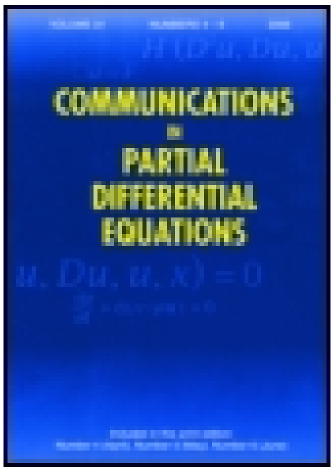

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THE PROBLEM OF UNIQUENESS OF THE LIMIT IN A SEMILINEAR HEAT EQUATION

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1 Introduction

Let us consider a solution $u(x,t)$, globally defined in time of a semi-linear parabolic equation of the form

$$u_t = \Delta u + f(u) \tag{1.1}$$

in a domain Ω in $\mathbf{R}^N$ or in entire space, plus boundary or decay at infinity conditions.

A natural question is whether the solution approaches a time-independent or *steady state* solution of equation (1.1) as time goes to infinity.

If boundary conditions are homogeneous Dirichlet, Neumann or the domain is entire $\mathbf{R}^N$, this equation corresponds formally to the L^2 gradient flow associated to the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u),$$

where $F(u) = \int_0^u f(s)ds$. Provided that the functional is well-defined along the trajectory, one formally sees at once that J is decreasing in time along it, for

$$\frac{d}{dt}J(u(\cdot,t)) = -\frac{1}{2}\int_\Omega u_t^2(x,t)dx. \tag{1.2}$$

In other words, J is a Lyapunov functional for this flow. Assume that the trajectory $u(\cdot,t)$ is bounded in some suitable space norm, say, H^1-norm. Then every sequence $t_n \to \infty$ has a subsequence which we still denote the same way, so that

$$u(x,t_n) \rightharpoonup w(x)$$

weakly, in H^1-sense. If suitable growth is assumed on $f(s)$, then we also have that $J(u(\cdot,t))$ is uniformly bounded from below and hence has a limit as $t \to \infty$. A standard consequence of this fact and (1.2) is that w is a steady state of (1.1). See for instance the proof of Lemma 2.1 below.

A natural question here is the problem of *uniqueness of the limit*, namely whether the entire trajectory approaches just a single steady state as time goes to infinity, or else, it remains oscillating between different steady states.

The former is obviously the case if there is a unique steady state of (1.1) for the given boundary (or decay) conditions, or if the set of such solutions is discrete. However, this is not at all clear if steady states are not isolated. This property occurs naturally if the space domain is entire $\mathbf{R}^N$ since all space translations of a given steady state are also steady states.

Important insight into this question was gained by L. Simon in [9]. Among other results, he proves that for a broad class of parabolic L^2-gradient flows of functionals of the form $I(u) = \int_{\mathcal{M}} \mathcal{F}(\nabla u, u, x)dx$ under suitable ellipticity conditions, the answer to this question is affirmative, provided that $\mathcal{F}$ admit real analytic expansions in its arguments, around steady states, with coefficients uniform in the space variable. Here $\mathcal{M}$ is a compact manifold without boundary.

Assume that for a globally defined solution u and a sequence $t_n \to +\infty$ one has that $u(\cdot, t_n) \to w$, strongly in H^1-sense. Then $I(u(\cdot,t)) \downarrow I(w)$ as $t \uparrow +\infty$. In the analytic setting, it is found then that

$$\int_t^\infty \|u_t(\cdot,s)\|_{L^2(\mathcal{M})} ds \le C\{I(u(\cdot,t) - I(w)\}^\theta$$

for certain numbers $C > 0$ and $0 < \theta < 1$ dependent only on the data. From here it easily follows that

$$\|u(\cdot,t) - w\|_{L^2(\mathcal{M})} ds \to 0$$

as $t \to +\infty$, so that the entire trajectory approaches w. The proof of this estimate uses the uniform analyticity assumption in essential way.

The main purpose of the present work is to analyze the problem of uniqueness of the limit in an equation in entire $\mathbf{R}^N$ for a nonlinearity, with a continuum of steady states, not included in the class covered by the results of [9], but for which good information on the degeneracy of the steady states is available. The problem we consider is

$$u_t = \Delta u - u + u^p \quad \text{in } \mathbf{R}^N \times (0,\infty) \tag{1.3}$$

$$u(x,0) = u_0(x) \tag{1.4}$$

where $1 < p < \frac{N+2}{N-2}$ if $N \ge 3$ or $1 < p < +\infty$ if $N = 1,2$. We assume in what follows that the initial data u_0 is in $H^1(\mathbf{R}^N) \cap C(\mathbf{R}^N)$ and is nonnegative and compactly supported.

By a (globally defined) solution of (1.3)-(1.4) we understand a function $u \in C([0,\infty), H^1(\mathbf{R}^N))$ such that

$$\int_0^\infty \int (\nabla u \nabla \varphi + u\varphi - u^p\varphi - u\varphi_t) dx dt = \int u_0(x)\varphi(x,0)dx$$

for all $\varphi \in C_c^1([0,\infty), H^1(\mathbf{R}^N))$. Here and henceforth the symbol $\int$ with no limits specified denotes integration on entire $\mathbf{R}^N$. Let us observe that the integral quantities involved in the above expression are indeed well defined thanks to the usual Sobolev embedding of $H^1(\mathbf{R}^N)$ into $L^{p+1}(\mathbf{R}^N)$ for $p < \frac{N+2}{N-2}$.

Our main result states that a solution globally defined in time for this equation converges, as time goes to infinity, either to zero or to a solution w of the problem

$$\Delta w - w + w^p = 0, \; w > 0 \quad \text{in } \mathbf{R}^N \tag{1.5}$$

$$w(x) \to 0 \quad \text{as } |x| \to +\infty. \tag{1.6}$$

This elliptic problem is by now well understood. In fact, it follows from the work by Gidas, Ni and Nirenberg [4] that any solution to this problem is radially symmetric around some point. We observe that even though the nonlinearity $f(u) = u^p - u$ has a local analytic expansion around $w(x)$, its coefficients do not remain uniformly bounded in x if p is not an integer, so that the results in [9] do not apply.

On the other hand, Kwong [7] established that the radially symmetric solution for $1 < p < \frac{N+2}{N-2}$ is unique. An intermediate step in the uniqueness proof is the fact that the linearized equation around a radial solution w of (1.5)-(1.6)

$$\Delta h - h + pw^{p-1}h = 0, \quad \text{in } \mathbf{R}^N \tag{1.7}$$

$$h(x) \to 0 \quad \text{as } |x| \to +\infty, \tag{1.8}$$

does not admit nontrivial radial solutions. An observation due to Ni and Takagi [8] shows that this fact implies that the only solutions of (1.7)-(1.8) are linear combinations of the functions $\frac{\partial w}{\partial x_i}$, $i = 1, \ldots, N$.

This is a key fact in the proof of our main result

Theorem 1.1 *Let $u(x,t)$ be a globally defined solution of (1.3)-(1.4). Then either $u(\cdot,t)$ goes to zero in H^1-sense as $t \to +\infty$ or there exists a solution $w(x)$ to problem (1.5)-(1.6) such that*

$$\lim_{t\to+\infty} \|u(\cdot,t) - w(\cdot)\|_{H^1(\mathbf{R}^N)} = 0.$$

While proving that limit points of the trajectory are steady states is, as we mentioned, relatively standard, our proof of uniqueness of the limit takes strong advantage of the invariance under space and time translations of (1.3) as well as of the radial nondegeneracy property above mentioned of its steady states. This allows the use of an iterative procedure induced by Lemma 2.2 which roughly states that once u becomes close to a w in an interval $[t^*, t^* + T]$ at "distance", say, η, then a small translation of w in an amount proportional to η reduces this error by half on the later time interval $[t^* + T, t^* + 2T]$ provided that T is larger than certain universal constant. This implies, after iteration, that u at later times never gets farther from w than a constant times η.

This idea has been used in the work [6] in the context of establishing uniqueness of asymptotic profiles of singularities of solutions of an elliptic equation at the critical exponent.

The special form of the nonlinearity $-u + u^p$ does not play an essential role in our setting and could be replaced by an $f(u)$ satisfying appropriate growth assumptions, in particular non analytic. We can mention for instance conditions appearing in the work [3] which guarantee the radial nondegeneracy property.

It is natural to attempt to characterize the set of initial data for which the solution of (1.3)-(1.4) is globally defined and tends to a non-trivial steady state as $t \to \infty$. In this direction we have the following result.

Theorem 1.2 *Given $\psi \in H^1(\mathbf{R}^N) \cap C(\mathbf{R}^N)$, compactly supported, nonnegative and not identically zero, there exists a unique number $\lambda_0 > 0$ such that the solution of (1.3)-(1.4) for $u_0 = \lambda_0\psi$ converges as $t \to +\infty$ to a solution w of (1.5)-(1.6). Moreover, for $u_0 = \lambda\psi$, we have that the solution goes to zero in H^1-sense if $\lambda < \lambda_0$ and blows-up in finite time if $\lambda > \lambda_0$. The number λ_0 depends continuously on ψ in the H^1 norm.*

Finally, we would like to mention that while the condition that the initial data is compactly supported is used in the proof to control the size of the solution outside of a bounded region, we do not believe this requirement is essential.

The rest of this paper is devoted to the proof of the above results. In §2 we carry out the proof of Theorem 1.1, leaving the main technical steps for sections §3-§5. Finally, we prove Theorem 1.2 in §6.

2 The proof of Theorem 1.1

In this section, we denote by $u(x,t)$ a globally defined solution of (1.3)-(1.4), with $u_0(x)$ a compactly supported initial data in $H^1(\mathbf{R}^N)$ and prove Theorem 1.1 leaving the main technical steps for later sections.

In order to prove the existence of the limit we need the following lemma which we will prove in the next section.

Lemma 2.1

(a)

$$\sup_{t>0} \|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} < +\infty$$

(b)

$$\sup_{t>0} \int (|\nabla u(x,t)|^2 + |u(x,t)|^2)dx < +\infty$$

(c) Let $t_n \to +\infty$ be a sequence such that

$$u(\cdot,t_n) \rightharpoonup w(x) \quad \textit{weakly in } H^1(\mathbf{R}^N).$$

Then for all $\tau > 0$ $u(\cdot,t_n+\tau) \rightharpoonup w(x)$ weakly in $H^1(\mathbf{R}^N)$ and $w(x)$ is either *zero or a solution of problem (1.5)-(1.6). Moreover, the convergence is also uniform and strong in H^1-sense.*

A result analogous to this one in the case of a bounded domain under Dirichlet boundary conditions has been established in [2] Let t_n, $n \in \mathbf{N}$, be such that $t_n \to \infty$ as $n \to \infty$. By Lemma 2.1, part (b), the sequence $u(\cdot, t_n)$ is uniformly bounded in $H^1(\mathbf{R}^N)$. Hence, up to a subsequence, there exists $w \in H^1(\mathbf{R}^N)$ such that $u(\cdot, t_n) \rightharpoonup w(x)$ weakly in $H^1(\mathbf{R}^N)$. It follows now from part c) that $w(x)$ is either zero or a solution of problem (1.5)-(1.6) and the convergence is also uniform and strong in H^1-sense.

Therefore the theorem will be established as soon as we prove that the limit w is the same no matter which sequence t_n we start with. The case $w \equiv 0$ follows from the fact that small constants are super-solutions of (1.3). The case when w is non-trivial is more subtle. The main tool we use is the following lemma, whose proof will be postponed until Section 4.

Lemma 2.2 *(Main Lemma) There are positive numbers $T_0 > 1$, δ, C and $\bar{t}$, with the following property. Assume that $w(x)$ is a solution of problem (1.5)-(1.6), and $t^* > \bar{t}$ are such that*

$$\eta \equiv \Big(\int_0^{T_0} \|u(\cdot, t+t^*) - w(\cdot)\|^2_{H^1(\mathbf{R}^N)} dt\Big)^{1/2} < \delta.$$

Then, there is a vector $\bar{x} \in \mathbf{R}^N$ with $|\bar{x}| \le C\eta$ with the property that

$$\Big(\int_{T_0}^{2T_0} \|u(\cdot, t+t^*) - w(\cdot\, + \bar{x})\|^2_{H^1(\mathbf{R}^N)} dt\Big)^{1/2} \le \frac{1}{2}\eta.$$

We continue now with the proof of Theorem 1.1.

Let $\delta > 0$ and T_0 be as in the above lemma. Let $0 < \epsilon < \delta$. Then we can find $t_0 =: t_{n_0} > \bar{t}$ such that

$$\eta = \Big(\int_0^{T_0} \|u(\cdot, t_0 + t) - w(\cdot)\|^2_{H^1(\mathbf{R}^N)} dt\Big)^{1/2} < \epsilon.$$

Let $\bar{x}_1$ be, as predicted by Lemma 2.2, so that

$$\Big(\int_{T_0}^{2T_0} \|u(\cdot, t_0 + t) - w(\cdot\, + \bar{x}_1)\|^2_{H^1(\mathbf{R}^N)} dt\Big)^{1/2} \le \frac{1}{2}\eta$$

and $|\bar{x}_1| \le C\eta$. Now, applying Lemma 2.3 again we find $\bar{x}_2$ with $|\bar{x}_2| \le \frac{C\eta}{2}$ such that

$$\Big(\int_{2T_0}^{3T_0} \|u(\cdot, t_0 + t) - w(\cdot\, + \bar{x}_1 + \bar{x}_2)\|^2_{H^1(\mathbf{R}^N)} dt\Big)^{1/2} \le \frac{1}{2^2}\eta.$$

Iterating this procedure, we find a sequence $\bar{x}_1,\ \bar{x}_2, \ldots$ such that $|\bar{x}_k| \leq \frac{C}{2^k}\eta$, and

$$(\int_{kT_0}^{(k+1)T_0} \|u(\cdot, t_0 + t) - w(\cdot\ +\ \bar{x}_1 + \ldots + \bar{x}_k)\|^2_{H^1(\mathbf{R}^N)} dt)^{1/2} \leq \frac{1}{2^k}\eta.$$

Now,

$$(\int_{kT_0}^{(k+1)T_0} \|u(\cdot, t_0 + t) - w(\cdot)\|^2_{H^1(\mathbf{R}^N)} dt)^{1/2}$$
$$\leq \frac{1}{2^k}\eta + {T_0}^{1/2}\|w(\cdot) - w(\ \cdot\ +\ \bar{x}_1 + \ldots + \bar{x}_k)\|_{H^1(\mathbf{R}^N)}.$$

It is known that $w(x)$ and its derivatives decay exponentially as $|x| \to \infty$. It follows then that for some $M > 0$,

$$\|w(\cdot) - w(\cdot\ + \bar{x}_1 + \ldots + \bar{x}_k)\|_{H^1(\mathbf{R}^N)} \leq M|\sum_{i=1}^{k} \bar{x}_i| \leq MC\eta \sum_{i=1}^{k} \frac{1}{2^i}.$$

Hence

$$(\int_t^{t+T_0} \|u(\cdot, \tau) - w(\cdot)\|^2_{H^1(\mathbf{R}^N)} dt)^{1/2} \leq D\epsilon,$$

for some number $D > 0$, independent of ϵ for all $t > t_0$. From here, using part c) of Lemma 2.1, the uniqueness of the limit point immediately follows. □

As we mentioned in the introduction, in the above proof we have borrowed an argument contained in the paper [6], in a different context.

3 Proof of Lemma 2.1

A main role in the proof will be played by the functional

$$J(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\int |u|^2 - \frac{1}{p+1}\int |u|^{p+1}.$$

It is a standard fact that

$$\frac{d}{dt}J(u(\cdot, t)) = -\frac{1}{2}\int u_t^2(t,x)dx. \tag{3.1}$$

and hence $J(u(\cdot, t))$ is a non-increasing function of t.

We need first some preliminary results.

The following lemma is essentially contained in [1], We omit its proof, based on Alexandroff's reflection principle and pointwise comparison.

Lemma 3.1 *Assume that* $\operatorname{supp} u_0 \subset B(0, R_0)$. *Then for all* $R > 0$ *one has*

$$\sup_{x \in \partial B(0, R+2R_0)} u(x,t) \leq \inf_{x \in B(0,R)} u(x,t) \text{ for all } t \geq 0.$$

The next lemma and its corollary help us to control the size of the solution for large values of $|x|$.

Lemma 3.2 *Assume* u_0 *is compactly supported. Then given* $A > 1$ *there exists* $R > 0$ *such that* $u(x,t) \leq A$ *for all* $|x| \geq R$ *and all* $t > 0$.

Proof. We argue by contradiction. If the lemma fails, using Lemma 3.1, we see that there exists $A_0 > 1$ and sequences t_n and R_n with $R_n \to \infty$ as $n \to \infty$ such that

$$u(x, t_n) \geq A_0 \quad \text{for all} \quad x \in B(0, R_n).$$

Let us consider the problem

$$v_t = \Delta v + v^p - v \text{ in } B(0, R_n) \times [t_n, \infty)$$

$$v(x,t) = 0 \text{ on } \partial B(0, R_n) \times [t_n, \infty) \tag{3.2}$$

$$v(x, t_n) = A_0 \text{ on } B(0, R_n). \tag{3.3}$$

We claim that if n is large enough, then v blows up in finite time. Indeed, let ϕ_n be the first eigenfunction of the Laplacian with homogeneous Dirichlet boundary data on $B_n = B(0, R_n)$, normalized such that $\int_{B_n} \phi_n(x)dx = 1$. Let λ_n be its corresponding eigenvalue. Set $\psi_n(t) = \int_{B_n} v(x,t)\phi_n(x)dx$. Then from (3.3),

$$\int_{B_n} v_t \phi_n = \int_{B_n} \phi_n(\Delta v - v) + \int_{B_n} \phi_n v^p$$

Integrating by parts twice in space variable, using the fact that $\frac{\partial \phi_n}{\partial n} \leq 0$ on ∂B_n and Jensen's inequality one gets

$$\psi_n'(t) \geq \psi_n^p(t) - (\lambda_n + 1)\psi_n(t) \text{ for } t \geq t_n.$$

Since $\lambda_n \to 0$ and $\psi_n(t_n) = A_0 > 1$ one obtains that $\psi_n(t)$ blows up in finite time if n is large enough. This proves the claim. We obtain the desired contradiction now by comparison, since u is a super solution, globally defined in time, of problem (3.2). □

Corollary 3.1 *There exist constants* $M > 0$ *and* $R > 0$ *such that*

$$u(x,t) \leq Me^{-(|x|-R)} \text{ for all } |x| \geq R \text{ and all } t \geq 0.$$

Proof. The proof will be performed by comparison with a suitable barrier. Let $g(s)$ be the positive solution of

$$g''(s) + g^p(s) - g(s) = 0 \text{ in } \mathbf{R}$$

which is symmetric with respect to the origin and tends to zero as $|s| \to \infty$.

It is well known that g is decreasing in $[0, \infty)$, $g(0) > 1$ and g decays exponentially at infinity. By Lemma 3.2 there exists $R > 0$ such that $u(x,t) \leq g(0)$ for all $|x| \geq R$ and all $t > 0$. Without loss of generality we may assume that the support of u_0 is contained in $B(0, R)$. Set $h(|x|) = g(|x| - R)$. Then

$$h'' + \frac{N-1}{r}h' + h^p - h \leq 0, \quad r > R$$

since $g'(s) \leq 0$ for $s \geq 0$. In other words

$$\Delta h + h^p - h \leq 0$$

for $|x| > R$. Since $u(x,t) \leq h(x)$ for all $|x| = R$ and all $t > 0$ the lemma follows by comparison. □

Lemma 3.3 $J(u(\cdot, t)) \geq 0$ *for all* $t > 0$.

Proof. We argue by contradiction. Assume $J(u(\cdot, t_0)) < 0$ for some $t_0 > 0$. Let $\varepsilon > 0$. By Corollary 3.1 there exists $R > 0$ such that $u(x,t) \leq \varepsilon$ for all $|x| \geq R$ and all $t > 0$.

Consider the problem

$$\begin{aligned} v_t = \Delta v + v^p - v \text{ in } B(0,R) \times [0, \infty) \\ v \equiv 0 \text{ on } \partial B(0,R) \times [0, \infty) \\ v(x, 0) = (u(x, t_0) - \varepsilon)_+ =: v_0(x) \end{aligned} \tag{3.4}$$

and define

$$J_R(v) = \frac{1}{2} \int_{B(0,R)} |\nabla v|^2 + \frac{1}{2} \int_{B(0,R)} v^2 - \frac{1}{p+1} \int_{B(0,R)} v^{p+1}.$$

We claim that we can choose ε small and R large such that $J_R(v_0) < 0$. In fact one has

$$J_R(v_0) \leq J(u(\cdot, t_0)) + \frac{1}{p+1} \left(\int u^{p+1}(x, t_0)dx - \int_{B(0,R)} (u - \varepsilon)_+^{p+1}(x, t_0)dx \right).$$

But
$$\int u^{p+1}(x, t_0)dx - \int_{B(0,R)} (u - \varepsilon)_+^{p+1}(x, t_0)dx =$$

$$\int_{|x| \geq R} u^{p+1}(x, t_0)dx + \int_{B(0,R)} (u^{p+1} - (u - \varepsilon)_+^{p+1})(x, t_0)dx \leq$$

$$Ce^{-R} + \varepsilon \int u^p(x, t_0)dx.$$

This proves the claim since ε can be chosen arbitrarily small, R arbitrarily large and, by Corollary 3.1, $\int u^p(x, t_0)dx$ is a fixed finite number.

Since $J_R(v(\cdot, t))$ is non increasing in t, it follows that $J_R(v(\cdot, t)) < 0$ for all $t \geq t_0$. Now we have

$$\frac{1}{2}\frac{\partial}{\partial t} \int_{B(0,R)} v^2(x,t)dx = - \int_{B(0,R)} (|\nabla v|^2 - v^{p+1} + v^2)(x,t)dx =$$

$$-2J_R(v(\cdot, t)) + (1 - \frac{2}{p+1}) \int_{B(0,R)} v^{p+1}(x,t)dx \geq C(\int_{B(0,R)} v^2(x,t)dx)^{(p+1)/2}$$

and hence v blows up in finite time. This is a contradiction since u is a super-solution of problem (3.4) which is globally defined in time. The lemma is proved. □

The following lemma is essentially due to Giga and Kohn [5]. We sketch its proof for completeness.

Lemma 3.4 *Let L be such that $0 < L \leq \infty$. Assume that $J(u(\cdot, t)) \geq 0$ for all $t \in (0, L)$. Then, there exists a constant C such that*

$$u(x,t) \leq C \text{ for all } x \in R^n \text{ and all } t \in (0, L).$$

Proof. If the lemma was false, then there would exist a sequence $t_n \in (0, L)$ converging to L such that

$$M_n =: ||u(\cdot, t_n)||_{L^\infty} \to \infty.$$

Let x_n be such that

$$\frac{M_n}{2} \leq u(x_n, t_n) \leq M_n.$$

Set

$$v_n(y, \tau) =: \varepsilon_n^{2/(p-1)} u(x_n + \varepsilon_n y, t_n + \varepsilon_n^2 \tau)$$

where $\varepsilon_n^{2/(p-1)} = M_n^{-1}$.

It is easily checked that v satisfies

$$\frac{\partial v_n}{\partial \tau} = \Delta v_n + v_n^p - \varepsilon_n^2 v_n \text{ on } \mathbf{R}^N \times (-\frac{t_n}{\varepsilon_n^2}, 0].$$

By the same argument as in the last paragraph of page 18 of [5] there exists a subsequence, which we still denote by v_n, and a solution v of

$$v_\tau = \Delta v + v^p \text{ in } \mathbf{R}^N \times (-\infty, 0]$$

such that

$$v_n \to v \text{ in } C^{2,1}_{\mathrm{loc}}(\mathbf{R}^N \times (-\infty, 0]).$$

Now, since

$$\int_{t_0}^{t_1}\int |u_t(x,t)|^2 dxdt \le J(u(\cdot,t_0)) - J(u(\cdot,t_1))$$

and $J(u(\cdot,t)) \ge 0$ for all $t \in (0,L)$, we have

$$\int\int |\frac{\partial}{\partial\tau} v_n|^2 dy d\tau =$$

$$\varepsilon_n^{4p/(p-1)} \int\int |u_t(x_n + \varepsilon_n y, t_n + \varepsilon_n^2 \tau)|^2 dy d\tau \le$$

$$\varepsilon_n^{4p/(p-1)-N-2} \int_0^\infty \int |u_t(x,t)|^2 dxdt \le$$

$$\varepsilon_n^{4p/(p-1)-N-2} J(u_0).$$

Since $4p/(p-1) - N - 2 > 0$ if and only if $p < \frac{N+2}{N-2}$ it follows that $v_\tau \equiv 0$ and hence v is a nontrivial, since $v(0) \ge \frac{1}{2}$, nonnegative solution of

$$\Delta v + v^p = 0 \text{ in } \mathbf{R}^N.$$

This contradicts a well known result by Gidas and Spruck about nonexistence of such a solution if $1 < p < \frac{N+2}{N-2}$. This concludes the proof of the lemma. □

We can finish now the proof of Lemma 2.1.

Part (a) follows immediately from Lemma 3.3 and Lemma 3.4 with $L = \infty$. As for part (b) we note that by part (a), just proved, and Corollary 3.1 that there exists a constant C, independent of t, such that

$$\int u^{p+1} \le C \text{ for all } t > 0.$$

Now, since $J(u(\cdot,t))$ decreases as t increases, we get

$$\frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\int |u|^2 - \frac{1}{p+1}\int |u|^{p+1} = J(u(\cdot,t)) \le J(u_0)$$

and (b) follows.

Finally we prove statement (c). Let $t_n \to +\infty$ be a sequence such that

$$u(\cdot,t_n) \rightharpoonup w(x) \quad \text{weakly in } H^1(\mathbf{R}^N).$$

Let us fix a number $\tau > 0$. Integrating relation (3.1) in time from t_n to $t_n + \tau$ and using Cauchy-Schwarz's inequality we obtain the estimate

$$\int_\Omega |u(x,t_n+\tau)-u(x,t_n)|^2dx \le \tau\{J(u(\cdot,t_n))-J(u(\cdot,t_n+\tau))\} \tag{3.5}$$

Since $J(u(\cdot,t))$ is decreasing and bounded below, it follows that $u(x,t_n+\tau)\to w(x)$ in L^2. By part (b) any subsequence of $u(\cdot,t_n+\tau)$ has a subsequence convergent weakly in H^1. Its limit must therefore be w. We conclude that $u(\cdot,t_n+\tau)\rightharpoonup w$ weakly in H^1 for any fixed τ. Let $\varphi\in C_c^\infty(\mathbf{R}^N)$. Then, since u satisfies (1.1), we have

$$\int (u(x,t_n+\tau)-u(x,t_n))\varphi(x)dx =$$

$$\int_{t_n}^{t_n+\tau}\int\{-\nabla u(x,t_n+s)\nabla\varphi(x)+(u^p-u)(x,t_n+s)\varphi(x)\}dsdx,$$

so that letting $n\to\infty$ we get

$$\int\{-\nabla w\nabla\varphi+(w^p-w)\varphi\}dx=0,$$

namely w satisfies

$$\Delta w+w^p-w=0 \tag{3.6}$$

in the weak sense. Since the sequence $u(\cdot,t_n)$ is uniformly bounded and $u(\cdot,t_n)\to w$ in L^2, we obtain from standard parabolic estimates that this convergence is uniform on bounded sets, see for instance the argument in the last paragraph of page 18 of [5]. Finally, the uniform exponential decay of u in space variable provided by Corollary 3.1 imply that the convergence is uniform in entire space.

It only remains to prove that the convergence is strong in H^1. The uniform convergence and Corollary 3.1 yield that $u(\cdot,t_n)\to w$ in $L^q(\mathbf{R}^N)$ for all $q>1$. The same holds for $u(\cdot,t_n+s)$ for any $s\in(0,1)$.

Now let $h(t)$ be a smooth function supported in $(0,1)$ such that $\int_0^1 h(t)dt=1$. Using the weak equations satisfied by u and w with suitable test functions we obtain that

$$\lim_{n\to\infty}\int_0^1 h(s)ds\int(\nabla u(x,t_n+s)\cdot\nabla(u(x,t_n+s)-w(x))dx=0.$$

This implies the existence of $s_n\in(0,1)$ such that

$$\lim_{n\to\infty}\int|\nabla u(x,t_n+s_n)|^2=\int|\nabla w(x)|^2.$$

Hence $u(x,t_n+s_n)\to w(x)$ strongly in H^1. But

$$J((u(x,t_n+s_n))-J(u(x,t_n))\to 0,$$

from where it follows that the convergence of $u(\cdot,t_n)$ to w is strong in H^1. □

4 The proof of Lemma 2.2

In this section we prove the main step in the proof of uniqueness of the limit, Lemma 2.2, leaving for the next section the proof of a technical step, Proposition 4.1 below.

An indirect argument shows that proving Lemma 2.2 amounts to establishing the validity of the following statement:

There exist numbers C and T_0 such that if $t_n \to +\infty$ and $y_n \in \mathbf{R}^N$ are any sequences such that for a solution $w(x)$ of problem (1.5)-(1.6) one has

$$\eta_n \equiv \Big(\int_0^{T_0} \|u(\cdot, t_n + t) - w(\cdot + y_n)\|^2_{H^1(\mathbf{R}^N)}\Big)^{1/2} \to 0,$$

then there exists a subsequence of t_n, still denoted by t_n, and vectors $\bar{x}_n$ with $|\bar{x}_n| \le C\eta_n$ such that

$$\Big(\int_{T_0}^{2T_0} \|u(\cdot, t_n + t) - w(\cdot + y_n + \bar{x}_n)\|^2_{H^1(\mathbf{R}^N)}\Big)^{1/2} \le \frac{1}{2}\eta_n \tag{4.1}$$

for all large n.

Indeed, assume that this statement hold true. If the result of Lemma 2.2 did not hold for the constants T_0 and C of the above statement, then there would exist sequences $t_n \to \infty$ and $y_n \in \mathbf{R}^N$ as $n \to \infty$ such that

$$\eta_n \equiv \Big(\int_0^{T_0} \|u(\cdot, t_n + t) - w(\cdot + y_n)\|^2_{H^1(\mathbf{R}^N)}\Big)^{1/2} \to 0,$$

but

$$\Big(\int_{T_0}^{2T_0} \|u(\cdot, t_n + t) - w(\cdot + y_n + \bar{x})\|^2_{H^1(\mathbf{R}^N)}\Big)^{1/2} > \frac{1}{2}\eta_n \tag{4.2}$$

for all $\bar{x}$ such that $|\bar{x}| \le C\eta_n$, a contradiction which shows the validity of Lemma 2.2. □

Next we prove the above statement. Let t_n y_n be sequences as stated. The proof consists of analyzing the convergence of the sequence

$$\phi_n(x, t) \equiv \frac{u_n(x,t) - w(x)}{\eta_n}, \tag{4.3}$$

where

$$u_n(x, t) \equiv u(x - y_n, t_n + t).$$

We see that ϕ_n satisfies

$$(\phi_n)_t = \Delta\phi_n - \phi_n + p(w + \mu(u_n - w))^{p-1}\phi_n \quad \text{in } \mathbf{R}^N \times (0, \infty), \tag{4.4}$$

where $0 < \mu < 1$ depends on the values of u_n and w. We observe that the sequence y_n is bounded thanks to Corollary 3.1. Now, possibly passing to a subsequence, Lemma 2.1 implies that u_n converges uniformly, with no loss of the generality, to w. Also $\int_0^{T_0} \|\phi_n\|_{H^1}^2 = 1$. Therefore it is reasonable to expect that ϕ_n converges in some sense to a solution ϕ of the linearized problem

$$\phi_t = \Delta\phi - \phi + pw^{p-1}\phi \quad \text{in } \mathbf{R}^N \times (0, \infty). \tag{4.5}$$

This is indeed the case. We say that a function ϕ is a *weak-H^1* solution of (4.5) if $\phi \in L^2([0, K), H^1(\mathbf{R}^N))$ for all $K > 0$, and for all $\varphi \in C_0^\infty((0, K) \times \mathbf{R}^N)$ we have

$$\int_0^\infty \Big(\int \phi\varphi_t - \nabla\phi\nabla\varphi - (1 - pw^{p-1})\phi\varphi \Big)\, dx\, dt = 0.$$

It follows from standard linear parabolic theory that a weak-H^1 solution ϕ of (4.5) is actually classical, of class $C^{2,1}$, and furthermore, $\phi \in C((0, \infty), L^2(\mathbf{R}^N))$.

The next result describes the convergence of ϕ_n to a weak-H^1 solution of (4.5) up to subsequences. We postpone its proof for the next section.

Proposition 4.1 *There is a subsequence of ϕ_n, which we denote the same way, and a weak-H^1 solution ϕ of (4.5) such that $\phi_n \rightharpoonup \phi$ weakly in $L^2([0, K), H^1(\mathbf{R}^N))$, for all $K > 0$. Moreover*

(a) Set

$$\theta_n(x, t) = \phi_n(x, t) - \phi(x, t),$$

then

$$\int_T^{2T} \|\theta_n(\cdot, t)\|_{H^1(\mathbf{R}^N)}^2 dt \le \frac{12}{T}$$

for all sufficiently large n.

(b) The function ϕ has the form

$$\phi(x, t) = \sum_{i=1}^N C_i \frac{\partial w}{\partial x_i}(x) + \hat{\theta}(x, t),$$

where $\sum_{i=1}^N C_i^2 \le D$ with D depending only on p and N. Moreover, there exists $T_0 > 0$ such that if $T > T_0$, then

$$\int_T^{2T} \|\hat{\theta}(\cdot, t)\|_{H^1(\mathbf{R}^N)}^2 dt \le e^{-\alpha T}$$

for some $\alpha > 0$ depending only on p and N.

Next we conclude the proof of the claim assuming the validity of this result. From Proposition 4.1 we can write, passing to a subsequence if necessary,

$$u(x, t_n + t) = w(x) + \eta_n \sum_{i=1}^N C_i \frac{\partial w}{\partial x_i}(x) + \eta_n \hat{\theta}(x, t) + \eta_n \theta_n(x, t).$$

Let us set $\bar{x}_n = \eta_n(C_1, \dots, C_N)$. Then $|\bar{x}_n| \le D\eta_n$ with D dependent only on p and N. A Taylor's expansion yields

$$u(x, t_n + t) - w(x + \bar{x}_n) = \eta_n^2 \xi_n(x) + \eta_n \hat{\theta}(x,t) + \eta_n \theta_n(x,t)$$

where $\xi_n(x)$ has a uniformly bounded H_1-norm. Then, integrating we get

$$(\int_T^{2T} \|u(\cdot, t_n + t) - w(\cdot\ + \bar{x}_n)\|_{H^1}^2)^{1/2} \le$$

$$\eta_n \left\{ T^{1/2}\eta_n \|\xi_n\|_{H^1} + (\int_T^{2T} \|\theta_n(\cdot, t)\|_{H^1}^2 dt)^{1/2} + (\int_T^{2T} \|\hat{\theta}(\cdot, t)\|_{H^1}^2 dt)^{1/2} \right\} \le$$

$$\eta_n \left\{ (\frac{12}{T})^{1/2} + T^{1/2} K \eta_n + e^{-\alpha T/2} \right\}.$$

Finally, pick $T_0 > 1$ such that $(\frac{12}{T_0})^{1/2} + e^{-\alpha T_0/2} < \frac{1}{4}$. Then the expression between brackets in the last inequality becomes less than $1/2$ for all large n since K depends only on p and N. This proves inequality (4.1) and the proof of the claim, and hence of Lemma 2.2, is complete. □

5 The proof of Proposition 4.1

The following result contains in particular that of part (a) of Proposition 4.1.

Proposition 5.1 *Let ϕ_n be the sequence defined in (4.3). Assume $T > 1$. Then there is a subsequence of ϕ_n which we denote the same way, and ϕ a weak-H^1 solution of (4.5) such that*

(i) $\phi_n \rightharpoonup \phi$ weakly in $L^2([0,K), H^1(\mathbf{R}^N))$. for all $K > 0$, that is

$$\int_0^\infty \int \nabla(\phi_n - \phi)\nabla\varphi + (\phi_n - \phi)\varphi dx ds \to 0$$

for all $\varphi \in C_0^\infty([0,K) \times (0,\infty)$.

(ii) For each $t > 1$ and B compact subset of $\mathbf{R}^N$

$$\int_1^t \|\phi_n(\cdot, s) - \phi(\cdot, s)\|_{H^1(B)}^2 ds \to 0$$

as $n \to \infty$.

(iii) Given $t > 1$ there is an integer $n_0(t)$ such that for $n \ge n_0(t)$

$$\int_1^t \|\phi_n - \phi\|_{H^1(\mathbf{R}^N)}^2 ds \le 10$$

(iv) There exists an integer $n_0(T)$ *such that*

$$\int_T^{2T} ||\phi_n - \phi||^2_{H^1(\mathbf{R}^N)} ds \leq \frac{12}{T}$$

for $n \geq n_0(T)$.

Proof. Since $\int_0^{T_0} ||\phi_n(\cdot, s)||^2_{H^1} ds \leq 1$, there exists a sequence $\tau_n \in [1/2, 1]$ such that $\tau_n \to \tau$ and $||\phi_n(\cdot, \tau_n)||_{H^1} \leq 2$. Therefore we can assume, passing to a subsequence, that $\phi_n(\cdot, \tau_n)$ converges weakly in $H^1(\mathbf{R}^N)$ to a function $\phi_0(x) \in H^1(\mathbf{R}^N)$. Moreover, by Sobolev embeddings and a diagonal argument, we can assume also that $\phi_n(\cdot, \tau_n)$ converges to ϕ_0 strongly in L^2 of any compact subset of $\mathbf{R}^N$. Now we prove statement (i). Fix $K > 0$. We claim that $\int_0^K ||\phi_n(\cdot, s)||^2_{H^1} ds$ is uniformly bounded. In fact, multiplying (4.4) by ϕ_n and integrating we get

$$\frac{\partial}{\partial t} \int {\phi_n}^2 dx + 2 \int (|\nabla \phi_n|^2 + {\phi_n}^2) dx \leq D \int {\phi_n}^2 dx \tag{5.1}$$

where $D = 2p(||u||_\infty)^{p-1}$. Hence, recalling that $\int {\phi_n}^2(x, \tau_n) dx \leq 2$, we have that for $t \geq \tau_n$,

$$\int {\phi_n}^2(x, t) dx \leq 2e^{Dt}. \tag{5.2}$$

Combining this estimate with (5.1) we get

$$\frac{\partial}{\partial t} \int {\phi_n}^2 dx + 2 \int (|\nabla \phi_n|^2 + {\phi_n}^2) dx \leq 2De^{Dt}.$$

Integrating this last inequality from τ_n to K and using that $\int_0^T ||\phi_n(\cdot, s)||^2_{H^1} ds \leq 1$, we obtain

$$\int_0^t ||\phi_n(\cdot, s)||^2_{H^1} ds \leq 1 + e^{Dt} \leq 2e^{Dt}. \tag{5.3}$$

and the claim is proved. It follows that there exists a subsequence, which we relabel as ϕ_n, weakly convergent in $L^2([0, K), H^1(\mathbf{R}^N))$ to a function ϕ. Let us fix a φ in $C_0^\infty((0, K) \times \mathbf{R}^N)$. Multiplying relation (4.4) by φ and integrating we obtain

$$\int_{\tau_n}^K \int \phi_n \varphi_t - \nabla \phi_n \nabla \varphi - \phi_n \varphi + p(w + \mu(u_n - w))^{p-1}) \phi_n \varphi) = \int \phi_n(x, \tau_n) \varphi(x, \tau_n) dx. \tag{5.4}$$

Now, using the fact that

$$|\int_{\tau_n}^{\tau} \int (\phi_n \varphi_t - \nabla \phi_n \nabla \varphi - \phi_n \varphi + p(w + \mu(u_n - w))^{p-1} + \phi_n \varphi)| \leq$$

$$C(\varphi)(\int_0^1 ||\phi_n(\cdot,s)||^2_{H^1}ds)^{1/2}|\tau-\tau_n|^{1/2}$$

we see that if we let $n\to\infty$ in (5.4), then

$$\int_\tau^K\int(\phi\varphi_t-\nabla\phi\nabla\varphi-\phi\nabla\varphi+pw^{p-1}\phi\nabla\varphi)=$$

$$\int\phi_0(x)\varphi(x,\tau)dx.$$

A similar argument, integrating as in (5.4) but from 0 to τ_n yields that ϕ solves (4.5) up to time K. Finally, a ϕ as in statement (i) is constructed by letting $K\to\infty$ using a standard diagonal argument.

Fix now $t_0>0$ and let B be a compact subset of $\mathbf{R}^N$. For a given $\delta>0$, let $h\in C_0^\infty(\mathbf{R}^N)$ be a compactly supported function such that $0\le h\le 1$, $h\equiv 1$ on B and $|\nabla h|<\delta$. Set $\theta_n=\phi_n-\phi$. Subtracting (4.5) from (4.4), multiplying by $\theta_n h^2$ and integrating we obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int\theta_n^2h^2dx+\int(|\nabla\theta_n|^2+\theta_n^2)h^2dx=-\frac{1}{2}\int\nabla\theta_n^2\nabla h^2dx\\ +\int pw^{p-1}\theta_n^2h^2+(p(w+\mu(u_n-w))^{p-1}-pw^{p-1})\phi_n\theta_nh^2dx.\end{aligned}\tag{5.5}$$

A standard argument then gives

$$|\int\nabla\theta_n^2\nabla h^2dx|\le\int|\nabla\theta_n|^2h^2dx+4\int|\nabla h|^2\theta_n^2dx$$

and hence from (5.5)

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int\theta_n^2h^2dx+\frac{1}{2}\int(|\nabla\theta_n|^2+\theta_n^2)h^2dx\le 4\int|\nabla h|^2\theta_n^2dx\\ +\int pw^{p-1}\theta_n^2h^2+(p(w+\mu(u_n-w))^{p-1}-pw^{p-1})\phi_n\theta_nh^2dx.\end{aligned}\tag{5.6}$$

From (5.2), we know that $\int{\phi_n}^2(x,t)dx\le 2e^{Dt}$, hence the same is true for $\int\phi^2dx$. Moreover, by Lemma 2.1, the functions u_n are uniformly bounded. This implies the existence of a constant $C>0$ such that for any given $\varepsilon>0$

$$\frac{1}{2}\frac{\partial}{\partial t}\int\theta_n^2h^2dx+\frac{1}{2}\int(|\nabla\theta_n|^2+\theta_n^2)h^2dx\le$$

$$+4\int|\nabla h|^2\theta_n^2dx+C\int\theta_n^2h^2dx+\varepsilon$$

for all $n\ge n_1(\varepsilon)$. We can choose h with δ so small that

$$\frac{1}{2}\frac{\partial}{\partial t}\int \theta_n^2 h^2 dx + 1/2\int((\nabla\theta_n)^2 + \theta_n^2)h^2 dx \le C(\int \theta_n^2 h^2 dx + 2\varepsilon) \qquad (5.7)$$

for any $0 < t < t_0$. Gronwall's inequality then leads us to

$$\int \theta_n^2(x,t)h^2(x)dx + 2\varepsilon \le (\int \theta_n^2(x,\tau_n)h^2(x)dx + 2\varepsilon)e^{Ct}.$$

From this and relation (5.7) it follows that

$$\int_{\tau_n}^{t_0} ||\theta_n(\cdot,s)||^2_{H^1(B)}ds \le 2(\int \theta_n^2(x,\tau_n)h^2(x)dx + 2\varepsilon)e^{Ct_0}.$$

On the other hand,

$$\int \theta_n^2(x,\tau_n)h^2(x)dx \le$$
$$2\int(\phi_n(x,\tau_n) - \phi(x,\tau))^2 h^2(x)dx + 2\int(\phi(x,\tau_n) - \phi(x,\tau))^2 h^2(x)dx.$$

So since $\phi_n(\cdot,\tau_n)$ converges to $\phi(\cdot,\tau)$ strongly in L^2 of any compact subset of $\mathbf{R}^N$ and, $\phi \in C((0,\infty), L^2(\mathbf{R}^N))$ we obtain

$$\limsup_{n\to\infty}\int_{\tau_n}^{t_0} ||\theta_n(\cdot,s)||^2_{H^1(B)}ds \le 4\varepsilon e^{Ct_0}.$$

Thus, since ε is arbitrary,

$$\lim_{n\to\infty}\int_{\tau_n}^{t_0} ||\theta_n(\cdot,s)||^2_{H^1(B)}ds = 0 \qquad (5.8)$$

This yields (ii) since $\tau_n \le 1$.

To establish statement (iii), we subtract again (4.4) from (4.3) and multiply by θ_n. Setting $\delta_n = p((w + \mu(u_n - w))^{p-1} - w^{p-1}$ and integrating we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int \theta_n^2 dx + \int |\nabla\theta_n|^2 + \theta_n^2(1 - p(w + \mu(u_n - w))^{p-1})dx = \int \delta_n\phi\theta_n dx.$$

Let $B(R)$ be a ball of radius R centered at the origin. By Corollary 3.1 we can assume R big enough so that $|p(w + \mu(u_m - w))^{p-1}| \le 1/2$ outside $B(R)$. Then by standard arguments,using the square of the binomial, we have for $\epsilon > 0$

$$\frac{1}{2}\frac{\partial}{\partial t}\int \theta_n^2 dx + \int |\nabla\theta_n|^2 + \frac{1}{2}\theta_n^2 dx \le$$
$$\int_{B(R)}(p(w + \mu(u_n - w))^{p-1} - 1/2)\theta_n^2 dx + 1/\epsilon\int \delta_n^2\phi^2 + \epsilon\int \theta_n^2 dx.$$

Taking $\epsilon = \frac{1}{4}$ we obtain a constant C so that

$$\frac{1}{2}\frac{\partial}{\partial t}\int \theta_n^2 dx + \frac{1}{4}||\theta_n||_{H^1}^2 \leq C\int_{B(R)} \theta_n^2 dx + 4\int \delta_n^2\phi^2.$$

Integrating in time

$$\int \theta_n^2 dx + \frac{1}{2}\int_{\tau_n}^t ||\theta_n||_{H^1}^2 ds \leq C\int_{\tau_n}^t \int_{B(R)} \theta_n^2 dx ds + 8\int_{\tau_n}^t \int \delta_n^2\phi^2 dx ds \tag{5.9}$$
$$+ \int \theta_n^2(x, \tau_n)dx.$$

From the uniform convergence of $u_n \to w$ it follows that $\delta_n \to 0$ uniformly. On the other hand the continuity of ϕ plus the fact $||\phi_n(\cdot, \tau_n)||_{H^1} \leq 1$ and $||\phi(\cdot, \tau)||_{H^1} \leq 1$ imply that $\int \theta_n^2(x, \tau_n)dx < 4.5$ for n large. Now part (iii) follows from (5.9) after an application of (5.8).

To prove (iv) pick $t_n \in [0, T]$ such that

$$\int \theta_n^2(x, t_n)dx \leq \frac{10}{T}.$$

Now the same argument as above but with t_n instead of τ_n gives that there exists $n_0(T)$ such that

$$\int_{t_n}^{2T} ||\theta_n||_{H^1}^2 ds < \frac{12}{T},$$

and hence

$$\int_T^{2T} ||\phi_n - \phi||_{H^1}^2 ds < \frac{12}{T}$$

if $n \geq n_0(T)$. This concludes the proof of Proposition 5.1. □

Now we prove now part (b) of Proposition 4.1. Let ϕ be the solution of equation (4.5) previouly found as a weak limit of ϕ_n in $L^2([0, K), H^1(\mathbf{R}^N))$ for all $K > 0$. We assume in the remaining of this section that $T > 1$, and will identify the form of ϕ.

As we mentioned, ϕ is smooth for $t > 0$, and also $\phi \in C((0, \infty), L^2(\mathbf{R}^N))$.

Let us consider the linear eigenvalue problem

$$\Delta\psi - \psi + pw^{p-1}\psi = \lambda\psi, \quad \psi \in L^2(\mathbf{R}^N). \tag{5.10}$$

This problem has a first eigenvalue λ_1 which is positive and variationally characterized as

$$\lambda_1 = -\inf\{A(\psi) \,|\, \psi \in H^1(\mathbf{R}^N), \int \psi^2 = 1\}.$$

where

$$A(\psi) = \int |\nabla\psi|^2 + (1 - pw^{p-1})\psi^2.$$

We call ψ_1 its unique associated positive eigenfunction with $\int \psi_1^2 = 1$. It is known, see the appendix in [8], that just one positive eigenvalue exists, being the second eigenvalue $\lambda_2 = 0$ with associated eigenspace spanned by the functions $\frac{\partial w}{\partial x_i}$, $i = 1, \dots, N$.

Let us now consider the quantity

$$\bar{\lambda} = \inf\{A(\psi) \mid \psi \in H^1(\mathbf{R}^N),\ \int \psi\psi_1 = 0,\ \int \psi \frac{\partial w}{\partial x_i} = 0\ \forall i,\ \int \psi^2 = 1\}. \tag{5.11}$$

We claim that $\bar{\lambda} > 0$. In fact, on the one hand, the variational characterization of the second eigenvalue $\lambda_2 = 0$ implies that $\bar{\lambda} \geq 0$. Let us assume by contradiction that $\bar{\lambda} = 0$. Then there is a sequence $g_n \in H^1$ with $\int g_n^2 = 1$ such that $A(g_n) \to 0$ and also $\int g_n\psi_1 = 0$, $\int g_n \frac{\partial w}{\partial x_i} = 0\ \forall i$. Then,

$$p \int w^{p-1} g_n^2 = 1 + \int |\nabla g_n|^2 + o(1). \tag{5.12}$$

It follows in particular that g_n is uniformly bounded in H^1, and passing to a subsequence we may assume that $g_n \rightharpoonup g$ in H^1, weakly, and also strongly in L^2 on compact sets. Since $w(x) \to 0$ as $|x| \to \infty$ and $\int g_n^2 = 1$, we have that there exists an $R > 0$ such that $\int_{|x|>R} p w^{p-1} g_n^2 < 1/4$. It then follows from (5.12) that for all sufficiently large n

$$\int_{|x|<R} g_n^2 \geq \frac{1}{4}.$$

Letting n go to infinity we find that $g \not\equiv 0$. Using the weak convergence in H^1, we find that $A(g) \leq 0$ and from the definition of $\bar{\lambda} = 0$, that $A(g) = 0$. Hence the infimum (5.11) is attained and equals zero. But this implies that 0 is an eigenvalue of problem (5.10) with eigenfunction g. This is impossible, since $\int g \frac{\partial w}{\partial x_i} = 0$ for all i. This completes the proof of the claim.

Fix now any $0 < t_0 < 1$. Then there are numbers $C_0, C_1, \dots, C_N$ such that

$$\phi(x, t_0) = C_0 e^{\lambda_1 t_0} \psi_1(x) + \sum_{i=1}^N C_i \frac{\partial w}{\partial x_i}(x) + \hat{\theta}_0(x), \tag{5.13}$$

and all terms in the right hand side are mutually orthogonal in L^2. Define next

$$\hat{\theta}(x, t) \equiv \phi(x, t) - C_0 e^{\lambda_1 t} \psi_1(x) - \sum_{i=1}^N C_i \frac{\partial w}{\partial x_i}(x).$$

Then clearly $\hat{\theta}$ solves

$$\hat{\theta}_t = \Delta\hat{\theta} - \hat{\theta} + p w^{p-1} \hat{\theta} \quad \text{in } \mathbf{R}^N \times (0, \infty), \tag{5.14}$$

and furthermore

$$\int \hat{\theta}(\cdot,0)\psi_1 = \int \hat{\theta}(\cdot,0)\frac{\partial w}{\partial x_i} = 0 \text{ for all } i. \tag{5.15}$$

We claim that

$$\int \hat{\theta}(\cdot,t)\psi_1\,dx = \int \hat{\theta}(\cdot,t)\frac{\partial w}{\partial x_i} = 0 \text{ for all } i, \tag{5.16}$$

for all $t > t_0$. In fact, set

$$\eta(t) = \int \hat{\theta}(\cdot,t)\psi_1\,dx = 0,$$

Then an integration by parts shows that

$$\frac{d}{dt}\eta(t) = \lambda_1\eta(t),$$

from where it follows that $\eta(t) = 0$ for all $t > t_0$ since $\eta(t_0) = 0$. The other orthogonality relations follow similarly.

Next, we want to estimate some norms of $\hat{\theta}(x,t)$. Since $\hat{\theta}$ solves (5.14), we have that for all $t > 0$,

$$\frac{d}{dt}\int \hat{\theta}^2(\cdot,t)\,dx = -\{\int(|\nabla\hat{\theta}|^2 + \hat{\theta}^2)\,dx - \int pw^{p-1}\hat{\theta}^2\,dx\}.$$

Then, using the orthogonality relations (5.15) and the definition of the number $\bar{\lambda}$ in (5.11), we have that

$$\frac{d}{dt}\int \hat{\theta}^2(\cdot,t)dx \le -\bar{\lambda}\int \hat{\theta}^2(\cdot,t)dx$$

Hence

$$\int \hat{\theta}^2(\cdot,t)dx \le e^{-\bar{\lambda}(t-t_0)}\int \hat{\theta}^2(\cdot,t_0)dx.$$

Using this and the identity

$$\int \hat{\theta}^2(\cdot,T)dx + \int_T^{2T}\int(|\nabla\hat{\theta}|^2 + \hat{\theta}^2)\,dxdt =$$

$$\int \hat{\theta}^2(\cdot,T)dx + \int_T^{2T}\int pw^{p-1}\hat{\theta}^2dx\,dt$$

we also find that

$$\int_T^{2T}\int(|\nabla\hat{\theta}|^2 + \hat{\theta}^2)\,dx\,dt \le C(p,N)\int \hat{\theta}^2(\cdot,t_0)dx\,e^{-\bar{\lambda}T}. \tag{5.17}$$

Finally, we have, from the orthogonal decomposition (5.13),

$$\int \phi^2(\cdot, t_0) = C_0^2 + C(p, N) \sum_{i=1}^{N} C_i^2 + \int \hat{\theta}^2(\cdot, t_0). \tag{5.18}$$

Since, by definition of ϕ we have that $\int_0^T \|\phi\|_{H^1}^2 dt = 1$, and we assume $T > 1$, then there is a $0 < t_0 < 1$ such that $\|\hat{\theta}(\cdot, t_0)\|_{H^1}^2 \leq 1$. We fix such a t_0 in the above computation. Thus from (5.17) we find

$$\int_T^{2T} \int \|\hat{\theta}(\cdot, t)\|_{H^1}^2 dt \leq C(p, N) e^{-\bar{\lambda} T} \leq e^{-\alpha T},$$

with $\alpha = \bar{\lambda}/2$, provided that $T > T_0 = T_0(p, N)$. Equality (5.18) also estimates only in terms of p and N the numbers C_i. Thus we have established part (b) of Proposition 4.1 except for the following important fact:

Lemma 5.1 *$C_0 = 0$ in the decomposition (5.13).*

Proof.

The idea of the proof is the following: If $C_0 \neq 0$, then the Lyapunov functional J takes values less than $J(w)$ along the solution $u(x, t)$ for a sufficiently large t, contradicting the fact that $J(w) \leq J(u(\cdot, t))$ for all $t \geq 0$.

We will denote by $\langle \cdot\,, \cdot \rangle$ the usual inner product in $H^1(\mathbf{R}^N)$. We will assume, by contradiction, that $C_0 \neq 0$. Since $J'(w) = 0$, a Taylor expansion of J around w gives

$$\int_1^{t_0} (J(u(\cdot, t_n + s)) - J(w)) ds = \frac{\eta_n^2}{2} \int_1^{t_0} \langle J''(w)\phi_n, \phi_n \rangle ds \tag{5.19}$$

$$+\eta_n^2 \int_1^{t_0} \int_0^1 (1 - \mu) \langle (J''(w + \mu(u_n - w)) - J''(w))\phi_n, \phi_n \rangle d\mu ds.$$

By continuity of J'', the uniform convergence of u_n and (5.3) we get

$$\int_1^{t_0} \int_0^1 (1 - \mu) \langle (J''(w + \mu(u_n - w)) - J''(w))\phi_n, \phi_n \rangle d\mu ds$$

$$\leq o(1) \int_1^{t_0} \|\phi_n\|_{H^1}^2 ds \leq o(1) c(t_0).$$

On the other hand, if as in the proof of (5.1) $\theta_n = \phi_n - \phi$, then

$$\int_1^{t_0} \langle J''(w)\phi_n, \phi_n \rangle ds =$$

$$\int_1^{t_0} (\langle J''(w)\phi, \phi \rangle + 2\langle J''(w)\phi, \theta_n \rangle + \langle J''(w)\theta_n, \theta_n \rangle) ds.$$

Now, using (5.1) part (iii), we get

$$\int_1^{t_0} \langle J''(w)\theta_n, \theta_n\rangle ds = \int_0^{t_0} \int (|\nabla\theta_n|^2 + \theta_n^2 - pw^{p-1}\theta_n^2) dx ds$$

$$\leq c \int_1^{t_0} ||\theta_n||_{H^1}^2 ds \leq c10,$$

for $n > n(t_0)$. By (5.1) part (i)

$$\int_0^{t_0} \langle J''(w)\phi, \theta_n\rangle ds = \int_0^{t_0} \int (\nabla\phi\nabla\theta_n + \phi\theta_n - pw^{p-1}\phi\theta_n) dx ds \leq o(1).$$

Finally, using the decomposition

$$\phi(x,t) = C_0 e^{\lambda_1 t}\psi_1(x) + \sum_{i=1}^{N} C_i \frac{\partial w}{\partial x_i}(x) + \hat{\theta}(x,t),$$

we have

$$\int_1^{t_0} \langle J''(w)\phi, \phi\rangle ds =$$

$$\langle J''(w)\psi_1, \psi_1\rangle \int_1^{t_0} {C_0}^2 e^{2\lambda_1 s} ds + \int_1^{t_0} C_0 e^{\lambda_1 s} \langle J''(w)\psi_1, \sum_{i=0}^{N} C_i \frac{\partial w}{\partial x_i} + \hat{\theta}\rangle ds$$

$$+ \int_1^{t_0} \langle J''(w)(\sum_{i=1}^{N} C_i \frac{\partial w}{\partial x_i} + \hat{\theta}), \sum_{i=1}^{N} C_i \frac{\partial w}{\partial x_i} + \hat{\theta}\rangle ds.$$

So there is a constant $K > 0$ independent of t_0, such that

$$\int_1^{t_0} \langle J''(w)\phi, \phi\rangle ds \leq {C_0}^2(-\lambda_1)\frac{(e^{2\lambda_1 t_0} - e^{2\lambda_1})}{2\lambda_1} + KC_0 e^{\lambda_1 t_0},$$

and hence for $n > n_0(t_0)$ we get,

$$\int_1^{t_0} J(u(\cdot, t_n + s)) - J(w) ds \leq \frac{\eta_n^2}{2}((-\frac{1}{2}){C_0}^2(e^{2\lambda_1 t_0} - e^{2\lambda_1})$$

$$+\bar{K} D e^{\lambda_1 t_0} + o(1)c(t_0) + K).$$

Since $C_0 \neq 0$ we can choose t_0 big enough so that

$$-\frac{1}{2}{C_0}^2(e^{2\lambda_1 t_0} - e^{2\lambda_1}) + KC_0 e^{\lambda_1 t_0} + K < 0,$$

and now, with t_0 so fixed, we can pick n_0 sufficiently large so that, for $n \geq n_0$,

$$\int_1^{t_0} J(u(\cdot, t_n + s)) - J(w) ds < 0.$$

On the other hand, since $J(u_n(\cdot, t))$ is decreasing in t,

$$0 < \int_1^{t_0} (J(u(\cdot, t_n + s)) - J(w)) ds,$$

a contradiction, so $C_0 = 0$. The lemma, and hence all conclusions of Proposition 4.1 are thus proved.□

6 The proof of Theorem 1.2

Let ψ be as in the statement of the theorem and $\lambda > 0$. We will denote by u_λ the solution of (1.3)-(1.4) with initial condition $u_0 = \lambda\psi$. There are two possibilities for u_λ, one: it blows up in finite time, the other: it is globally defined in time. In the second case, according to Theorem 1.1, u_λ converges uniformly to either zero or, to a nonnegative non-trivial solution of (1.5). Let us define the sets

$$\mathcal{A} = \{\lambda \in (0,\infty)/u_\lambda \text{ blows up in finite time.}\}.$$
$$\mathcal{B} = \{\lambda \in (0,\infty)/u_\lambda \text{ converges to a non trivial solution of (1.5).}\}.$$
$$\mathcal{C} = \{\lambda \in (0,\infty)/u_\lambda \text{ converges to zero.}\}.$$

By comparison we have that these three sets are intervals and moreover their union is $(0,\infty)$.

We claim that $\mathcal{A}$ is open. Indeed, using standard parabolic estimates it can be proved that, for a fixed t, the mapping

$$\lambda \to J(u_\lambda(\cdot,t))$$

is continuous at points where it is defined in a neighborhood of that point. On the other hand it follows, from Lemma 3.3 and Lemma 3.4, that u_λ blows up in finite time if and only if there exists a time t_0 such that $J(u_\lambda(\cdot,t_0)) < 0$. These two facts imply the claim.

We also claim that $\mathcal{C}$ is open. This is a consequence of the facts that any constant less than 1 is a super-solution of (1.3), that non trivial steady state solutions of (1.3) have a maximum strictly bigger than 1 and, that if $\lambda_n \to \lambda$ then, for a fixed t for which they are all defined, $u_{\lambda_n}(\cdot,t) \to u_\lambda(\cdot,t)$ uniformly.

Again, since small constants are super-solutions, it follows that $\mathcal{C}$ is not empty.

Let ϕ be the first eigenfunction for the Dirichlet problem for $-\Delta$ in a ball B such that $\int_B \psi\phi > 0$. Let μ be the corresponding eigenvalue and assume ϕ normalized so that $\int_B \phi = 1$. We then have, after integration by parts and an application of Jensen's inequality,

$$\frac{\partial}{\partial t}\int_B u_\lambda(x,t)\phi(x)dx \geq -(1+\mu)\int_B u_\lambda(x,t)\phi(x)dx + (\int_B u_\lambda(x,t)\phi(x)dx)^p.$$

Since $\int_B u_\lambda(x,0)\phi(x)dx = \lambda \int_B \psi(x)\phi(x)dx$ we conclude that u_λ blows up in finite time for large enough λ. It follows that $\mathcal{A}$ is not empty.

Since $(0,\infty)$ is connected we have that $\mathcal{B}$ is not empty. It remains to check that it consists of a single point. Let $\lambda_0 \in \mathcal{B}$ and let $\lambda > \lambda_0$. Assume $u_{\lambda_0}(x,t) \to w(x)$ as $t \to \infty$ where w is a solution of (1.5). Define

$$v(x,t) = \frac{\lambda}{\lambda_0} u_{\lambda_0}(x,t).$$

It is easily checked that

$$v_t \leq \Delta v - v + v^p.$$

Since $v(x,0) = u_\lambda(x,0)$, by comparison we get

$$v \leq u_\lambda.$$

Therefore, if u_λ converges to a steady state solution, that must be of the form $w(\cdot + \bar{x})$ for some vector $\bar{x} \in \mathbf{R}^N$, then $\frac{\lambda}{\lambda_0} w(x) \leq w(x+\bar{x})$, a contradiction proving that $(\lambda_0, \infty) \subset \mathcal{A}$. This ends the proof of uniqueness of the number λ_0, which we denote $\lambda_0(\psi)$.

For the continuity assertion, let us take a sequence $\psi_n \to \psi$ in H^1-sense, where ψ_n and $\psi \not\equiv 0$ are continuous, nonnegative and compactly supported. Let us set $\lambda_n = \lambda_0(\psi_n)$. We claim that λ_n is bounded. Indeed, otherwise we assume $\lambda_n \to \infty$. Since $\psi \not\equiv 0$, it is easily checked then that $J(\lambda_n \psi_n) \to -\infty$. But, on the other hand, the definition of λ_n implies that $J(\lambda_n \psi_n) > 0$, a contradiction. With no loss of generality we may assume then that $\lambda_n \to \lambda_* \geq 0$. We need to show that $\lambda_* = \lambda_0(\psi)$. Let $u(x,t)$ be the solution of (1.3)-(1.4) with initial condition $u_0 = \lambda_* \psi$, and $u_n(x,t)$ the solution with $u_0 = \lambda_n \psi_n$. Let us assume, by contradiction, that $\lambda_* \neq \lambda_0(\psi)$. Then $u(x,t)$ either goes to zero in H^1 or blows-up in finite time. In either case, we must then have the existence of a time t_0 at which $J(u(\cdot,t_0)) < J(w)$. But, continuity of the solution of the initial value problem in the initial data implies that also $J(u_n(\cdot,t_0)) < J(w)$ for all n sufficiently large. But, again, the definition of λ_n implies that $J(u_n(\cdot,t)) \geq J(w)$ for all t, a contradiction which finishes the proof of the theorem. □

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