Point Ruptures for a MEMS Equation with Fringing Field

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We construct solutions of the equation

$$-\Delta u = \lambda (1 + |\nabla u|^2) \left( 1 - u \right), \quad 0 < u < 1$$

in a bounded smooth domain of $\mathbb{R}^2$ with Dirichlet boundary condition, for $\lambda > 0$ small. These solutions approach 1 as $\lambda \to 0$ at one point, and if $\Omega$ is not simply connected we find solutions forming singularities at many points. The equation arises in the modeling of a MEMS with fringing field. A surprising connection with plasma problem is found.

Keywords Fringing-field; MEMS; Singularity formation.

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1. Introduction

The following elliptic equation arises in the modeling of electrostatic Micro-Electromechanical Systems (MEMS),

$$\begin{cases}
-\Delta u = \frac{\lambda}{(1 - u)^2}, & 0 < u < 1 \text{ in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

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where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary and $\lambda > 0$. Taking into account a fringing field in the modeling of the MEMS yields an extra term:

$$
\begin{cases}
-\Delta u = \frac{1 + \delta |\nabla u|^2}{(1 - u)^2}, & 0 < u < 1 \text{ in } \Omega \\
u = 0 & \text{ on } \partial \Omega,
\end{cases}
$$

(1.2)

where $\delta > 0$, see [13–16].

Of special interest are solutions that give rise to singularities in the equation, that is such that $u \approx 1$ in some region, which in the physical model represents a rupture in the device. We will say that a family of classical solutions $u_\lambda$ of (1.1) or (1.2) develops ruptures as $\lambda$ approaches a critical parameter if $\sup_\Omega u_\lambda \to 1$.

As observed numerically by Pelesko and Driscoll [16] and rigorously by Ye and Wei [18], there is a striking difference between (1.2) with $\delta > 0$ and (1.1). On one hand, for (1.1) in the unit ball, one knows that there is a family of solutions $u_\lambda$ developing a rupture at the origin for $\lambda \to \lambda_0 \neq 0$. In this case $\lambda_0$ and the limit function $u_0$ are explicit, see [12]. More properties on equation (1.1) can be found in [6, 7, 9–11]. On the other hand, for (1.2) with $\delta > 0$, if $\lambda$ has a fixed positive lower bound, then there is an a-priori estimate $u \leq C < 1$ for any solution $u$ of (1.2), see Theorem 4 in [18]. Then the implicit function theorem and the global bifurcation theorem of Rabinowitz [17], imply that there is a family $(\lambda, u)$ of solutions of (1.2) with $\lambda \to 0$ and $\sup_\Omega u \to 1$.

The precise behavior of solutions developing ruptures as $\lambda \to 0$ and the set of possible ruptures is not known so far. We give an answer to this question by constructing families of solutions that develop one or more ruptures as $\lambda \to 0$, and obtain a precise asymptotic description. The analysis also reveals connections with the plasma problem and the Liouville equation. This seems to be the first result for the construction of multiple point ruptures.

For simplicity we will work with $\delta = 1$, namely we work with

$$
\begin{cases}
-\Delta u = \frac{1 + |\nabla u|^2}{(1 - u)^2}, & 0 < u < 1 \text{ in } \Omega \\
u = 0 & \text{ on } \partial \Omega,
\end{cases}
$$

(1.3)

but the results are valid for any $\delta > 0$.

Let $m \geq 1$ be an integer. We say that a family of solutions $u_\lambda$ of (1.3) defined for all $\lambda > 0$ small develops $m$ isolated ruptures as $\lambda \to 0$, if there are points $\xi_{1, \lambda}, \ldots, \xi_{m, \lambda} \in \Omega$, uniformly separated between them and from the boundary, such that for any $\delta > 0$

$$
\limsup_{\lambda \to 0} \sup_{\Omega \setminus \bigcup_{j=1}^m B_\delta(\xi_{j, \lambda})} u < 1
$$

and for any $\delta > 0$ and all $i = 1, \ldots, m$:

$$
\sup_{B_\delta(\xi_{i, \lambda})} u \to 1 \text{ as } \lambda \to 0.
$$

Our main results are the following.
Theorem 1.1. There exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$ there is a solution $u_\lambda$ of (1.3) developing one isolated rupture as $\lambda \to 0$.

Theorem 1.2. If $\Omega$ is not simply connected, for any integer $m \geq 1$ there exists $\lambda_m > 0$ such that for $\lambda \in (0, \lambda_m)$ there is a solution $u_\lambda$ of (1.3) developing $m$ isolated ruptures as $\lambda \to 0$.

The location of the ruptures of the solutions constructed in the previous results is determined by a function $\varphi_m$ that depends on the Green function of the Laplacian in $\Omega$ and its regular part. More precisely, let $G$ denote the Green function for the Laplacian with Dirichlet boundary condition:

\[
\begin{cases}
-\Delta G(x, y) = \delta_x & \text{in } \Omega \\
G(x, y) = 0 & \text{for all } x \in \partial \Omega
\end{cases}
\]

and $H$ its regular part, given by

\[
H(x, y) = G(x, y) - \frac{1}{2\pi} \log \left( \frac{1}{|x - y|} \right).
\]

For an integer $m \geq 1$ and $\xi_1, \ldots, \xi_m$ different points in $\Omega$ we define

\[
\varphi_m(\xi_1, \ldots, \xi_m) = \sum_{i=1}^{m} H(\xi_i, \xi_i) + \sum_{j \neq i} G(\xi_i, \xi_j).
\]

Then in Theorems 1.1 and 1.2, after passing to a subsequence, the rupture points $\xi_{1,j}, \ldots, \xi_{m,j} \in \Omega$ of the solution $u_\lambda$, converge to a critical point of $\varphi_m$.

For multiple ruptures, i.e. $m > 1$, the condition that $\Omega$ is not simply connected guarantees existence of stable critical points of $\varphi_m$, see [5]. By stable we mean that these critical points also exist for functions close to $\varphi_m$ in $C^1$ norm.

As a byproduct of the analysis, we also have the following result.

Theorem 1.3. For any non-degenerate critical point $\xi = (\xi_1, \ldots, \xi_m)$ of $\varphi_m$ there is $\lambda_0 > 0$ such that $\lambda \in (0, \lambda_0)$ there is a solution $u_\lambda$ of (1.3) developing $m$ isolated ruptures as $\lambda \to 0$ at points $\xi_{1,j}, \ldots, \xi_{m,j} \in \Omega$ that converge to $(\xi_1, \ldots, \xi_m)$.

Regarding the asymptotic behavior of the solutions we construct, if $u_\lambda$ is the solution of any of the three theorems above developing $m$ ruptures $\xi_{1,j}, \ldots, \xi_{m,j}$ in $\Omega$, then

\[
u(x) = \frac{2\pi}{|\log \lambda|} \sum_{j=1}^{m} G(x, \xi_{j,j})(1 + o(1))
\]

as $\lambda \to 0$, for points $x \in \Omega$ away from $\xi_{1,j}, \ldots, \xi_{m,j}$. Very close to the points $\xi_{j,j}$ we have an expansion of the form

\[
u(x) = 1 - \frac{\lambda}{|\log \lambda|} - \frac{\lambda \log |\log \lambda|}{|\log \lambda|^2} + \frac{\lambda}{|\log \lambda|^2} \log \left( \zeta_0 V_0 \left( \frac{|\log \lambda|^2}{\lambda} |x - \xi_{j,j}| \right) \right) + \text{l.o.t.}
\]

(1.6)
where l.o.t. contains smaller order terms and the function $V_0$ is the unique radial function satisfying

$$-\Delta V_0 = (V_0)_+ \quad \text{in } \mathbb{R}^2, \quad \max_{\mathbb{R}^2} V_0 = 1. \quad (1.7)$$

The number $z_0$ is given by $z_0 = (\frac{1}{2\pi} \int_{\mathbb{R}^2} (V_0)_+)^{-1}$. The expansion (1.6) is valid for points $x \in \Omega$ such that $|x - \bar{z}_{x, 0}| \leq \frac{1}{|\log 2|^2} R$, where $0 < R < R_0$ and $R_0 > 0$ is the radius for which $V_0(R_0) = 0$.

The function $V_0$ can be written more explicitly as follows. The number $R_0 > 0$ is the radius such that the principal Dirichlet eigenvalue of $-\Delta$ in $B_{R_0}(0)$ is 1. Let $\varphi_1$ be the first eigenfunction in $B_{R_0}(0)$:

$$\begin{cases}
-\Delta \varphi_1 = \varphi_1, & \varphi_1 > 0 \quad \text{in } B_{R_0}(0) \\
\varphi_1 = 0 & \text{on } \partial B_{R_0}(0)
\end{cases}$$

normalized so that $\varphi_1(0) = 1$ and let

$$d_0 = -\varphi_1'(R_0). \quad (1.8)$$

Then

$$V_0(x) = \begin{cases}
\varphi_1(x) & \text{if } x \in B_{R_0}(0) \\
-d_0 R_0 \log(|x|/R_0) & \text{if } x \not\in B_{R_0}(0).
\end{cases} \quad (1.9)$$

The key behind the main results is a change of variables similar to the one introduced in Ye and Wei [18], which allows us to rewrite problem (1.3) in the form

$$\begin{cases}
-\Delta v = f_\lambda(v - |\log \lambda|) & \text{in } \Omega_\lambda = \frac{|\log \lambda|^2}{\lambda} \Omega \\
v = 0 & \text{on } \partial \Omega_\lambda,
\end{cases} \quad (1.10)$$

with a nonlinearity $f_\lambda$ that satisfies

$$f_\lambda(t) \rightarrow t_+ \quad \text{as } \lambda \rightarrow 0$$

for all $t \in \mathbb{R}$, where $t_+ = \max(t, 0)$. See the derivation in Section 2 and the definition of $f_\lambda$ in (2.6).

Formally, replacing $f_\lambda(t)$ in (1.10) by $t_+$ and setting

$$\tilde{v}(x) = \frac{1}{|\log \lambda|^2} v \left( \frac{|\log \lambda|^2}{\lambda} x \right), \quad x \in \Omega,$$

we are lead to

$$\begin{cases}
-\varepsilon^2 \Delta \tilde{v} = (\tilde{v} - 1)_+ & \text{in } \Omega \\
\tilde{v} = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\varepsilon = \frac{i}{|\log 2|^2}$. This problem is, after some transformations, the same as the plasma problem studied in [2, 3] and also presents similarities with the Liouville
equation [1, 4, 5, 8]. For both problems there are existence and classification results of solutions exhibiting point concentration. In both problems the location of the concentration points is determined by an energy expansion that at main nontrivial order involves the same function \( \phi_m \).

The results of this article give some information on the bifurcation diagram associated to (1.3). In fact, we can say that there are branches of solutions of the form \((\lambda, u_\lambda)\) defined for small \( \lambda > 0 \) such that

\[
\max u_\lambda = 1 - \frac{\lambda}{|\log \lambda|} (1 + o(1)) \quad \text{as} \quad \lambda \to 0.
\]

In the problem (1.2) with an arbitrary \( \delta > 0 \), the same branches exist but it is natural to expect that there is a non-uniform behavior as \( \delta > 0 \). In particular, the smaller the value \( \delta > 0 \) is, the more that the bifurcation diagram of (1.2) should resemble the one of (1.1), in particular with an increasing number of foldings, up to some point in the \( \max u_\lambda \) axis that depends on \( \delta \), and after which one can expect the branches constructed in this article. This can be seen in numerical calculations in [16] for the unit disk. Regarding the applications, a question of interest is the first folding point, that is related to the maximum voltage that can be applied to the device. In [13] the authors study formally the effect of small \( \delta > 0 \) in the first folding point.

The proof of all theorems is carried out with equation (1.10), by a Lyapunov-Schmidt finite dimensional reduction. In Section 3, given \( m \) uniformly separated points \( \xi_1, \ldots, \xi_m \) in \( \Omega \), we construct an approximate solution \( V_\lambda(\xi_1, \ldots, \xi_m) \) of (1.10), in the same spirit as in [3–5]. We also compute in this section an expansion of the energy \( I_\lambda \) of \( V_\lambda(\xi_1, \ldots, \xi_m) \), where \( I_\lambda \) is the natural energy functional associated to (1.10). We seek a true solution of (1.10) of the form \( V_\lambda + \phi \), where \( \phi \) is small in an appropriate norm. For this, in Section 4 we study the linearization of (1.10) around \( V_\lambda \) and prove its bounded invertibility in appropriate spaces, except natural orthogonality conditions. In Section 5 we show existence of solutions to a projected version of (1.10). After the reduction, the problem becomes one of finding critical points of a function that is close in \( C^1 \) norm to \( \phi_m \) on compact subsets of its domain of definition, and this is done in Section 6. In the case of two or more points of concentration in a non-simply connected domain, this is the guaranteed by a result in [5], which has also appeared in [3].

The main difficulty in this process is that the nonlinearity \( f_\lambda \) converges to \( x_+ \), which is only Lipschitz and hence, some estimates of derivatives of the solution, which involve \( f_\lambda'' \), become delicate as \( \lambda \to 0 \). This is similar to the difficulty in [3], except that for us the nonlinearity is not explicit but smooth.

2. Change of Variables

Equation (1.3) is equivalent to

\[
\begin{cases}
\Delta u_\lambda = \lambda \frac{1 + |\nabla u_1|^2}{u_1^2}, & 0 < u_1 < 1 \quad \text{in} \quad \Omega \\
u_1 = 1 & \text{on} \quad \partial \Omega,
\end{cases}
\]
where $u_1 = 1 - u$. Let us write $u_1$ in the form $u_1(x) = \lambda w(x/\lambda), \ x \in \Omega$, so that $w$ satisfies

\[
\begin{cases}
\Delta w = \frac{1 + |\nabla w|^2}{w^2}, & 0 < w < \frac{1}{\lambda} \text{ in } \Omega/\lambda \\
w = \frac{1}{\lambda} & \text{ on } \partial\Omega/\lambda.
\end{cases}
\]

Motivated by [18], we can eliminate the gradient term by introducing the change of variables

\[v_1 = g(w) \quad \text{where } g(w) = \int_w^1 e^{1/s} \, ds. \quad (2.1)\]

We compute

\[
\Delta v_1 = -\frac{1}{w^2} e^{1/w},
\]

and note that $g : (0, +\infty) \to \mathbb{R}$ is a decreasing convex function with range equal to $\mathbb{R}$. Therefore $g^{-1} : \mathbb{R} \to (0, +\infty)$ is well defined. Let

\[h(w) = \frac{1}{w^2} e^{1/w} \quad \text{for all } w > 0\]

and

\[f(t) = h(g^{-1}(t)). \quad (2.2)\]

Then $v_1$ satisfies

\[
\begin{cases}
-\Delta v_1 = f(v_1) & \text{in } \Omega/\lambda \\
v_1 = -c_\lambda & \text{on } \partial\Omega/\lambda
\end{cases}
\]

where

\[c_\lambda = -g(1/\lambda) = \int_{1/\lambda}^{1} e^{1/s} \, ds.
\]

Note that

\[c_\lambda = \frac{1}{\lambda} + |\log \lambda| + O(1) \quad \text{as } \lambda \to 0\]

and that $f : \mathbb{R} \to (0, \infty)$ is a convex increasing function. Moreover

\[f(v) = v(\log v)^4 + o(v(\log v)^4) \quad \text{as } v \to +\infty \quad (2.3)\]

\[f(v) = O\left(\frac{1}{v^2}\right) \quad \text{as } v \to -\infty. \quad (2.4)\]
Let $v_2 = v_1 + c_\lambda$ so that it satisfies
\[\begin{cases} -\Delta v_2 = f(v_2 - c_\lambda) & \text{in } \Omega / \lambda \\ v_2 = 0 & \text{on } \partial \Omega / \lambda. \end{cases}\]

Finally, we write $v_2$ as
\[v_2(x) = \frac{c_\lambda}{|\log \lambda|} v(|\log \lambda|^2 x) \quad (2.5)\]
which leads to
\[\begin{cases} -\Delta v = f_\lambda(v - |\log \lambda|) & \text{in } \Omega_\lambda = \frac{|\log \lambda|^2}{\lambda} \Omega \\ v = 0 & \text{on } \partial \Omega_\lambda, \end{cases}\]
where
\[f_\lambda(t) = \frac{1}{c_\lambda |\log \lambda|^3} f\left(\frac{c_\lambda}{|\log \lambda|} t\right). \quad (2.6)\]

This is the formulation used in the proofs of the main results. Observe that thanks to (2.3), (2.4) we have
\[f_\lambda(t) \to t_+ \quad \text{as } \lambda \to 0 \quad \text{for all } t \in \mathbb{R}.\]

We collect here some useful estimates for the non-linearity (we omit the computations).

\textbf{Lemma 2.1.} Let $f$ be defined by (2.2) and $f_\lambda$ be the function (2.6). Then the following properties hold:

\[f(t) = t \left( (\log t)^4 + O((\log t)^3 \log \log t) \right) \quad \text{as } t \to +\infty\]
\[f(t) \leq \frac{C}{1 + t^2} \quad \text{for all } t \leq 0\]
\[f_\lambda(t) = t_+ + O\left( \frac{|\log \log \lambda|}{|\log \lambda|} \right) \quad \text{as } \lambda \to 0, \quad \text{uniformly for } t \text{ on bounded sets} \quad (2.7)\]
\[f_\lambda(t) \leq C \frac{\lambda^3}{|\log \lambda|(t^3 + \lambda^3 |\log \lambda|^3)} \quad \text{for } t \leq 0 \quad (2.8)\]
\[f_\lambda'(t) = O(1) \quad \text{uniformly for } t \text{ in bounded sets} \quad (2.9)\]
\[f_\lambda'(t) \leq C \frac{\lambda^3}{|\log \lambda|(t^3 + \lambda^3 |\log \lambda|^3)} \quad \text{for all } t \leq 0 \quad (2.10)\]
\[f_\lambda''(t) = O(1/\lambda) \quad \text{uniformly for } t \text{ in bounded sets} \quad (2.11)\]
\[f_\lambda''(t) \leq \frac{C}{t} \quad \text{if } t / (\lambda |\log \lambda|) \to \infty \quad (2.12)\]
\[f_\lambda''(t) \leq C \frac{\lambda^3}{|\log \lambda|(t^4 + \lambda^4 |\log \lambda|^4)} \quad \text{for all } t \leq 0 \quad (2.13)\]
Remark 2.2. The choice of the scaling of the domain in (2.5) is motivated by the following computations. At points where \( u \) develops ruptures, the function \( v_1 \) (c.f. (2.1)) blows up. Therefore it is natural to introduce a new variable \( \tilde{v} \) so that

\[
v_1(x) = M\tilde{v}((\log M)^2 x) \quad x \in \frac{1}{\lambda} \Omega,
\]

(2.14)

where \( M > 0 \) is a new large parameter to be chosen later on. We find

\[
-\Delta \tilde{v} = \frac{1}{M(\log M)^4} f(M\tilde{v}) \quad \text{in} \quad \frac{(\log M)^2}{\lambda} \Omega.
\]

By (2.3), (2.4)

\[
\frac{1}{M(\log M)^4} f(Mt) \to t_+ \quad \text{as} \quad M \to +\infty
\]

for any \( t \in \mathbb{R} \), which is the motivation to introduce \((\log M)^2\) as scaling in (2.14). If we choose \( M \) such that \( \max \tilde{v} = 1 \), then \( \tilde{v} \) solves at main order

\[
-\Delta \tilde{v} = \tilde{v}_+ \quad \text{in} \quad \frac{(\log M)^2}{\lambda} \Omega
\]

\[
\max \tilde{v} = 1.
\]

Assuming that \( \tilde{v} \) has a maximum at the origin, heuristically we can expect that

\[
\tilde{v}(x) \approx V_0(x).
\]

We use this information to obtain an estimate of the size of \( M \). By definition of \( \tilde{v} \),

\[
v_1(x) \approx MV_0((\log M)^2 x).
\]

Evaluating this relation on the boundary of \( \frac{1}{\lambda} \Omega \) we find

\[
c_\lambda = Md_0R_0 \log \left( \frac{(\log M)^2 D}{\lambda R_0} \right)
\]

where \( D \) represents the diameter of the domain. This gives the relation

\[
M_\lambda = \frac{1}{d_0R_0\lambda|\log \lambda|} + o\left( \frac{1}{\lambda|\log \lambda|} \right) \quad \text{as} \quad \lambda \to 0,
\]

so \( \log(M_\lambda) = |\log \lambda| + o(|\log \lambda|) \) as \( \lambda \to 0 \). This motivates (2.5).
3. First Approximation and Its Energy

3.1. Setup

We will work mainly with the following reformulation of problem (1.3),

\[
\begin{aligned}
-\Delta v &= f_\lambda(v - \log |\lambda|) \quad \text{in } \Omega_\lambda \\
v &= 0 \quad \text{on } \partial \Omega_\lambda
\end{aligned}
\]

(3.1)

where

\[
f_\lambda(x) = \frac{1}{c_\lambda |\log \lambda|^3} f \left( \frac{c_\lambda}{|\log \lambda|} x \right)
\]

with \( f \) defined in (2.2) and

\[
\Omega_\lambda = \frac{|\log \lambda|^2}{\lambda} \Omega.
\]

We will define an initial approximation of a solution to (3.1) based on the solutions \( w_{\lambda, \alpha} \) of the following problem

\[
\begin{aligned}
w''_{\lambda, \alpha}(r) + \frac{1}{r} w'_{\lambda, \alpha}(r) &= -f_\lambda(w_{\lambda, \alpha}) \quad r > 0 \\
w_{\lambda, \alpha}(0) &= \alpha, \quad w'_{\lambda, \alpha}(0) = 0,
\end{aligned}
\]

(3.2)

where \( \alpha > 0 \).

Lemma 3.1. We have:

(a)

\[
w_{\lambda, \alpha} = \alpha V_0 + O \left( \frac{\log |\log \lambda|}{|\log \lambda|} \right)
\]

as \( \lambda \to 0 \) in the \( C^1 \) norm over compact sets of \( \mathbb{R}^2 \), where \( V_0 \) is the function defined in (1.9).

(b) There is a unique \( R_{\lambda, \alpha} > 0 \) such that

\[
w_{\lambda, \alpha}(R_{\lambda, \alpha}) = 0,
\]

and it satisfies

\[
R_{\lambda, \alpha} = R_0 + O \left( \frac{\log |\log \lambda|}{|\log \lambda|} \right), \quad \frac{d}{d\alpha} R_{\lambda, \alpha} = O \left( \frac{\log |\log \lambda|}{|\log \lambda|} \right)
\]

(3.4)

as \( \lambda \to 0 \) where \( R_0 \) is the number defined just before (1.9) and \( O \left( \frac{\log |\log \lambda|}{|\log \lambda|} \right) \) is uniform for \( \alpha \) in compact sets of \( (0, +\infty) \).

(c)

\[-w'_{\lambda, \alpha}(R_{\lambda, \alpha}) = d_0 \alpha + O \left( \frac{\log |\log \lambda|}{|\log \lambda|} \right) \quad \text{as } \lambda \to 0
\]

(3.5)
For the proof see section 3.2.

We proceed now with the construction of an initial approximation of a solution to (3.1). Let \( m \geq 1 \) be a fixed integer, \( \xi_1, \ldots, \xi_m \in \Omega \), and \( \mu_1, \ldots, \mu_m > 0 \). We will always assume that for some small \( \delta > 0 \)

\[
|\xi_i - \xi_j| \geq \delta \quad \text{for all } i \neq j \tag{3.6}
\]

\[
\text{dist}(\xi_j, \partial \Omega) \geq \delta, \quad \text{for all } 1 \leq j \leq m,
\]

\[
\delta \leq \mu_j \leq \delta^{-1} \quad \text{for all } 1 \leq j \leq m. \tag{3.7}
\]

Let us use the notation

\[
\hat{\xi}_j = \log \frac{\lambda \xi_j}{\delta}, \quad \lambda \in \Omega_j.
\]

The parameters \( \mu_j \) will be chosen later on, but let us comment that they are used to decrease the error of the approximation. We will see that to choose the numbers \( \mu_j \) is equivalent to choose the value of \( \xi \) in (3.2). In the limit as \( \lambda \to 0 \) the solution of this ODE is just \( \alpha \chi_0 \) as stated in Lemma 3.1. Note that \( \alpha \chi_0 \) is also a solution of (1.7) for any \( \alpha > 0 \). Actually the same approach is taken in the plasma problem [3] and in the Liouville equation [1, 4, 5, 8].

Thanks to (3.5), we can find for \( \lambda > 0 \) small a unique positive number \( \alpha_j \) such that

\[
-w_{\lambda, \alpha_j}(R_{\lambda, \alpha_j}) R_{\lambda, \alpha_j} = \mu_j. \tag{3.9}
\]

Let us write

\[
w_j(r) = w_{\lambda, \alpha_j}(r), \quad R_j = R_{\lambda, \alpha_j}.
\]

We define

\[
V_j(x) = \begin{cases} 
w_j(|x - \tilde{\xi}_j|) + |\log \lambda| & \text{if } |x - \tilde{\xi}_j| \leq R_j \\
-\mu_j \log \left( \frac{|x - \tilde{\xi}_j|}{R_j} \right) + |\log \lambda| & \text{if } |x - \tilde{\xi}_j| > R_j.
\end{cases}
\]

The function \( V_j \) is radial about the point \( \tilde{\xi}_j \) and it is \( C^1 \) across the boundary of the ball \( B_{R_j}(\tilde{\xi}_j) \), thanks to (3.9). It satisfies

\[
\begin{cases}
-\Delta V_j = f_{\lambda}(V_j - |\log \lambda|) \chi_{B_{R_j}(\tilde{\xi}_j)} \quad \text{on } \mathbb{R}^2, \\
\max_{\mathbb{R}^2} V_j = V_j(\tilde{\xi}_j) = \alpha_j + |\log \lambda|.
\end{cases}
\]

Since the function \( V_j \) does not satisfy the boundary condition on \( \partial \Omega \) we consider \( V_j - H_j \) where \( H_j \) is the solution of the problem

\[
\begin{cases}
\Delta H_j = 0 \quad \text{in } \Omega_j \\
H_j = V_j \quad \text{on } \partial \Omega_j.
\end{cases}
\]
Then

\[ H_j(x) = -\mu_j \log \left( \frac{|x - \bar{x}_j|}{R_j} \right) + |\log \lambda| \quad \text{for all } x \in \partial\Omega_j, \]

and we get the formula

\[ H_j(x) = -2\pi \mu_j H \left( \frac{\lambda}{|\log \lambda|^2}, x, \bar{x}_j \right) - 2\mu_j \log |\log \lambda| + \mu_j \log R_j + (1 - \mu_j)|\log \lambda| \]

where \( H \) is the regular part of the Green function, c.f. (1.4). We define the initial approximation as:

\[ V_\lambda = \sum_{j=1}^{m} V_j - H_j. \] (3.13)

We look for a solution \( v \) of (3.1) of the form \( v = V_\lambda + \phi \) where \( \phi \) is small compared to \( V_\lambda \). Then problem (3.1) gets reformulated in terms of \( \phi \) as follows:

\[
\begin{cases}
\Delta \phi + f_j(V_\lambda - |\log \lambda|) \phi + E_\lambda \phi + N(\phi) = 0 \quad \text{in } \Omega_j, \\
\phi = 0 \quad \text{on } \partial\Omega_j,
\end{cases}
\]

(3.14)

where

\[ E_\lambda = \Delta V_\lambda + f_j(V_\lambda - |\log \lambda|) \]

\[ N(\phi) = f_j(V_\lambda + \phi - |\log \lambda|) - f_j(V_\lambda - |\log \lambda|) - f_j(V_\lambda - |\log \lambda|) \phi. \]

Up to now the parameters \( \mu_j \) were free in the interval \( (\delta, \delta^{-1}) \) (\( \delta > 0 \) a small constant). To ensure that \( E_\lambda \) is small in an appropriate norm to be introduced later it is necessary to adjust the numbers \( \mu_j \) in a suitable way.

**Lemma 3.2.** Assume that \( \mu_1, \ldots, \mu_m > 0 \) satisfy the system of equations

\[ \mu_i = 1 - 2\mu_i \log |\log \lambda| + \mu_i \frac{\log R_i}{|\log \lambda|} - \frac{2\pi}{|\log \lambda|} \left[ \mu_i H(\xi_i, \xi_i) + \sum_{i \neq j} \mu_j G(\xi_i, \xi_j) \right] \]

(3.15)

for all \( i = 1, \ldots, m \). Then for all \( R > 0 \), we have

\[ V_\lambda(x) = V_j(x) + O \left( \frac{\lambda}{|\log \lambda|^2} \right), \quad \text{for all } x \in B_R(\xi_i) \] (3.16)

as \( \lambda \to 0 \), where the term \( O(\frac{\lambda}{|\log \lambda|^2}) \) is in \( C^1 \) norm in \( B_R(\xi_i) \).

The proof of this lemma is given in section 3.2.

We note that the system of equations (3.15) is nonlinear since the functions \( R_i \) depend on \( \mu_i \). Nevertheless this system is solvable for small \( \lambda > 0 \) and we obtain the
following expansion for the solution
\[
\mu_i = 1 - 2 \frac{\log |z_i|}{|z_i|} + \frac{\log R_0}{|z_i|} - \frac{2\pi}{|z_i|} \left[ H(\xi_i, \xi_j) + \sum_{j \neq i} G(\xi_i, \xi_j) \right] + O(\frac{1}{|z_i|^2})
\]
(3.17)
as \lambda \to 0.

Because of Lemma 3.2 we will work in the sequel only with \(\mu_i\) satisfying (3.15) and in particular we will always assume that
\[
\mu_i = 1 + O(\frac{\log |z_i|}{|z_i|}) \quad \text{as } \lambda \to 0. \tag{3.18}
\]

Also, thanks to Lemma 3.1, we will assume
\[
R_i = R_0 + O(\frac{\log |z_i|}{|z_i|}) \quad \text{as } \lambda \to 0.
\]

To solve (3.14) with \(\phi\) small in a convenient sense we need to choose the points \(\xi_1, \ldots, \xi_m \in \Omega\) appropriately, and for this we use a variational formulation of (3.1): \(v\) is a solution of (3.1) if and only if \(v \in H^1_0(\Omega)\) is a critical point of the energy functional
\[
I_\lambda(v) = \frac{1}{2} \iint_{\Omega} |\nabla v|^2 - \iint_{\Omega} F_\lambda(v - |z_i|)
\]
where
\[
F_\lambda(v) = \int_{0}^{v} f_\lambda(s) \, ds.
\]

We note that \(I_\lambda\) is a \(C^1\) functional on \(H^1_0(\Omega)\). If \(\phi\) is a small solution of (3.14) one may expect that \(I_\lambda(V_\lambda + \phi)\) is critical with respect to \(\xi_1, \ldots, \xi_m\). Therefore, in order to find the good choice of the points \(\xi_1, \ldots, \xi_m\) it becomes important to compute \(I_\lambda(V_\lambda)\).

**Lemma 3.3.** Suppose \(\xi_1, \ldots, \xi_m \in \Omega\) satisfy the separation conditions (3.6), (3.7), and let \(\mu_j\) satisfy (3.15). Then
\[
I_\lambda(V_\lambda) = \pi m |z| - 2\pi m \log |z| + \pi m \log R_0 - 2\pi^2 \phi_m(\xi_1, \ldots, \xi_m) + o(1)
\]
(3.20)
as \lambda \to 0, where \(\phi_m\) is defined in (1.5), and \(o(1)\) is uniform with respect to the \(C^1\) norm in the region (3.6), (3.7).

**Proof.** We compute
\[
I_\lambda(V_\lambda) = \frac{1}{2} \iint_{\Omega} |\nabla V_\lambda|^2 - \iint_{\Omega} F_\lambda(V_\lambda - |z|).
\]
Using (3.11) and (3.12) we have
\[ \int_{\Omega_1} |\nabla V_j|^2 = \int_{\Omega_2} \left| \sum_{j=1}^{m} \nabla (V_j - H_j) \right|^2 = \sum_{j=1}^{m} \int_{B_{R_j}(\xi_j)} f_j(w_j(|x - \xi_j|))V_j. \]

Therefore, by Lemma 3.2
\[ \int_{\Omega_2} |\nabla V_j|^2 = \sum_{j=1}^{m} \int_{B_{R_j}(\xi_j)} f_j(w_j(|x - \xi_j|)) \left( w_j(|x - \xi_j|) + |\log \lambda| + O\left(\frac{\lambda}{|\log \lambda|^2}\right) \right) dx 
= |\log \lambda| \sum_{j=1}^{m} \int_{B_{R_j}(0)} f_j(w_j(x)) \, dx + \sum_{j=1}^{m} \int_{B_{R_j}(0)} f_j(w_j(x))w_j(x) \, dx + O\left(\frac{\lambda}{|\log \lambda|^2}\right) \]  
(3.21)

as \( \lambda \to 0 \). Integrating (3.11) in \( B_{R_j}(\xi_j) \) and then using (3.4) and (3.17) we obtain
\[ \int_{B_{R_j}(0)} f_j(w_j(x)) \, dx = 2\pi \mu_j = 2\pi - 4\pi \frac{\log |\log \lambda|}{|\log \lambda|} + 2\pi \frac{\log R_0}{|\log \lambda|} \]  
(3.22)

For the terms \( \int_{B_{R_j}(0)} f_j(w_j)w_j \), we note that multiplying (3.11) by \( w_j \) and integrating in \( B_{R_j} \) we obtain
\[ \int_{B_{R_j}(0)} f_j(w_j)w_j = \int_{B_{R_j}(0)} |\nabla w_j|^2. \]

But by (3.3)
\[ w_j = \xi_j V_0 + O\left(\frac{\log |\log \lambda|}{|\log \lambda|}\right) \]
in the \( C^1 \) norm over compact sets of \( \mathbb{R}^2 \), where \( \xi_j \) satisfies (3.9), and we get
\[ \mu_j = d_0 R_0 \xi_j + O\left(\frac{\log |\log \lambda|}{|\log \lambda|}\right). \]

This gives
\[ |\nabla w_j|^2 = \frac{1}{d_0^2 R_0^2} |\nabla V_0|^2 + O\left(\frac{\log |\log \lambda|}{|\log \lambda|}\right). \]

Using this, (3.4) and (3.18) we see that
\[ \int_{B_{R_j}(0)} f_j(w_j)w_j = \frac{1}{d_0^2 R_0^2} \int_{B_{R_j}(0)} |\nabla V_0|^2 + O\left(\frac{\log |\log \lambda|}{|\log \lambda|}\right). \]
Therefore we find
\[
\frac{1}{2} \int_{\Omega} |\nabla V_2| = \pi m |\log \lambda| - 2\pi m \log |\log \lambda| + \pi m \log R_0 - 2\pi^2 \varphi_m(\xi_1, \ldots, \xi_m)
+ \frac{m}{2d_0^2 R_0^4} \int_{R_0} |\nabla V_0|^2 + O \left( \frac{(\log |\log \lambda|)^2}{|\log \lambda|} \right).
\]

(3.23)

Now we compute
\[
\int_{\Omega} \mathcal{F}_j(V_2 - |\log \lambda|) = \sum_{j=1}^m \int_{B_{R_0}(\tilde{z}_j)} \mathcal{F}_j(V_2 - |\log \lambda|) + \int_{\Omega} \mathcal{F}_j(V_2 - |\log \lambda|)
\]
where \( \tilde{\Omega} = \Omega \setminus \cup_{j=1}^m B_{R_0}(\tilde{z}_j) \). First we have, using (2.7) and (3.16),
\[
\int_{B_{R_0}(\tilde{z}_j)} \mathcal{F}_j(V_2 - |\log \lambda|) = \int_{B_{R_0}(0)} \mathcal{F}_j(w_j) + O \left( \frac{\lambda}{|\log \lambda|^2} \right).
\]

(3.24)

Using Lemma 3.1, \( \mathcal{F}_j(t) = \frac{1}{2} t^2 + O \left( \frac{\log |\log \lambda|}{|\log \lambda|} \right) \), (3.4) and (3.18), we obtain
\[
\int_{B_{R_0}(\tilde{z}_j)} \mathcal{F}_j(V_2 - |\log \lambda|) = \frac{1}{2d_0^2 R_0^4} \int_{R_0} V_0^2 + O \left( \frac{\log |\log \lambda|}{|\log \lambda|} \right)
\]
(3.25)
as \( \lambda \to 0 \). To estimate the integral in \( \tilde{\Omega} \) we use the inequality (2.8) which implies that
\[
|\mathcal{F}_j(v)| \leq C \frac{\lambda}{|\log \lambda|^2} \quad \text{for all } -1 \leq v \leq 0
\]
(3.26)
and
\[
|\mathcal{F}_j(v)| \leq C \frac{\lambda^3}{|\log \lambda|} \quad \text{for all } v \leq -1.
\]
(3.27)

We write
\[
\int_{\Omega} \mathcal{F}_j(V_2 - |\log \lambda|) = \sum_{j=1}^m \int_{B_{R_0}(\tilde{z}_j) \setminus B_{R_0}(\tilde{z}_j)} \mathcal{F}_j(V_2 - |\log \lambda|)
+ \int_{\Omega \setminus \cup_{j=1}^m B_{R_0}(\tilde{z}_j)} \mathcal{F}_j(V_2 - |\log \lambda|)
\]
and estimate, using Lemma 3.2 and (3.26),
\[
\int_{B_{R_0}(\tilde{z}_j) \setminus B_{R_0}(\tilde{z}_j)} \mathcal{F}_j(V_2 - |\log \lambda|) = \int_{B_{R_0}(\tilde{z}_j) \setminus B_{R_0}(\tilde{z}_j)} \mathcal{F}_j(V_2 - |\log \lambda|) + O \left( \frac{\lambda}{|\log \lambda|^2} \right)
= \int_{B_{R_0}(\tilde{z}_j) \setminus B_{R_0}(\tilde{z}_j)} \mathcal{F}_j(V_2 - |\log \lambda|) + O \left( \frac{\lambda}{|\log \lambda|^2} \right)
= O \left( \frac{\lambda}{|\log \lambda|^2} \right).
\]

(3.28)
Far away from the points $\bar{\xi}_j$, we argue as follows. For points $x$ in $\partial B_{10R_j}(\bar{\xi}_j)$ we have, because of Lemma 3.2, that $V_\lambda(x) - |\log \lambda| \leq -M$ where $M > 0$ is a fixed constant. But $V_\lambda - |\log \lambda|$ is harmonic in $\Omega \setminus \bigcup_{j=1}^m B_{10R_j}(\bar{\xi}_j)$ and equal to $-|\log \lambda|$ on $\partial \Omega_j$, and therefore $V_\lambda(x) - |\log \lambda| \leq -M$ holds for all points in $\Omega \setminus \bigcup_{j=1}^m B_{10R_j}(\bar{\xi}_j)$. Thus we can use (3.27) and deduce

$$\int_{\Omega \setminus \bigcup_{j=1}^m B_{10R_j}(\bar{\xi}_j)} F_j(V_\lambda - |\log \lambda|) = O(\lambda|\log \lambda|^3).$$

Combining the estimates for each term, and noting that

$$3\lambda \partial_{\lambda} R_{\lambda,\alpha} + R_{\lambda,\alpha} = O(1),$$

we conclude that formula (3.20) is valid with $o(1)$ in the $C^0$ norm over the region (3.6), (3.7).

Regarding the estimate in $C^1$ norm, the error term in (3.23) is also $O(\lambda|\log \lambda|^2)$ in $C^1$ norm for the parameters in the region (3.6), (3.7), since we can use (3.16) for the error in (3.21) and the dependence on the $\bar{\xi}_j$ in the terms appearing in (3.21) is through the $\mu_i$ in expression (3.22). Similarly, the errors in (3.24), (3.25) and (3.28) are also $C^1$ with respect to $\bar{\xi}_j$.

3.2. Proof of Lemmas 3.1 and 3.2

Proof of Lemma 3.1. First we remark that for $\lambda > 0$ the nonlinearity $f_j$ is smooth, and that the solution $w_{j,\alpha}$ of (3.2) exists for all $r \geq 0$. It is also smooth with respect to $\alpha > 0$, $\lambda > 0$.

a) Estimate (3.3) follows from $f_j(t) = t_+ + O(\frac{|\log |\lambda||}{|\log \lambda|})$ as $\lambda \to 0$ uniformly on compact subsets of $\mathbb{R}$ (c.f. (2.7)).

b) The existence of $R_{j,\alpha}$ follows from the convergence in part a), and the uniqueness because $\frac{\partial}{\partial \lambda} w_{j,\alpha}(r) < 0$. The estimate for $R_{j,\alpha}$ in (3.4) follows by the implicit function theorem, noting that $w_{j,\alpha}(r)$ is $C^1$ in $r$, that $w_{j,\alpha}$ and $\frac{\partial}{\partial \lambda} w_{j,\alpha}(r)$ have continuous extensions to $\lambda = 0$, and $\frac{\partial}{\partial \lambda} w_{j,\alpha}(R_0) < 0$ (with a uniform distance as $\lambda \to 0$). What needs to be verified is that the expansion

$$w_{j,\alpha}(r) = w_{j,\alpha}(R_0) + \frac{\partial}{\partial \lambda} w_{j,\alpha}(R_0)(r - R_0) + o(|r - R_0|)$$

as $r \to R_0$, has an error $o(|r - R_0|)$ which is uniform as $\lambda \to 0$. This estimate can be obtained from elliptic estimates, since $\Delta w_{j,\alpha}$ remains bounded, and so is uniformly $C^{1,\alpha}$ on compact sets.

We prove now the estimate $\frac{d}{d \alpha} R_{j,\alpha} = O(\frac{|\log |\lambda||}{|\log \lambda|})$ in (3.4). Differentiating the relation $w_{j,\alpha}(R_{j,\alpha}) = 0$ with respect to $\alpha$ we obtain

$$\frac{d}{d \alpha} R_{j,\alpha} = -\frac{\partial}{\partial \lambda} w_{j,\alpha}(R_{j,\alpha}) \frac{\partial}{\partial r} w_{j,\alpha}(R_{j,\alpha}).$$
Since \( \frac{\partial}{\partial r} w_{j,a}(R_{j,a}) \) is bounded away from zero, we need only to estimate \( \frac{\partial}{\partial r} w_{j,a}(R_{j,a}) \).

Let \( z_{j,a} = \frac{1}{\mu_j} w_{j,a} \) which is smooth and satisfies

\[
-\Delta z_{j,a} = f'_j(w_{j,a})z_{j,a} \quad \text{in} \quad \mathbb{R}^2.
\] (3.29)

We claim that

\[
|z_{j,a}(R_0)| \leq C \frac{\log |\log \lambda|}{|\log \lambda|^\mu}.
\] (3.30)

To prove this we let \( \phi(r) = z_{j,a}(r) - V_0(r) \) and note that

\[
-\Delta \phi = f'_j(w_{j,a})z_{j,a} - \chi_{B_{R_0}}(0) = \chi_{B_{R_0}} \phi + (f'_j(w_{j,a}) - \chi_{B_{R_0}})z_{j,a}.
\]

Multiplying this equation by \( V_0 \) and integrating in \( B_{R_0} \) gives:

\[
2\pi R_0 \phi(R_0)V_0(R_0) = \int_{B_{R_0}} (f'_j(w_{j,a}) - \chi_{B_{R_0}})z_{j,a} V_0
\]

and the integral on the right hand side can be verified to be \( O\left(\frac{\log |\log \lambda|}{|\log \lambda|^\mu}\right) \) as \( \lambda \to 0 \).

Equation (3.29) shows that \( z_{j,a} \) is \( C^{1,\mu} \) on compact set, for any \( \mu \in (0, 1) \). Using this property and (3.30) we deduce

\[
|z_{j,a}(R_0)| \leq C \frac{\log |\log \lambda|}{|\log \lambda|^\mu}.
\]

\( \square \)

c) This property follows from the convergence \( w_{j,a} \to \lambda V_0 \) as \( \lambda \to 0 \) in \( C^{1,\mu} \) on compact sets of \( \mathbb{R}^2 \) and \( R_{j,a} \to R_0 \) as \( \lambda \to 0 \).

\textbf{Proof of Lemma 3.2.} Fix \( i = 1, \ldots, m \) and \( R > 0 \) be fixed. For \( x \in B_R(\bar{\zeta}_i) \) we have

\[
V_j(x) - H_j(x) = V_i(x) + 2\pi \mu_i H\left(\frac{\lambda}{|\log \lambda|^\mu}, x, \zeta_i\right) + 2\mu_i \log |\log \lambda| - \mu_i \log R_i - (1 - \mu_i)|\log \lambda|.
\]

For \( x \in B_R(\bar{\zeta}_i) \), if we take \( j \neq i \) then

\[
V_j(x) - H_j(x) = 2\pi \mu_j G\left(\frac{\lambda}{|\log \lambda|^\mu}, x, \zeta_j\right).
\]

(This is valid actually for \( |x - \zeta_j| > R_j \).

Therefore, for \( x \in B_R(\bar{\zeta}_i) \)

\[
V_j(x) = \sum_{j=1}^m (V_j(x) - H_j(x)) = V_i(x) + 2\pi \mu_i H\left(\frac{\lambda}{|\log \lambda|^\mu}, x, \zeta_i\right) + 2\mu_i \log |\log \lambda| - \mu_i \log R_i - (1 - \mu_i)|\log \lambda| + 2\pi \sum_{j \neq i} \mu_j G\left(\frac{\lambda}{|\log \lambda|^\mu}, x, \zeta_j\right)
\]
\[ V(x) + 2\pi \mu H(\xi_i, \zeta_i) + 2\mu \log |\log \lambda| 
- \mu \log R_i - (1 - \mu_i)|\log \lambda| + 2\pi \sum_{j \neq i} \mu_j G(\xi_i, \zeta_j) + O\left(\frac{\lambda}{\log \lambda}^2\right), \]

where the \( O\left(\frac{\lambda}{\log \lambda}^2\right) \) is in \( C^1 \) norm. Using the equations satisfied by \( \mu_j, (3.15) \), we find the desired estimate (3.16). \( \square \)

4. Linear Theory

Here we study the invertibility of the operator

\[ L\phi = \Delta \phi + W\phi \]

in \( \Omega_\lambda \) where

\[ W = f_\lambda(V_\lambda - |\log \lambda|). \]

Furthermore this section we assume that \( \bar{\xi}_1, \ldots, \bar{\xi}_m \in \Omega \) satisfy the separation conditions (3.6), (3.7) and \( \tilde{\xi}_j = \frac{1}{|\log \lambda|^2} \bar{\xi}_j \in \Omega_\lambda \). We also assume that \( \mu_1, \ldots, \mu_m \) satisfy (3.8).

We consider the linear problem of given \( h \) in an appropriate space, finding \( \phi \) and \( c_{ij}, i = 1, \ldots, m, j = 1, 2 \), such that

\[ \begin{cases} L\phi = h + \sum_{i=1}^m \sum_{j=1,2} c_{ij} Z_{ij} & \text{in } \Omega_\lambda, \\ \phi = 0 & \text{on } \partial \Omega_\lambda. \end{cases} \] \( (4.1) \)

\[ \int_{\Omega_\lambda} \phi Z_{ij} = 0 \quad \text{for all } i = 1, 2, j = 1, \ldots, m, \] \( (4.2) \)

where the functions \( Z_{ij} \) are defined by

\[ Z_{ij}(x) = z_{ij}(x) \eta_0(x - \bar{\xi}_i) \]

with

\[ z_{ij}(x) = \frac{\partial V_0}{\partial x_j}(x - \bar{\xi}_j), \] \( (4.3) \)

and \( \eta_0 \) is a smooth radial function in \( \mathbb{R}^2 \) with support in \( B_{R_0}(0) \) and identically 1 in \( B_{R_0/2}(0) \), and \( V_0 \) is defined in (1.9). Note that \( \frac{\partial V_0}{\partial x_j}(x - \bar{\xi}_j) \) is continuous but not \( C^1 \).

The choice of \( \eta_0 \) makes \( Z_{ij} \) a smooth function with compact support.

Let \( Y \) be the space of measurable functions \( h : \Omega_\lambda \to \mathbb{R} \) such that

\[ h \in L^\infty(\Omega_\lambda) \text{ and } h \in L^p(B_{2R_0(\bar{\xi}_j)}) \text{ for all } j = 1, \ldots, m, \]
where

\[ \hat{\Omega}_j = \Omega_j \setminus \bigcup_{j=1}^{m} B_{2R_0} (\hat{\xi}_j) \]

and \( 2 < p < +\infty \) is fixed. We will consider the following norm on \( Y \):

\[ \|h\|_Y = \sup_{x \in \hat{\Omega}_j} \left( \sum_{j=1}^{m} |x - \hat{\xi}_j|^{-2+\sigma} \right)^{-1} |h(x)| + \sum_{j=1}^{m} \|h\|_{L^p(\Omega_j \setminus B_{2R_0} (\hat{\xi}_j))}, \]

where \( 0 < \sigma < 1 \) is a small constant. Note that since \( p > 2 \), if \( h \in Y \) then any solution \( \phi \in H^1_0 (\Omega_j) \) of (4.1) is \( C^1 (\Omega_j) \).

**Proposition 4.1.** There is \( \lambda_0 > 0 \) such that for any \( 0 < \lambda \leq \lambda_0 \) and all \( h \in Y \) there is a unique \( \phi \in L^\infty (\Omega_j) \) and unique \( c_{ij} \in \mathbb{R} \) that solve (4.1), (4.2). Moreover

\[ \|\phi\|_{L^\infty(\Omega_j)} + \sum_{i=1}^{m} \sum_{j=1,2} |c_{ij}| \leq C \|h\|_Y. \] (4.4)

In addition, the maps \( \hat{\xi}_1, \ldots, \hat{\xi}_m \mapsto \phi, c_{ij} \) are differentiable and

\[ \|\partial_{\hat{\xi}_i} \phi\|_{L^\infty(\Omega_j)} + |\partial_{\hat{\xi}_i} c_{ij}| \leq C \lambda^{1/2-1} \|h\|_Y. \] (4.5)

The proof of this result relies on the non-degeneracy of \( V_0 \) (defined in (1.9)), which satisfies

\[ -\Delta V_0 = (V_0)_+ \text{ in } \mathbb{R}^2, \]

and is radially symmetric with \( \max_{\mathbb{R}^2} V_0 = 1 \).

**Proposition 4.2.** Let \( \phi \in L^\infty (\mathbb{R}^2) \) be a solution of

\[ -\Delta \phi = \chi_{[V_0 > 0]} \phi \text{ in } \mathbb{R}^2, \]

where \( \chi_{[V_0 > 0]} \) is the characteristic function of the set \( [V_0 > 0] = B_{R_0} (0) \). Then \( \phi \) is a linear combination of

\[ \frac{\partial V_0}{\partial x_1}, \quad \frac{\partial V_0}{\partial x_2}. \]

For the proof see [3, Proposition 3.1].

To prove Proposition 4.1 we start with an apriori estimate.

**Lemma 4.3.** There is \( C > 0 \) such that for all \( \lambda > 0 \) small, and for any \( h \in Y \), \( \phi \in L^\infty (\Omega_j) \), and \( c_{ij} \in \mathbb{R} \) that verify (4.1), (4.2) we have

\[ \|\phi\|_{L^\infty(\Omega_j)} + \sum_{i=1}^{m} \sum_{j=1,2} |c_{ij}| \leq C \|h\|_Y. \]
Proof. We first prove that

$$|c_{ij}| \leq C \|h\|_Y + o(1) \|\phi\|_{L^\infty(\Omega_2)} \quad (4.6)$$

as $\lambda \to 0$. For this let $\eta$ be a radial function in $C^\infty(\mathbb{R}^2)$ with support in $B_2(0)$ and $\eta \equiv 1$ in $B_1(0)$. Let $\eta_j(x) = \eta(\lambda^{1/2}(x - \tilde{z}_j))$ and

$$\tilde{Z}_{ij} = z_{ij} \eta_j$$

where $z_{ij}$ is defined in (4.3). Multiplying (4.1) by $\tilde{Z}_{ij}$ we find

$$\int_{\Omega_2} \phi(\Delta \tilde{Z}_{ij} + W \tilde{Z}_{ij}) = \int_{\Omega_2} h \tilde{Z}_{ij} + c_{ij} \int_{\Omega_2} Z_{ij} \tilde{Z}_{ij}. $$

This gives

$$|c_{ij}| \leq C \|h\|_Y + \|\phi\|_{L^\infty(\Omega_2)} \int_{\Omega_2} |\Delta \tilde{Z}_{ij} + W \tilde{Z}_{ij}|. $$

We compute

$$\Delta \tilde{Z}_{ij} + W \tilde{Z}_{ij} = \eta_j(\Delta z_{ij} + W_{z_{ij}}) + 2\nabla \eta_j \nabla z_{ij} + \Delta \eta_j z_{ij}. $$

Using that $|z_{ij}(x)| \leq C|x - \tilde{z}_i|^{-1}$ and $|\nabla z_{ij}(x)| \leq C|x - \tilde{z}_i|^{-2}$ for $|x - \tilde{z}_i| \geq 2R_0$, we see that

$$\int_{\Omega_2} |2\nabla \eta_j \nabla z_{ij} + \Delta \eta_j z_{ij}| = O(\lambda^{1/2})$$

as $\lambda \to 0$. The other term can be estimated as follows:

$$\int_{\Omega_2} |\eta_j(\Delta z_{ij} + W_{z_{ij}})| \leq \|\eta_j\|_{L^\infty} \|z_{ij}\|_{L^\infty} \int_{\Omega_2} |\chi_{B_{R_0}(\tilde{z}_i)} - f'_\lambda(V - |\log \lambda|)| \to 0$$

as $\lambda \to 0$. Therefore

$$\int_{\Omega_2} |\Delta \tilde{Z}_{ij} + W \tilde{Z}_{ij}| = o(1)$$

as $\lambda \to 0$ and this proves (4.6).

Now we claim that if $R > 0$ is large enough, then

$$\|\phi\|_{L^\infty(\Omega_2)} \leq C(\|\phi\|_Y + \|h\|_Y + \sum_{i=1}^m \sum_{j=1,2} |c_{ij}|) \quad (4.7)$$

where

$$\|\phi\|_i = \sup_{\cup_{B_{R_0}(\tilde{z}_i)}} |\phi|. $$
For this we use a barrier argument. Let
\[ \psi(x) = \sum_{j=1}^{m} (1 - |x - \tilde{\zeta}_j|^{-\sigma}). \]

Fix \( R > 2R_0 \). By (2.10), \( W(x) \leq C \tilde{\lambda}^{\frac{3}{2}}/\log \tilde{\lambda} \) for \( x \in \Omega \setminus \bigcup_{i=1}^{m} B_\rho(\tilde{\zeta}_i) \). Then
\[ \Delta \psi + W(x)\psi < 0 \text{ in } \Omega \setminus \bigcup_{i=1}^{m} B_\rho(\tilde{\zeta}_i), \]
provided \( \tilde{\lambda} > 0 \) is small. This shows that the operator \( \Delta + W \) satisfies the maximum principle in this region. Applying the maximum principle to \( C \|h\|_Y + \sum |c_{ij}| + \|\phi\|_i \psi \pm \phi \), where \( C \) is a large constant, we arrive at (4.7).

Using (4.6) and (4.7) we obtain
\[ \|\phi\|_{L^{\infty}(\Omega_1)} \leq C(\|\phi\|_i + \|h\|_Y). \]

Therefore to prove the lemma it suffices to show
\[ \|\phi\|_i \leq C \|h\|_Y. \tag{4.8} \]

We prove this estimate by contradiction. Assume that there are sequences \( \tilde{\lambda}_n \to 0 \), \( (\phi_n) \) in \( L^{\infty}(\Omega_1) \), \( (c_{ij}^{(n)}) \) in \( \mathbb{R} \) and \( (h_n) \) in \( Y \) that satisfy (4.1), (4.2) and
\[ \|\phi_n\|_i > n\|h_n\|_Y. \]

By linearity we can assume that \( \|\phi_n\|_i = 1 \). Then (4.6) implies that \( \|h_n\|_Y \to 0 \) and \( c_{ij}^{(n)} \to 0 \) as \( n \to +\infty \). Then for a fixed \( i \in \{1, \ldots, m\} \) and a subsequence (denoted the same as the original sequence)
\[ \sup_{B_{\rho}(\tilde{\zeta}_i)} |\phi_n| \geq c \]
for some \( c > 0 \). By translating we can assume that \( \tilde{\zeta}_i = 0 \). Using the equation, we get that up to another subsequence, \( \phi_n \to \phi \) uniformly on compact sets of \( \mathbb{R}^2 \) and that \( \phi \not\equiv 0 \) is a bounded solution of
\[ \Delta \phi + \chi_{B_{\rho_n}(0)} \phi = 0 \text{ in } \mathbb{R}^2. \]

By Proposition 4.2, \( \phi = a_1z_{i1} + a_2z_{i2} \) for some \( a_1, a_2 \in \mathbb{R} \). But \( \phi \) also satisfies
\[ \int_{\mathbb{R}^2} \phi Z_{ij} = 0 \quad j = 1, 2, \]
which shows that \( a_1 = a_2 = 0 \) so \( \phi \equiv 0 \), a contradiction. This proves (4.8) and finishes the proof of the lemma. \( \square \)

Proof of Proposition 4.1. Consider the Hilbert space
\[ H = \left\{ \phi \in H^1_0(\Omega_j) : \int_{\Omega_j} Z_{ij}\phi = 0 \ i = 1, \ldots, m, \ j = 1, 2 \right\} \]
with inner product \( \langle \phi_1, \phi_2 \rangle = \int_{\Omega} \nabla \phi_1 \nabla \phi_2 \). For \( h \in Y \), the variational problem: find \( \phi \in H \) such that

\[
\langle \phi, \psi \rangle = \int_{\Omega} (W\phi - h)\phi \quad \text{for all } \phi \in H
\]

is a weak formulation of (4.1), (4.2). Using the Riesz representation theorem, this variational problem is equivalent to solve

\[
\phi + K(\phi) = \tilde{h} \tag{4.9}
\]

where \( \tilde{h} \in H \) and \( K : H \to H \) is a compact operator. When \( h = 0 \) then \( \tilde{h} = 0 \) and by Lemma 4.3 \( \phi = 0 \). By the Fredholm alternative there is a solution \( \phi \in H \) of (4.9) giving a weak solution of (4.1), (4.2). By standard regularity theory \( \phi \in C(\overline{\Omega}_j) \) and we get the estimate (4.4) from Lemma 4.3.

Now we proceed with the differentiability properties of \( \phi, c_{ij} \) with respect to \( \xi_1, \ldots, \xi_m \). For this we proceed formally, assuming the differentiability, and obtain estimate (4.5). This argument can then later be justified by applying it to finite differences instead of derivatives. We recall that \( \phi \) is the unique solution of

\[
L\phi = h + \sum_{i=1}^{m} \sum_{j=1,2} c_{ij}Z_{ij} \quad \text{in } \Omega_j
\]

satisfying \( \phi = 0 \) on \( \partial\Omega_j \) and the orthogonality conditions (4.2), and that the \( c_{ij} \) are uniquely determined. Assuming that \( \phi, c_{ij} \) are differentiable and setting \( \psi = \tilde{\partial}_{\xi_1} \phi \) we find

\[
L\psi = -\tilde{\partial}_{\xi_1} W\phi + h + \sum_{i=1}^{m} d_{ij}Z_{ij} + \sum_{i=1}^{m} c_{ij}\tilde{\partial}_{\xi_1} Z_{ij}
\]

where \( d_{ij} = \tilde{\partial}_{\xi_1} c_{ij} \). Differentiating the orthogonality condition (4.2) we get

\[
\int_{\Omega_j} \psi Z_{ij} + \phi \tilde{\partial}_{\xi_1} Z_{ij} = 0.
\]

Setting

\[
\tilde{\psi} = \psi + b_{ij}Z_{ij} \quad \text{where } b_{ij} = \frac{\int_{\Omega_j} \phi \tilde{\partial}_{\xi_1} Z_{ij}}{\int_{\Omega_j} \text{Z}_{ij}}
\]

we get that \( \tilde{\psi} \) satisfies (4.2) and

\[
L\tilde{\psi} = -\tilde{\partial}_{\xi_1} W\phi + h + \sum_{i=1}^{m} d_{ij}Z_{ij} + \sum_{i=1}^{m} c_{ij}\tilde{\partial}_{\xi_1} Z_{ij} + \sum_{i=1}^{m} b_{ij}L(Z_{ij}) \quad \text{in } \Omega_j.
\]

Hence, applying the apriori estimate of Lemma 4.3 we deduce

\[
\|\psi\|_{L^\infty(\Omega_j)} + |d_{ij}| \leq C(\|\tilde{\partial}_{\xi_1} W\phi\|_Y + \|h\|_Y + \sum |c_{ij}|\|\tilde{\partial}_{\xi_1} L(Z_{ij})\|_Y + \sum |b_{ij}|\|L(Z_{ij})\|_Y).
\]
We claim that each term is bounded by \( C\lambda^{\frac{1}{2}} \|h\|_Y \). Let us verify this explicitly for \( \partial_{\lambda} W\phi \), because the others are direct. Since \( \|\phi\|_{L^\infty} \leq C\|h\|_Y \), it suffices to verify that
\[
\|\partial_{\lambda} W\|_Y \leq C\lambda^{\frac{1}{2}}.
\] (4.10)

But \( \partial_{\lambda} W = f''_x(V_\lambda - |\log \hat{\lambda}|)\partial_{\lambda} V_\lambda \). One can verify directly that \( \|\partial_{\lambda} V_\lambda\|_{L^\infty(\Omega_\lambda)} \leq C \). We then conclude the validity of (4.10) by using the next lemma. \( \square \)

**Lemma 4.4.** We have
\[
\|f''_x(V_\lambda - |\log \hat{\lambda}|)\|_Y \leq C\lambda^{\frac{1}{2}}
\] (4.11)
for some \( C > 0 \) and all \( \lambda > 0 \) small.

**Proof.** Let us write \( \hat{\Omega}_\lambda = \Omega_\lambda \setminus \bigcup_{j=1}^m B_{2R_0}(\tilde{\xi}_j) \). Using (2.13) and that \( V_\lambda - |\log \hat{\lambda}| \) is \( -a \) for some \( a > 0 \) on \( \hat{\Omega}_\lambda \) we get
\[
\sup_{x \in \hat{\Omega}_\lambda} \left( \sum_{j=1}^m |x - \tilde{\xi}_j|^{-2-\sigma} \right)^{-1} |f''_x(V_\lambda - |\log \hat{\lambda}|)| \leq C\lambda^{1-2\sigma}.
\]

To estimate \( \|f''_x(V_\lambda - |\log \hat{\lambda}|)\|_{L^p(B_{2R_0}(\tilde{\xi}))} \) we split the integral
\[
\int_{B_{2R_0}(\tilde{\xi})} |f''_x(V_\lambda - |\log \hat{\lambda}|)|^p = \int_{D_1} \cdots + \int_{D_2} \cdots
\]
where \( D_1 = B_{R_1}(\tilde{\xi}) \setminus B_{R_1}(\tilde{\xi}) \), \( D_2 = B_{2R_0}(\tilde{\xi}) \setminus D_1 \) and \( L > 0 \) is some large fixed number. In \( D_1 \), by (2.11) we estimate \( |f''_x| \leq C/\lambda \) and we get
\[
\int_{D_1} |f''_x(V_\lambda - |\log \hat{\lambda}|)|^p \leq C\lambda^{1-p}.
\]

To estimate the integral over \( D_2 \) we recall that \( V_j(x) - |\log \hat{\lambda}| = w_j(|x - \tilde{\xi}_j|) + O(|\log \hat{\lambda}|) \) for \( x \in B_{2R_0}(\tilde{\xi}_j) \) (c.f. (3.16)) in \( C^1 \) norm. Since \( w_j(R_j) = 0, V_j - |\log \hat{\lambda}| \) has a zero set that is at distance \( O(\lambda) \) from \( \partial B_{R_j}(\tilde{\xi}_j) \), and \( V_j(x) - |\log \hat{\lambda}| \) separates linearly from this curve. So thanks to (2.12), \( |f''_x(V_j(x) - |\log \hat{\lambda}|)| \leq C\text{dist}(x, \partial B_{R_j}(\tilde{\xi}_j))^{-1} \) on \( D_2 \). Therefore
\[
\int_{D_2} |f''_x(V_\lambda - |\log \hat{\lambda}|)|^p \leq C \int_{\text{dist}} y^{-p} dy \leq C\lambda^{1-p}.
\]

It follows that
\[
\|f''_x(V_\lambda - |\log \hat{\lambda}|)\|_{L^p(B_{2R_0}(\tilde{\xi}))} \leq C\lambda^{\frac{1}{2}}
\]
and taking \( \sigma > 0 \) small we get the stated estimate (4.11). \( \square \)
5. A Nonlinear Projected Problem

In this section we solve the nonlinear problem

\[
\begin{cases}
L\phi + N(\phi) + E_\lambda = \sum_{i=1}^{m} \sum_{j=1}^{2} c_{ij} Z_{ij} & \text{in } \Omega,
\phi = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{5.1}
\]

with \(\phi\) satisfying

\[
\int_{\Omega} \phi Z_{ij} = 0 \quad \text{for all } i = 1, 2, j = 1, \ldots, m.
\tag{5.2}
\]

We always assume that \(\xi_1, \ldots, \xi_m \in \Omega\) satisfy the separation conditions (3.6), (3.7), and \(\bar{x}_j = \frac{\log \lambda}{\lambda} x_j\).

**Proposition 5.1.** Assume \(\mu_1, \ldots, \mu_m\) satisfy (3.15). Then there is \(\lambda_0 > 0\) such that for \(\lambda \in (0, \lambda_0)\), (1.5), (2.2) has a unique solution \(\phi = \phi(\xi_1, \ldots, \xi_m)\) and \(c_{ij} = c_{ij}(\xi_1, \ldots, \xi_m)\) such that

\[
\|\phi\|_{L^\infty} + \sum_{i=1}^{m} \sum_{j=2}^{2} |c_{ij}| \leq C \lambda^{1-2\sigma}.
\tag{5.3}
\]

**Proof.** For the argument it is better to work with the space \(X\) defined as the set of continuous functions \(\phi\) on \(\bar{\Omega}\) such that \(\phi\) restricted to \(B_{2R_0}(\bar{x}_j)\) belongs to \(W^{1,\infty}(B_{2R_0}(\bar{x}_j))\) for all \(j = 1, \ldots, m\), equipped with the norm

\[
\|\phi\|_X = \|\phi\|_{L^\infty(\Omega)} + \sum_{j=1}^{m} \|\phi\|_{W^{1,\infty}(B_{2R_0}(\bar{x}_j))}.
\]

We rewrite problem (5.1), (5.2) as the fixed point problem

\[
\phi = F(\phi)
\]

where \(F = -T(N(\phi) + E_\lambda)\) and \(T\) is the linear operator defined in Proposition 4.1, which by estimate (4.4) satisfies

\[
\|T(h)\|_{\infty} \leq C \|h\|_Y \quad \text{for all } h \in Y.
\]

By elliptic estimates we also deduce

\[
\|T(h)\|_X \leq C \|h\|_Y
\]

with a constant \(C\) independent of \(\lambda\), and here it is important that \(p > 2\).

Let us estimate \(\|E_\lambda\|_Y\). Using (2.8) we have

\[
\sup_{x \in \Omega} \left( \sum_{j=1}^{m} |x - \bar{x}_j|^{-2-\sigma} \right)^{-1} |E_\lambda(x)| \leq C \lambda^{1-\sigma} |\log \lambda|^{1+2\sigma} \leq C \lambda^{1-2\sigma}
\]
for $\lambda > 0$ small, where $\Omega_\lambda = \Omega_\lambda \setminus \cup_{j=1}^m B_{2R_j}(\tilde{\xi}_j)$. Also, using Lemma 3.2 we see that

$$|E_\lambda| \leq C \frac{\lambda}{|\log \lambda|^3} \quad \text{in } B_{2R_0}(\tilde{\xi}_j).$$

So we deduce

$$\|E_\lambda\|_Y \leq C\lambda^{1-2\sigma}. \quad (5.4)$$

For fixed $\gamma > 0$, let $B = \{ \phi \in X : \|\phi\|_X \leq \gamma \lambda^{1-2\sigma} \}$. We claim that there exist constants $C, a > 0$ such that for $\phi_1, \phi_2 \in B$

$$\|N(\phi_1) - N(\phi_2)\|_Y \leq C\lambda^a\|\phi_1 - \phi_2\|_X, \quad (5.5)$$

and we can prove this with $a = \frac{1-2\sigma}{\rho}$.

Indeed,

$$N(\phi_1) - N(\phi_2) = (\phi_1 - \phi_2) \int_0^1 N'(\phi_2 + s(\phi_1 - \phi_2)) \, ds.$$

Therefore, to obtain (5.5) it is enough to show that for all $s \in [0, 1]$

$$\|N'(\phi_2 + s(\phi_1 - \phi_2))\|_Y \leq C\lambda^a.$$

Part of this norm is $\|N'(\phi_2 + s(\phi_1 - \phi_2))\|_{L^p(B_{2R_0}(\tilde{\xi}_j))}$. Write $\phi = \phi_2 + s(\phi_1 - \phi_2)$ and note that $\|\phi\|_X \leq \gamma \lambda^{1-2\sigma}$. Then

$$\int_{B_{2R_0}(\tilde{\xi}_j)} |N'(\phi)|^p = \int_{B_{2R_0}(\tilde{\xi}_j)} |f'_x(V_\lambda + \phi - |\log \lambda|) - f'_x(V_\lambda - |\log \lambda|)|^p = I_1 + I_2,$$

where

$$I_k = \int_{A_k} |f'_x(V_\lambda + \phi - |\log \lambda|) - f'_x(V_\lambda - |\log \lambda|)|^p, \quad k = 1, 2$$

$$A_1 = B_{R_1}^c(\tilde{\xi}_j) \setminus B_{R_2}^c(\tilde{\xi}_j), \quad A_2 = B_{2R_0}(\tilde{\xi}_j) \setminus A_1,$$

the numbers $R_j$ are given in (3.10) and $m > 0$ is a parameter to be chosen. Since $f'_x$ is uniformly bounded in the range of its arguments (see (2.9)), we find

$$|I_k| \leq C|A_k| \leq C\lambda^m. \quad (5.6)$$

On $A_2$ we obtain

$$|I_2| \leq \int_{A_2} \int_0^1 |f''_x(V_\lambda + \tau \phi - |\log \lambda|)|^p |\phi|^p \, d\tau \, dx.$$
We recall that $V_j(x) - |\log \lambda| = w_j(|x - \tilde{\zeta}_j|) + O(\frac{1}{|x|})$ for $x \in B_{2R_0}(\tilde{\zeta}_j)$ (c.f. (3.16)) in $C^1$ norm, and note that the gradient of $\tau \phi$ is small in uniform norm in $B_{2R_0}(\tilde{\zeta}_j)$, because $\|\phi\|_x \leq \gamma^{1-2\sigma}$. This and $w_j(R) = 0$ yield
\[ |V_j + \tau \phi - |\log \lambda| | \geq c\lambda^m \text{ in } A_2 \]
for some constant $c > 0$. Therefore, from estimate (2.12)
\[ |f''_j(V_j + \tau \phi - |\log \lambda|) | \leq C \text{dist}(x, \partial B_R(\tilde{\zeta}_j))^{-1} \]
on $A_2$ and we get
\[ |I_2| \leq C\lambda^{m(1-p)}\|\phi\|_p^p \leq C\lambda^{m(1-p)+(1-2\sigma)p} \]
Combining this last estimate with (5.6) we obtain
\[ \| N'(\phi_2 + s(\phi_1 - \phi_2)) \|_{L^p(B_{2R_0}(\tilde{\zeta}_j))} \leq C(\lambda^{m/p} + \lambda^{m(1/p-1)+1-2\sigma}) \]
We choose $m = 1 - 2\sigma$ and we obtain
\[ \| N'(\phi) \|_{L^p(B_{2R_0}(\tilde{\zeta}_j))} \leq C\lambda^{\frac{1-2\sigma}{p}}. \tag{5.7} \]
Next we estimate the weighted $L^\infty$ norm of $N'(\phi)$ away from the points $\tilde{\zeta}_j$. For this we recall that if $|x - \tilde{\zeta}_j| \geq 2R_0$ for all $j$, then $V_j - |\log \lambda| \leq -M$ for some fixed $M > 0$ and the same holds for $V_j + s\phi - |\log \lambda|$. Therefore (2.10) yields
\[ \sup_{x \in \Omega_2} \left( \sum_{j=1}^m |x - \tilde{\zeta}_j|^{-2-\sigma} \right)^{-1} |N'(\phi)| \leq C\lambda^{1-2\sigma}. \tag{5.8} \]
The combination of (5.7), (5.8) proves (5.5).

Using (5.4) and (5.5), we see that choosing $\gamma > 0$ fixed and large in the definition of $\mathcal{B}$, for $\lambda > 0$ sufficiently small, $F$ is a contraction in $\mathcal{B}$, and by the contraction mapping principle $F$ has a unique fixed point in $\mathcal{B}$.

**Proposition 5.2.** For $\lambda > 0$ sufficiently small, the maps $\tilde{\zeta}_1, \ldots, \tilde{\zeta}_m \to \phi, c_{ij}$ constructed in Proposition 5.1 are differentiable in the region defined by (3.6), (3.7) and for any $k = 1, \ldots, m$, and $\lambda > 0$ small
\[ \| \partial_{\tilde{\zeta}_k} \phi \|_{L^\infty} + \| \partial_{\tilde{\zeta}_k} c_{ij} \|_{L^\infty} \leq C\lambda^{\frac{1-2\sigma}{p}}. \tag{5.9} \]

For the proof we need the following lemma.

**Lemma 5.3.** We have
\[ \| \partial_{\tilde{\zeta}_k} E \|_Y \leq C\lambda^{\min(1-2\sigma, 1/p)}. \]
**Proof.** We compute $\partial_{\tilde{\zeta}_k} E = \Delta \partial_{\tilde{\zeta}_k} V_j + f''_j(V_j - |\log \lambda|) \partial_{\tilde{\zeta}_k} V_j$. It can be verified that $|\partial_{\tilde{\zeta}_k} V_j | \leq C$ in $\Omega_2$ for some fixed constant.
Let us compute $\partial_{\bar{\zeta}_i} E_i(x)$ for $x \in \Omega_j \setminus \cup_{j=1}^m B_{R_i}(\bar{\zeta}_j)$. In this region $V_\lambda$ is harmonic and therefore $\partial_{\bar{\zeta}_i} E_i = f_j(V_\lambda - |\log \lambda|) \partial_{\bar{\zeta}_i} V_j$. Therefore, using (2.10) and that $\|\partial_{\bar{\zeta}_i} V_j\|_{L^\infty(\Omega_j)} \leq C$ we get

$$\sup_{x \in I_j} \left( \sum_{j=1}^m |x - \bar{\zeta}_j|^{-2-\sigma} \right)^{-1} \left| \partial_{\bar{\zeta}_i} E_i \right| \leq C \lambda^{1-\sigma} |\log \lambda|^{3+2\sigma} \leq C \lambda^{1-\sigma}$$

(5.10)

for $\lambda > 0$ small, where $\Omega_j = \Omega_j \setminus \cup_{j=1}^m B_{2R_i}(\bar{\zeta}_j)$.

Let us estimate $\|\partial_{\bar{\zeta}_i} E_i\|_p(\Omega_j(\xi_i)) = I_1 + I_2$ where

$$I_1 = \int_{R_{2R_i}(\xi_i) \setminus B_{2R_i}(\xi_i)} \left| \partial_{\bar{\zeta}_i} E_i \right|^p, \quad I_2 = \int_{B_{R_i}(\xi_i) \setminus B_{R_i}(\xi_i)} \left| \partial_{\bar{\zeta}_i} E_i \right|^p,$$

and $L > 0$ is some large constant. Using inequality (2.10) we can estimate

$$|f_j(V_\lambda - |\log \lambda|)| \leq \frac{C \lambda^3}{\text{dist}(x, \partial B_{R_i}(\xi_i))^3}$$

(5.11)

for $x \in B_{2R_i}(\bar{\zeta}_j) \setminus B_{R_i}(\bar{\zeta}_j)$. Therefore

$$I_1 \leq C \lambda^3 \int_{L_i} \frac{1}{y^{3p}} dy \leq C \lambda.$$

For $I_2$ we have, using the uniform bound for $f_j$ and $\partial_{\bar{\zeta}_i} V_j$, $|I_2| \leq C \lambda$. Hence we find

$$\int_{B_{R_i}(\xi_i) \setminus B_{R_i}(\xi_i)} \left| \partial_{\bar{\zeta}_i} E_i \right|^p \leq C \lambda.$$ 

(5.12)

To estimate inside $B_{R_i}(\bar{\zeta}_j)$ we use Lemma 3.2 to write

$$E_i = f_j(V_i - |\log \lambda| + g(x)) = f_j(V_i - |\log \lambda|)$$

for $x \in B_{R_i}(\bar{\zeta}_j)$, where

$$g(x) = 2\pi \mu_i \left[ H \left( \frac{\lambda}{|\log \lambda|^2} x, \bar{\zeta}_j \right) - H(\xi_i, \xi_j) \right] + 2\pi \sum_{j \neq i} \mu_j \left[ G \left( \frac{\lambda}{|\log \lambda|^2} x, \xi_j \right) - G(\bar{\zeta}_j, \xi_j) \right].$$

We recall that $u_j$ depends on $\mu_i$ because the initial condition $z_i$ in the ODE (3.2) is determined by $\mu_i$ from the relation (3.9). We make the dependence of the solution $u$ of (3.2) explicit by writing $w = w(r, \varphi)$. We also note that $\mu_i$ depends on $\bar{\zeta}_j$, $j = 1, \ldots, m$ through the formula (3.15). Then

$$\partial_{\bar{\zeta}_i} E_i = E_1 + E_2.$$
where
\begin{align*}
E_1 &= \left[ f'_i(w_i(|x - \tilde{\zeta}_i|) + g(x)) - f'_i(w_i(|x - \tilde{\zeta}_i|)) \right] \partial_{\tilde{\zeta}_i} w_i(|x - \tilde{\zeta}_i|, z_i) \\
E_2 &= f'_i(w_i(|x - \tilde{\zeta}_i|) + g(x)) \partial_{\tilde{\zeta}_i} g,
\end{align*}

Then we compute
\[ \partial_{\tilde{\zeta}_i} w_i(|x - \tilde{\zeta}_i|, z_i) = -\delta_{ik} \frac{\partial w_i}{\partial r} \frac{x - \tilde{\zeta}_i}{|x - \tilde{\zeta}_i|} + \frac{\partial w_i}{\partial x} \sum_j \frac{\partial x_i}{\partial \mu_j} \partial_{\mu_j} \]
and we get
\[ |\partial_{\tilde{\zeta}_i} w_i(|x - \tilde{\zeta}_i|, z_i)| \leq C \quad \text{for } x \in B_R(\tilde{\zeta}_i). \quad (5.13) \]

To estimate \( E_1 \) we write \( A_1 = B_R(\tilde{\zeta}_i) \setminus B_{R-Ld}(\tilde{\zeta}_i), \quad A_2 = B_{R-Ld}(\tilde{\zeta}_i) \) and \( L > 0 \) is a large constant. Then, using the uniform bounds \((5.13)\) and \((2.9)\) we get \( \int_{A_1} |E_1|^p \leq C\lambda \).

Since \( g(x) = O(\lambda / \log \lambda^2) \), we estimate using \((2.12)\)
\[ \int_{A_2} |E_1|^p \leq C \int_{A_2} \left| f'_i(w_i(|x - \tilde{\zeta}_i|) + g(x)) - f'_i(w_i(|x - \tilde{\zeta}_i|)) \right|^p \]
\[ \leq C \int_{A_2} \int_0^1 \left| f''_i(w_i(|x - \tilde{\zeta}_i|) + \tau g(x)) \right|^p |g|^p d\tau d\lambda \leq C\lambda. \]

Also \( \int_{B_{R(\tilde{\zeta}_i)}} |E_2|^p \leq C\lambda^p \) and we conclude that
\[ \int_{B_{R(\tilde{\zeta}_i)}} |\partial_{\tilde{\zeta}_i} E_2|^p \leq C\lambda. \quad (5.14) \]

Combining \((5.10)\), \((5.12)\) and \((5.14)\) we obtain the result of the lemma. \( \square \)

**Proof of Proposition 5.2.** The proof that \( \phi, \ c_{ij} \) are differentiable with respect to \( \tilde{\zeta}_1, \ldots, \tilde{\zeta}_m \) can be done with the contraction mapping principle, using that the linear operator defined in Proposition 4.1 is differentiable with respect to \( \tilde{\zeta}_1, \ldots, \tilde{\zeta}_m \). We proceed to prove estimate \((5.9)\). Differentiating the equation in \((5.1)\) with respect to \( \tilde{\zeta}_k \) we obtain, for \( \tilde{\psi} = \partial_{\tilde{\zeta}_k} \phi \),
\[ \Delta \psi + \Psi + (\partial_{\tilde{\zeta}_k} W) \phi + \partial_{\tilde{\zeta}_k} E_2 + \partial_{\tilde{\zeta}_k} N(\phi) - \sum (\partial_{\tilde{\zeta}_k} c_{ij}) Z_{ij} - \sum c_{ij} \partial_{\tilde{\zeta}_k} Z_{ij} = 0 \]
in \( \Omega_2 \). Let \( \tilde{\psi} = \psi - \sum d_{ij} Z_{ij} \) where \( d_{ij} = -f_{\Omega_1} \phi \partial_{\tilde{\zeta}_k} Z_{ij} / f_{\Omega_1} Z_{ij} \). In this way, \( \tilde{\psi} \) satisfies the orthogonality conditions \((4.2)\). Applying Proposition 4.1 we can estimate
\[ \|\tilde{\psi}\|_{L^\infty} + \sum |\partial_{\tilde{\zeta}_k} c_{ij}| \leq C \left( \|\partial_{\tilde{\zeta}_k} E_2\|_{L^1} + \|\partial_{\tilde{\zeta}_k} W\| + \|\partial_{\tilde{\zeta}_k} N(\phi)\|_{L^1} + \sum |c_{ij}| \|\partial_{\tilde{\zeta}_k} Z_{ij}\|_{L^1} + \sum |d_{ij}| \|\Delta Z_{ij}\|_{L^1} + WZ_{ij} \right). \]

Estimate \((5.3)\) gives that \( |c_{ij}| = O(\lambda^{1-2\varepsilon}) \) and \( |d_{ij}| = O(\lambda^{1-2\varepsilon}) \) as \( \lambda \to 0 \).

Lemma 5.3 yields
\[ \|\partial_{\tilde{\zeta}_k} E_2\|_{Y} \leq C\lambda^{\min(1-2\varepsilon,1/p)}. \]
We compute
\[(\partial_{\xi_i} W)\phi + \partial_{\xi_i} N(\phi) = \left[f'_{\lambda}(V_\lambda + \phi - |\log \lambda|) - f'_{\lambda}(V_\lambda - |\log \lambda|)\right](\partial_{\xi_i} V_\lambda + \psi).\]

The same computations that lead to (5.7) and (5.8) show that
\[\|f'_{\lambda}(V_\lambda + \phi - |\log \lambda|) - f'_{\lambda}(V_\lambda - |\log \lambda|)\|_Y \leq C\lambda^{\frac{1}{r'}}.\]
Therefore
\[\|(\partial_{\xi_i} W)\phi + \partial_{\xi_i} N(\phi)\|_Y \leq C(\lambda^{\frac{1}{r'}} + \lambda^{\frac{1}{r'}} \|\psi\|_{L^\infty}).\]
Collecting the estimates above, we obtain
\[\|\psi\|_{L^\infty} \leq C(\lambda^{\frac{1}{r'}} + \lambda^{\frac{1}{r'}} \|\psi\|_{L^\infty}).\]

For \(\lambda > 0\) small we deduce (5.9).

6. Proof of the Main Results

We work with points \(\xi_j \in \Omega\) satisfying (3.6), (3.7), that is, in the set
\[\Omega_m = \{(\xi_1, \ldots, \xi_m) \in \Omega^m : |\xi_i - \xi_j| \geq \delta \forall i \neq j, \text{dist}(\xi_j, \partial \Omega) \geq \delta \forall 1 \leq j \leq m\},\]
where \(\delta > 0\) is small. Recall that \(\tilde{\xi}_j = \frac{\log \lambda}{\lambda} \xi_j \in \Omega_j\). We also work with \(\mu_j\) given by (3.15).

For \(\xi = (\xi_1, \ldots, \xi_m) \in \Omega_m\), let \(\phi(\xi)\) denote the solution of (5.1), (5.2) that satisfies \(\|\phi\|_\infty \leq C\lambda^{1-2\alpha}\) constructed in Proposition 5.1 and let \(c_{ij}(\xi)\) denote the constants appearing in equation (5.1).

Writing the initial approximation \(V_\lambda = V_\lambda(\xi)\) (defined in (3.13)), we set
\[J_\lambda(\xi) = I_\lambda(V_\lambda(\xi) + \phi(\xi))\]
where \(I_\lambda\) is the functional given in (3.19).

**Lemma 6.1.** If \(\xi = (\xi_1, \ldots, \xi_m) \in \Omega_m\) is a critical point of \(J_\lambda\) then \(c_{ij}(\xi) = 0\) for all \(i = 1, \ldots, m\) and all \(j = 1, 2\), so that \(V_\lambda(\xi) + \phi(\xi)\) is a solution of (3.1).

The proof of this lemma is very similar to Lemma 4.1 in [3] or Lemma 5.1 in [5].

**Lemma 6.2.** We have the expansion
\[I_\lambda(V_\lambda + \phi_\lambda) = I_\lambda(V_\lambda) + \Theta_\lambda(\xi)\]
as \(\lambda \to 0\), where \(\Theta_\lambda(\xi) = o(1)\) in the \(C^1\) norm in \(\Omega_m\).
Proof. Using that

$$DI_x(V_\bar{z} + \phi_\bar{z})[\phi_\bar{z}] = - \sum c_{ij} \int_{\Omega}\bar{Z}_j\phi_x = 0$$

we compute

$$I_\bar{z}(V_\bar{z} + \phi_\bar{z}) - I_\bar{z}(V_\bar{z}) = - \int_0^1 sD^2I_x(V_\bar{z} + s\phi_\bar{z})[\phi_\bar{z}, \phi_\bar{z}] dx ds$$

$$= - \int_0^1 s \int_{\Omega}\bar{Z}_j(V_\bar{z} + \phi_\bar{z} - |\log \bar{z}|) - f_\bar{z}(V_\bar{z} - |\log \bar{z}|) \phi_\bar{z}$$

$$+ \int_0^1 s \int_{\Omega}\bar{Z}_j(V_\bar{z} + s\phi_\bar{z} - |\log \bar{z}|)\phi_\bar{z} - E_\bar{z}) \phi_\bar{z}.$$ 

We compute with detail the estimate for the derivative of $I_\bar{z}(V_\bar{z} + \phi_\bar{z}) - I_\bar{z}(V_\bar{z})$ with respect to $\bar{z}$. The estimate for the $C^0$ norm is similar. Differentiating with respect to $\bar{z}$,

$$\frac{\partial}{\partial \bar{z}}(I_\bar{z}(V_\bar{z} + \phi_\bar{z}) - I_\bar{z}(V_\bar{z})) = \int_0^1 s(I_1 + I_2 + \ldots) ds$$

where

$$I_1 = - \int_{\Omega}\bar{Z}_j(V_\bar{z} + \phi_\bar{z} - |\log \bar{z}|) - f_\bar{z}(V_\bar{z} - |\log \bar{z}|)(\bar{z}_\bar{z} V_\bar{z})\phi_\bar{z}$$

$$I_2 = - \int_{\Omega}\bar{Z}_j(V_\bar{z} + \phi_\bar{z} - |\log \bar{z}|) - 2f_\bar{z}(V_\bar{z} + s\phi_\bar{z} - |\log \bar{z}|)(\bar{z}_\bar{z} \phi_\bar{z})\phi_\bar{z}$$

$$I_3 = \int_{\Omega}f_\bar{z}^2(V_\bar{z} + s\phi_\bar{z} - |\log \bar{z}|)(\bar{z}_\bar{z} \phi_\bar{z}) (\bar{z}_\bar{z} \phi_\bar{z})^2$$

$$I_4 = - \int_{\Omega}(\bar{z}_\bar{z} E_\bar{z})\phi_\bar{z} + E_\bar{z}(\bar{z}_\bar{z} \phi_\bar{z})$$

$$I_5 = - \int_{\Omega}(f_\bar{z}(V_\bar{z} + \phi_\bar{z} - |\log \bar{z}|) - f_\bar{z}(V_\bar{z} - |\log \bar{z}|))(\bar{z}_\bar{z} \phi_\bar{z})$$

Each term $I_i$ will be estimated below, but before, we remark that

$$\int_{\Omega}|f_\bar{z}(V_\bar{z} + s\phi_\bar{z} - |\log \bar{z}|)| \leq C,$$  \hspace{1cm} (6.1)

by (2.9), (2.10), and

$$\int_{\Omega}|f_\bar{z}(V_\bar{z} + s\phi_\bar{z} - |\log \bar{z}|)| \leq C|\log \bar{z}|.$$  \hspace{1cm} (6.2)

which is proved by a computation similar to Lemma 4.4.

We start with

$$|I_1| \leq \int_{\Omega}|f_\bar{z}(V_\bar{z} + \phi_\bar{z} - |\log \bar{z}|) - f_\bar{z}(V_\bar{z} - |\log \bar{z}|)||\bar{z}_\bar{z} V_\bar{z}||\phi_\bar{z}||$$

$$\leq ||\bar{z}_\bar{z} V_\bar{z}||_{L^\infty}||\phi_\bar{z}||_{L^\infty} \int_{\Omega}|f_\bar{z}(V_\bar{z} + \phi_\bar{z} - |\log \bar{z}|) - f_\bar{z}(V_\bar{z} - |\log \bar{z}|)|$$
Point Ruptures for a MEMS Equation

\[
\begin{align*}
\leq C\|\phi_\xi\|_L^2 \int_0^1 \int_{\mathcal{N}_\xi} \left| f'_{\xi}(V_\xi + \tau \phi - \log(\lambda)) \right| d\tau \\
\leq C\lambda^{2-4\varepsilon} \left| \log(\lambda) \right|,
\end{align*}
\]

where we have used (5.3) and (6.2). Next we estimate

\[
|I_3| \leq \int_0^1 \int_{\mathcal{N}_\xi} |f'_{\xi}(V_\xi + \phi_\xi - \log(\lambda)) + 2|f'_{\xi}(V_\xi + s\phi_\xi - \log(\lambda))||\partial_{\xi_i} \phi_\xi| |\phi_\xi| \\
\leq C\|\partial_{\xi_i} \phi_\xi\|_{L^\infty} \|\phi_\xi\|_{L^\infty} \leq C\lambda^{1+\frac{1}{2} - 2\varepsilon - \frac{\varepsilon}{2}}
\]

by (5.3), (5.9) and (6.1). Similarly

\[
|I_3| \leq \int_0^1 \int_{\mathcal{N}_\xi} |f'_{\xi}(V_\xi + s\phi_\xi - \log(\lambda))|\partial_{\xi_i} (V_\xi + s\phi_\xi) \phi_\xi | \leq C\lambda^{2-4\varepsilon} \left| \log(\lambda) \right|
\]

and

\[
|I_4| \leq \|\phi_\xi\|_{L^\infty} \int_{\mathcal{N}_\xi} |\partial_{\xi_i} E_\xi| + \|\partial_{\xi_i} \phi_\xi\|_{L^\infty} \int_{\mathcal{N}_\xi} |E_\xi| \leq C\lambda^{1+\frac{1}{2} - \frac{\varepsilon}{2}} \left| \log(\lambda) \right|^3,
\]

since outside a big ball around the \( \xi_j \), \(|f_{\xi_j}|, |f'_{\xi_j}| \leq C\lambda^3 / \left| \log(\lambda) \right| \) and the area of \( \Omega_\xi \) is proportional to \( \left| \log(\lambda) \right|^{4/\varepsilon^2} \). In summary

\[
|\partial_{\xi_i} (I_3(V_\xi + \phi_\xi) - I_3(V_\xi))| \leq C\lambda^{1+\frac{1}{2} - 2\varepsilon - \frac{\varepsilon}{2}}.
\]

We choose now \( p > 2 \) close to 2 and \( \sigma > 0 \) small. Since \( \partial_{\xi_i} = \frac{|\log(\lambda)|^2}{\lambda} \partial_{\xi_i} \lambda \) we deduce

\[
|\partial_{\xi_i} I_3(V_\xi + \phi_\xi) - \partial_{\xi_i} I_3(V_\xi)| \leq C\lambda^a
\]

for some \( a > 0 \).

Using the same arguments as before, we can prove the same estimates for \( J_\lambda(\xi) = I_3(V_\xi + \phi(\xi)) \).

We have

\[
\tilde{J}_\lambda(\xi) = -\frac{1}{2\pi} \int_{\mathcal{N}_\xi} \lambda \left| I_3(V_\xi) - \pi m \left| \log(\lambda) \right| + 2\pi m \log \left| \log(\lambda) \right| - \pi m \log R_\lambda \right|.
\]

Thanks to Lemmas 3.3 and 6.2 for \( \xi \in \tilde{\Omega}_m \) we have

\[
\tilde{J}_\lambda(\xi) = \varphi_m(\xi) + \Theta_\lambda(\xi).
\]

where \( \Theta_\lambda = o(1) \) as \( \lambda \to 0 \) in the \( C^1 \) norm of \( \tilde{\Omega}_m \).

To prove Theorem 1.1, we note that when \( m = 1 \), \( \varphi_1(\xi) = H(\tilde{\xi}, \tilde{\xi}) \) and therefore

\[
\varphi_1(\xi) \to -\infty \quad \text{as} \quad \xi \to \partial \Omega.
\]

Hence, choosing \( \delta > 0 \) small, \( \varphi_1 \) has a strict absolute maximum in \( \tilde{\Omega}_1 \). By (6.3), \( \tilde{J}_\lambda \) has also an absolute maximum in \( \tilde{\Omega}_1 \) for \( \lambda > 0 \) small, and this yields the existence of a solution to problem (3.1).
Theorem 1.3 is a direct consequence of (6.3), since a non-degenerate critical point of $\varphi_m$ gives rise to a critical point of a small $C^1$ perturbation of this function.

To prove Theorem 1.2 we invoke the work [5], where it is proved that if $\Omega$ is not simply connected, then $\varphi_m$ and any sufficiently $C^1$ close map have critical points in $\tilde{\Omega}_m$, for $\delta > 0$ small enough. This result also appears in [3]. □

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