_\varepsilon \le \bar{u}$ in Ω.*

Proof. Let

$$g_M(x,u) = -u^{-\beta} + \lambda f(x,u) - M(u - \bar{u}(x)),$$

so that

$$\frac{\partial g_M}{\partial u}(x,u) = \beta u^{-1-\beta} + \lambda f_u(x,u) - M.$$

Let $\varphi = (u_\varepsilon - \bar{u})^+ \in K$. The goal is to prove that $\varphi \equiv 0$. Since $\bar{u}$ solves (1.1) we have

$$\int_\Omega \nabla \bar{u} \nabla \varphi = \int_\Omega g_{M_\varepsilon}(x,\bar{u})\varphi. \tag{3.3}$$

Note that $\varphi \le u_\varepsilon$ and therefore we can use (3.2) to obtain

$$\int_\Omega \nabla u_\varepsilon \nabla \varphi \le \int_\Omega g_{M_\varepsilon}(x,u_\varepsilon)\varphi. \tag{3.4}$$

Subtracting (3.3) from (3.4) yields

$$\int_\Omega |\nabla\varphi|^2 \le \int_\Omega (g_{M_\varepsilon}(x,u_\varepsilon) - g_{M_\varepsilon}(x,\bar{u}))\varphi. \tag{3.5}$$

But

$$\int_\Omega |\nabla\varphi|^2 \ge \int_\Omega \frac{\partial g_{M_\varepsilon}}{\partial u}(x,\bar{u})\varphi^2, \tag{3.6}$$

by (1.6). So, from (3.5) and (3.6), we deduce that

$$0 \le \int_\Omega \left(g_{M_\varepsilon}(x,u_\varepsilon) - g_{M_\varepsilon}(x,\bar{u}) - \frac{\partial g_{M_\varepsilon}}{\partial u}(x,\bar{u})(u_\varepsilon - \bar{u})\right)(u_\varepsilon - \bar{u})^+.$$

But the integrand above is negative if $u_\varepsilon > \bar{u}$ because g_{M_ε} is strictly concave, and therefore we conclude $u_\varepsilon \le \bar{u}$ a.e. in Ω. □

Lemma 3.4. *Let $p \in \Omega$, $r \in (0, r_0)$ and $D = D_{p,r}$. Then there exists c_0, $c_1 > 0$ depending only on Ω, β and $\lambda\|f(x,\bar{u}(x))\|_\infty$ such that if*

$$\fint_{\partial D} u_\varepsilon \ge c_0 r^\alpha, \tag{3.7}$$

then

$$u_\varepsilon(x) \ge c_1 \left(\fint_{\partial D} u_\varepsilon\right) \operatorname{dist}(x,\partial D)/r, \quad \forall x \in D.$$

To prove this lemma, we shall construct a solution to a nonlocal problem.

Lemma 3.5. *Assume the hypotheses of Lemma* 3.4. *For* $v \in H^1(\Omega)$, *consider*

$$M(v) = \frac{4}{\varepsilon^2}\left(\int_\Omega |v - \bar{u}|^2 - \varepsilon\right)^+ .$$

Then there exists $w \in H^1(\Omega)$ *with* $w \equiv u_\varepsilon$ *in* $\Omega \setminus D$, $u_\varepsilon \le w \le \bar{u}$ *in* Ω, *which satisfies*

$$\begin{cases} -\Delta w + w^{-\beta} = f(x,w) + M(w)(\bar{u} - w) & \text{in } D\,, \\ w = u_\varepsilon & \text{on } \partial D \end{cases} \tag{3.8}$$

and

$$w(x) \ge c_1 \left(⨍_{\partial D} u_\varepsilon\right) \operatorname{dist}(x, \partial D)/r\,, \quad \forall x \in D\,. \tag{3.9}$$

Proof. For $m \ge 0$ consider the problem

$$\begin{cases} -\Delta w + w^{-\beta} = f(x,w) + m(\bar{u} - w) & \text{in } D\,, \\ w = u_\varepsilon & \text{on } \partial D\,. \end{cases} \tag{$\mathcal{P}_m$}$$

Let $\underline{w}$ the function obtained in Lemma 2.3 properly rescaled to be defined in D, with $\underline{w} = u_\varepsilon$ on ∂D. We recall that $\underline{w}$ satisfies $\Delta\underline{w} \ge \underline{w}^{-\beta}$ and

$$\underline{w}(x) \ge c_1 \left(⨍_{\partial D} u_\varepsilon\right) \operatorname{dist}(x, D)/r\,. \tag{3.10}$$

We will establish the following properties:

(i) For any $m \ge 0$ there is a unique maximal solution w_m of $(\mathcal{P}_m)$ such that $\underline{w} \le w_m \le \bar{u}$.
(ii) w_m is nondecreasing with respect to m.
(iii) The map $m \in [0,\infty) \mapsto w_m$ is continuous in $H^1(D)$.

In fact (i) follows from the method of sub and supersolutions, noting that $\underline{w}$ is a subsolution and $\bar{u}$ is a supersolution. Observe that by the maximal property of $\bar{u}$ we have $\underline{w} \le \bar{u}$.

Property (ii) follows easily from the definition of w_m.

For (iii) suppose that $m_k \ge 0$ is a sequence such that $m_k \to m$ and let $w_k = w_{m_k}$. Since $\underline{w} \le w_k \le \bar{u}$ we have from the equation $(\mathcal{P}_{m_k})$ that Δw_k is bounded in $L^\infty_{\text{loc}}(D)$, and hence w_k is bounded in $C^{1,\alpha}_{\text{loc}}(D)$. It also follows from $(\mathcal{P}_{m_k})$, the lower bound $w_k \ge \underline{w}$, (3.10) and Hardy's inequality on the domain D, that w_k is bounded in $H^1(D)$. For a subsequence (denoted the same) w_k converges in $C^{1,\alpha}_{\text{loc}}(D)$ to some function $w \in H^1(D)$ with $\underline{w} \le w \le \bar{u}$. Passing to the limit in the equations $(\mathcal{P}_{m_k})$ we see that w satisfies $(\mathcal{P}_m)$ and it only rests to verify that w is the maximal solution to that problem. To accomplish this, we observe that the functions w_k satisfy the stability property

$$\int_D (\beta w_k^{-1-\beta} + \lambda f_u(x, w_k) - m_k)\varphi^2 \le \int_D |\nabla\varphi|^2\,, \quad \forall \varphi \in C_0^\infty(D)\,.$$

Hence w satisfies

$$\int_D (\beta w^{-1-\beta} + \lambda f_u(x,w) - m)\varphi^2 \le \int_D |\nabla\varphi|^2 , \quad \forall \varphi \in C_0^\infty(D)$$

and this property, together with the fact that the function $-u^{-\beta} + \lambda f(x,u) - m(u - \bar u(x))$ is concave for a.e. x implies that w is indeed the maximal solution to $(\mathcal{P}_m)$ (the proof of this is standard, and it closely follows that of Lemma 3.3). Finally note that since w_k is bounded in $H^1(D)$ it converges weakly on $H^1(D)$ to w. Thus, to prove that $w \to w$ in $H^1(D)$ it suffices to verify that $\|w_k\|_{H^1(D)} \to \|w\|_{H^1(D)}$. But from the equation $(\mathcal{P}_{m_k})$, we see that

$$\int_D |\nabla w_k|^2 = \int_{\partial D} u_\varepsilon \frac{\partial w_k}{\partial \nu} + \int_D -w_k^{1-\beta} + \lambda f(x,w_k)w_k + m_k(\bar u - w_k)w_k . \quad (3.11)$$

Since $w_k \rightharpoonup w$ in $H^1(D)$ weakly and $u_\varepsilon|_{\partial D} \in H^{1/2}(\partial D)$, we have that

$$\int_{\partial D} u_\varepsilon \frac{\partial w_k}{\partial \nu} \to \int_{\partial D} u_\varepsilon \frac{\partial w}{\partial \nu} .$$

Hence, the right hand side of (3.11) converges to

$$\int_{\partial D} u_\varepsilon \frac{\partial w}{\partial \nu} + \int_D -w^{1-\beta} + \lambda f(x,w)w + m(\bar u - w)w = \int_D |\nabla w|^2 .$$

To complete the proof of this lemma, we extend the functions w_m to Ω by setting $w_m \equiv u_\varepsilon$ in $\Omega \setminus D$. Now consider the map $m \in [0,\infty) \mapsto M(w_m)$. By (iii) this map is continuous. We also have that this function is nonincreasing, because $w_m \le \bar u$ and (ii). We conclude that there exists $m \ge 0$ (unique) such that $m = M(w_m)$. □

Proof of Lemma 3.4. We shall show that by taking c_0 larger if necessary, under condition (3.7) the function u_ε cannot minimize Ψ_ε unless it coincides with the function w constructed in Lemma 3.5. For this purpose, let us write

$$\Psi_\varepsilon(u_\varepsilon) = \int_\Omega \frac{1}{2}|\nabla u|^2 - G(x,u_\varepsilon) + P_\varepsilon(u_\varepsilon) ,$$

where

$$G(x,u) = -\frac{u^{1-\beta}}{1-\beta} + \lambda \int_0^u f(x,t)dt .$$

Writing

$$\frac{1}{2}|\nabla u_\varepsilon|^2 = \frac{1}{2}|\nabla w|^2 + \frac{1}{2}|\nabla(u_\varepsilon - w)|^2 + \nabla w \nabla(u_\varepsilon - w)$$

we see that

$$\Psi_\varepsilon(u_\varepsilon) = \Psi_\varepsilon(w) + \frac{1}{2}\int_\Omega |\nabla(u_\varepsilon - w)|^2 + \int_\Omega \nabla w \nabla(u_\varepsilon - w)$$
$$+ \int_\Omega G(x,w) - G(x,u_\varepsilon) + P_\varepsilon(u_\varepsilon) - P_\varepsilon(w) . \quad (3.12)$$

Multiplying equation (3.8) with $u_\varepsilon - w$ and integrating by parts on D we obtain

$$\int_D \nabla w \nabla (u_\varepsilon - w) = \int_D (g(x,w) - M(w)(w-\bar{u}))(u_\varepsilon - w)\,, \tag{3.13}$$

where

$$g(x,u) = -u^{-\beta} + \lambda f(x,u) = G_u(x,u)\,. \tag{3.14}$$

But $w \equiv u_\varepsilon$ on $\Omega \setminus D$, so combining (3.13) and (3.12) we get

$$\begin{aligned}\Psi_\varepsilon(u_\varepsilon) = \Psi_\varepsilon(w) + \frac{1}{2}\int_\Omega |\nabla(u_\varepsilon - w)|^2 + \int_\Omega G(x,w) + g(x,w)(u_\varepsilon - w) - G(x,u_\varepsilon)\\ + P_\varepsilon(u_\varepsilon) - P_\varepsilon(w) - M(w)\int_\Omega (w-\bar{u})(u_\varepsilon - w)\,.\end{aligned} \tag{3.15}$$

Observe now that the derivative of P_ε at w in the direction of $u_\varepsilon - w$ is given by

$$DP_\varepsilon(w)(u_\varepsilon - w) = M(w)\int_\Omega (w-\bar{u})(u_\varepsilon - w)\,.$$

Since the function P_ε is convex, we have

$$P_\varepsilon(w) + DP_\varepsilon(w)(u_\varepsilon - w) \le P_\varepsilon(u_\varepsilon)\,, \tag{3.16}$$

and combining (3.15) with (3.16), we obtain the inequality

$$\Psi_\varepsilon(u_\varepsilon) \ge \Psi_\varepsilon(w) + \frac{1}{2}\int_\Omega |\nabla(u_\varepsilon - w)|^2 + \int_\Omega G(x,w) + g(x,w)(u_\varepsilon - w) - G(x,u_\varepsilon)\,.$$

We will show now that by taking c_0 larger if necessary, condition (3.7) implies that

$$\int_\Omega G(x,u_\varepsilon) - G(x,w) - g(x,w)(u_\varepsilon - w) \le \frac{1}{4}\int_\Omega |\nabla(u_\varepsilon - w)|^2\,. \tag{3.17}$$

For this purpose we translate so that p is at the origin and rescale our functions

$$\tilde{u}_\varepsilon(y) = r^{-\alpha} u_\varepsilon(ry)\,,$$

$$\tilde{w}(y) = r^{-\alpha} w(ry)\,,$$

for $y \in \tilde{D} = \frac{1}{r}D$. A computation then shows that (3.17) is equivalent to the estimate

$$\int_{\tilde{D}} \tilde{G}(x,\tilde{u}_\varepsilon) - \tilde{G}(x,\tilde{w}) - \tilde{g}(x,\tilde{w})(\tilde{u}_\varepsilon - \tilde{w}) \le \frac{1}{4}\int_{\tilde{D}} |\nabla(\tilde{u}_\varepsilon - \tilde{w})|^2\,.$$

where the functions $\tilde{G}$, $\tilde{g}$ are given respectively by

$$\tilde{G}(y,u) = -\frac{u^{1-\beta}}{1-\beta} + \lambda r^{2-\alpha}\int_0^u f(ry, r^\alpha t)dt\,,$$

$$\tilde{g}(y,u) = \tilde{G}_u(y,u) = -u^{-\beta} + \lambda r^{2-\alpha} f(ry, r^\alpha u)\,.$$

Let us define

$$m = \fint_{\partial\tilde{D}} \tilde{u}_\varepsilon\,,$$

and observe that condition (3.7) is equivalent to $m \geq c_0$, and that estimate (3.9) becomes

$$\tilde{w}(y) \geq c_1 m \operatorname{dist}(y, \partial\tilde{D}) \quad \forall y \in \tilde{D}\,. \tag{3.18}$$

Let us write

$$\tilde{G}(x, \tilde{u}_\varepsilon) - \tilde{G}(x, \tilde{w}) - \tilde{g}(x, \tilde{w})(\tilde{u}_\varepsilon - \tilde{w}) = A(y) + B(y)\,,$$

where

$$A(y) = -\frac{\tilde{u}_\varepsilon^{1-\beta}}{1-\beta} - \left(-\frac{\tilde{w}_\varepsilon^{1-\beta}}{1-\beta} - \tilde{w}^{-\beta}(\tilde{u}_\varepsilon - \tilde{w})\right)$$

$$B(y) = \tilde{F}(y, \tilde{u}_\varepsilon) - \tilde{F}(y, \tilde{w}) - \tilde{f}(y, \tilde{w})(\tilde{u}_\varepsilon - \tilde{w})\,.$$

We claim that

$$A(y) \leq Cm^{-1-\beta}\operatorname{dist}(y, \partial\tilde{D})^{-1-\beta}(\tilde{u}_\varepsilon - \tilde{w})^2 \quad \forall y \in \tilde{D}\,, \tag{3.19}$$

for some $C > 0$ depending only on c_1. Indeed, if $\tilde{u}_\varepsilon < \frac{1}{2}\tilde{w}$, then

$$\begin{aligned} A(y) &\leq \frac{\tilde{w}_\varepsilon^{1-\beta}}{1-\beta} \\ &\leq C\tilde{w}^{-1-\beta}(\tilde{u}_\varepsilon - \tilde{w})^2 \end{aligned}$$

and using (3.18)

$$A(y) \leq Cm^{-1-\beta}\operatorname{dist}(y, \partial\tilde{D})^{-1-\beta}(\tilde{u}_\varepsilon - \tilde{w})^2\,.$$

If, on the contrary, $\tilde{u}_\varepsilon \geq \frac{1}{2}\tilde{w}$, then

$$A(y) \leq C\beta(1+\beta)\xi(y)^{-1-\beta}(\tilde{u}_\varepsilon - \tilde{w})^2$$

where $\xi(y)$ is in the interval with endpoints $\tilde{u}_\varepsilon(y)$ and $\tilde{w}(y)$. But then, using (3.18) we find (3.19).

Now we estimate $B(y)$. When $\tilde{u}_\varepsilon < \frac{1}{2}\tilde{w}$ we have

$$\begin{aligned} B(y) &\leq \tilde{f}(y, \tilde{w})(\tilde{w} - \tilde{u}_\varepsilon) \\ &\leq r^{2-\alpha}\|f(x, w(x))\|_\infty(\tilde{w} - \tilde{u}_\varepsilon) \\ &\leq r^{2-\alpha}\|f(x, w(x))\|_\infty \frac{2}{\tilde{w}}(\tilde{w} - \tilde{u}_\varepsilon)^2 \\ &\leq Cm^{-1}r^{2-\alpha}\|f(x, w(x))\|_\infty \operatorname{dist}(y, \partial\tilde{D})^{-1}(\tilde{w} - \tilde{u}_\varepsilon)^2\,. \end{aligned}$$

When $\tilde{u}_\varepsilon(y) < \frac{1}{2}\tilde{w}(y)$ we estimate

$$B(y) = \tilde{f}_u(y, \xi(y))(\tilde{u}_\varepsilon - \tilde{w})^2 \tag{3.20}$$

where $\xi(y)$ is in the interval with endpoints $\tilde{u}_\varepsilon(y)$ and $\tilde{w}(y)$. Using that $\tilde{f}$ is concave in u and that $\tilde{f} \geq 0$, we have

$$\tilde{f}_u(y, \xi) \leq \frac{\tilde{f}(y, \xi)}{\xi}\,. \tag{3.21}$$

Observe that since $\tilde{u}_\varepsilon(y) \geq \tilde{w}(y)$ (3.18) implies that $\xi(y) \geq \frac{1}{2}c_1\, m\, \mathrm{dist}(y, \partial\tilde{D})$. Hence, from (3.20) and (3.21) we obtain

$$B(y) \leq Cm^{-1}\mathrm{dist}(y, \partial\tilde{D})^{-1}(\tilde{w} - \tilde{u}_\varepsilon)^2\,,$$

where C depends only on c_1, $\|f(x, w(x))\|_\infty$ and $\|f(x, u_\varepsilon(x))\|_\infty$. Thus

$$B(y) \leq Cm^{-1}\mathrm{dist}(y, \partial\tilde{D})^{-1}(\tilde{w} - \tilde{u}_\varepsilon)^2 \quad \forall y \in \tilde{D}\,. \tag{3.22}$$

Putting together (3.19) and (3.22), we find (for $m \geq 1$)

$$\int_{\tilde{D}} \tilde{G}(x, \tilde{u}_\varepsilon) - \tilde{G}(x, \tilde{w}) - \tilde{g}(x, \tilde{w})(\tilde{u}_\varepsilon - \tilde{w}) \leq Cm^{-1}\int_{\tilde{D}} \mathrm{dist}(y, \partial\tilde{D})^{-1-\beta}(\tilde{u}_\varepsilon - \tilde{w})^2\,.$$

By Hardy's inequality

$$\int_{\tilde{D}} \tilde{G}(x, \tilde{u}_\varepsilon) - \tilde{G}(x, \tilde{w}) - \tilde{g}(x, \tilde{w})(\tilde{u}_\varepsilon - \tilde{w}) \leq C'm^{-1}\int_{\tilde{D}} |\nabla(\tilde{u}_\varepsilon - \tilde{w})|^2\,.$$

For m large enough this yields (3.17). □

4. Proof of Theorem 1.5

Lemma 4.1. *For $\lambda > \lambda^*$, $\bar{u}_\lambda$ is a strict local minimum of Φ in the topology of $C^1(\bar{\Omega})$.*

Before the proof of this lemma we need some observations. From now on we will use the notation $\bar{u} = \bar{u}_\lambda$.

Remark 4.2. If $\lambda > \lambda^*$ then there exists $\mu > 0$ such that

$$\int_\Omega |\nabla\varphi|^2 - g_u(x, \bar{u})\varphi^2 \geq \mu\int_\Omega |\nabla\varphi|^2 \quad \forall\varphi \in C_0^\infty(\Omega)\,, \tag{4.1}$$

where $g(x, u)$ is given by (3.14).

Indeed, using (1.4) and $f_u(x, u) \leq f(x, u)/u$, we see that

$$g_u(x, \bar{u}) \leq \frac{C}{\delta^{1+\beta}}$$

for some $C > 0$. Hence, using Hardy's and then Young's inequality we find

$$\int_\Omega g_u(x, \bar{u})\varphi^2 \leq \frac{1}{2}\int_\Omega |\nabla\varphi|^2 + C\int_\Omega \varphi^2 \quad \forall\varphi \in C_0^\infty(\Omega)\,.$$

Now choose

$$\mu = \frac{\Lambda(\bar{u})}{2(\Lambda(\bar{u}) + C)}\,,$$

(recall that $\Lambda(\bar{u}) > 0$). Then for any $\varphi \in C_0^\infty(\Omega)$

$$2\mu \int_\Omega g_u(x,\bar{u})\varphi^2 \le \mu \int_\Omega |\nabla\varphi|^2 + 2\mu C \int_\Omega \varphi^2$$
$$= \mu \int_\Omega |\nabla\varphi|^2 + \Lambda(\bar{u})(1-2\mu) \int_\Omega \varphi^2 . \tag{4.2}$$

On the other hand, by definition of $\Lambda(\bar{u})$

$$\int_\Omega |\nabla\varphi|^2 - g_u(x,\bar{u})\varphi^2 \ge \Lambda(\bar{u}) \int_\Omega \varphi^2 \tag{4.3}$$

and multiplying (4.3) by $1-2\mu$ we find

$$\int_\Omega |\nabla\varphi|^2 - g_u(x,\bar{u})\varphi^2 \ge -2\mu \int_\Omega g_u(x,\bar{u})\varphi^2 + \Lambda(\bar{u})(1-2\mu) \int_\Omega \varphi^2 + 2\mu \int_\Omega |\nabla\varphi|^2$$
$$\ge \mu \int_\Omega |\nabla\varphi|^2$$

by (4.2).

We also need the following property:

Lemma 4.3. *Let $0 < m < 2$. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that if $E \subset \Omega$ is measurable and $|E| < \delta$, then*

$$\int_E \frac{\varphi^2}{\delta^m} \le \varepsilon \int_\Omega |\nabla\varphi|^2 \quad \forall \varphi \in C_0^\infty(\Omega) .$$

Proof. By contradiction, if the statement of the lemma is not true, then there is some $\varepsilon > 0$ such that for all $i = 1, 2, \dots$, one can find $E_i \subset \Omega$ with $|E_i| < 1/i$ and some $\varphi_i \in C_0^\infty(\Omega)$ such that

$$\int_{E_i} \frac{\varphi_i^2}{\delta^m} > \varepsilon \int_\Omega |\nabla\varphi_i|^2 .$$

We can assume that $\|\varphi_i\|_{H_0^1} = 1$ and hence (for a subsequence) $\varphi_i \to \varphi$ in L^2. But then, using Hardy's inequality

$$\varepsilon \le \int_{E_i} \frac{\varphi_i^2}{\delta^m} \le \left(\int_\Omega \frac{\varphi_i^2}{\delta^2}\right)^{m/2} \left(\int_{E_i} \varphi_i^2\right)^{1-m/2} \le C \left(\int_{E_i} \varphi_i^2\right)^{1-m/2} .$$

But φ_i converges in $L^2(\Omega)$ and therefore there is some $\bar{\varphi} \in L^1(\Omega)$ such that (for a subsequence) $\varphi_i^2 \le \bar{\varphi}$. Hence by dominated convergence $\int_{E_i} \varphi_i^2 \to 0$ as $i \to \infty$, a contradiction. □

Proof of Lemma 4.1. Let $\rho > 0$ and $v \in C^1(\bar{\Omega})$ with $\|v - \bar{u}\|_{C^1(\bar{\Omega})} \le \rho$. Note that since $\bar{u}$ satisfies (1.4), for $\rho > 0$ small $v \in K$.

Expanding Φ around $\bar{u}$ and using (1.3) we find

$$\Phi(v) = \Phi(\bar{u}) + \frac{1}{2}\int_\Omega |\nabla(v-\bar{u})|^2 - g_u(x,\bar{u})(v-\bar{u})^2 + \frac{1}{6}\beta(\beta+1)\int_\Omega \xi^{-\beta-2}(v-\bar{u})^3 + \int_\Omega\int_{\bar{u}}^v (v-\tau)(f_u(x,\tau)-f_u(x,\bar{u}))d\tau dx\,, \tag{4.4}$$

where $\xi = \xi(x)$ is in the interval with endpoints $\bar{u}(x)$ and $v(x)$. Using (4.1) combined with (4.4) yields

$$\Phi(v) \geq \Phi(\bar{u}) + \mu\int_\Omega |\nabla(v-\bar{u})|^2 + \frac{1}{6}\beta(\beta+1)\int_\Omega \xi^{-\beta-2}(v-\bar{u})^3 + \int_\Omega\int_{\bar{u}}^v (v-\tau)(f_u(x,\tau)-f_u(x,\bar{u}))d\tau dx\,. \tag{4.5}$$

Since $\bar{u}$ satisfies (1.4), for $\rho > 0$ small, we have the estimate

$$\xi(x) \geq \frac{1}{C}\delta(x)\,,$$

for some $C > 0$ independent of ρ. Combining this fact with $|v(x)-\bar{u}(x)| \leq C\rho\delta(x)$ we have

$$\int_\Omega \xi^{-\beta-2}|v-\bar{u}|^3 \leq C\rho\int_\Omega \frac{(v-\bar{u})^2}{\delta^{1+\beta}} \leq C\rho\int_\Omega |\nabla(v-\bar{u})|^2\,. \tag{4.6}$$

We use now Lemma 4.3 with $\varepsilon = \sigma$ ($\sigma > 0$ to be chosen below) and $m = 1$ to find a $\delta_1 > 0$ such that if $E \subset \Omega$ and $|E| < \delta_1$ then

$$\int_E \frac{\varphi^2}{\delta} \leq \sigma\int_\Omega |\nabla\varphi|^2 \quad \forall\varphi \in C_0^\infty(\Omega)\,. \tag{4.7}$$

Using again (1.4) we can find $\varepsilon > 0$ small so that

$$|\{x\in\Omega|\bar{u}(x) < \varepsilon\}| < \delta_1/2\,, \tag{4.8}$$

and also

$$\max_{\bar{\Omega}} \bar{u} \leq \frac{1}{\varepsilon}\,.$$

On the other hand, since for a.e. $x \in \Omega$, $f_u(x,\cdot)$ is continuous on $(0,\infty)$, the sequence

$$h_j(x) = \sup\{|f_u(x,\eta)-f_u(x,\theta)|\,|\eta,\theta\in[\varepsilon,1/\varepsilon], |\eta-\theta| < 1/j\}$$

converges to 0 as $j \to \infty$ for a.e. $x \in \Omega$. By Egorov's theorem there is a measurable subset $F \subset \Omega$ with

$$|\Omega\setminus F| < \delta_1/2 \tag{4.9}$$

such that $h_j \to 0$ uniformly on F. Therefore, there is some $\delta_2 > 0$ such that for all $x \in F$ and all η, $\theta \in [\varepsilon, 1/\varepsilon]$, $|\eta-\theta| < \delta_2$ one has

$$|f_u(x,\eta)-f_u(x,\theta)| < \varepsilon\,.$$

Let $E = \{\bar{u} < \varepsilon\} \cup (\Omega \setminus F)$ and split the integral

$$\int_\Omega \int_{\bar{u}}^{v} (v-\tau)(f_u(x,\tau) - f_u(x,\bar{u}))d\tau dx = \int_E \cdots + \int_{\Omega\setminus E} \cdots .$$

We first estimate the integral over E, using the fact that $f_u(x,u) \le f(x,u)/u$ and $\bar{u} \ge a\delta$, $\delta < Cv$

$$\left| \int_E \int_{\bar{u}}^{v} (v-\tau)(f_u(x,\tau) - f_u(x,\bar{u}))d\tau dx \right| \le C \int_E \frac{(v-\bar{u})^2}{\delta} .$$

Note that $|E| < \delta_1$ by (4.8) and (4.9) and therefore we can apply (4.7)

$$\left| \int_E \int_{\bar{u}}^{v} (v-\tau)(f_u(x,\tau) - f_u(x,\bar{u}))d\tau dx \right| \le C\sigma \int_\Omega |\nabla(v-\bar{u})|^2 . \tag{4.10}$$

The integral on $\Omega \setminus E$ can be estimated as well, if $\rho > 0$ is small enough so that $|v(x) - \bar{u}(x)| < \delta_2$:

$$\left| \int_{\Omega\setminus E} \int_{\bar{u}}^{v} (v-\tau)(f_u(x,\tau) - f_u(x,\bar{u}))d\tau dx \right| \le C\varepsilon \int_\Omega |\nabla(v-\bar{u})|^2 . \tag{4.11}$$

Hence, putting together (4.5), (4.6), (4.9) and (4.10) we obtain, for $\rho > 0$ small

$$\Phi(v) \ge \Phi(\bar{u}) + (\mu - C\rho - C\sigma - C\varepsilon) \int_\Omega |\nabla(v-\bar{u})|^2 .$$

We choose first $\sigma > 0$, then $\varepsilon > 0$ small and then ρ_0 so that for $0 < \rho < \rho_0$ and $\|v - \bar{u}\|_{C^1(\bar{\Omega})} < \rho$ we have

$$\Phi(v) \ge \Phi(\bar{u}) + \frac{\mu}{4} \int_\Omega |\nabla(v-\bar{u})|^2 ,$$

which proves the lemma. □

Remark 4.4. The proof of Lemma 4.1 is simpler if one assumes that f is C^2 with respect to u and satisfies

$$\sup_{x\in\Omega, u>0} |f_{uu}(x,u)| < \infty .$$

Indeed, in this case one can estimate

$$\begin{aligned} \left| \int_\Omega \int_{\bar{u}}^{v} (v-\tau)(f_u(x,\tau) - f_u(x,\bar{u}))d\tau dx \right| &\le C \sup_{x\in\Omega, u>0} |f_{uu}(x,u)| \int_\Omega |v-\bar{u}|^3 \\ &\le C\rho \int_\Omega |\nabla(v-\bar{u})|^2 . \end{aligned}$$

Proof of Theorem 1.5. We prove this theorem by contradiction. Let C_0 be such that $\|w\|_{L^2}^2 \le C_0 \|w\|_{H_0^1}^2$ $\forall w \in H_0^1$. If $\bar{u}$ is not a strict local minimum of Φ in the H^1 topology, then for all $\varepsilon > 0$ there exists $v_\varepsilon \in K$, with $0 < \|v_\varepsilon - \bar{u}\|_{H_0^1}^2 < \varepsilon/C_0$ and

$$\Phi(v_\varepsilon) \le \Phi(\bar{u}) .$$

Let u_ε be a minimizer of Ψ_ε. Then

$$\Psi_\varepsilon(u_\varepsilon) \le \Psi(v_\varepsilon) = \Phi(v_\varepsilon) \le \Phi(\bar{u})\,,$$

because $\|v_\varepsilon - \bar{u}\|^2_{L^2} < \varepsilon$ so $P_\varepsilon(v_\varepsilon) = 0$. If $u_\varepsilon \equiv \bar{u}$ then

$$\min_K \Psi_\varepsilon = \Psi_\varepsilon(\bar{u}) = \Phi(\bar{u}) \ge \Phi(v_\varepsilon) = \Psi_\varepsilon(v_\varepsilon)\,,$$

and we replace u_ε by v_ε. This shows that for all $\varepsilon > 0$ there exists a minimizer u_ε of Ψ_ε, such that $u_\varepsilon \not\equiv \bar{u}$.

Clearly $u_\varepsilon \to \bar{u}$ in $L^2(\Omega)$ and by Theorem 1.6 $u_\varepsilon \to \bar{u}$ in $C^1(\bar{\Omega})$. But this and $\Phi(u_\varepsilon) \le \Psi_\varepsilon(u_\varepsilon) \le \Phi(\bar{u})$ contradict Lemma 4.1. □

Remark 4.5. Without using Lemma 4.1 one can still show, using a standard argument, that for $\lambda > \lambda^* \bar{u}_\lambda$ is a local minimum of Φ on K in the C^1 topology, and therefore (using Theorem 1.6) also a local minimum of Φ in the H^1 topology.

Indeed, following [1], we first construct a subsolution $\underline{U} > 0$ and supersolution $\bar{U}$ to (1.1) such that $\underline{U} \le \bar{U}$. Let ζ solve

$$\begin{cases} -\Delta\zeta = 1 & \text{in } \Omega\,, \\ \zeta = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Then if $K > 0$ is large enough $\bar{U} = K\zeta$ is a supersolution. We get a positive subsolution $\underline{U}$ by taking $\underline{U} = \bar{u}_{\lambda'}$ with $\lambda' \in (\lambda^*, \lambda)$. We also see that neither $\underline{U}$ nor $\bar{U}$ are solutions to (1.1). Then the same approach as in [1] shows that there exists a minimizer u_0 of Φ in the class

$$\{u \in H^1_0 | \underline{U} \le u \le \bar{U}\}\,,$$

and that u_0 is a local minimizer of Φ in the C^1 topology.

We claim that $u_0 = \bar{u}$. Indeed, u_0 is a solution of (1.1) and since it is local minimizer of Φ it is stable. Then by [2, Theorem 2.3] (or an argument similar to the proof of Lemma 3.3) we conclude that $u_0 = \bar{u}$.

5. Some Examples

In this section we exhibit different examples where the following situations occur:

Example 5.1. $\bar{u}_\lambda \not\equiv 0$ and $\operatorname{supp}(\bar{u}_\lambda)$ is compact.

Example 5.2(a). $\operatorname{supp}(\bar{u}_\lambda)$ is not compact and not equal to Ω, and the behavior of $\bar{u}_\lambda$ near the boundary of the set $\omega = \{x \in \Omega | \bar{u}_\lambda(x) > 0\}$ is of the form $\operatorname{dist}(x, \partial\omega)^\alpha$.

Example 5.2(b). This a variation of the previous example, in which $\operatorname{supp}(\bar{u}_\lambda)$ is not compact and not equal to Ω, but $\nabla\bar{u}_\lambda(x) \ne 0$ for some points of $\partial\Omega$, that is $\bar{u}_\lambda \sim \delta$ near some parts of $\partial\Omega$.

Example 5.3. The set $\{x \in \Omega | u(x) = 0\}$ is compact.

We recall that if $v : \Omega \to \mathbb{R}$ then its support, which is denoted by $\operatorname{supp}(v)$, is defined as the closure in Ω of the set $\{ x \in \Omega | v(x) \neq 0\}$.

In all these examples the function f depends on x (and it turns out that is independent of u). In contrast with these constructions, when $f = f(u)$ we can rule out some of the previous situations.

Lemma 5.4. *Suppose that $f = f(u)$. Then $\operatorname{supp}(\bar{u}_\lambda)$ can not be compact unless $\bar{u}_\lambda \equiv 0$.*

If, in addition to the hypothesis $f = f(u)$, Ω is a ball, then $\bar{u}_\lambda \equiv 0$ for $0 < \lambda < \lambda^*$ and $\bar{u}_\lambda > 0$ in Ω for $\lambda \geq \lambda^*$.

Putting together some of the above constructions, we obtain the following.

Example 5.5. Take $f = \chi_{B_1}$ and Ω the ball B_R with $R > 1$ sufficiently large. Then there exists $0 < \lambda_0 < \lambda^*$ such that:

$$\begin{aligned} &\bar{u}_\lambda \equiv 0 && \text{for } \lambda < \lambda_0 \,, \\ &\bar{u}_\lambda \neq 0 && \text{for } \lambda_0 \leq \lambda < \lambda^* \,, \\ &\bar{u}_\lambda > 0 \text{ in } \Omega \,, && \text{for } \lambda^* < \lambda \,. \end{aligned}$$

For the constructions we need some preliminary results. We first mention a basic observation (a proof can be obtained from the results in [3]).

Lemma 5.6. *Let Ω, U be bounded, smooth domains with $\Omega \subset U$. Let u be a solution of (1.3) in the domain Ω and define*

$$v(x) = \begin{cases} u(x) & \text{if } x \in \Omega \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

Then v is a subsolution of (1.3) in the domain U.

Next we show how to get a maximal solution with compact support.

Lemma 5.7. *Let $f \in L^\infty(\mathbb{R}^n)$, $f \geq 0$ with compact support. Then there exist $R_1 > 0$, $R_0 > 0$ such that for all $R > R_1$ the maximal solution to*

$$\begin{cases} -\Delta u = \chi_{\{u>0\}}(-u^{-\beta} + f(x)) & \text{in } B_R \,, \\ \quad u = 0 & \text{on } \partial B_R \,, \end{cases} \tag{5.1}$$

has support contained in B_{R_0}.

Proof. Let $\rho > 0$, $C_1 > 0$ such that $f \leq C_1 \chi_{B_\rho}$.

We claim that it is sufficient to establish the result with $f = C_1 \chi_{B_\rho}$. In fact, if v is the maximal solution with f replaced by $C_1 \chi_{B_\rho}$, then the maximal solution u of (5.1) satisfies $u \leq v$ so that $\operatorname{supp}(u) \subset \operatorname{supp}(v) \subset B_{R_0}$.

We assume now that $f = C_1 \chi_{B_\rho}$. Take a sequence $R_k \to \infty$ and let $\bar{u}_k$ denote the maximal solution for the problem (5.1) in the domain B_{R_k}. Observe that $\bar{u}_k$ is radial (the maximal solution is unique), so that $\operatorname{supp}(\bar{u}_k)$ is a ball. If the conclusion of the lemma fails, then for a subsequence (denoted the same) $\mathrm{meas}(\mathrm{supp}(\bar u_k)) \to \infty$. We can assume that R_k is the radius of the ball $\mathrm{supp}(\bar u_k)$. Define

$$v_k(x) = R_k^{-\alpha} \bar u_k(R_k x)\,,$$

so that it satisfies

$$\begin{cases} -\Delta v_k = -v_k^{-\beta} + f_k & \text{in } B_1\,, \\ \quad v_k > 0 & \text{in } B_1\,, \\ \quad v_k = 0 & \text{on } \partial B_1\,, \end{cases}$$

where $f_k(x) = R_k^{2-\alpha} f(R_k x)$. Integrating the equation in B_1 we find

$$0 \le -\int_{\partial B_1} \frac{\partial v_k}{\partial \nu} = -\int_{B_1} v_k^{-\beta} + R_k^{-n+2-\alpha} \int_{\mathbb{R}^n} f\,.$$

So we deduce on one hand that

$$\int_{B_1} v_k^{-\beta} \to 0 \quad \text{as } k \to \infty\,. \tag{5.2}$$

But on the other hand there exists $C > 0$ independent of k such that

$$v_k(x) \le C\delta(x) \quad \forall x \in B_1 \setminus B_{1/4}\,. \tag{5.3}$$

Indeed $v_k \le \zeta_k$ where ζ_k solves

$$\begin{cases} -\Delta \zeta_k = f_k & \text{in } B_1\,, \\ \quad \zeta_k = 0 & \text{on } \partial B_1\,. \end{cases}$$

Since the functions f_k are bounded in $L^1(B_1)$ (actually $\int_{B_1} f_k \to 0$ as $k \to \infty$), and $f_k \equiv 0$ in $B_1 \setminus B_{1/4}$, by standard elliptic estimates we deduce the validity of (5.3). Hence $\int_{B_1} v_k^{-\beta}$ is bounded away from zero, which contradicts (5.2). □

Construction for Example 5.1. Fix $f \in L^\infty(\mathbb{R}^n)$, $f \ge 0$, $f \ne 0$, f with compact support. Now we fix $\lambda > 0$ large enough so that the maximal solution $\bar v$ to

$$\begin{cases} -\Delta v = \chi_{\{v>0\}}(-v^{-\beta} + \lambda f(x)) & \text{in } B_1\,, \\ \quad v = 0 & \text{on } \partial B_1\,, \end{cases}$$

is positive in B_1. Then using Lemma 5.7 we find $R > 0$ large enough so that the maximal solution $\bar u$ in $\Omega = B_R$ has compact support. Note that $\bar u \ge \bar v$ by Lemma 5.6, and therefore $\bar u \ne 0$.

Construction for Example 5.2(a). Take the solution found in the previous example and restrict it to a domain U, such that U contains the set $\{\bar u > 0\}$ and such that $\partial U \cap \partial\{\bar u > 0\} \ne \emptyset$ and $\partial U \setminus \partial\{\bar u > 0\} \ne \emptyset$. If the regularity of $\partial\{\bar u > 0\}$ is a concern, we may take f to be radial, so that $\{\bar u > 0\}$ is a ball.

For the next construction we need a modification of Lemma 5.7, which is a direct consequence of Lemmas 5.6 and 5.7.

Lemma 5.8. *Let $f \in L^\infty(\mathbb{R}^n)$, $f \geq 0$ with compact support. Then there exist $R_1 > 0$, $R_0 > 0$ such that for all $R > R_1$ and any smooth, bounded domain Ω such that Ω is contained in the half space $H := \{x = (x_1, \ldots, x_n) | x_1 > 0\}$ and $H \cap B_R \subset \Omega$, the maximal solution to*

$$\begin{cases} -\Delta u = \chi_{\{u>0\}}(-u^{-\beta} + f(x)) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

has support contained in B_{R_0}.

Construction for Example 5.2(b). Let $B = B_1(z_0)$ be the ball of radius 1 centered at a the point $z_0 = (1, 0, \ldots, 0)$ so that $B \subset H$ and $\bar{B} \cap \partial H = \{0\}$. Let $\bar{v}_\lambda$ denote the maximal solution to

$$\begin{cases} -\Delta v = \chi_{\{v>0\}}(-v^{-\beta} + \lambda) & \text{in } B\,, \\ v = 0 & \text{on } \partial B\,. \end{cases} \tag{5.4}$$

We fix a value $\lambda > \lambda^*$ where λ^* is the critical parameter for the above problem. Set

$$f = \lambda \chi_B\,.$$

By (1.4) the maximal solution $\bar{v}_\lambda$ to (5.4) satisfies $\frac{\partial \bar{v}_\lambda}{\partial \nu}(0) < 0$ (ν denotes the exterior unit normal vector to $\partial\Omega$). Take a smooth domain Ω satisfying the conditions of Lemma 5.7. Then the maximal solution $\bar{u}$ for the problem in Ω has support contained in B_{R_0}. Hence the support of $\bar{u}$ is different from Ω but $\bar{u} \geq \bar{v}$ so that $\frac{\partial \bar{u}}{\partial \nu}(0) < 0$.

Construction for Example 5.3. In this construction we consider the sequence of functions $f_k = \chi_{A_k}$ where A_k is the annulus $A_k = B_k \setminus B_{k-2}$. We shall show that there exist constants $\lambda > 0$ and $k > 0$, such that the maximal solution $\bar{u}_k$ of

$$\begin{cases} -\Delta u = \chi_{\{u>0\}}(-u^{-\beta} + \lambda f_k) & \text{in } B_k\,, \\ u = 0 & \text{on } \partial B_k\,, \end{cases}$$

satisfies the two following properties

$$\begin{cases} \bar{u}_k > 0 & \text{in } A_k\,, \\ \bar{u}_k \equiv 0 & \text{in } B_\rho\,, \end{cases}$$

for some $\rho > 0$.

To accomplish the first goal, we fix $\lambda > 0$ so that the maximal solution $\bar{v}$ to

$$\begin{cases} -\Delta v = \chi_{\{v>0\}}(-v^{-\beta} + \lambda) & \text{in } B_1\,, \\ v = 0 & \text{on } \partial B_1 \end{cases} \tag{5.5}$$

is positive in B_1. Then we deduce that $\bar{u}_k > 0$ in A_k by comparison with a suitable translation of $\bar{v}$.

It remains to verify the second property. Actually we will show that for any $\rho > 0$, $\bar{u}_k \equiv 0$ in B_ρ for k large enough. We argue by contradiction, assuming that there exists $\rho > 0$, so that for a sequence $k \to \infty$ we have $\bar{u}_k \not\equiv 0$ in B_ρ. Observe that $\bar{u}_k$ is radial. We claim that

$$\bar{u}_k > 0 \quad \text{in } B_k \setminus \bar{B}_\rho \,. \tag{5.6}$$

To see this, suppose that $\bar{u}_k(r) = 0$ for some $r \in (\rho, k)$. Recall that $\bar{u}_k \not\equiv 0$ in B_ρ so there is $r_0 \in [0, \rho)$ such that $\bar{u}_k(r_0) > 0$. Define

$$r_1 = \inf\{r \in (r_0, k) | \bar{u}_k(r) = 0\} \,.$$

Then $r_1 > r_0$, $\bar{u}_k(r_1) = 0$ and $\bar{u}_k(r) > 0$ for all $r \in (r_0, r_1)$. Let

$$w(r) = \begin{cases} \bar{u}_k(r) & \text{if } 0 \le r \le r_1 \\ 0 & \text{otherwise} \end{cases} .$$

We see that w is a solution of (5.5). Comparing $\bar{u}_k$ with $w(\cdot + \tau)$ with $|\tau|$ small, we get that $\bar{u}_k(r_1) > 0$, which is not possible and proves (5.6).

Define

$$v_k(x) = k^{-\alpha} \bar{u}_k(kx) \quad \text{and} \quad \tilde{f}_k(x) = k^{2-\alpha} f_k(kx) = k^{2-\alpha} \chi_{B_1 \setminus B_{1-2/k}}(x) \,.$$

Then

$$\begin{cases} -\Delta v_k = \chi_{\{v_k > 0\}}(-v_k^{-\beta} + \lambda \tilde{f}_k) & \text{in } B_1 \,, \\ \quad v_k = 0 & \text{on } \partial B_1 \,. \end{cases}$$

From this equation we conclude that

$$\int_{\{v_k > 0\}} v_k^{-\beta} \le \lambda \int_{B_1} \tilde{f}_k = C k^{1-\alpha} \to 0 \,,$$

as $k \to \infty$ (recall that $\alpha = \frac{2}{1+\beta} \in (1, 2)$). On the other hand $v_k \le \zeta_k$ where

$$\begin{cases} -\Delta \zeta_k = \lambda \tilde{f}_k & \text{in } B_1 \,, \\ \quad \zeta_k = 0 & \text{on } \partial B_1 \,. \end{cases}$$

Since $\tilde{f}_k \equiv 0$ in $B_{3/4}$ for k large we deduce that $v_k \le \zeta_k \le C$ in $B_{1/2}$ for some constant C independent of k. On the other hand $v_k > 0$ in $B_1 \setminus \bar{B}_{\rho/k}$ so $v_k^{-\beta} \ge C^{-\beta}$ in $B_{1/2} \setminus \bar{B}_{1/4}$ for k large, which shows that $\int_{\{v_k > 0\}} v_k^{-\beta}$ is bounded away from zero. This contradiction finishes the proof of our claim.

We now proceed with the proof of Lemma 5.4.

Proof of Lemma 5.4. Suppose that $\bar{u}_\lambda$ has compact support and $\bar{u}_\lambda \not\equiv 0$. Then for any $\tau \in \mathbb{R}^n$ with $|\tau|$ small $\bar{u}_\lambda(\cdot + \tau)$ is also a nontrivial solution. Therefore $\max(\bar{u}_\lambda, \bar{u}_\lambda(\cdot + \tau))$ is a nontrivial subsolution, but this contradicts the maximality of $\bar{u}_\lambda$.

Now suppose additionally that Ω is a ball. By uniqueness of the maximal solution $\bar{u}_\lambda$ is radial. We shall show that if $\bar{u}_\lambda(r_0) = 0$ for some $r_0 \in [0, R)$ then $\bar{u}_\lambda$ has compact support. In fact, we claim that: the set $I := \{r \in (0, R) | u(r) > 0\}$ is an interval of the form $(0, \rho)$ for some ρ.

To prove this, consider a nonempty connected component (r_0, r_2) of I and suppose that $r_0 > 0$. Then $\bar{u}_\lambda(r_0) = \bar{u}'_\lambda(r_0) = 0$. Since $\bar{u}_\lambda$ is radial let us write the equation (1.1) in the form

$$-\frac{1}{r^{n-1}} \frac{d}{dr}(r^{n-1}\bar{u}'_\lambda) = g(\bar{u}_\lambda)\,,$$

where $g(u) = -u^{-\beta} + \lambda f(u)$. Let $r_1 \in [r_0, r_2]$. Multiplying by $r^{2(n-1)}\bar{u}'_\lambda$ and integrating on $[r_0, r_1]$, we obtain

$$-\frac{1}{2}(r_1^{n-1}\bar{u}'_\lambda(r_1))^2 = r_1^{2n-2}G(\bar{u}_\lambda(r_1)) - 2(n-1)\int_{r_0}^{r_1} r^{2n-1}G(\bar{u}_\lambda(r))dr\,, \tag{5.7}$$

where

$$G(u) = -\frac{u^{1-\beta}}{1-\beta} + \lambda \int_0^u f(t)dt\,.$$

Let $\theta > 0$ be the unique positive number satisfying $G(\theta) = 0$. Note that $G(u) < 0$ for $u \in (0, \theta)$ and $G(u) > 0$ for $u > \theta$. If $\bar{u}_\lambda(r) < \theta$ for all $r \in (r_0, r_2)$, we choose $r_1 = r_2$, and then $\bar{u}_\lambda(r_1) = 0$. Otherwise, we select $r_1 \in (r_0, r_2)$ as the smallest value in (r_0, r_2), such that $\bar{u}_\lambda(r_1) = \theta$ and $\bar{u}_\lambda(r) < \theta$ for all $r \in (r_0, r_1)$. With this choice we see that (5.7) implies

$$\frac{1}{2}(r_1^{n-1}\bar{u}'_\lambda(r_1))^2 = 2(n-1)\int_{r_0}^{r_1} r^{2n-1}G(\bar{u}_\lambda(r))dr\,.$$

But the left hand side of the previous equation is nonnegative, while the right hand side is negative. This contradiction shows that $\{r \in (0, R) | u(r) > 0\} = (0, \rho)$ for some ρ.

If $\bar{u}_\lambda(0) = 0$ the same argument as above (used with $r_0 \to 0^+$) also leads to a contradiction.

Now consider $\lambda < \lambda^*$. The previous argument shows that if $\bar{u}_\lambda(r_0) = 0$ for some r_0, then $\bar{u}_\lambda$ would have compact support, which is impossible by the the first part of the lemma, unless $\bar{u}_\lambda \equiv 0$, which is the desired conclusion. □

Proof of the statements for Example 5.4. We start by fixing $R > 0$ large enough so that by Lemma 5.7 the maximal solution of

$$\begin{cases} -\Delta u = \chi_{\{u>0\}}(-u^{-\beta} + \chi_{B_1}) & \text{in } B_R\,, \\ u = 0 & \text{on } \partial B_R\,, \end{cases}$$

has compact support in B_R. We set $\Omega = B_R$.

Let

$$\lambda_0 = \inf\{\lambda > 0 | \bar{u}_\lambda \neq 0\}\,.$$

Then $\lambda_0 \le 1 < \lambda^*$ and we shall show that $\lambda_0 > 0$. Arguing by contradiction, assume that $\lambda_0 = 0$. Then for all $\lambda > 0$ we have $\bar{u}_\lambda \neq 0$.

We first observe that $\mathrm{supp}(\bar{u}_\lambda) \subset \bar{B}_1$ for $\lambda > 0$ small enough. Otherwise, we would have

$$\int_{B_1} \bar{u}_\lambda^{-\beta} \le \lambda \,\mathrm{meas}(B_1) \to 0 \quad \text{as } \lambda \to 0\,.$$

But on the other hand $\bar{u}_\lambda \le \bar{u}_{\lambda^*}$ for $\lambda \le \lambda^*$ so that $\int_{B_1} \bar{u}_\lambda^{-\beta}$ is bounded away from zero. This contradiction shows that $\mathrm{supp}(\bar{u}_\lambda) \subset \bar{B}_1$ for $\lambda > 0$ small enough. Hence for $\lambda > 0$ small, $\bar{u}_\lambda$ also solves

$$\begin{cases} -\Delta u = \chi_{\{u>0\}}(-u^{-\beta} + \lambda) & \text{in } B_1\,, \\ \quad u = 0 & \text{on } \partial B_1\,. \end{cases}$$

But now we see that $\bar{u}_\lambda$ solves a problem with a right hand side independent of x and therefore, by Lemma 5.4 $\bar{u}_\lambda \equiv 0$ for $\lambda > 0$ small. This contradicts the assumption $\lambda_0 = 0$. □

Acknowlegment

The author is supported in part by H. J. Sussmann's NSF Grant DMS01-03901 and FONDECYT Project 1020815.

References

[1] H. Brezis and L. Nirenberg, H^1 versus C^1 local minimizers, *C. R. Acad. Sci. Paris, Série I* **317** (1993), 465–472.
[2] J. Dávila and M. Montenegro, Positive versus free boundary solutions to a singular elliptic equation, Preprint (2002).
[3] J. Dávila and A. C. Ponce, Variants of Kato's inequality and removable singularities, to appear in *J. Anal. Math.* 2001.
[4] M. Giaquinta, "Multiple integrals in the calculus of variations and nonlinear elliptic systems", *Annals of Math. Stud.*, 105. Princeton University Press, 1983.
[5] M. Giaquinta and E. Giusti, Differentiability of minima of nondifferentiable functionals, *Invent. Math.* **72** (1983), 285–298.
[6] M. Giaquinta and E. Giusti, Sharp estimates for the derivatives of local minima of variational integrals, *Boll. Un. Mat. Ital. A* (6) **3** (1984), 239–248.
[7] C. Gui and F. H. Lin, Regularity of an elliptic problem with a singular nonlinearity, *Proc. Roy. Soc. Edinburgh Sect. A* **123** (1993), 1021–1029.
[8] D. Phillips, A minimization problem and the regularity of solutions in the presence of a free boundary, *Indiana Univ. Math. J.* **32** (1983), 1–17.