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Point Ruptures for a MEMS Equation with Fringing Field

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We construct solutions of the equation

$$-\Delta u = \frac{\lambda(1+|\nabla u|^2)}{(1-u)^2}, \quad 0 < u < 1$$

in a bounded smooth domain of $\mathbb{R}^2$ *with Dirichlet boundary condition, for* $\lambda > 0$ *small. These solutions approach 1 as* $\lambda \to 0$ *at one point, and if* Ω *is not simply connected we find solutions forming singularities at many points. The equation arises in the modeling of a MEMS with fringing field. A surprising connection with plasma problem is found.*

Keywords Fringing-field; MEMS; Singularity formation.

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1. Introduction

The following elliptic equation arises in the modeling of electrostatic Micro-Electromechanical Systems (MEMS),

$$\begin{cases} -\Delta u = \dfrac{\lambda}{(1-u)^2}, \quad 0 < u < 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary and $\lambda > 0$. Taking into account a fringing field in the modeling of the MEMS yields an extra term:

$$\begin{cases} -\Delta u = \lambda \dfrac{1 + \delta|\nabla u|^2}{(1-u)^2}, \quad 0 < u < 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\delta > 0$, see [13–16].

Of special interest are solutions that give rise to singularities in the equation, that is such that $u \approx 1$ in some region, which in the physical model represents a rupture in the device. We will say that a family of classical solutions u_λ of (1.1) or (1.2) develops ruptures as λ approaches a critical parameter if $\sup_\Omega u_\lambda \to 1$.

As observed numerically by Pelesko and Driscoll [16] and rigorously by Ye and Wei [18], there is a striking difference between (1.2) with $\delta > 0$ and (1.1). On one hand, for (1.1) in the unit ball, one knows that there is a family of solutions u_λ developing a rupture at the origin for $\lambda \to \lambda_0 \neq 0$. In this case λ_0 and the limit function u_0 are explicit, see [12]. More properties on equation (1.1) can be found in [6, 7, 9–11]. On the other hand, for (1.2) with $\delta > 0$, if λ has a fixed positive lower bound, then there is an *a-priori* estimate $u \le C < 1$ for any solution u of (1.2), see Theorem 4 in [18]. Then the implicit function theorem and the global bifurcation theorem of Rabinowitz [17], imply that there is a family (λ, u) of solutions of (1.2) with $\lambda \to 0$ and $\sup_\Omega u \to 1$.

The precise behavior of solutions developing ruptures as $\lambda \to 0$ and the set of possible ruptures is not known so far. We give an answer to this question by constructing families of solutions that develop one or more ruptures as $\lambda \to 0$, and obtain a precise asymptotic description. The analysis also reveals connections with the plasma problem and the Liouville equation. This seems to be the first result for the construction of multiple point ruptures.

For simplicity we will work with $\delta = 1$, namely we work with

$$\begin{cases} -\Delta u = \lambda \dfrac{1 + |\nabla u|^2}{(1-u)^2}, \quad 0 < u < 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

but the results are valid for any $\delta > 0$.

Let $m \ge 1$ be an integer. We say that a family of solutions u_λ of (1.3) defined for all $\lambda > 0$ small develops m isolated ruptures as $\lambda \to 0$, if there are points $\xi_{1,\lambda}, \dots, \xi_{m,\lambda} \in \Omega$, uniformly separated between them and from the boundary, such that for any $\delta > 0$

$$\limsup_{\lambda \to 0} \sup_{\Omega \setminus \bigcup_{j=1}^m B_\delta(\xi_{j,\lambda})} u < 1$$

and for any $\delta > 0$ and all $i = 1, \dots, m$:

$$\sup_{B_\delta(\xi_{i,\lambda})} u \to 1 \quad \text{as } \lambda \to 0.$$

Our main results are the following.

Theorem 1.1. *There exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$ there is a solution u_λ of* (1.3) *developing one isolated rupture as $\lambda \to 0$.*

Theorem 1.2. *If Ω is not simply connected, for any integer $m \geq 1$ there exists $\lambda_m > 0$ such that for $\lambda \in (0, \lambda_m)$ there is a solution u_λ of* (1.3) *developing m isolated ruptures as $\lambda \to 0$.*

The location of the ruptures of the solutions constructed in the previous results is determined by a function φ_m that depends on the Green function of the Laplacian in Ω and its regular part. More precisely, let G denote the Green function for the Laplacian with Dirichlet boundary condition:

$$\begin{cases} -\Delta G(\cdot, y) = \delta_y & \text{in } \Omega \\ G(x, y) = 0 & \text{for all } x \in \partial\Omega \end{cases}$$

and H its regular part, given by

$$H(x, y) = G(x, y) - \frac{1}{2\pi} \log\left(\frac{1}{|x - y|}\right). \tag{1.4}$$

For an integer $m \geq 1$ and $\xi_1, \ldots, \xi_m$ different points in Ω we define

$$\varphi_m(\xi_1, \ldots, \xi_m) = \sum_{i=1}^{m} H(\xi_i, \xi_i) + \sum_{j \neq i} G(\xi_i, \xi_j). \tag{1.5}$$

Then in Theorems 1.1 and 1.2, after passing to a subsequence, the rupture points $\xi_{1,\lambda}, \ldots, \xi_{m,\lambda} \in \Omega$ of the solution u_λ, converge to a critical point of φ_m.

For multiple ruptures, i.e. $m > 1$, the condition that Ω is not simply connected guarantees existence of *stable* critical points of φ_m, see [5]. By stable we mean that these critical points also exist for functions close to φ_m in C^1 norm.

As a byproduct of the analysis, we also have the following result.

Theorem 1.3. *For any non-degenerate critical point $\xi = (\xi_1, \ldots, \xi_m)$ of φ_m there is $\lambda_0 > 0$ such that $\lambda \in (0, \lambda_0)$ there is a solution u_λ of* (1.3) *developing m isolated ruptures as $\lambda \to 0$ at points $\xi_{1,\lambda}, \ldots, \xi_{m,\lambda} \in \Omega$ that converge to $(\xi_1, \ldots, \xi_m)$.*

Regarding the asymptotic behavior of the solutions we construct, if u_λ is the solution of any of the three theorems above developing m ruptures $\xi_{1,\lambda}, \ldots, \xi_{m,\lambda} \in \Omega$, then

$$u(x) = \frac{2\pi}{|\log \lambda|} \sum_{j=1}^{m} G(x, \xi_{j,\lambda})(1 + o(1))$$

as $\lambda \to 0$, for points $x \in \Omega$ away from $\xi_{1,\lambda}, \ldots, \xi_{m,\lambda}$. Very close to the points $\xi_{j,\lambda}$ we have an expansion of the form

$$u(x) = 1 - \frac{\lambda}{|\log \lambda|} - \frac{\lambda \log|\log \lambda|}{|\log \lambda|^2} + \frac{\lambda}{|\log \lambda|^2} \log\left(\alpha_0 V_0\left(\frac{|\log \lambda|^2}{\lambda}|x - \xi_{j,\lambda}|\right)\right) + \text{l.o.t.} \tag{1.6}$$

where l.o.t. contains smaller order terms and the function V_0 is the unique radial function satisfying

$$-\Delta V_0 = (V_0)_+ \quad \text{in } \mathbb{R}^2, \quad \max_{\mathbb{R}^2} V_0 = 1. \tag{1.7}$$

The number α_0 is given by $\alpha_0 = \left(\frac{1}{2\pi}\int_{\mathbb{R}^2}(V_0)_+\right)^{-1}$. The expansion (1.6) is valid for points $x \in \Omega$ such that $|x - \xi_{j,\lambda}| \leq \frac{\lambda}{|\log\lambda|^2}R$, where $0 < R < R_0$ and $R_0 > 0$ is the radius for which $V_0(R_0) = 0$.

The function V_0 can be written more explicitly as follows. The number $R_0 > 0$ is the radius such that the principal Dirichlet eigenvalue of $-\Delta$ in $B_{R_0}(0)$ is 1. Let φ_1 be the first eigenfunction in $B_{R_0}(0)$:

$$\begin{cases} -\Delta\varphi_1 = \varphi_1, \quad \varphi_1 > 0 & \text{in } B_{R_0}(0) \\ \varphi_1 = 0 & \text{on } \partial B_{R_0}(0) \end{cases}$$

normalized so that $\varphi_1(0) = 1$ and let

$$d_0 = -\varphi_1'(R_0). \tag{1.8}$$

Then

$$V_0(x) = \begin{cases} \varphi_1(x) & \text{if } x \in B_{R_0}(0) \\ -d_0R_0\log(|x|/R_0) & \text{if } x \notin B_{R_0}(0). \end{cases} \tag{1.9}$$

The key behind the main results is a change of variables similar to the one introduced in Ye and Wei [18], which allows us to rewrite problem (1.3) in the form

$$\begin{cases} -\Delta v = f_\lambda(v - |\log\lambda|) & \text{in } \Omega_\lambda = \dfrac{|\log\lambda|^2}{\lambda}\Omega \\ v = 0 & \text{on } \partial\Omega_\lambda, \end{cases} \tag{1.10}$$

with a nonlinearity f_λ that satisfies

$$f_\lambda(t) \to t_+ \quad \text{as } \lambda \to 0$$

for all $t \in \mathbb{R}$, where $t_+ = \max(t, 0)$. See the derivation in Section 2 and the definition of f_λ in (2.6).

Formally, replacing $f_\lambda(t)$ in (1.10) by t_+ and setting

$$\bar{v}(x) = \frac{1}{|\log\lambda|}v\left(\frac{|\log\lambda|^2}{\lambda}x\right), \quad x \in \Omega,$$

we are lead to

$$\begin{cases} -\varepsilon^2\Delta\bar{v} = (\bar{v} - 1)_+ & \text{in } \Omega \\ \bar{v} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\varepsilon = \frac{\lambda}{|\log\lambda|^2}$. This problem is, after some transformations, the same as the *plasma* problem studied in [2, 3] and also presents similarities with the Liouville equation [1, 4, 5, 8]. For both problems there are existence and classification results of solutions exhibiting point concentration. In both problems the location of the concentration points is determined by an energy expansion that at main nontrivial order involves the same function φ_m.

The results of this article give some information on the bifurcation diagram associated to (1.3). In fact, we can say that there are branches of solutions of the form (λ, u_λ) defined for small $\lambda > 0$ such that

$$\max u_\lambda = 1 - \frac{\lambda}{|\log \lambda|}(1 + o(1)) \quad \text{as } \lambda \to 0.$$

In the problem (1.2) with an arbitrary $\delta > 0$, the same branches exist but it is natural to expect that there is a non-uniform behavior as $\delta > 0$. In particular, the smaller the value $\delta > 0$ is, the more that the bifurcation diagram of (1.2) should resemble the one of (1.1), in particular with an increasing number of foldings, up to some point in the $\max u$ axis that depends on δ, and after which one can expect the branches constructed in this article. This can be seen in numerical calculations in [16] for the unit disk. Regarding the applications, a question of interest is the first folding point, that is related to the maximum voltage that can be applied to the device. In [13] the authors study formally the effect of small $\delta > 0$ in the first folding point.

The proof of all theorems is carried out with equation (1.10), by a Lyapunov-Schmidt finite dimensional reduction. In Section 3, given m uniformly separated points $\xi_1, \ldots, \xi_m$ in Ω, we construct an approximate solution $V_\lambda(\xi_1, \ldots, \xi_m)$ of (1.10), in the same spirit as in [3–5]. We also compute in this section an expansion of the energy I_λ of $V_\lambda(\xi_1, \ldots, \xi_m)$, where I_λ is the natural energy functional associated to (1.10). We seek a true solution of (1.10) of the form $V_\lambda + \phi$, where ϕ is small in an appropriate norm. For this, in Section 4 we study the linearization of (1.10) around V_λ and prove its bounded invertibility in appropriate spaces, except natural orthogonality conditions. In Section 5 we show existence of solutions to a projected version of (1.10). After the reduction, the problem becomes one of finding critical points of a function that is close in C^1 norm to φ_m on compact subsets of its domain of definition, and this is done in Section 6. In the case of two or more points of concentration in a non-simply connected domain, this is the guaranteed by a result in [5], which has also appeared in [3].

The main difficulty in this process is that the nonlinearity f_λ converges to x_+, which is only Lipschitz and hence, some estimates of derivatives of the solution, which involve f_λ'', become delicate as $\lambda \to 0$. This is similar to the difficulty in [3], except that for us the nonlinearity is not explicit but smooth.

2. Change of Variables

Equation (1.3) is equivalent to

$$\begin{cases} \Delta u_1 = \lambda \dfrac{1 + |\nabla u_1|^2}{u_1^2}, \ 0 < u_1 < 1 & \text{in } \Omega \\ u_1 = 1 & \text{on } \partial\Omega, \end{cases}$$

where $u_1 = 1 - u$. Let us write u_1 in the form $u_1(x) = \lambda w(x/\lambda)$, $x \in \Omega$, so that w satisfies

$$\begin{cases} \Delta w = \dfrac{1 + |\nabla w|^2}{w^2}, \quad 0 < w < \dfrac{1}{\lambda} & \text{in } \Omega/\lambda \\ w = \dfrac{1}{\lambda} & \text{on } \partial\Omega/\lambda. \end{cases}$$

Motivated by [18], we can eliminate the gradient term by introducing the change of variables

$$v_1 = g(w) \quad \text{where } g(w) = \int_w^1 e^{1/s}\, ds. \tag{2.1}$$

We compute

$$\Delta v_1 = -\frac{1}{w^2} e^{1/w},$$

and note that $g : (0, +\infty) \to \mathbb{R}$ is a decreasing convex function with range equal to $\mathbb{R}$. Therefore $g^{-1} : \mathbb{R} \to (0, +\infty)$ is well defined. Let

$$h(w) = \frac{1}{w^2} e^{1/w} \quad \text{for all } w > 0$$

and

$$f(t) = h(g^{-1}(t)). \tag{2.2}$$

Then v_1 satisfies

$$\begin{cases} -\Delta v_1 = f(v_1) & \text{in } \Omega/\lambda \\ v_1 = -c_\lambda & \text{on } \partial\Omega/\lambda \end{cases}$$

where

$$c_\lambda = -g(1/\lambda) = \int_1^{1/\lambda} e^{1/s}\, ds.$$

Note that

$$c_\lambda = \frac{1}{\lambda} + |\log \lambda| + O(1) \quad \text{as } \lambda \to 0$$

and that $f : \mathbb{R} \to (0, \infty)$ is a convex increasing function. Moreover

$$f(v) = v(\log v)^4 + o(v(\log v)^4) \quad \text{as } v \to +\infty \tag{2.3}$$

$$f(v) = O\left(\frac{1}{v^2}\right) \quad \text{as } v \to -\infty. \tag{2.4}$$

Let $v_2 = v_1 + c_\lambda$ so that it satisfies

$$\begin{cases} -\Delta v_2 = f(v_2 - c_\lambda) & \text{in } \Omega/\lambda \\ v_2 = 0 & \text{on } \partial\Omega/\lambda. \end{cases}$$

Finally, we write v_2 as

$$v_2(x) = \frac{c_\lambda}{|\log\lambda|} v(|\log\lambda|^2 x) \tag{2.5}$$

which leads to

$$\begin{cases} -\Delta v = f_\lambda(v - |\log\lambda|) & \text{in } \Omega_\lambda = \dfrac{|\log\lambda|^2}{\lambda}\Omega \\ v = 0 & \text{on } \partial\Omega_\lambda, \end{cases}$$

where

$$f_\lambda(t) = \frac{1}{c_\lambda |\log\lambda|^3} f\left(\frac{c_\lambda}{|\log\lambda|} t\right). \tag{2.6}$$

This is the formulation used in the proofs of the main results. Observe that thanks to (2.3), (2.4) we have

$$f_\lambda(t) \to t_+ \quad \text{as } \lambda \to 0 \quad \text{for all } t \in \mathbb{R}.$$

We collect here some useful estimates for the non-linearity (we omit the computations).

Lemma 2.1. *Let f be defined by* (2.2) *and f_λ be the function* (2.6). *Then the following properties hold*:

$$f(t) = t\big((\log t)^4 + O((\log t)^3 \log\log t)\big) \quad \textit{as } t \to +\infty$$

$$f(t) \le \frac{C}{1+t^2} \quad \textit{for all } t \le 0$$

$$f_\lambda(t) = t_+ + O\left(\frac{\log|\log\lambda|}{|\log\lambda|}\right) \quad \textit{as } \lambda \to 0, \quad \textit{uniformly for } t \textit{ on bounded sets} \tag{2.7}$$

$$f_\lambda(t) \le C \frac{\lambda^3}{|\log\lambda|(t^2 + \lambda^2|\log\lambda|^2)} \quad \textit{for } t \le 0 \tag{2.8}$$

$$f_\lambda'(t) = O(1) \quad \textit{uniformly for } t \textit{ in bounded sets} \tag{2.9}$$

$$f_\lambda'(t) \le \frac{C\lambda^3}{|\log\lambda|(|t|^3 + \lambda^3|\log\lambda|^3)} \quad \textit{for all } t \le 0 \tag{2.10}$$

$$f_\lambda''(t) = O(1/\lambda) \quad \textit{uniformly for } t \textit{ in bounded sets} \tag{2.11}$$

$$f_\lambda''(t) \le \frac{C}{t} \quad \textit{if } t/(\lambda|\log\lambda|) \to \infty \tag{2.12}$$

$$f_\lambda''(t) \le C \frac{\lambda^3}{|\log\lambda|(|t|^4 + \lambda^4|\log\lambda|^4)} \quad \textit{for all } t \le 0 \tag{2.13}$$

Remark 2.2. The choice of the scaling of the domain in (2.5) is motivated by the following computations. At points where u develops ruptures, the function v_1 (c.f. (2.1)) blows up. Therefore it is natural to introduce a new variable $\tilde{v}$ so that

$$v_1(x) = M\tilde{v}((\log M)^2 x) \quad x \in \frac{1}{\lambda}\Omega, \tag{2.14}$$

where $M > 0$ is a new large parameter to be chosen later on. We find

$$-\Delta\tilde{v} = \frac{1}{M(\log M)^4} f(M\tilde{v}) \quad \text{in } \frac{(\log M)^2}{\lambda}\Omega.$$

By (2.3), (2.4)

$$\frac{1}{M(\log M)^4} f(Mt) \to t_+ \quad \text{as } M \to +\infty$$

for any $t \in \mathbb{R}$, which is the motivation to introduce $(\log M)^2$ as scaling in (2.14). If we choose M such that $\max \tilde{v} = 1$, then $\tilde{v}$ solves at main order

$$\begin{aligned} -\Delta\tilde{v} &= \tilde{v}_+ \quad \text{in } \frac{(\log M)^2}{\lambda}\Omega \\ \max \tilde{v} &= 1. \end{aligned}$$

Assuming that $\tilde{v}$ has a maximum at the origin, heuristically we can expect that

$$\tilde{v}(x) \approx V_0(x).$$

We use this information to obtain an estimate of the size of M. By definition of $\tilde{v}$,

$$v_1(x) \approx MV_0((\log M)^2 x).$$

Evaluating this relation on the boundary of $\frac{1}{\lambda}\Omega$ we find

$$c_\lambda = Md_0R_0 \log\left(\frac{(\log M)^2 D}{\lambda R_0}\right)$$

where D represents the diameter of the domain. This gives the relation

$$M_\lambda = \frac{1}{d_0R_0\lambda|\log\lambda|} + o\left(\frac{1}{\lambda|\log\lambda|}\right) \quad \text{as } \lambda \to 0,$$

so $\log(M_\lambda) = |\log\lambda| + o(|\log\lambda|)$ as $\lambda \to 0$. This motivates (2.5).

3. First Approximation and Its Energy

3.1. *Setup*

We will work mainly with the following reformulation of problem (1.3),

$$\begin{cases} -\Delta v = f_\lambda(v - |\log\lambda|) & \text{in } \Omega_\lambda \\ v = 0 & \text{on } \partial\Omega_\lambda \end{cases} \tag{3.1}$$

where

$$f_\lambda(x) = \frac{1}{c_\lambda |\log\lambda|^3} f\left(\frac{c_\lambda}{|\log\lambda|} x\right)$$

with f defined in (2.2) and

$$\Omega_\lambda = \frac{|\log\lambda|^2}{\lambda}\Omega.$$

We will define an initial approximation of a solution to (3.1) based on the solutions $w_{\lambda,\alpha}$ of the following problem

$$\begin{cases} w''_{\lambda,\alpha}(r) + \frac{1}{r} w'_{\lambda,\alpha}(r) = -f_\lambda(w_{\lambda,\alpha}) & r > 0 \\ w_{\lambda,\alpha}(0) = \alpha, \quad w'_{\lambda,\alpha}(0) = 0, \end{cases} \tag{3.2}$$

where $\alpha > 0$.

Lemma 3.1. *We have*:

(a)

$$w_{\lambda,\alpha} = \alpha V_0 + O\left(\frac{\log|\log\lambda|}{|\log\lambda|}\right) \tag{3.3}$$

as $\lambda \to 0$ in the C^1 norm over compact sets of $\mathbb{R}^2$, where V_0 is the function defined in (1.9).

(b) *There is a unique $R_{\lambda,\alpha} > 0$ such that*

$$w_{\lambda,\alpha}(R_{\lambda,\alpha}) = 0,$$

and it satisfies

$$R_{\lambda,\alpha} = R_0 + O\left(\frac{\log|\log\lambda|}{|\log\lambda|}\right), \quad \frac{d}{d\alpha} R_{\lambda,\alpha} = O\left(\frac{\log|\log\lambda|}{|\log\lambda|}\right) \tag{3.4}$$

as $\lambda \to 0$ where R_0 is the number defined just before (1.9) *and $O(\frac{\log|\log\lambda|}{|\log\lambda|})$ is uniform for α in compact sets of* $(0, +\infty)$.

(c)

$$-w'_{\lambda,\alpha}(R_{\lambda,\alpha}) = d_0\alpha + O\left(\frac{\log|\log\lambda|}{|\log\lambda|}\right) \quad \textit{as } \lambda \to 0 \tag{3.5}$$

uniformly for α *in compact sets of* $(0, +\infty)$, *where* d_0 *is defined in* (1.8).

For the proof see section 3.2.

We proceed now with the construction of an initial approximation of a solution to (3.1). Let $m \geq 1$ be a fixed integer, $\xi_1, \ldots, \xi_m \in \Omega$, and $\mu_1, \ldots, \mu_m > 0$. We will always assume that for some small $\delta > 0$

$$|\xi_i - \xi_j| \geq \delta \quad \text{for all } i \neq j \tag{3.6}$$

$$dist(\xi_j, \partial\Omega) \geq \delta, \quad \text{for all } 1 \leq j \leq m, \tag{3.7}$$

$$\delta \leq \mu_j \leq \delta^{-1} \quad \text{for all } 1 \leq j \leq m. \tag{3.8}$$

Let us use the notation

$$\bar{\xi}_j = \frac{|\log\lambda|^2}{\lambda}\xi_j \in \Omega_\lambda.$$

The parameters μ_j will be chosen later on, but let us comment that they are used to decrease the error of the approximation. We will see that to choose the numbers μ_j is equivalent to choose the value of α in (3.2). In the limit as $\lambda \to 0$ the solution of this ODE is just αV_0 as stated in Lemma 3.1. Note that αV_0 is also a solution of (1.7) for any $\alpha > 0$. Actually the same approach is taken in the plasma problem [3] and in the Liouville equation [1, 4, 5, 8].

Thanks to (3.5), we can find for $\lambda > 0$ small a unique positive number α_j such that

$$-w'_{\lambda,\alpha_j}(R_{\lambda,\alpha_j})R_{\lambda,\alpha_j} = \mu_j, \tag{3.9}$$

Let us write

$$w_j(r) = w_{\lambda,\alpha_j}(r), \quad R_j = R_{\lambda,\alpha_j}. \tag{3.10}$$

We define

$$V_j(x) = \begin{cases} w_j(|x - \bar{\xi}_j|) + |\log\lambda| & \text{if } |x - \bar{\xi}_j| \leq R_j \\ -\mu_j \log\left(\dfrac{|x - \bar{\xi}_j|}{R_j}\right) + |\log\lambda| & \text{if } |x - \bar{\xi}_j| > R_j. \end{cases}$$

The function V_j is radial about the point $\bar{\xi}_j$ and it is C^1 across the boundary of the ball $B_{R_j}(\bar{\xi}_j)$, thanks to (3.9). It satisfies

$$\begin{cases} -\Delta V_j = f_\lambda(V_j - |\log\lambda|)\,\chi_{B_{R_j}(\bar{\xi}_j)} & \text{on } \mathbb{R}^2, \\ \max_{\mathbb{R}^2} V_j = V_j(\bar{\xi}_j) = \alpha_j + |\log\lambda|. \end{cases} \tag{3.11}$$

Since the function V_j does not satisfy the boundary condition on $\partial\Omega_\lambda$ we consider $V_j - H_j$ where H_j is the solution of the problem

$$\begin{cases} \Delta H_j = 0 & \text{in } \Omega_\lambda \\ H_j = V_j & \text{on } \partial\Omega_\lambda. \end{cases} \tag{3.12}$$

Then

$$H_j(x) = -\mu_j \log\left(\frac{|x-\bar{\xi}_j|}{R_j}\right) + |\log\lambda| \quad \text{for all } x \in \partial\Omega_\lambda,$$

and we get the formula

$$H_j(x) = -2\pi\mu_j H\left(\frac{\lambda}{|\log\lambda|^2}x, \xi_j\right) - 2\mu_j \log|\log\lambda| + \mu_j \log R_j + (1-\mu_j)|\log\lambda|$$

where H is the regular part of the Green function, c.f. (1.4). We define the initial approximation as:

$$V_\lambda = \sum_{j=1}^{m} V_j - H_j. \tag{3.13}$$

We look for a solution v of (3.1) of the form $v = V_\lambda + \phi$ where ϕ is small compared to V_λ. Then problem (3.1) gets reformulated in terms of ϕ as follows:

$$\begin{cases} \Delta\phi + f_\lambda'(V_\lambda - |\log\lambda|)\phi + E_\lambda + N(\phi) = 0 & \text{in } \Omega_\lambda \\ \phi = 0 & \text{on } \partial\Omega_\lambda, \end{cases} \tag{3.14}$$

where

$$E_\lambda = \Delta V_\lambda + f_\lambda(V_\lambda - |\log\lambda|)$$
$$N(\phi) = f_\lambda(V_\lambda + \phi - |\log\lambda|) - f_\lambda(V_\lambda - |\log\lambda|) - f_\lambda'(V_\lambda - |\log\lambda|)\phi.$$

Up to now the parameters μ_j were free in the interval (δ, δ^{-1}) ($\delta > 0$ a small constant). To ensure that E_λ is small in an appropriate norm to be introduced later it is necessary to adjust the numbers μ_j in a suitable way.

Lemma 3.2. *Assume that $\mu_1, \ldots, \mu_m > 0$ satisfy the system of equations*

$$\mu_i = 1 - 2\mu_i \frac{\log|\log\lambda|}{|\log\lambda|} + \mu_i \frac{\log R_i}{|\log\lambda|} - \frac{2\pi}{|\log\lambda|}\left[\mu_i H(\xi_i, \xi_i) + \sum_{i\neq j} \mu_j G(\xi_i, \xi_j)\right] \tag{3.15}$$

for all $i = 1, \ldots, m$. Then for all $R > 0$, we have

$$V_\lambda(x) = V_i(x) + O\left(\frac{\lambda}{|\log\lambda|^2}\right), \quad \textit{for all } x \in B_R(\bar{\xi}_i) \tag{3.16}$$

as $\lambda \to 0$, where the term $O(\frac{\lambda}{|\log\lambda|^2})$ is in C^1 norm in $B_R(\bar{\xi}_i)$.

The proof of this lemma is given in section 3.2.

We note that the system of equations (3.15) is nonlinear since the functions R_i depend on μ_i. Nevertheless this system is solvable for small $\lambda > 0$ and we obtain the following expansion for the solution

$$\mu_i = 1 - 2\frac{\log|\log\lambda|}{|\log\lambda|} + \frac{\log R_0}{|\log\lambda|} - \frac{2\pi}{|\log\lambda|}\left[H(\xi_i,\xi_i) + \sum_{j\neq i} G(\xi_i,\xi_j)\right] + O\left(\frac{(\log|\log\lambda|)^2}{|\log\lambda|^2}\right) \tag{3.17}$$

as $\lambda \to 0$.

Because of Lemma 3.2 we will work in the sequel only with μ_i satisfying (3.15) and in particular we will always assume that

$$\mu_i = 1 + O\left(\frac{\log|\log\lambda|}{|\log\lambda|}\right) \quad \text{as } \lambda \to 0. \tag{3.18}$$

Also, thanks to Lemma 3.1, we will assume

$$R_i = R_0 + O\left(\frac{\log|\log\lambda|}{|\log\lambda|}\right) \quad \text{as } \lambda \to 0.$$

To solve (3.14) with ϕ small in a convenient sense we need to choose the points $\xi_1, \ldots, \xi_m \in \Omega_\lambda$ appropriately, and for this we use a variational formulation of (3.1): v is a solution of (3.1) if and only if $v \in H_0^1(\Omega)$ is a critical point of the energy functional

$$I_\lambda(v) = \frac{1}{2}\int_{\Omega_\lambda} |\nabla v|^2 - \int_{\Omega_\lambda} F_\lambda(v - |\log\lambda|) \tag{3.19}$$

where

$$F_\lambda(v) = \int_{-1}^{v} f_\lambda(s)\,ds.$$

We note that I_λ is a C^1 functional on $H_0^1(\Omega)$. If ϕ is a small solution of (3.14) one may expect that $I_\lambda(V_\lambda + \phi)$ is critical with respect to $\xi_1, \ldots, \xi_m$. Therefore, in order to find the good choice of the points $\xi_1, \ldots, \xi_m$ it becomes important to compute $I_\lambda(V_\lambda)$.

Lemma 3.3. *Suppose $\xi_1, \ldots, \xi_m \in \Omega$ satisfy the separation conditions* (3.6), (3.7), *and let μ_j satisfy* (3.15). *Then*

$$I_\lambda(V_\lambda) = \pi m|\log\lambda| - 2\pi m\log|\log\lambda| + \pi m\log R_0 - 2\pi^2\varphi_m(\xi_1, \ldots, \xi_m) + o(1) \tag{3.20}$$

as $\lambda \to 0$, where φ_m is defined in (1.5), *and $o(1)$ is uniform with respect to the C^1 norm in the region* (3.6), (3.7).

Proof. We compute

$$I_\lambda(V_\lambda) = \frac{1}{2}\int_{\Omega_\lambda} |\nabla V_\lambda|^2 - \int_{\Omega_\lambda} F_\lambda(V_\lambda - |\log\lambda|).$$

Using (3.11) and (3.12) we have

$$\int_{\Omega_\lambda} |\nabla V_\lambda|^2 = \int_{\Omega_\lambda} \left| \sum_{j=1}^m \nabla(V_j - H_j) \right|^2 = \sum_{j=1}^m \int_{B_{R_j}(\bar{\xi}_j)} f_\lambda(w_j(|x - \bar{\xi}_j|))V_\lambda.$$

Therefore, by Lemma 3.2

$$\begin{aligned}
&\int_{\Omega_\lambda} |\nabla V_\lambda|^2 \\
&\quad = \sum_{j=1}^m \int_{B_{R_j}(\bar{\xi}_j)} f_\lambda(w_j(|x-\bar{\xi}_j|)) \left(w_j(|x-\bar{\xi}_j|) + |\log\lambda| + O(\frac{\lambda}{|\log\lambda|^2}) \right) dx \\
&\quad = |\log\lambda| \sum_{j=1}^m \int_{B_{R_j}(0)} f_\lambda(w_j(x))\, dx + \sum_{j=1}^m \int_{B_{R_j}(0)} f_\lambda(w_j(x))w_j(x)\, dx + O\left(\frac{\lambda}{|\log\lambda|^2}\right)
\end{aligned} \tag{3.21}$$

as $\lambda \to 0$. Integrating (3.11) in $B_{R_j}(\bar{\xi}_j)$ and then using (3.4) and (3.17) we obtain

$$\begin{aligned}
\int_{B_{R_j}(0)} f_\lambda(w_j(x))\, dx = 2\pi\mu_j = 2\pi - 4\pi\frac{\log|\log\lambda|}{|\log\lambda|} + 2\pi\frac{\log R_0}{|\log\lambda|} \\
- \frac{4\pi^2}{|\log\lambda|}\left[H(\xi_j,\xi_j) + \sum_{i\neq j} G(\xi_j,\xi_i) \right] + O\left(\frac{(\log|\log\lambda|)^2}{|\log\lambda|^2}\right).
\end{aligned} \tag{3.22}$$

For the terms $\int_{B_{R_j}(0)} f_\lambda(w_j)w_j$, we note that multiplying (3.11) by w_j and integrating in B_{R_j} we obtain

$$\int_{B_{R_j}(0)} f_\lambda(w_j)w_j = \int_{B_{R_j}(0)} |\nabla w_j|^2.$$

But by (3.3)

$$w_j = \alpha_j V_0 + O\left(\frac{\log|\log\lambda|}{|\log\lambda|}\right)$$

in the C^1 norm over compact sets of $\mathbb{R}^2$, where α_j satisfies (3.9), and we get

$$\mu_j = d_0 R_0 \alpha_j + O\left(\frac{\log|\log\lambda|}{|\log\lambda|}\right).$$

This gives

$$|\nabla w_j|^2 = \frac{1}{d_0^2 R_0^2}|\nabla V_0|^2 + O\left(\frac{\log|\log\lambda|}{|\log\lambda|}\right).$$

Using this, (3.4) and (3.18) we see that

$$\int_{B_{R_j}(0)} f_\lambda(w_j)w_j = \frac{1}{d_0^2 R_0^2}\int_{B_{R_0}(0)} |\nabla V_0|^2 + O\left(\frac{\log|\log\lambda|}{|\log\lambda|}\right).$$

Therefore we find

$$\frac{1}{2}\int_{\Omega_\lambda} |\nabla V_\lambda|^2 = \pi m|\log\lambda| - 2\pi m\log|\log\lambda| + \pi m\log R_0 - 2\pi^2\varphi_m(\xi_1,\ldots,\xi_m)$$
$$+ \frac{m}{2d_0^2R_0^2}\int_{B_{R_0}} |\nabla V_0|^2 + O\Big(\frac{(\log|\log\lambda|)^2}{|\log\lambda|}\Big). \tag{3.23}$$

Now we compute

$$\int_{\Omega_\lambda} F_\lambda(V_\lambda - |\log\lambda|) = \sum_{j=1}^m \int_{B_{R_j}(\bar\xi_j)} F_\lambda(V_\lambda - |\log\lambda|) + \int_{\widetilde\Omega_\lambda} F_\lambda(V_\lambda - |\log\lambda|)$$

where $\widetilde\Omega_\lambda = \Omega\setminus\cup_{j=1}^m B_{R_j}(\bar\xi_j)$. First we have, using (2.7) and (3.16),

$$\int_{B_{R_j}(\bar\xi_j)} F_\lambda(V_\lambda - |\log\lambda|) = \int_{B_{R_j}(0)} F_\lambda(w_j) + O\bigg(\frac{\lambda}{|\log\lambda|^2}\bigg). \tag{3.24}$$

Using Lemma 3.1, $F_\lambda(t) = \frac{1}{2}t_+^2 + O(\frac{\log|\log\lambda|}{|\log\lambda|})$, (3.4) and (3.18), we obtain

$$\int_{B_{R_j}(\bar\xi_j)} F_\lambda(V_\lambda - |\log\lambda|) = \frac{1}{2d_0^2R_0^2}\int_{B_{R_0}} V_0^2 + O\bigg(\frac{\log|\log\lambda|}{|\log\lambda|}\bigg) \tag{3.25}$$

as $\lambda\to 0$. To estimate the integral in $\widetilde\Omega_\lambda$ we use the inequality (2.8) which implies that

$$|F_\lambda(v)| \le C\frac{\lambda}{|\log\lambda|^3} \quad \text{for all } -1\le v\le 0 \tag{3.26}$$

and

$$|F_\lambda(v)| \le C\frac{\lambda^3}{|\log\lambda|} \quad \text{for all } v\le -1. \tag{3.27}$$

We write

$$\int_{\widetilde\Omega_\lambda} F_\lambda(V_\lambda - |\log\lambda|) = \sum_{j=1}^m \int_{B_{10R_j(\bar\xi_j)}\setminus B_{R_j(\bar\xi_j)}} F_\lambda(V_\lambda - |\log\lambda|)$$
$$+ \int_{\Omega_\lambda\setminus\cup B_{10R_j(\bar\xi_j)}} F_\lambda(V_\lambda - |\log\lambda|)$$

and estimate, using Lemma 3.2 and (3.26),

$$\int_{B_{10R_j(\bar\xi_j)}\setminus B_{R_j(\bar\xi_j)}} F_\lambda(V_\lambda - |\log\lambda|) = \int_{B_{10R_j(\bar\xi_j)}\setminus B_{R_j(\bar\xi_j)}} F_\lambda\bigg(V_j - |\log\lambda| + O\bigg(\frac{\lambda}{|\log\lambda|^2}\bigg)\bigg)$$
$$= \int_{B_{10R_j(\bar\xi_j)}\setminus B_{R_j(\bar\xi_j)}} F_\lambda(V_j - |\log\lambda|) + O\bigg(\frac{\lambda}{|\log\lambda|^2}\bigg)$$
$$= O\bigg(\frac{\lambda}{|\log\lambda|^2}\bigg). \tag{3.28}$$

Far away from the points $\bar{\xi}_j$ we argue as follows. For points x in $\partial B_{10R_j}(\bar{\xi}_j)$ we have, because of Lemma 3.2, that $V_\lambda(x) - |\log\lambda| \le -M$ where $M > 0$ is a fixed constant. But $V_\lambda - |\log\lambda|$ is harmonic in $\Omega\setminus\cup_{j=1}^m B_{10R_j}(\bar{\xi}_j)$ and equal to $-|\log\lambda|$ on $\partial\Omega_\lambda$, and therefore $V_\lambda(x) - |\log\lambda| \le -M$ holds for all points in $\Omega\setminus\cup_{j=1}^m B_{10R_j}(\bar{\xi}_j)$. Thus we can use (3.27) and deduce

$$\int_{\Omega_\lambda\setminus\cup B_{10R_j(\bar{\xi}_j)}} F_\lambda(V_\lambda - |\log\lambda|) = O(\lambda|\log\lambda|^3).$$

Combining the estimates for each term, and noting that

$$\int_{B_{R_0}} |\nabla V_0|^2 = \int_{B_{R_0}} V_0^2,$$

we conclude that formula (3.20) is valid with $o(1)$ in the C^0 norm over the region (3.6), (3.7).

Regarding the estimate in C^1 norm, the error term in (3.23) is also $O(\frac{(\log|\log\lambda|)^2}{|\log\lambda|})$ in C^1 norm for the parameters in the region (3.6), (3.7), since we can use (3.16) for the error in (3.21) and the dependence on the ξ_j in the terms appearing in (3.21) is through the μ_i in expression (3.22). Similarly, the errors in (3.24), (3.25) and (3.28) are also C^1 with respect to ξ_i. □

3.2. *Proof of Lemmas* 3.1 *and* 3.2

Proof of Lemma 3.1. Firs we remark that for $\lambda > 0$ the nonlinearity f_λ is smooth, and that the solution $w_{\lambda,\alpha}$ of (3.2) exists for all $r \ge 0$. It is also smooth with respect to $\alpha > 0$, $\lambda > 0$.

a) Estimate (3.3) follows from $f_\lambda(t) = t_+ + O(\frac{\log|\log\lambda|}{|\log\lambda|})$ as $\lambda \to 0$ uniformly on compact subsets of $\mathbb{R}$ (c.f. (2.7)).

b) The existence of $R_{\lambda,\alpha}$ follows from the convergence in part a), and the uniqueness because $\frac{\partial}{\partial r}w_{\lambda,\alpha}(r) < 0$. The estimate for $R_{\lambda,\alpha}$ in (3.4) follows by the implicit function theorem, noting that $w_{\lambda,\alpha}(r)$ is C^1 in r, that $w_{\lambda,\alpha}$ and $\frac{\partial}{\partial r}w_{\lambda,\alpha}(r)$ have continuous extensions to $\lambda = 0$, and $\frac{\partial}{\partial r}w_{\lambda,\alpha}(R_0) < 0$ (with a uniform distance as $\lambda \to 0$). What needs to be verified is that the expansion

$$w_{\lambda,\alpha}(r) = w_{\lambda,\alpha}(R_0) + \frac{\partial}{\partial r}w_{\lambda,\alpha}(R_0)(r - R_0) + o(|r - R_0|)$$

as $r \to R_0$, has an error $o(|r - R_0|)$ which is uniform as $\lambda \to 0$. This estimate can be obtained from elliptic estimates, since $\Delta w_{\lambda,\alpha}$ remains bounded, and so is uniformly $C^{1,\mu}$ on compact sets.

We prove now the estimate $\frac{d}{d\alpha}R_{\lambda,\alpha} = O(\frac{\log|\log\lambda|}{|\log\lambda|})$ in (3.4). Differentiating the relation $w_{\lambda,\alpha}(R_{\lambda,\alpha}) = 0$ with respect to α we obtain

$$\frac{d}{d\alpha}R_{\lambda,\alpha} = -\frac{\frac{\partial}{\partial\alpha}w_{\lambda,\alpha}(R_{\lambda,\alpha})}{\frac{\partial}{\partial r}w_{\lambda,\alpha}(R_{\lambda,\alpha})}.$$

Since $\frac{\partial}{\partial r} w_{\lambda,\alpha}(R_{\lambda,\alpha})$ is bounded away from zero, we need only to estimate $\frac{\partial}{\partial \alpha} w_{\lambda,\alpha}(R_{\lambda,\alpha})$. Let $z_{\lambda,\alpha} = \frac{\partial}{\partial \alpha} w_{\lambda,\alpha}$ which is smooth and satisfies

$$-\Delta z_{\lambda,\alpha} = f'_\lambda(w_{\lambda,\alpha}) z_{\lambda,\alpha} \quad \text{in } \mathbb{R}^2. \tag{3.29}$$

We claim that

$$|z_{\lambda,\alpha}(R_0)| \leq C \frac{\log|\log\lambda|}{|\log\lambda|}. \tag{3.30}$$

To prove this we let $\phi(r) = z_{\lambda,\alpha}(r) - V_0(r)$ and note that

$$-\Delta\phi = f'_\lambda(w_{\lambda,\alpha}) z_{\lambda,\alpha} - \chi_{B_{R_0}} V_0 = \chi_{B_{R_0}} \phi + (f'_\lambda(w_{\lambda,\alpha}) - \chi_{B_{R_0}}) z_{\lambda,\alpha}.$$

Multiplying this equation by V_0 and integrating in B_{R_0} gives:

$$2\pi R_0 \phi(R_0) V'_0(R_0) = \int_{B_{R_0}} (f'_\lambda(w_{\lambda,\alpha}) - \chi_{B_{R_0}}) z_{\lambda,\alpha} V_0$$

and the integral on the right hand side can be verified to be $O(\frac{\log|\log\lambda|}{\log\lambda|})$ as $\lambda \to 0$. Equation (3.29) shows that $z_{\lambda,\alpha}$ is $C^{1,\mu}$ on compact set, for any $\mu \in (0, 1)$. Using this property and (3.30) we deduce

$$|z_{\lambda,\alpha}(R_{\lambda,\alpha})| \leq C \frac{\log|\log\lambda|}{|\log\lambda|}.$$

c) This property follows from the convergence $w_{\lambda,\alpha} \to \alpha V_0$ as $\lambda \to 0$ in $C^{1,\mu}$ on compact sets of $\mathbb{R}^2$ and $R_{\lambda,\alpha} \to R_0$ as $\lambda \to 0$. □

Proof of Lemma 3.2. Fix $i = 1, \dots, m$ and $R > 0$ be fixed. For $x \in B_R(\bar{\xi}_i)$ we have

$$V_i(x) - H_i(x) = V_i(x) + 2\pi\mu_i H\left(\frac{\lambda}{|\log\lambda|^2} x, \xi_i\right) + 2\mu_i \log|\log\lambda| - \mu_i \log R_i - (1 - \mu_i)|\log\lambda|.$$

For $x \in B_R(\bar{\xi}_i)$, if we take $j \neq i$ then

$$V_j(x) - H_j(x) = 2\pi\mu_j G\left(\frac{\lambda}{|\log\lambda|^2} x, \xi_j\right).$$

(This is valid actually for $|x - \bar{\xi}_j| > R_j$.) Therefore, for $x \in B_R(\bar{\xi}_i)$

$$V_\lambda(x) = \sum_{j=1}^{m} (V_j(x) - H_j(x)) = V_i(x) + 2\pi\mu_i H\left(\frac{\lambda}{|\log\lambda|^2} x, \xi_i\right) + 2\mu_i \log|\log\lambda| - \mu_i \log R_i - (1 - \mu_i)|\log\lambda| + 2\pi \sum_{j\neq i} \mu_j G\left(\frac{\lambda}{|\log\lambda|^2} x, \xi_j\right)$$

$$= V_i(x) + 2\pi\mu_i H(\xi_i, \xi_i) + 2\mu_i \log|\log\lambda|$$
$$- \mu_i \log R_i - (1-\mu_i)|\log\lambda| + 2\pi\sum_{j\neq i}\mu_j G(\xi_i, \xi_j) + O\left(\frac{\lambda}{|\log\lambda|^2}\right),$$

where the $O(\frac{\lambda}{|\log\lambda|^2})$ is in C^1 norm. Using the equations satisfied by μ_j, (3.15), we find the desired estimate (3.16). □

4. Linear Theory

Here we study the invertibility of the operator

$$L\phi = \Delta\phi + W\phi$$

in Ω_λ where

$$W = f_\lambda'(V_\lambda - |\log\lambda|).$$

Throughout this section we assume that $\xi_1, \ldots, \xi_m \in \Omega$ satisfy the separation conditions (3.6), (3.7) and $\bar{\xi}_j = \frac{|\log\lambda|^2}{\lambda}\xi_j \in \Omega_\lambda$. We also assume that $\mu_1, \ldots, \mu_m$ satisfy (3.8).

We consider the linear problem of given h in an appropriate space, finding ϕ and c_{ij}, $i = 1, \ldots, m$, $j = 1, 2$, such that

$$\begin{cases} L\phi = h + \sum_{i=1}^{m}\sum_{j=1,2} c_{ij}Z_{ij} & \text{in } \Omega_\lambda \\ \phi = 0 & \text{on } \partial\Omega_\lambda \end{cases} \tag{4.1}$$

$$\int_{\Omega_\lambda} \phi Z_{ij} = 0 \quad \text{for all } i = 1, 2, j = 1, \ldots, m, \tag{4.2}$$

where the functions Z_{ij} are defined by

$$Z_{ij}(x) = z_{ij}(x)\eta_0(x - \bar{\xi}_i)$$

with

$$z_{ij}(x) = \frac{\partial V_0}{\partial x_j}(x - \bar{\xi}_i), \tag{4.3}$$

and η_0 is a smooth radial function in $\mathbb{R}^2$ with support in $B_{R_0}(0)$ and identically 1 in $B_{R_0/2}(0)$, and V_0 is defined in (1.9). Note that $\frac{\partial V_0}{\partial x_j}(x - \bar{\xi}_j)$ is continuous but not C^1. The choice of η_0 makes Z_{ij} a smooth function with compact support.

Let Y be the space of measurable functions $h : \Omega_\lambda \to \mathbb{R}$ such that

$$h \in L^\infty(\hat{\Omega}_\lambda) \text{ and } h \in L^p(B_{2R_0(\bar{\xi}_j)}) \quad \text{for all } j = 1, \ldots, m,$$

where

$$\hat{\Omega}_\lambda = \Omega_\lambda \setminus \cup_{j=1}^m B_{2R_0}(\bar{\xi}_j)$$

and $2 < p < +\infty$ is fixed. We will consider the following norm on Y:

$$\|h\|_Y = \sup_{x\in\hat{\Omega}_\lambda} \left(\sum_{j=1}^m |x-\bar{\xi}_j|^{-2-\sigma} \right)^{-1} |h(x)| + \sum_{j=1}^m \|h\|_{L^p(B_{2R_0}(\bar{\xi}_j))},$$

where $0 < \sigma < 1$ is a small constant. Note that since $p > 2$, if $h \in Y$ then any solution $\phi \in H_0^1(\Omega_\lambda)$ of (4.1) is $C^1(\overline{\Omega}_\lambda)$.

Proposition 4.1. *There is $\lambda_0 > 0$ such that for any $0 < \lambda \le \lambda_0$ and for all $h \in Y$ there is a unique $\phi \in L^\infty(\Omega_\lambda)$ and unique $c_{ij} \in \mathbb{R}$ that solve* (4.1), (4.2). *Moreover*

$$\|\phi\|_{L^\infty(\Omega_\lambda)} + \sum_{i=1}^m \sum_{j=1,2} |c_{ij}| \le C\|h\|_Y. \tag{4.4}$$

In addition, the maps $\xi_1, \ldots, \xi_m \mapsto \phi, c_{ij}$ are differentiable and

$$\|\partial_{\bar{\xi}_k}\phi\|_{L^\infty(\Omega_\lambda)} + |\partial_{\bar{\xi}_k} c_{ij}| \le C\lambda^{\frac{1}{p}-1}\|h\|_Y. \tag{4.5}$$

The proof of this result relies on the non-degeneracy of V_0 (defined in (1.9)), which satisfies

$$-\Delta V_0 = (V_0)_+ \quad \text{in } \mathbb{R}^2,$$

and is radially symmetric with $\max_{\mathbb{R}^2} V_0 = 1$.

Proposition 4.2. *Let $\phi \in L^\infty(\mathbb{R}^2)$ be a solution of*

$$-\Delta\phi = \chi_{[V_0>0]}\phi \quad \textit{in } \mathbb{R}^2,$$

where $\chi_{[V_0>0]}$ is the characteristic function of the set $[V_0 > 0] = B_{R_0}(0)$. Then ϕ is a linear combination of

$$\frac{\partial V_0}{\partial x_1}, \quad \frac{\partial V_0}{\partial x_2}.$$

For the proof see [3, Proposition 3.1].

To prove Proposition 4.1 we start with an apriori estimate.

Lemma 4.3. *There is $C > 0$ such that for all $\lambda > 0$ small, and for any $h \in Y$, $\phi \in L^\infty(\Omega_\lambda)$, and $c_{ij} \in \mathbb{R}$ that verify* (4.1), (4.2) *we have*

$$\|\phi\|_{L^\infty(\Omega_\lambda)} + \sum_{i=1}^m \sum_{j=1,2} |c_{ij}| \le C\|h\|_Y.$$

Proof. We first prove that

$$|c_{ij}| \le C\|h\|_Y + o(1)\|\phi\|_{L^\infty(\Omega_\lambda)} \tag{4.6}$$

as $\lambda \to 0$. For this let η be a radial function in $C^\infty(\mathbb{R}^2)$ with support in $B_2(0)$ and $\eta \equiv 1$ in $B_1(0)$. Let $\eta_\lambda(x) = \eta(\lambda^{1/2}(x - \bar{\xi}_i))$ and

$$\widetilde{Z}_{ij} = z_{ij}\eta_\lambda$$

where z_{ij} is defined in (4.3). Multiplying (4.1) by $\widetilde{Z}_{ij}$ we find

$$\int_{\Omega_\lambda} \phi(\Delta\widetilde{Z}_{ij} + W\widetilde{Z}_{ij}) = \int_{\Omega_\lambda} h\widetilde{Z}_{ij} + c_{ij}\int_{\Omega_\lambda} Z_{ij}\widetilde{Z}_{ij}.$$

This gives

$$|c_{ij}| \le C\|h\|_Y + \|\phi\|_{L^\infty(\Omega_\lambda)}\int_{\Omega_\lambda} |\Delta\widetilde{Z}_{ij} + W\widetilde{Z}_{ij}|.$$

We compute

$$\Delta\widetilde{Z}_{ij} + W\widetilde{Z}_{ij} = \eta_\lambda(\Delta z_{ij} + Wz_{ij}) + 2\nabla\eta_\lambda\nabla z_{ij} + \Delta\eta_\lambda z_{ij}.$$

Using that $|z_{ij}(x)| \le C|x - \bar{\xi}_i|^{-1}$ and $|\nabla z_{ij}(x)| \le C|x - \bar{\xi}_i|^{-2}$ for $|x - \bar{\xi}_i| \ge 2R_0$, we see that

$$\int_{\Omega_\lambda} |2\nabla\eta_\lambda\nabla z_{ij} + \Delta\eta_\lambda z_{ij}| = O(\lambda^{1/2})$$

as $\lambda \to 0$. The other term can be estimated as follows:

$$\int_{\Omega_\lambda} |\eta_\lambda(\Delta z_{ij} + Wz_{ij})| \le \|\eta_\lambda\|_{L^\infty}\|z_{ij}\|_{L^\infty}\int_{\Omega_\lambda} |\chi_{B_{R_0}(\bar{\xi}_i)} - f_\lambda'(V_\lambda - |\log\lambda)| \to 0$$

as $\lambda \to 0$. Therefore

$$\int_{\Omega_\lambda} |\Delta\widetilde{Z}_{ij} + W\widetilde{Z}_{ij}| = o(1)$$

as $\lambda \to 0$ and this proves (4.6).

Now we claim that if $R > 0$ is large enough, then

$$\|\phi\|_{L^\infty(\Omega_\lambda)} \le C(\|\phi\|_i + \|h\|_Y + \sum_{i=1}^m \sum_{j=1,2} |c_{ij}|) \tag{4.7}$$

where

$$\|\phi\|_i = \sup_{\cup_{i=1}^m B_R(\bar{\xi}_i)} |\phi|.$$

For this we use a barrier argument. Let

$$\psi(x) = \sum_{j=1}^{m}(1 - |x - \bar{\xi}_j|^{-\sigma}).$$

Fix $R > 2R_0$. By (2.10), $W(x) \leq C\lambda^3/|\log\lambda|$ for $x \in \Omega_\lambda \setminus \cup_{i=1}^m B_R(\bar{\xi}_i)$. Then

$$\Delta\psi + W(x)\psi < 0 \quad \text{in } \Omega_\lambda \setminus \cup_{i=1}^m B_R(\bar{\xi}_i),$$

provided $\lambda > 0$ is small. This shows that the operator $\Delta + W$ satisfies the maximum principle in this region. Applying the maximum principle to $C(\|h\|_Y + \sum|c_{ij}| + \|\phi\|_i)\psi \pm \phi$, where C is a large constant, we arrive at (4.7).

Using (4.6) and (4.7) we obtain

$$\|\phi\|_{L^\infty(\Omega_\lambda)} \leq C(\|\phi\|_i + \|h\|_Y).$$

Therefore to prove the lemma it suffices to show

$$\|\phi\|_i \leq C\|h\|_Y. \tag{4.8}$$

We prove this estimate by contradiction. Assume that there are sequences $\lambda_n \to 0$, (ϕ_n) in $L^\infty(\Omega_{\lambda_n})$, $(c_{ij}^{(n)})$ in $\mathbb{R}$ and (h_n) in Y that satisfy (4.1), (4.2) and

$$\|\phi_n\|_i > n\|h_n\|_Y.$$

By linearity we can assume that $\|\phi_n\|_i = 1$. Then (4.6) implies that $\|h_n\|_Y \to 0$ and $c_{ij}^{(n)} \to 0$ as $n \to +\infty$. Then for a fixed $i \in \{1, \ldots, m\}$ and a subsequence (denoted the same as the original sequence)

$$\sup_{B_R(\bar{\xi}_i)} |\phi_n| \geq c$$

for some $c > 0$. By translating we can assume that $\bar{\xi}_i = 0$. Using the equation, we get that up to another subsequence, $\phi_n \to \phi$ uniformly on compact sets of $\mathbb{R}^2$ and that $\phi \not\equiv 0$ is a bounded solution of

$$\Delta\phi + \chi_{B_{R_0}(0)}\phi = 0 \quad \text{in } \mathbb{R}^2.$$

By Proposition 4.2, $\phi = a_1 z_{i1} + a_2 z_{i2}$ for some $a_1, a_2 \in \mathbb{R}$. But ϕ also satisfies

$$\int_{\mathbb{R}^2} \phi Z_{ij} = 0 \quad j = 1, 2,$$

which shows that $a_1 = a_2 = 0$ so $\phi \equiv 0$, a contradiction. This proves (4.8) and finishes the proof of the lemma. □

Proof of Proposition 4.1. Consider the Hilbert space

$$H = \left\{\phi \in H_0^1(\Omega_\lambda) : \int_{\Omega_\lambda} Z_{ij}\phi = 0 \; i = 1, \ldots, m, \; j = 1, 2\right\}$$

with inner product $\langle \phi_1, \phi_2 \rangle = \int_{\Omega_\lambda} \nabla\phi_1 \nabla\phi_2$. For $h \in Y$, the variational problem: find $\phi \in H$ such that

$$\langle \phi, \psi \rangle = \int_{\Omega_\lambda} (W\phi - h)\phi \quad \text{for all } \phi \in H$$

is a weak formulation of (4.1), (4.2). Using the Riesz representation theorem, this variational problem is equivalent to solve

$$\phi + K(\phi) = \tilde{h} \tag{4.9}$$

where $\tilde{h} \in H$ and $K : H \to H$ is a compact operator. When $h = 0$ then $\tilde{h} = 0$ and by Lemma 4.3 $\phi = 0$. By the Fredholm alternative there is a solution $\phi \in H$ of (4.9) giving a weak solution of (4.1), (4.2). By standard regularity theory $\phi \in C(\overline{\Omega}_\lambda)$ and we get the estimate (4.4) from Lemma 4.3.

Now we proceed with the differentiability properties of ϕ, c_{ij} with respect to $\xi_1, \ldots, \xi_m$. For this we proceed formally, assuming the differentiability, and obtain estimate (4.5). This argument can then later be justified by applying it to finite differences instead of derivatives. We recall that ϕ is the unique solution of

$$L\phi = h + \sum_{i=1}^{m} \sum_{j=1,2} c_{ij} Z_{ij} \quad \text{in } \Omega_\lambda$$

satisfying $\phi = 0$ on $\partial\Omega_\lambda$ and the orthogonality conditions (4.2), and that the c_{ij} are uniquely determined. Assuming that ϕ, c_{ij} are differentiable and setting $\psi = \partial_{\xi_k}\phi$ we find

$$L\psi = -\partial_{\xi_k} W\phi + h + \sum d_{ij} Z_{ij} + \sum c_{ij} \partial_{\xi_k} Z_{ij}$$

where $d_{ij} = \partial_{\xi_k} c_{ij}$. Differentiating the orthogonality condition (4.2) we get

$$\int_{\Omega_\lambda} \psi Z_{ij} + \phi \partial_{\xi_k} Z_{ij} = 0.$$

Setting

$$\tilde{\psi} = \psi + b_{ij} Z_{ij} \quad \text{where} \quad b_{ij} = \frac{\int_{\Omega_\lambda} \phi \partial_{\xi_k} Z_{ij}}{\int_{\Omega_\lambda} Z_{ij}^2}$$

we get that $\tilde{\psi}$ satisfies (4.2) and

$$L\tilde{\psi} = -\partial_{\xi_k} W\phi + h + \sum d_{ij} Z_{ij} + \sum c_{ij} \partial_{\xi_k} Z_{ij} + \sum b_{ij} L(Z_{ij}) \quad \text{in } \Omega_\lambda.$$

Hence, applying the apriori estimate of Lemma 4.3 we deduce

$$\|\psi\|_{L^\infty(\Omega_\lambda)} + |d_{ij}| \le C(\|\partial_{\xi_k} W\phi\|_Y + \|h\|_Y + \sum |c_{ij}| \|\partial_{\xi_k} L(Z_{ij})\|_Y + \sum |b_{ij}| \|L(Z_{ij})\|_Y).$$

We claim that each term is bounded by $C\lambda^{\frac{1}{p}-1}\|h\|_Y$.Let us verify this explicitly for $\|\partial_{\bar{\xi}_k} W\phi\|_Y$, because the others are direct. Since $\|\phi\|_{L^\infty} \leq C\|h\|_Y$, it suffices to verify that

$$\|\partial_{\bar{\xi}_k} W\|_Y \leq C\lambda^{\frac{1}{p}-1}. \tag{4.10}$$

But $\partial_{\bar{\xi}_k} W = f''_\lambda(V_\lambda - |\log\lambda|)\partial_{\bar{\xi}_k} V_\lambda$. One can verify directly that $\|\partial_{\bar{\xi}_k} V_\lambda\|_{L^\infty(\Omega_\lambda)} \leq C$. We then conclude the validity of (4.10) by using the next lemma. □

Lemma 4.4. *We have*

$$\|f''_\lambda(V_\lambda - |\log\lambda|)\|_Y \leq C\lambda^{\frac{1}{p}-1} \tag{4.11}$$

for some $C > 0$ and all $\lambda > 0$ small.

Proof. Let us write $\hat{\Omega}_\lambda = \Omega_\lambda \setminus \cup_{j=1}^m B_{2R_0}(\bar{\xi}_j)$. Using (2.13) and that $V_\lambda - |\log\lambda| \leq -a$ for some $a > 0$ on $\hat{\Omega}_\lambda$ we get

$$\sup_{x\in\hat{\Omega}_\lambda}\left(\sum_{j=1}^m |x-\bar{\xi}_j|^{-2-\sigma}\right)^{-1} |f''_\lambda(V_\lambda - |\log\lambda|)| \leq C\lambda^{1-2\sigma}.$$

To estimate $\|f''_\lambda(V_\lambda - |\log\lambda|)\|_{L^p(B_{2R_0}(\bar{\xi}_j))}$ we split the integral

$$\int_{B_{2R_0}(\bar{\xi}_j)} |f''_\lambda(V_\lambda - |\log\lambda|)|^p = \int_{D_1}\cdots + \int_{D_2}\cdots$$

where $D_1 = B_{R_j+L\lambda}(\bar{\xi}_j)\setminus B_{R_j-L\lambda}(\bar{\xi}_j)$, $D_2 = B_{2R_0}(\bar{\xi}_j)\setminus D_1$ and $L > 0$ is some large fixed number. In D_1, by (2.11) we estimate $|f''_\lambda| \leq C/\lambda$ and we get

$$\int_{D_1} |f''_\lambda(V_\lambda - |\log\lambda|)|^p \leq C\lambda^{1-p}.$$

To estimate the integral over D_2 we recall that $V_\lambda(x) - |\log\lambda| = w_j(|x-\bar{\xi}_j|) + O(\frac{\lambda}{|\log\lambda|^2})$ for $x \in B_{2R_0}(\bar{\xi}_j)$ (c.f. (3.16)) in C^1 norm. Since $w_j(R_j) = 0$, $V_\lambda - |\log\lambda|$ has a zero set that is at distance $O(\lambda)$ from $\partial B_{R_j}(\bar{\xi}_j)$, and $V_\lambda(x) - |\log\lambda|$ separates linearly from this curve. So thanks to (2.12), $|f''_\lambda(V_\lambda(x) - |\log\lambda|)| \leq C dist(x, \partial B_{R_j}(\bar{\xi}_j))^{-1}$ on D_2. Therefore

$$\int_{D_2} |f''_\lambda(V_\lambda - |\log\lambda|)|^p \leq C\int_{L\lambda}^C y^{-p}\,dy \leq C\lambda^{1-p}.$$

It follows that

$$\|f''_\lambda(V_\lambda - |\log\lambda|)\|_{L^p(B_{2R_0}(\bar{\xi}_j))} \leq C\lambda^{\frac{1}{p}-1},$$

and taking $\sigma > 0$ small we get the stated estimate (4.11). □

5. A Nonlinear Projected Problem

In this section we solve the nonlinear problem

$$\begin{cases} L\phi + N(\phi) + E_\lambda = \sum_{i=1}^{m} \sum_{j=1,2} c_{ij} Z_{ij} & \text{in } \Omega_\lambda \\ \phi = 0 & \text{on } \partial\Omega_\lambda \end{cases} \tag{5.1}$$

with ϕ satisfying

$$\int_{\Omega_\lambda} \phi Z_{ij} = 0 \quad \text{for all } i = 1, 2, \ \ j = 1, \ldots, m. \tag{5.2}$$

We always assume that $\xi_1, \ldots, \xi_m \in \Omega$ satisfy the separation conditions (3.6), (3.7), and $\bar{\xi}_j = \frac{|\log \lambda|^2}{\lambda} \xi_j$.

Proposition 5.1. *Assume $\mu_1, \ldots, \mu_m$ satisfy (3.15). Then there is $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$ (5.1), (5.2) has a unique solution $\phi = \phi(\xi_1, \ldots, \xi_m)$ and $c_{ij} = c_{ij}(\xi_1, \ldots, \xi_m)$ such that*

$$\|\phi\|_{L^\infty} + \sum_{i=1}^{m} \sum_{j=2}^{2} |c_{ij}| \le C\lambda^{1-2\sigma}. \tag{5.3}$$

Proof. For the argument it is better to work with the space X defined as the set of continuous functions ϕ on $\overline{\Omega}_\lambda$ such that ϕ restricted to $B_{2R_0}(\bar{\xi}_j)$ belongs to $W^{1,\infty}(B_{2R_0}(\bar{\xi}_j))$ for all $j = 1, \ldots, m$, equipped with the norm

$$\|\phi\|_X = \|\phi\|_{L^\infty(\Omega_\lambda)} + \sum_{j=1}^{m} \|\phi\|_{W^{1,\infty}(B_{2R_0}(\bar{\xi}_j))}.$$

We rewrite problem (5.1), (5.2) as the fixed point problem

$$\phi = F(\phi)$$

where $F = -T(N(\phi) + E_\lambda)$ and T is the linear operator defined in Proposition 4.1, which by estimate (4.4) satisfies

$$\|T(h)\|_\infty \le C\|h\|_Y \quad \text{for all } h \in Y.$$

By elliptic estimates we also deduce

$$\|T(h)\|_X \le C\|h\|_Y$$

with a constant C independent of λ, and here it is important that $p > 2$.

Let us estimate $\|E_\lambda\|_Y$. Using (2.8) we have

$$\sup_{x \in \hat{\Omega}_\lambda} \left(\sum_{j=1}^{m} |x - \bar{\xi}_j|^{-2-\sigma} \right)^{-1} |E_\lambda(x)| \le C\lambda^{1-\sigma} |\log \lambda|^{3+2\sigma} \le C\lambda^{1-2\sigma}$$

for $\lambda > 0$ small, where $\hat{\Omega}_\lambda = \Omega_\lambda \setminus \cup_{j=1}^m B_{2R_0}(\bar{\xi}_j)$. Also, using Lemma 3.2 we see that

$$|E_\lambda| \le C \frac{\lambda}{|\log \lambda|^3} \quad \text{in } B_{2R_0}(\bar{\xi}_j).$$

So we deduce

$$\|E_\lambda\|_Y \le C\lambda^{1-2\sigma}. \tag{5.4}$$

For fixed $\gamma > 0$, let $\mathcal{B} = \{\phi \in X : \|\phi\|_X \le \gamma\lambda^{1-2\sigma}\}$. We claim that there exist constants C, $a > 0$ such that for $\phi_1, \phi_2 \in \mathcal{B}$

$$\|N(\phi_1) - N(\phi_2)\|_Y \le C\lambda^a \|\phi_1 - \phi_2\|_X, \tag{5.5}$$

and we can prove this with $a = \frac{1-2\sigma}{p}$.

Indeed,

$$N(\phi_1) - N(\phi_2) = (\phi_1 - \phi_2) \int_0^1 N'(\phi_2 + s(\phi_1 - \phi_2))\, ds.$$

Therefore, to obtain (5.5) it is enough to show that for all $s \in [0, 1]$

$$\|N'(\phi_2 + s(\phi_1 - \phi_2))\|_Y \le C\lambda^a.$$

Part of this norm is $\|N'(\phi_2 + s(\phi_1 - \phi_2))\|_{L^p(B_{2R_0}(\bar{\xi}_j))}$. Write $\phi = \phi_2 + s(\phi_1 - \phi_2)$ and note that $\|\phi\|_X \le \gamma\lambda^{1-2\sigma}$. Then

$$\int_{B_{2R_0}(\bar{\xi}_j)} |N'(\phi)|^p = \int_{B_{2R_0}(\bar{\xi}_j)} |f'_\lambda(V_\lambda + \phi - |\log \lambda|) - f'_\lambda(V_\lambda - |\log \lambda|)|^p = I_1 + I_2,$$

where

$$I_k = \int_{A_k} |f'_\lambda(V_\lambda + \phi - |\log \lambda|) - f'_\lambda(V_\lambda - |\log \lambda|)|^p, \quad k = 1, 2$$
$$A_1 = B_{R_j+\lambda^m}(\bar{\xi}_j) \setminus B_{R_j-\lambda^m}(\bar{\xi}_j), \quad A_2 = B_{2R_0}(\bar{\xi}_j) \setminus A_1,$$

the numbers R_j are given in (3.10) and $m > 0$ is a parameter to be chosen. Since f'_λ is uniformly bounded in the range of its arguments (see (2.9)), we find

$$|I_1| \le C|A_1| \le C\lambda^m. \tag{5.6}$$

On A_2 we obtain

$$\begin{aligned} |I_2| &\le \int_{A_2} |f'_\lambda(V_\lambda + \phi - |\log \lambda|) - f'_\lambda(V_\lambda - |\log \lambda|)|^p \\ &\le \int_{A_2} \int_0^1 |f''_\lambda(V_\lambda + \tau\phi - |\log \lambda|)|^p |\phi|^p \, d\tau \, dx. \end{aligned}$$

We recall that $V_\lambda(x) - |\log\lambda| = w_j(|x-\bar{\xi}_j|) + O(\frac{\lambda}{|\log\lambda|^2})$ for $x \in B_{2R_0}(\bar{\xi}_j)$ (c.f. (3.16)) in C^1 norm, and note that the gradient of $\tau\phi$ is small in uniform norm in $B_{2R_0}(\bar{\xi}_j)$, because $\|\phi\|_X \le \gamma\lambda^{1-2\sigma}$. This and $w_j(R_j) = 0$ yield

$$|V_\lambda + \tau\phi - |\log\lambda|| \ge c\lambda^m \quad \text{in } A_2$$

for some constant $c > 0$. Therefore, from estimate (2.12)

$$|f_\lambda''(V_\lambda + \tau\phi - |\log\lambda|)| \le C dist(x, \partial B_{R_j}(\bar{\xi}_j))^{-1}$$

on A_2 and we get

$$|I_2| \le C\lambda^{m(1-p)}\|\phi\|_\infty^p \le C\lambda^{m(1-p)+(1-2\sigma)p}$$

Combining this last estimate with (5.6) we obtain

$$\|N'(\phi_2 + s(\phi_1 - \phi_2))\|_{L^p(B_{2R_0}(\bar{\xi}_j))} \le C(\lambda^{m/p} + \lambda^{m(1/p-1)+1-2\sigma}).$$

We choose $m = 1-2\sigma$ and we obtain

$$\|N'(\phi)\|_{L^p(B_{2R_0}(\bar{\xi}_j))} \le C\lambda^{\frac{1-2\sigma}{p}}. \tag{5.7}$$

Next we estimate the weighted L^∞ norm of $N'(\phi)$ away from the points $\bar{\xi}_j$. For this we recall that if $|x-\bar{\xi}_j| \ge 2R_0$ for all j, then $V_\lambda - |\log\lambda| \le -M$ for some fixed $M > 0$ and the same holds for $V_\lambda + s\phi - |\log\lambda|$. Therefore (2.10) yields

$$\sup_{x\in\hat{\Omega}_\lambda}\left(\sum_{j=1}^m |x-\bar{\xi}_j|^{-2-\sigma}\right)^{-1}|N'(\phi)| \le C\lambda^{1-2\sigma}. \tag{5.8}$$

The combination of (5.7), (5.8) proves (5.5).

Using (5.4) and (5.5), we see that choosing $\gamma > 0$ fixed and large in the definition of $\mathscr{B}$, for $\lambda > 0$ sufficiently small, F is a contraction in $\mathscr{B}$, and by the contraction mapping principle F has a unique fixed point in $\mathscr{B}$. □

Proposition 5.2. *For $\lambda > 0$ sufficiently small, the maps $\xi_1, \ldots, \xi_m \to \phi, c_{ij}$ constructed in Proposition* 5.1 *are differentiable in the region defined by* (3.6), (3.7) *and for any $k = 1, \ldots, m$, and $\lambda > 0$ small*

$$\|\partial_{\bar{\xi}_k}\phi\|_{L^\infty} + \|\partial_{\bar{\xi}_k}c_{ij}\|_{L^\infty} \le C\lambda^{\frac{1-2\sigma}{p}}. \tag{5.9}$$

For the proof we need the following lemma.

Lemma 5.3. *We have*

$$\|\partial_{\bar{\xi}_k}E\|_Y \le C\lambda^{\min(1-2\sigma,1/p)}.$$

Proof. We compute $\partial_{\bar{\xi}_k}E_\lambda = \Delta\partial_{\bar{\xi}_k}V_\lambda + f_\lambda'(V_\lambda - |\log\lambda|)\partial_{\bar{\xi}_k}V_\lambda$. It can be verified that $|\partial_{\bar{\xi}_k}V_\lambda| \le C$ in Ω_λ for some fixed constant.

Let us compute $\partial_{\bar{\xi}_k} E_\lambda(x)$ for $x \in \Omega_\lambda \setminus \cup_{j=1}^m B_{R_j}(\bar{\xi}_j)$. In this region V_λ is harmonic and therefore $\partial_{\bar{\xi}_k} E_\lambda = f'_\lambda(V_\lambda - |\log\lambda|)\partial_{\bar{\xi}_k} V_\lambda$. Therefore, using (2.10) and that $\|\partial_{\bar{\xi}_k} V_\lambda\|_{L^\infty(\Omega_\lambda)} \le C$ we get

$$\sup_{x\in\hat{\Omega}_\lambda} \left(\sum_{j=1}^m |x - \bar{\xi}_j|^{-2-\sigma} \right)^{-1} |\partial_{\bar{\xi}_k} E_\lambda| \le C\lambda^{1-\sigma} |\log\lambda|^{3+2\sigma} \le C\lambda^{1-2\sigma} \tag{5.10}$$

for $\lambda > 0$ small, where $\hat{\Omega}_\lambda = \Omega_\lambda \setminus \cup_{j=1}^m B_{2R_0}(\bar{\xi}_j)$.

Let us estimate $\|\partial_{\bar{\xi}_k} E_\lambda\|^p_{L^p(B_{2R_0}(\bar{\xi}_i)\setminus B_{R_i}(\bar{\xi}_i))} = I_1 + I_2$ where

$$I_1 = \int_{B_{2R_0}(\bar{\xi}_i)\setminus B_{R_i+L\lambda}(\bar{\xi}_i)} |\partial_{\bar{\xi}_k} E_\lambda|^p, \quad I_2 = \int_{B_{R_i+L\lambda}(\bar{\xi}_i)\setminus B_{R_i}(\bar{\xi}_i)} |\partial_{\bar{\xi}_k} E_\lambda|^p,$$

and $L > 0$ is some large constant. Using inequality (2.10) we can estimate

$$|f'_\lambda(V_\lambda - |\log\lambda|)| \le \frac{C\lambda^3}{dist(x, \partial B_{R_i}(\bar{\xi}_i))^3} \tag{5.11}$$

for $x \in B_{2R_0}(\bar{\xi}_i)\setminus B_{R_i+L\lambda}(\bar{\xi}_i)$. Therefore

$$I_1 \le C\lambda^{3p} \int_{L\lambda}^{2R_0} \frac{1}{y^{3p}}\, dy \le C\lambda.$$

For I_2 we have, using the uniform bound for f'_λ and $\partial_{\bar{\xi}_k} V_\lambda$, $|I_2| \le C\lambda$. Hence we find

$$\int_{B_{2R_0}(\bar{\xi}_i)\setminus B_{R_i}(\bar{\xi}_i)} |\partial_{\bar{\xi}_k} E_\lambda|^p \le C\lambda. \tag{5.12}$$

To estimate inside $B_{R_i}(\bar{\xi}_i)$ we use Lemma 3.2 to write

$$E_\lambda = f_\lambda\big(V_i - |\log\lambda| + g(x)\big) - f_\lambda(V_i - |\log\lambda|)$$

for $x \in B_{R_i}(\bar{\xi}_i)$, where

$$g(x) = 2\pi\mu_i\left[H\left(\frac{\lambda}{|\log\lambda|^2}x, \xi_i\right) - H(\xi_i, \xi_i)\right] + 2\pi\sum_{j\neq i}\mu_j\left[G\left(\frac{\lambda}{|\log\lambda|^2}x, \xi_j\right) - G(\xi_i, \xi_j)\right].$$

We recall that w_i depends on μ_i because the initial condition α_i in the ODE (3.2) is determined by μ_i from the relation (3.9). We make the dependence of the solution w of (3.2) explicit by writing $w = w(r, \alpha)$. We also note that μ_i depends on $\bar{\xi}_j$, $j = 1, \dots, m$ through the formula (3.15). Then

$$\partial_{\bar{\xi}_k} E_\lambda = E_1 + E_2$$

where

$$E_1 = \left[f_\lambda'(w_i(|x-\bar\xi_i|) + g(x)) - f_\lambda'(w_i(|x-\bar\xi_i|)\right]\partial_{\bar\xi_k} w_i(|x-\bar\xi_i|,\alpha_i)$$
$$E_2 = f_\lambda'(w_i(|x-\bar\xi_i|) + g(x))\partial_{\bar\xi_k} g$$

Then we compute

$$\partial_{\bar\xi_k} w_i(|x-\bar\xi_i|,\alpha_i) = -\delta_{ik}\frac{\partial w_i}{\partial r}\frac{x-\bar\xi_i}{|x-\bar\xi_i|} + \frac{\partial w_i}{\partial \alpha}\sum_j \frac{\partial \alpha_i}{\partial \mu_j}\frac{\partial \mu_j}{\partial \bar\xi_k}$$

and we get

$$|\partial_{\bar\xi_k} w_i(|x-\bar\xi_i|,\alpha_i)| \le C \quad \text{for } x\in B_{R_i}(\bar\xi_i). \tag{5.13}$$

To estimate E_1 we write $A_1 = B_{R_i}(\bar\xi_i)\backslash B_{R_i-L\lambda}(\bar\xi_i)$, $A_2 = B_{R_i-L\lambda}(\bar\xi_i)$ and $L>0$ is a large constant. Then, using the uniform bounds (5.13) and (2.9) we get $\int_{A_1}|E_1|^p \le C\lambda$. Since $g(x) = O(\lambda/|\log\lambda|^2)$, we estimate using (2.12)

$$\begin{aligned}\int_{A_2}|E_1|^p &\le C\int_{A_2}|f_\lambda'(w_i(|x-\bar\xi_i|) + g(x)) - f_\lambda'(w_i(|x-\bar\xi_i|))|^p \\ &\le C\int_{A_2}\int_0^1 |f_\lambda''(w_i(|x-\bar\xi_i|) + \tau g(x))|^p|g|^p\,d\tau\,dx \le C\lambda.\end{aligned}$$

Also $\int_{B_{R_i(\bar\xi_i)}}|E_2|^p \le C\lambda^p$ and we conclude that

$$\int_{B_{R_i}(\bar\xi_i)}|\partial_{\bar\xi_k}E_\lambda|^p \le C\lambda. \tag{5.14}$$

Combining (5.10), (5.12) and (5.14) we obtain the result of the lemma. □

Proof of Proposition 5.2. The proof that ϕ, c_{ij} are differentiable with respect to $\xi_1,\dots,\xi_m$ can be done with the contraction mapping principle, using that the linear operator defined in Proposition 4.1 is differentiable with respect to $\xi_1,\dots,\xi_m$. We proceed to prove estimate (5.9). Differentiating the equation in (5.1) with respect to $\bar\xi_k$ we obtain, for $\psi = \partial_{\bar\xi_k}\phi$,

$$\Delta\psi + W\psi + (\partial_{\bar\xi_k}W)\phi + \partial_{\bar\xi_k}E_\lambda + \partial_{\bar\xi_k}N(\phi) - \sum(\partial_{\bar\xi_k}c_{ij})Z_{ij} - \sum c_{ij}\partial_{\bar\xi_k}Z_{ij} = 0$$

in Ω_λ. Let $\tilde\psi = \psi - \sum d_{ij}Z_{ij}$ where $d_{ij} = -\int_{\Omega_\lambda}\phi\partial_{\bar\xi_k}Z_{ij}/\int_{\Omega_\lambda}Z_{ij}^2$. In this way, $\tilde\psi$ satisfies the orthogonality conditions (4.2). Applying Proposition 4.1 we can estimate

$$\begin{aligned}\|\tilde\psi\|_{L^\infty} + \sum|\partial_{\bar\xi_k}c_{ij}| \le C\Big(&\|\partial_{\bar\xi_k}E\|_Y + \|(\partial_{\bar\xi_k}W) + \partial_{\bar\xi_k}N(\phi)\|_Y \\ &+ \sum|c_{ij}|\|\partial_{\bar\xi_k}Z_{ij}\|_Y + \sum|d_{ij}|\|\Delta Z_{ij} + WZ_{ij}\|_Y\Big).\end{aligned}$$

Estimate (5.3) gives that $|c_{ij}| = O(\lambda^{1-2\sigma})$ and $|d_{ij}| = O(\lambda^{1-2\sigma})$ as $\lambda\to 0$. Lemma 5.3 yields

$$\|\partial_{\bar\xi_k}E\|_Y \le C\lambda^{\min(1-2\sigma,1/p)}.$$

We compute

$$(\partial_{\bar{\xi}_k} W)\phi + \partial_{\bar{\xi}_k} N(\phi) = \Big[f'_\lambda(V_\lambda + \phi - |\log\lambda|) - f'_\lambda(V_\lambda - |\log\lambda|)\Big](\partial_{\bar{\xi}_k} V_\lambda + \psi).$$

The same computations that lead to (5.7) and (5.8) show that

$$\|(f'_\lambda(V_\lambda + \phi - |\log\lambda|) - f'_\lambda(V_\lambda - |\log\lambda|))\|_Y \le C\lambda^{\frac{1-2\sigma}{p}}.$$

Therefore

$$\|(\partial_{\bar{\xi}_k} W)\phi + \partial_{\bar{\xi}_k} N(\phi)\|_Y \le C(\lambda^{\frac{1-2\sigma}{p}} + \lambda^{\frac{1-2\sigma}{p}}\|\psi\|_{L^\infty}).$$

Collecting the estimates above, we obtain

$$\|\psi\|_{L^\infty} \le C(\lambda^{\frac{1-2\sigma}{p}} + C\lambda^{\frac{1-2\sigma}{p}}\|\psi\|_{L^\infty}).$$

For $\lambda > 0$ small we deduce (5.9). □

6. Proof of the Main Results

We work with points $\xi_j \in \Omega$ satisfying (3.6), (3.7), that is, in the set

$$\widetilde{\Omega}_m = \{(\xi_1, \dots, \xi_m) \in \Omega^m : |\xi_i - \xi_j| \ge \delta \ \forall i \ne j, \ dist(\xi_j, \partial\Omega) \ge \delta \ \forall 1 \le j \le m\},$$

where $\delta > 0$ is small. Recall that $\bar{\xi}_j = \frac{|\log\lambda|^2}{\lambda}\xi_j \in \Omega_\lambda$. We also work with μ_j given by (3.15).

For $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \widetilde{\Omega}_m$, let $\phi(\boldsymbol{\xi})$ denote the solution of (5.1), (5.2) that satisfies $\|\phi\|_\infty \le C\lambda^{1-2\sigma}$ constructed in Proposition 5.1 and let $c_{ij}(\boldsymbol{\xi})$ denote the constants appearing in equation (5.1).

Writing the initial approximation $V_\lambda = V_\lambda(\boldsymbol{\xi})$ (defined in (3.13)), we set

$$J_\lambda(\boldsymbol{\xi}) = I_\lambda(V_\lambda(\boldsymbol{\xi}) + \phi(\boldsymbol{\xi}))$$

where I_λ is the functional given in (3.19).

Lemma 6.1. *If $\boldsymbol{\xi} = (\xi_1, \dots, \xi_j) \in \widetilde{\Omega}_m$ is a critical point of J_λ then $c_{ij}(\boldsymbol{\xi}) = 0$ for all $i = 1, \dots, m$ and all $j = 1, 2$, so that $V_\lambda(\boldsymbol{\xi}) + \phi(\boldsymbol{\xi})$ is a solution of* (3.1).

The proof of this lemma is very similar to Lemma 4.1 in [3] or Lemma 5.1 in [5].

Lemma 6.2. *We have the expansion*

$$I_\lambda(V_\lambda + \phi_\lambda) = I_\lambda(V_\lambda) + \Theta_\lambda(\boldsymbol{\xi})$$

as $\lambda \to 0$, where $\Theta_\lambda(\boldsymbol{\xi}) = o(1)$ in the C^1 norm in $\widetilde{\Omega}_m$.

Proof. Using that

$$DI_\lambda(V_\lambda + \phi_\lambda)[\phi_\lambda] = -\sum c_{ij} \int_{\Omega_\lambda} Z_{ij}\phi_\lambda = 0$$

we compute

$$\begin{aligned} I_\lambda(V_\lambda + \phi_\lambda) - I_\lambda(V_\lambda) &= -\int_0^1 sD^2 I_\lambda(V_\lambda + s\phi_\lambda)[\phi_\lambda, \phi_\lambda]\, dx\, ds \\ &= -\int_0^1 s \int_{\Omega_\lambda} (f_\lambda(V_\lambda + \phi_\lambda - |\log\lambda|) - f_\lambda(V_\lambda - |\log\lambda))\, \phi_\lambda \\ &\quad + \int_0^1 s \int_{\Omega_\lambda} (f'_\lambda(V_\lambda + s\phi_\lambda - |\log\lambda|)\phi_\lambda - E_\lambda)\, \phi_\lambda. \end{aligned}$$

We compute with detail the estimate for the derivative of $I_\lambda(V_\lambda + \phi_\lambda) - I_\lambda(V_\lambda)$ with respect to $\bar{\xi}_k$. The estimate for the C^0 norm is similar. Differentiating with respect to $\bar{\xi}_k$,

$$\partial_{\bar{\xi}_k}(I_\lambda(V_\lambda + \phi_\lambda) - I_\lambda(V_\lambda)) = \int_0^1 s(I_1 + I_2 + \dots)\, ds$$

where

$$\begin{aligned} I_1 &= -\int_{\Omega_\lambda} (f'_\lambda(V_\lambda + \phi_\lambda - |\log\lambda|) - f'_\lambda(V_\lambda - |\log\lambda|))(\partial_{\bar{\xi}_k} V_\lambda)\phi_\lambda \\ I_2 &= -\int_{\Omega_\lambda} (f'_\lambda(V_\lambda + \phi_\lambda - |\log\lambda|) - 2f'_\lambda(V_\lambda + s\phi_\lambda - |\log\lambda|))(\partial_{\bar{\xi}_k}\phi_\lambda)\phi_\lambda \\ I_3 &= \int_{\Omega_\lambda} f''_\lambda(V_\lambda + s\phi_\lambda - |\log\lambda|)(\partial_{\bar{\xi}_k}(V_\lambda + s\phi_\lambda))\phi_\lambda^2 \\ I_4 &= -\int_{\Omega_\lambda} (\partial_{\bar{\xi}_k} E_\lambda)\phi_\lambda + E_\lambda(\partial_{\bar{\xi}_k}\phi_\lambda) \\ I_5 &= -\int_{\Omega_\lambda} (f_\lambda(V_\lambda + \phi_\lambda - |\log\lambda|) - f_\lambda(V_\lambda - |\log\lambda|))\partial_{\bar{\xi}_k}\phi_\lambda. \end{aligned}$$

Each term I_i will be estimated below, but before, we remark that

$$\int_{\Omega_\lambda} |f'_\lambda(V_\lambda + s\phi_\lambda - |\log\lambda|)| \le C, \tag{6.1}$$

by (2.9), (2.10), and

$$\int_{\Omega_\lambda} |f''_\lambda(V_\lambda + s\phi_\lambda - |\log\lambda|)| \le C|\log\lambda|, \tag{6.2}$$

which is proved by a computation similar to Lemma 4.4.

We start with

$$\begin{aligned} |I_1| &\le \int_{\Omega_\lambda} |f'_\lambda(V_\lambda + \phi_\lambda - |\log\lambda|) - f'_\lambda(V_\lambda - |\log\lambda|)||\partial_{\bar{\xi}_k} V_\lambda|\, |\phi_\lambda| \\ &\le \|\partial_{\bar{\xi}_k} V_\lambda\|_{L^\infty} \|\phi_\lambda\|_{L^\infty} \int_{\Omega_\lambda} |f'_\lambda(V_\lambda + \phi_\lambda - |\log\lambda|) - f'_\lambda(V_\lambda - |\log\lambda|)| \end{aligned}$$

$$\le C\|\phi_\lambda\|_{L^\infty}^2 \int_0^1 \int_{\Omega_\lambda} |f_\lambda''(V_\lambda + \tau\phi - |\log\lambda|)|\, d\tau$$
$$\le C\lambda^{2-4\sigma}|\log\lambda|,$$

where we have used (5.3) and (6.2). Next we estimate

$$|I_2| \le \int_0^1 s \int_{\Omega_\lambda} (|f_\lambda'(V_\lambda + \phi_\lambda - |\log\lambda|)| + 2|f_\lambda'(V_\lambda + s\phi_\lambda - |\log\lambda|)|)|\partial_{\bar\xi_k}\phi_\lambda||\phi_\lambda|$$
$$\le C\|\partial_{\bar\xi_k}\phi_\lambda\|_{L^\infty}\|\phi_\lambda\|_{L^\infty} \le C\lambda^{1+\frac{1}{p}-2\sigma-\frac{2\sigma}{p}}$$

by (5.3), (5.9) and (6.1). Similarly

$$|I_3| \le \int_0^1 \int_{\Omega_\lambda} |f_\lambda''(V_\lambda + s\phi_\lambda - |\log\lambda|)\partial_{\bar\xi_k}(V_\lambda + s\phi)\phi_\lambda^2| \le C\lambda^{2-4\sigma}|\log\lambda|$$

and

$$|I_4| \le \|\phi_\lambda\|_{L^\infty}\int_{\Omega_\lambda} |\partial_{\bar\xi_k}E_\lambda| + \|\partial_{\bar\xi_k}\phi_\lambda\|_{L^\infty}\int_{\Omega_\lambda}|E_\lambda| \le C\lambda^{1+\frac{1}{p}-\frac{2\sigma}{p}}|\log\lambda|^3,$$

since outside a big ball around the $\bar\xi_i$, $|f_\lambda|, |f_\lambda'| \le C\lambda^3/|\log\lambda|$ and the area of Ω_λ is proportional to $|\log\lambda|^4/\lambda^2$. In summary

$$|\partial_{\bar\xi_k}(I_\lambda(V_\lambda + \phi_\lambda) - I_\lambda(V_\lambda))| \le C\lambda^{1+\frac{1}{p}-2\sigma-\frac{2\sigma}{p}}.$$

We choose now $p > 2$ close to 2 and $\sigma > 0$ small. Since $\partial_{\xi_k} = \frac{|\log\lambda|^2}{\lambda}\partial_{\bar\xi_k}$ we deduce

$$|\partial_{\xi_k}I_\lambda(V_\lambda + \phi_\lambda) - \partial_{\xi_k}I_\lambda(V_\lambda)| \le C\lambda^a$$

for some $a > 0$. □

Proof of Theorems 1.1, 1.2, 1.3. According to Lemma 6.1, the function $V_\lambda(\boldsymbol\xi) + \phi(\boldsymbol\xi)$ is a solution of problem (3.1) if we adjust $\boldsymbol\xi = (\xi_1, \dots, \xi_m)$ so that it is a critical point of $J_\lambda(\boldsymbol\xi) = I_\lambda(V_\lambda(\boldsymbol\xi) + \phi(\boldsymbol\xi)))$. This is equivalent to finding a critical point of

$$\widetilde{J}_\lambda(\boldsymbol\xi) = -\frac{1}{2\pi}\left[I_\lambda(V_\lambda) - \pi m|\log\lambda| + 2\pi m\log|\log\lambda| - \pi m\log R_0\right].$$

Thanks to Lemmas 3.3 and 6.2 for $\boldsymbol\xi \in \widetilde\Omega_m$ we have

$$\widetilde{J}_\lambda(\boldsymbol\xi) = \varphi_m(\boldsymbol\xi) + \Theta_\lambda(\boldsymbol\xi). \tag{6.3}$$

where $\Theta_\lambda = o(1)$ as $\lambda \to 0$ in the C^1 norm of $\widetilde\Omega_m$.

To prove Theorem 1.1, we note that when $m = 1$, $\varphi_1(\xi) = H(\xi, \xi)$ and therefore

$$\varphi_1(\xi) \to -\infty \quad \text{as } \xi \to \partial\Omega.$$

Hence, choosing $\delta > 0$ small, φ_1 has a strict absolute maximum in $\widetilde\Omega_1$. By (6.3), $\widetilde{J}_\lambda$ has also an absolute maximum in $\widetilde\Omega_1$ for $\lambda > 0$ small, and this yields the existence of a solution to problem (3.1).

Theorem 1.3 is a direct consequence of (6.3), since a non-degenerate critical point of φ_m gives rise to a critical point of a small C^1 perturbation of this function.

To prove Theorem 1.2 we invoke the work [5], where it is proved that if Ω is not simply connected, then φ_m and any sufficiently C^1 close map have critical points in $\widetilde{\Omega}_m$, for $\delta > 0$ small enough. This result also appears in [3]. □

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