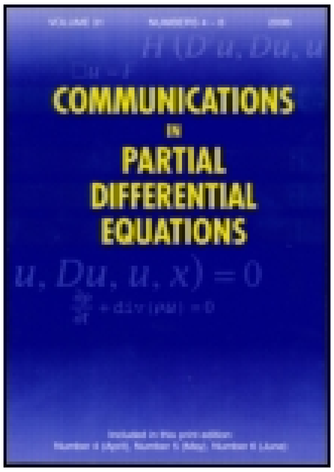

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The Supercritical Lane–Emden–Fowler Equation in Exterior Domains

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We consider the exterior problem

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}},$$
$$u = 0 \quad \text{on } \partial\mathcal{D}, \quad \lim_{|x|\to+\infty} u(x) = 0$$

where $\mathcal{D}$ is a bounded, smooth domain in $\mathbb{R}^N$, for supercritical powers $p > 1$. We prove that if $N \geq 4$ and $p > \frac{N+1}{N-3}$, then this problem admits infinitely many solutions. If $\mathcal{D}$ is symmetric with respect to N axes, this result holds whenever $N \geq 3$ and $p > \frac{N+2}{N-2}$.

Keywords Critical exponents; Linearized operators; Slow decay solutions.

Mathematics Subject Classification Primary 35J25; Secondary 35J60.

1. Introduction and Statement of the Main Results

A basic model of nonlinear elliptic boundary problem is the Lane–Emden–Fowler equation,

$$\Delta u + u^p = 0, \quad u > 0 \text{ in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

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where Ω is a domain with smooth boundary in $\mathbb{R}^N$ and $p > 1$. First formulated by Lane, an astrophysicist, in the mid 19th century, the role of this and related elliptic PDEs has been broad outside and inside mathematics. While simple looking, the structure of the solution set of this problem may be surprisingly complex. Much has been learned over the last decades, particularly thanks to the development of techniques from the calculus of variations (see Struwe, 1990). On the other hand, various basic issues are not yet fully understood.

A rather fascinating characteristic of this problem is the role played by the critical exponent $p = \frac{N+2}{N-2}$ in the solvability question. When Ω is *bounded* and $1 < p < \frac{N+2}{N-2}$, a solution can be found as a minimizer of the variational problem

$$\inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}},$$

which indeed exists thanks to compactness of Sobolev's embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$. When $p \geq \frac{N+2}{N-2}$, this minimization procedure fails, and so does existence in general: Pohozaev (1965) discovered that no solution exists if the domain is strictly star-shaped. On the other hand, in the critical case $p = \frac{N+2}{N-2}$, solvability is restored by the addition of linear terms, as discovered by Brezis and Nirenberg (1983). Coron (1984) found that (1.1)–(1.2) is solvable for p critical if Ω has a small hole, see also Rey (1989), while Bahri and Coron (1988) established that this is the case whenever Ω has a non-trivial topology. The presence of non-trivial topology in the domain does not suffice for existence of solutions to (1.1)–(1.2) if $N \geq 4$ and $p > \frac{N+1}{N-3}$ as found via examples by Passaseo (1993). If p is super-critical but close to critical, solvability still holds in domains with small holes (see del Pino et al., 2003).

A main reason why in most existence results for elliptic problems involving nonlinearities with power-like growth, the power is required to be at most critical is that variational approach naturally adapts to the problem, and certain control is typically possible on the forms of non-compactness arising. Loss of compactness appears in unbounded domains, say exterior domains, even in the subcritical situation, associated to invariance under translations. A model for which broad literature exists is

$$\Delta u - V(x)u + K(x)u^p = 0$$

with $V > 0$, $K > 0$, and p subcritical, $1 < p < \frac{N+2}{N-2}$ in entire space or in a exterior domain. We refer the reader for instance to Bahri and Lions (1997), Benci and Cerami (1987), Cerami and Molle (2003), Cerami and Passaseo (1995), and Esteban and Lions (1982/83), and references therein for various existence and multiplicity results in exterior domains via variational methods.

Except for results in domains involving symmetries or exponents close to critical (del Pino et al., 2003; Passaseo, 1998), solvability of supercritical problems is a widely open matter. In this paper we consider Problem (1.1)–(1.2) for exponents p above critical in an *exterior domain*. Thus we assume in what follows that the domain Ω has the form $\Omega = \mathbb{R}^N \setminus \mathcal{D}$ where $\mathcal{D}$ is a bounded domain with smooth boundary. We consider the problem of finding classical solutions of

$$\Delta u + u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^N \setminus \overline{\mathcal{D}}, \tag{1.3}$$

$$u = 0 \quad \text{on } \partial\mathcal{D}, \qquad \lim_{|x|\to+\infty} u(x) = 0 \tag{1.4}$$

where $p > \frac{N+2}{N-2}$. The supercritical case is particularly meaningful in this problem since Pohozaev's identity does not pose obstructions for its solvability. On the contrary, unlike the bounded domain case, existence of a finite-energy solution is ruled out if $p \le \frac{N+2}{N-2}$ and $\mathcal{D}$ is star-shaped. Not only this, if $\mathcal{D}$ is a ball and $p > \frac{N+2}{N-2}$, radial classical solutions *do exist*, as it can be seen by phase-plane analysis after a transformation of the problem found by Fowler (1931), see also for instance Johnson et al. (1993) and Joseph and Lundgren (1973).

Our main result asserts that for arbitrary domain $\mathcal{D}$, Problem (1.3)–(1.4) admits *infinitely many* solutions if the power p is above $\frac{N+1}{N-3}$, the critical exponent in one dimension less. This constraint is not needed if the domain is symmetric with respect to N coordinate axes, case in which pure supercriticality suffices.

To state our results precisely we consider the problem

$$\Delta w + w^p = 0 \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

which is well known to possess a positive radially symmetric solution $w(|x|)$ whenever $p > \frac{N+2}{N-2}$. Indeed, when looking for radial solutions $w(|x|)$ the classical change of variable $v(s) = r^{\frac{2}{p-1}} w(r)$, $r = e^s$ transforms the equation (1.5) into the autonomous ODE

$$v'' + \alpha v' - \beta v + v^p = 0 \tag{1.6}$$

where

$$\alpha = N - 2 - \frac{4}{p-1}, \quad \beta = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right).$$

Since α and β are positive, it is standard to check, using the Hamiltonian energy $\frac{1}{2}\dot{v}^2 + \frac{1}{p+1}v^{p+1} - \frac{\beta}{2}v^2$, the existence of a unique orbit connecting the equilibria $v = 0$ and $v = \beta^{\frac{1}{p-1}}$. This orbit gives rise to a family of solutions of the ODE by translation in s, which are in correspondence with the scaling

$$w_\lambda(r) = \lambda^{\frac{2}{p-1}} w(\lambda r). \tag{1.7}$$

We fix in what follows the solution w of (1.5) such that

$$w(0) = 1. \tag{1.8}$$

Asymptotics of w near $+\infty$, can be found by linearization of (1.6) at $v = \beta^{\frac{1}{p-1}}$ (see Gui et al., 1992). At main order one has

$$w(r) = \beta^{\frac{1}{p-1}} r^{-\frac{2}{p-1}} (1 + o(1)) \tag{1.9}$$

as $r \to +\infty$, which implies that this behavior is actually common to all solutions $w_\lambda(r)$. The idea is to consider w_λ as a first approximation for a solution of Problem (1.3)–(1.4), provided that $\lambda > 0$ is chosen small enough.

Our main results read as follows:

Theorem 1. *Assume that* $N \geq 4$ *and* $p > \frac{N+1}{N-3}$. *Then for all* $\lambda > 0$ *sufficiently small, problem* (1.3)–(1.4) *has a solution* u_λ *of the form*

$$u_\lambda = w_\lambda + \phi_\lambda$$

with

$$|\phi(x)| \leq C\lambda^{\frac{2}{p-1}} \quad \text{if } |x| \leq \frac{1}{\lambda} \quad \text{and} \quad |\phi(x)| = o(1)|x|^{-\frac{2}{p-1}} \quad \text{if } |x| \geq \frac{1}{\lambda},$$

where $o(1) \to 0$ *as* $\lambda \to 0$.

Since the location of the origin is arbitrary, one may conjecture the presence of a $N+1$-dimensional family of solutions, parametrized by a point in $\mathbb{R}^N$ and all small values of λ. However, our proof does not allow in principle to distinguish between solutions associated to different choices of the origin.

Theorem 2. *The result of Theorem* 1 *also holds true if* $N \geq 3$ *and* $p > \frac{N+2}{N-2}$, *provided that* $\mathcal{D}$ *is symmetric with respect to N coordinate axes, namely*

$$(x_1, \ldots, x_i, \ldots, x_N) \in \mathcal{D} \quad \text{if } (x_1, \ldots, -x_i, \ldots, x_N) \in \mathcal{D}, \quad \text{for all } i = 1, \ldots, N. \tag{1.10}$$

We do not know whether existence of solutions to (1.3)–(1.4) holds in the entire range $\frac{N+2}{N-2} < p \leq \frac{N+1}{N-3}$ without a further geometric constraint on $\mathcal{D}$.

As it has become standard for various problems of this type, it is first necessary to construct a right inverse of the linearization of (1.1) around w_λ in the whole of $\mathbb{R}^N$. This part does not require λ to be small and it is similar to the analysis by Mazzeo and Pacard (1996) where solutions with prescribed singular set for sub-critical problems are constructed (see also Pacard, 1992, 1993). Then for small λ a similar solvability property is established for the linearized operator around w_λ in the exterior domain $\mathbb{R}^N \setminus \overline{\mathcal{D}}$. As it will become apparent from the analysis in Section 2, if $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ the linearized operator is not surjective, having a range orthogonal to the generators of translations. This suggests that a further adjustment of the location of the origin may produce a family of solutions as in Theorem 1, and this is in fact the situation of Theorem 2.

The perturbation analysis in Mazzeo and Pacard (1996) and that carried out here shows a strong analogy between inner-subcritical and exterior-supercritical problems, and, at the same time, important differences like the fact that the phenomenon seems to break apart without further geometric assumptions in the domain to cover the full supercritical range. The analogy here revealed should be an interesting line to explore in searching for a better understanding of solvability for supercritical problems. For instance, we may observe that there exists a unique radial solution $u(r)$ to (1.3)–(1.4) with $u(r) \sim r^{2-N}$ when $\mathcal{D}$ is a ball. This solution is dual to the classical solution present in a ball in the sub-critical case. It would be interesting to understand whether this duality holds for other geometries.

2. The Operator $\Delta + pw^{p-1}$ in $\mathbb{R}^N$

Let w be the radial solution to (1.5), (1.8). In this section we study the linear equation

$$\Delta\phi + pw^{p-1}\phi = h \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

The main result concerns with solvability of this equation and estimates for the solution in appropriate norms. We choose to carry out this procedure in weighted L^∞ norms adapted to the problem. Since we will be interested in solving an equation of this type where ϕ will turn out to be a perturbation of w, it is rather natural to require that it has a decay at most the same as that of w, namely $O\big(|x|^{-\frac{2}{p-1}}\big)$ as $|x| \to +\infty$. Of course we would also like ϕ be bounded on compact sets. We shall however allow slightly more room near the origin, say $O(|x|^{-\sigma})$ near $x = 0$. Consistently, h should have a behavior like this but with two powers subtracted. In particular we will require $h = O\big(|x|^{-\frac{2}{p-1}-2}\big)$ at infinity.

Thus we define for a fixed small $\sigma > 0$ the following norms that give account of these requirements.

$$\|\phi\|_* = \sup_{|x|\le 1} |x|^\sigma |\phi(x)| + \sup_{|x|\ge 1} |x|^{\frac{2}{p-1}} |\phi(x)|, \tag{2.2}$$

and

$$\|h\|_{**} = \sup_{|x|\le 1} |x|^{2+\sigma} |h(x)| + \sup_{|x|\ge 1} |x|^{\frac{2}{p-1}+2} |h(x)|, \tag{2.3}$$

Proposition 2.1. *Assume $N \ge 4$ and $p > \frac{N+1}{N-3}$. There exist a small $\sigma > 0$ and a constant $C > 0$ such that for any h with $\|h\|_{**} < +\infty$, equation* (2.1) *has a solution $\phi = T(h)$ such that T defines a linear map and*

$$\|T(h)\|_* \le C\|h\|_{**}.$$

By the change of variables $\tilde\phi(y) = \phi(\frac{y}{\lambda})$ we deduce directly the solvability of

$$\Delta\phi + pw_\lambda^{p-1}\phi = h \quad \text{in } \mathbb{R}^N \tag{2.4}$$

in weighted L^∞ spaces as defined by the norms

$$\|\phi\|_{*,\lambda} = \lambda^\sigma \sup_{|x|\le\frac{1}{\lambda}} |x|^\sigma |\phi(x)| + \lambda^{\frac{2}{p-1}} \sup_{|x|\ge\frac{1}{\lambda}} |x|^{\frac{2}{p-1}} |\phi(x)|$$

$$\|h\|_{**,\lambda} = \lambda^\sigma \sup_{|x|\le\frac{1}{\lambda}} |x|^{2+\sigma} |h(x)| + \lambda^{\frac{2}{p-1}} \sup_{|x|\ge\frac{1}{\lambda}} |x|^{2+\frac{2}{p-1}} |h(x)|.$$

Theorem 3. *Assume that $N \ge 4$ and $p > \frac{N+1}{N-3}$. For any $\lambda > 0$ there is a linear map $h \mapsto \phi = T_\lambda(h)$ defined whenever $\|h\|_{**,\lambda} < \infty$ such that ϕ is a solution to* (2.4). *Moreover*

$$\|T_\lambda(h)\|_{*,\lambda} \le C\|h\|_{**,\lambda}.$$

The linear operator in (2.1) is of regular singular type and it is well known that it is Fredholm on weighted spaces provided the weight does not equal one of *indicial roots* (see Mazzeo, 1991; Mazzeo and Smale, 1991; Mazzeo and Pacard, 1996). Nonetheless, for completeness of the presentation, we prefer to include the main points of the argument and omit some technical computations.

We write h as

$$h(x) = \sum_{k=0}^{\infty} h_k(r)\Theta_k(\theta), \quad r > 0, \;\; \theta \in S^{N-1} \tag{2.5}$$

where Θ_k, $k \geq 0$ are the eigenfunctions of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ on the sphere S^{N-1}, normalized so that they constitute an othonormal system in $L^2(S^{N-1})$. We take Θ_0 to be a positive constant, associated to the eigenvalue 0 and Θ_i, $1 \leq i \leq N$ is an appropriate multiple of $\frac{x_i}{|x|}$ which has eigenvalue $\lambda_i = N-1$, $1 \leq i \leq N$. In general, λ_k denotes the eigenvalue associated to Θ_k, we repeat eigenvalues according to their multiplicity and we arrange them in an non-decreasing sequence. We recall that the set of eigenvalues is given by $\{j(N-2+j) \mid j \geq 0\}$.

We look for a solution ϕ to (2.1) in the form

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k(r)\Theta_k(\theta).$$

Then ϕ satisfies (2.1) if and only if

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi_k = h_k, \quad \text{for all } r > 0, \text{ for all } k \geq 0. \tag{2.6}$$

To construct solutions of this ODE we need to consider two linearly independent solutions $z_{1,k}$, $z_{2,k}$ of the homogeneous equation

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi_k = 0, \quad r \in (0, \infty). \tag{2.7}$$

Once these generators are identified, the general solution of the equation can be written through the variation of parameters formula as

$$\phi(r) = z_{1,k}(r)\int z_{2,k}h_k r^{N-1}dr - z_{2,k}(r)\int z_{1,k}h_k r^{N-1}dr$$

where the symbol $\int$ designates arbitrary antiderivatives, which we will specify in the choice of the operators. It is helpful to recall that if one solution $z_{1,k}$ to (2.7) is known, a second, linearly independent solution can be found in any interval where $z_{1,k}$ does not vanish as

$$z_{2,k}(r) = z_{1,k}(r)\int z_{1,k}(r)^{-2}r^{1-N}dr.$$

One can get the asymptotic behaviors of any solution z as $r \to 0$ and as $r \to +\infty$ by examining the indicial roots of the associated Euler equations. It is known that as $r \to +\infty$ $r^2 w(r)^{p-1} \to \beta$ where

$$\beta = \frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right).$$

Thus we get the limiting equation

$$r^2\phi'' + (N-1)r\phi' + (p\beta - \lambda_k)\phi = 0, \tag{2.8}$$

whose indicial roots are given by

$$\mu_k^{\pm} = \frac{N-2}{2} \pm \frac{1}{2}\sqrt{(N-2)^2 + 4(\lambda_k - p\beta)}.$$

When $k = 0$ the indicial roots are given by

$$\mu_0^{\pm} = \frac{N-2}{2} \pm \frac{1}{2}\sqrt{(N-2)^2 - 4p\beta}.$$

The situation depends of course on the sign of $D = (N-2)^2 - 4p\beta$. It is observed in Gui et al. (1992) that $D > 0$ if and only if $N > 10$ and $p > p_c$ where we set

$$p_c = \begin{cases} \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N > 10 \\ \infty & \text{if } N \leq 10. \end{cases}$$

Thus when $p < p_c$, $\mu_0^{\pm}$ are complex with negative real part, and the behavior of a solution $z(r)$ to (2.7) as $r \to +\infty$ is oscillatory and given by

$$z(r) = O\left(r^{-\frac{N-2}{2}}\right).$$

The behavior of w also depends on whether $p < p_c$ or $p \geq p_c$. Concerning the latter case it will be useful to recall asymptotic formulae derived in Gui et al. (1992) where it is shown that if $p = p_c$ (in which case $\mu_0^+ = \mu_0^- > \frac{2}{p-1}$),

$$w(r) = \frac{\beta^{\frac{1}{p-1}}}{r^{\frac{2}{p-1}}} + \frac{a_1 \log r}{r^{\mu_0^-}} + o\left(\frac{\log r}{r^{\mu_0^-}}\right), \quad r \to +\infty, \tag{2.9}$$

where $a_1 < 0$, and if $p > p_c$ (so that $\mu_0^+ > \mu_0^- > \frac{2}{p-1}$)

$$w(r) = \frac{\beta^{\frac{1}{p-1}}}{r^{\frac{2}{p-1}}} + \frac{a_1}{r^{\mu_0^-}} + o\left(\frac{1}{r^{\mu_0^-}}\right), \quad r \to +\infty. \tag{2.10}$$

In Fourier mode $k = 1$ the indicial roots are given by $\mu_1^+ = \frac{2}{p-1} + 1$ and $\mu_1^- = N - 3 - \frac{2}{p-1}$. Since we are looking for solutions that decay at a rate $r^{-\frac{2}{p-1}}$ as $r \to +\infty$ we will need $N - 3 - \frac{2}{p-1} \geq \frac{2}{p-1}$, which is equivalent to $p \geq \frac{N+1}{N-3}$.

As $r \to 0$ the limiting equation is given by

$$r^2\phi'' + (N-1)r\phi' - \lambda_k\phi = 0.$$

In this way the behavior will be ruled by $z(r) \sim r^{-\mu}$ as $r \to 0$ where μ satisfies

$$\mu^2 - (N-2)\mu - \lambda_k = 0.$$

Lemma 2.1. *Let $k = 0$ and $p > \frac{N+2}{N-2}$. Then equation (2.6) has a solution ϕ_0 which depends linearly on h_0 and satisfies*

$$\|\phi_0\|_* \leq C\|h_0\|_{**}. \tag{2.11}$$

Proof. Independently of the value of p, one can get immediately a solution of the homogeneous problem. Since equation (1.5) is invariant under the transformation $\lambda \mapsto \lambda^{\frac{2}{p-1}} w(\lambda r)$ we see by differentiation in λ that the function

$$z_{1,0} = rw' + \frac{2}{p-1} w$$

satisfies equation (2.7) for $k = 0$.

Using expansions (2.9), (2.10) and easily derived ones for w', we get that as $r \to +\infty$

$$\text{if } p < p_c : |z_{1,0}(r)| \leq Cr^{\frac{N-2}{2}} \tag{2.12}$$

$$\text{if } p = p_c : z_{1,0}(r) = cr^{-\frac{N-2}{2}} \log r(1 + o(1)) \tag{2.13}$$

$$\text{if } p > p_c : z_{1,0}(r) = cr^{-\mu_0^-}(1 + o(1)), \tag{2.14}$$

where $c \neq 0$.

Case $p < p_c$. We define $z_{2,0}(r)$ for small $r > 0$ by

$$z_{2,0}(r) = z_{1,0}(r) \int_{r_0}^{r} z_{1,0}(s)^{-2} s^{1-N} ds$$

where r_0 is small so that $z_{1,0} > 0$ in $(0, r_0)$ (which is possible because $z_{1,0}(r) \sim 1$ near 0). Then $z_{2,0}$ is extended to $(0, +\infty)$ so that it is a solution to the homogeneous equation (2.7) (with $k = 0$) in this interval. As mentioned earlier $z_{2,0}(r) = O\left(r^{-\frac{N-2}{2}}\right)$ as $r \to +\infty$.

We define

$$\phi_0(r) = z_{1,0}(r) \int_1^r z_{2,0} h_0 s^{N-1} ds - z_{2,0}(r) \int_0^r z_{1,0} h_0 s^{N-1} ds,$$

and omit a calculation that shows that this expression satisfies (2.11).

Case $p \geq p_c$. The strategy is the same as in the previous case, but this time it is more convenient to rewrite the variation of parameters formula in the form

$$\phi_0(r) = -z_{1,0}(r) \int_1^r z_{1,0}(s)^{-2} s^{1-N} \int_0^s z_{1,0}(\tau) h_0(\tau) \tau^{N-1} d\tau\, ds,$$

which is justified because when $p \geq p_c$ we have $z_{1,0}(r) > 0$ for all $r > 0$, which follows from the fact that $\lambda \mapsto \lambda^{\frac{2}{p-1}} w(\lambda r)$ is increasing for $\lambda > 0$, see Gui et al.

(1992). Again, a calculation using now (2.13) and (2.14) shows that ϕ_0 satisfies the estimate (2.11). □

Lemma 2.2. a) *Let $k = 1$ and $p \geq \frac{N+1}{N-3}$. Then equation* (2.6) *has a solution ϕ_1 which is linear with respect to h_1 and satisfies*

$$\|\phi_1\|_* \leq C\|h_1\|_{**}. \tag{2.15}$$

b) *Let $N \geq 3$ and $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ $\left(\frac{N+2}{N-2} < p \text{ if } N = 3\right)$. If $\|h\|_{**} < +\infty$ and*

$$\int_0^\infty h_1(r)w'(r)r^{N-1}\, dr = 0 \tag{2.16}$$

then (2.6) *has a solution ϕ_1 satisfying* (2.15) *and depending linearly on h_1* (*condition* (2.16) *makes sense when $p < \frac{N+1}{N-3}$ and $\|h_1\|_{**} < +\infty$*).

Proof. a) As mentioned before, the indicial roots in the case $k = 1$ are given by $\mu_1^+ = \frac{2}{p-1} + 1$ and $\mu_1^- = N - 3 - \frac{2}{p-1}$. Assuming $p \geq \frac{N+1}{N-3}$ we have $N - 3 - \frac{2}{p-1} \geq \frac{2}{p-1}$. On the other hand the behavior near 0 of $z(r)$ can be $z(r) \sim r$ or $z(r) \sim r^{1-N}$.

Similarly as in the case $k = 0$ we have a solution to (2.7), namely $z_1(r) = -w'(r)$ which is positive in all $(0, +\infty)$. With it we can build

$$\phi_1(r) = -z_1(r)\int_1^r z_1(s)^{-2}s^{1-N}\int_0^s z_1(\tau)h_1(\tau)\tau^{N-1}d\tau\, ds. \tag{2.17}$$

From this formula and using $p \geq \frac{N+1}{N-3}$ we obtain (2.15).

b) Since $z_1(r) \leq Cr^{-\frac{2}{p-1}-1}$ and $p < \frac{N+1}{N-3}$ it is not difficult to check that $z_1h_1\tau^{N-1}$ is integrable in $(0, +\infty)$ if $\|h_1\|_{**} < +\infty$. Thus, by (2.16) we can rewrite (2.17) as

$$\phi_1(r) = z_1(r)\int_1^r z_1(s)^{-2}s^{1-N}\int_s^\infty z_1(\tau)h_1(\tau)\tau^{N-1}d\tau\, ds$$

and from this formula (2.15) readily follows. □

Lemma 2.3. *Let $k \geq 2$ and $p > \frac{N+2}{N-2}$. If $\|h_k\|_{**} < \infty$ equation* (2.6) *has a unique solution ϕ_k with $\|\phi_k\|_{**} < \infty$ and there exists $C_k > 0$ such that*

$$\|\phi_k\|_* \leq C_k\|h_k\|_{**}. \tag{2.18}$$

Proof. Let us write L_k for the operator in (2.6), that is,

$$L_k\phi = \phi'' + \frac{N-1}{r}\phi' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi.$$

This operator satisfies the maximum principle in any interval of the form $(\delta, \frac{1}{\delta})$, $\delta > 0$. Indeed let $z = -w'$, so that $z > 0$ in $(0, +\infty)$ and it is a supersolution, because

$$L_kz = \frac{N-1-\lambda_k}{r^2}z < 0 \quad \text{in } (0, +\infty), \tag{2.19}$$

since $\lambda_k \geq 2N$ for $k \geq 2$. To prove solvability of (2.6) in the appropriate space we observe that

$$\psi = C_1 z + v, \quad v(r) = \frac{1}{r^\sigma + r^{\frac{2}{p-1}}},$$

(for some suitably large C_1) is a supersolution and then the method of sub and super-solutions yields the desired result. We omit the details. $\square$

To complete the proof of Theorem 3 we need to show that the sum of right inverses for each component is bounded in our weighted L^∞ space. This argument appears elsewhere (Caffarelli et al., 1984) but we shall reproduce it below.

Proof of Theorem 3. Let $m > 0$ be an integer. By Lemmas 2.1–2.3 we see that if $\|h\|_{**,1} < \infty$ and its Fourier series (2.5) has has $h_k \equiv 0 \ \forall k \geq m$ there exists a solution ϕ to (2.1) that depends linearly with respecto to h and moreover

$$\|\phi\|_{*,1} \leq C_m \|h\|_{**,1}$$

where C_m may depend only on m. We shall show that C_m may be taken independent of m. Assume on the contrary that there is sequence of functions h_j such that $\|h_j\|_{**,1} < \infty$, each h_j has only finitely many non-trivial Fourier modes and that the solution $\phi_j \not\equiv 0$ satisfies

$$\|\phi_j\|_{*,1} \geq C_j \|h_j\|_{**,1},$$

where $C_j \to +\infty$ as $j \to +\infty$. Replacing ϕ_j by $\frac{\phi_j}{\|\phi_j\|_{*,1}}$ we may assume that $\|\phi_j\|_{*,1} = 1$ and $\|h_j\|_{**,1} \to 0$ as $j \to +\infty$. We may also assume that the Fourier modes associated to $\lambda_0 = 0$ and $\lambda_1 = \cdots = \lambda_N = N - 1$ are zero.

Along a subsequence (which we write the same) we must have

$$\sup_{|x|>1} |x|^{\frac{2}{p-1}} |\phi_j(x)| \geq \frac{1}{2} \tag{2.20}$$

or

$$\sup_{|x|<1} |x|^\sigma |\phi_j(x)| \geq \frac{1}{2}. \tag{2.21}$$

Assume first that (2.20) occurs and let $x_j \in \mathbb{R}^N$ with $|x_j| > 1$ be such that

$$|x_j|^{\frac{2}{p-1}} |\phi_j(x_j)| \geq \frac{1}{4}.$$

Then again we have to distinguish two possibilities. Along a new subsequence (denoted the same) $x_j \to x_0 \in \mathbb{R}^N$ or $|x_j| \to +\infty$.

If $x_j \to x_0$ then $|x_0| \geq 1$ and by standard elliptic estimates $\phi_j \to \phi$ uniformly on compact sets of $\mathbb{R}^N$. Thus ϕ is a solution to (2.1) with right hand side equal to zero that also satisfies $\|\phi\|_{*,1} < +\infty$ and is such that the Fourier modes $\phi_0 = \cdots = \phi_N$ are zero. But the unique solution to this problem is $\phi \equiv 0$, contradicting $|\phi(x_0)| \geq \frac{1}{4}$.

If $|x_j| \to \infty$ consider $\tilde{\phi}_j(y) = |x_j|^{\frac{2}{p-1}}\phi_j(|x_j|y)$. Then $\tilde{\phi}_j$ satisfies

$$\Delta\tilde{\phi}_j + pw_j^{p-1}\tilde{\phi}_j = \tilde{h}_j \quad \text{in } \mathbb{R}^N$$

where $w_j(y) = |x_j|^{\frac{2}{p-1}}w(|x_j|y)$ and $\tilde{h}_j(y) = |x_j|^{2+\frac{2}{p-1}}h\left(\frac{y}{|x_j|}\right)$. But since $\|\phi_j\|_{*,1} = 1$ we have

$$|\tilde{\phi}_j(y)| \leq |y|^{-\frac{2}{p-1}}, \quad |y| > \frac{1}{|x_j|} \tag{2.22}$$

so $\tilde{\phi}_j$ is uniformly bounded on compact sets of $\mathbb{R}^N \backslash \{0\}$. Similarly, for $|y| > \frac{1}{|x_j|}$

$$|\tilde{h}_j(y)| \leq |y|^{-2-\frac{2}{p-1}}\|h_j\|_{**,1}$$

and hence $\tilde{h}_j \to 0$ uniformly on compact sets of $\mathbb{R}^N \setminus \{0\}$ as $j \to +\infty$. Also $w_j(y) \to C_{p,N}|y|^{-\frac{2}{p-1}}$ uniformly on compact sets of $\mathbb{R}^N \backslash \{0\}$. By elliptic estimates $\tilde{\phi}_j \to \phi$ uniformly on compact sets of $\mathbb{R}^N \backslash \{0\}$ and ϕ solves

$$\Delta\phi + C_{p,N}|y|^{-\frac{2}{p-1}}\phi = 0 \quad \text{in } \mathbb{R}^N \backslash \{0\}.$$

Moreover, since $\tilde{\phi}_j\left(\frac{x_j}{|x_j|}\right) \geq \frac{1}{4}$ we see that ϕ is non-trivial, and from (2.22) we have the bound

$$|\phi(y)| \leq |y|^{-\frac{2}{p-1}}, \quad |y| > 0. \tag{2.23}$$

Expanding ϕ as

$$\phi(x) = \sum_{k=N+1}^{\infty} \phi_k(r)\Theta_k(\theta)$$

(we assumed at the beginning that the first $N+1$ Fourier modes were zero) we see that ϕ_k has to be a solution to

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \frac{\beta p - \lambda_k}{r^2}\phi_k = 0, \quad \forall r > 0, \ \forall k \geq N+1.$$

The solutions to this equation are linear combinations of $r^{-\mu_k^{\pm}}$ where $\mu_k^+ > 0$ and $\mu_k^- < 0$. Thus ϕ_k can not have a bound of the form (2.23) unless it is identically zero, a contradiction.

The analysis of the case (2.21) is similar and this proves our claim. By density, for any h with $\|h\|_{**,1} < \infty$ a solution ϕ to (2.1) can be constructed and it satisfies $\|\phi\|_{*,1} \leq C\|h\|_{**,1}$. $\square$

It is important to emphasize that the constraint $p > \frac{N+1}{N-3}$ appears only at Fourier mode 1. We have

Remark 2.1. Suppose $N \geq 3$ and $p > \frac{N+2}{N-2}$. If $\|h\|_{**,\lambda} < \infty$ and in (2.5) we have $h_k \equiv 0$ for $1 \leq k \leq N$ then there exists a solution ϕ to (2.1), which defines a linear operator of h, with

$$\|\phi\|_{*,\lambda} \leq C\|h\|_{**,\lambda}.$$

Remark 2.2. Radial solutions to $\Delta w + w^p = 0$ in $\mathbb{R}^N$ with p sub and super-critical exhibit a certain duality with respect to p. The solution w to (1.5), (1.8) is smooth at the origin and $w(r) \sim r^{-\frac{2}{p-1}}$ as $r \to +\infty$. For sub-critical p there exists a radial solution $\hat{w}$ with the behavior $\hat{w}(r) \sim r^{-\frac{2}{p-1}}$ as $r \to 0$ and decaying as the fundamental solution when $r \to +\infty$. The former solution was used by Mazzeo and Pacard (1996) to construct singular solutions to $\Delta u + u^p = 0$ in bounded domains for sub-critical p. The analogy, however, breaks down when analyzing the linearized operators around these solutions. In Fourier mode 1, for super-critical p there are two elements in the kernel (i.e., two solutions to (2.7)) which have behaviors

$$z_1(r) \sim \begin{cases} r & r \to 0 \\ r^{-\frac{2}{p-1}-1} & r \to +\infty \end{cases} \quad z_2(r) \sim \begin{cases} r^{1-N} & r \to 0 \\ r^{3-N+\frac{2}{p-1}} & r \to +\infty \end{cases}$$

while their counterparts for sub-critical p behave as

$$\hat{z}_1(r) \sim \begin{cases} r^{-\frac{2}{p-1}-1} & r \to 0 \\ r^{1-N} & r \to +\infty \end{cases} \quad \hat{z}_2(r) \sim \begin{cases} r^{3-N+\frac{2}{p-1}} & r \to 0 \\ r & r \to +\infty. \end{cases}$$

We see that for sub-critical p none of these functions pose a problem for the invertibility of the operator. If p is super-critical we observe that an appropriate right inverse can be constructed without restrictions on the right hand side if $3 - N + \frac{2}{p-1} \leq -\frac{2}{p-1}$, while in the opposite case the orthogonality of the right hand side with respect to z_1 is needed.

3. The Operator $\Delta + pw_\lambda^{p-1}$ in $\mathbb{R}^N \setminus \overline{\mathcal{D}}$

Now we consider the linear problem

$$\begin{cases} \Delta\phi + pw_\lambda^{p-1}\phi = h & \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}} \\ \phi = 0 & \text{on } \partial\mathcal{D} \\ \phi(x) \to 0 & |x| \to +\infty. \end{cases} \tag{3.1}$$

which we intend to solve as a perturbation of the equation in entire space.

Proposition 3.1. (a) *Assume $N \geq 4$ and $p > \frac{N+1}{N-3}$. There exists a constant $C > 0$ such that for all sufficiently small $\lambda > 0$ and all h with $\|h\|_{**,\lambda} < +\infty$, Problem* (3.1) *has a solution $\phi = \mathcal{T}_\lambda(h)$ such that $\mathcal{T}_\lambda$ is a linear map and*

$$\|\mathcal{T}_\lambda(h)\|_{*,\lambda} \leq C\|h\|_{**,\lambda}.$$

(b) *If* $N = 3$, $\mathcal{D}$ *is symmetric with respect to all axes and* $p > \frac{N+2}{N-2}$, *then the same conclusion holds if h is symmetric in each variable, that is, if*

$$h(x_1, \ldots, x_i, \ldots, x_N) = h(x_1, \ldots, -x_i, \ldots, x_N) \quad \text{for all } 1 \le i \le N.$$

Proof. We first prove part (a). We shall solve (3.1) by writing $\phi = \eta\varphi + \psi$ where η is a radial cut-off function such that $\eta \in C^\infty(\mathbb{R}^N)$, $0 \le \eta \le 1$,

$$\eta(x) = 0 \quad \text{for } |x| \le R_0, \qquad \eta(x) = 1 \quad \text{for } |x| \ge R_0 + 1$$

and $R_0 > 0$ is fixed so that $\mathcal{D} \subseteq B_{R_0}$. We need another cut-off $\zeta \in C^\infty(\mathbb{R}^N)$, $0 \le \zeta \le 1$ such that

$$\zeta(x) = 0 \quad \text{for } |x| \le R_1, \qquad \zeta(x) = 1 \quad \text{for } |x| \ge R_1 + 1$$

where $R_1 > R_0 + 1$, in such a way that $\eta\zeta = \zeta$.

To find a solution of (3.1) it is sufficient to solve the following system

$$\Delta\varphi + pw_\lambda^{p-1}\varphi = -p\zeta w_\lambda^{p-1}\psi + \zeta h \quad \text{in } \mathbb{R}^N \tag{3.2}$$

$$\begin{cases} \Delta\psi + p(1-\zeta)w_\lambda^{p-1}\psi = -2\nabla\eta\nabla\varphi - \varphi\Delta\eta + (1-\zeta)h & \text{in } \mathbb{R}^N\setminus\overline{\mathcal{D}} \\ \psi = 0 \quad \text{on } \partial\mathcal{D} \\ \psi(x) \to 0 \quad |x| \to +\infty. \end{cases} \tag{3.3}$$

We assume $\|h\|_{**,\lambda} < \infty$. Let us consider the Banach space X consisting of functions φ such that $\|\varphi\|_{*,\lambda} < \infty$ and that are Lipschitz on $E = B_{R_1}\setminus B_{R_0}$ equipped with the norm

$$\|\varphi\|_X = \|\varphi\|_{*,\lambda} + \|\nabla\varphi\|_{L^\infty(E)}.$$

Given $\varphi \in X$ we solve first (3.3) and denote by $\psi(\varphi, h)$ the solution, which is clearly linear in its argument. Then note that $\zeta\psi$ is well defined in $\mathbb{R}^N$ and that $|\psi| \le \frac{C}{|x|^{N-2}}$ for large $|x|$ so hence the right hand side of (3.2) has a finite $\|\ \|_{**,\lambda}$ norm. We obtain a solution to the system, which defines a linear operator in h, if we solve the fixed point problem

$$\varphi = T_\lambda\big(-p\zeta w_\lambda^{p-1}\psi(\varphi, h) + \zeta h\big) \equiv F(\varphi).$$

where T_λ is the operator constructed in Theorem 3.

By Theorem 3 we have the estimate

$$\|F(\varphi)\|_{*,\lambda} \le C\| - p\zeta w_\lambda^{p-1}\psi + \zeta h\|_{**,\lambda} \le C\big(\|\zeta w_\lambda^{p-1}\psi\|_{**,\lambda} + \|h\|_{**,\lambda}\big). \tag{3.4}$$

But

$$\|\zeta w_\lambda^{p-1}\psi\|_{**,\lambda} = \lambda^\sigma \sup_{R_1 \le |x| \le \frac{1}{\lambda}} \big(|x|^{2+\sigma} w_\lambda(x)^{p-1}|\psi(x)|\big) + \lambda^{\frac{2}{p-1}} \sup_{|x| \ge \frac{1}{\lambda}} \big(|x|^{2+\sigma+\frac{2}{p-1}} w_\lambda(x)^{p-1}|\psi(x)|\big).$$

Using equation (3.3) and the fact that $w_\lambda(x) \to 0$ uniformly on compact sets we have

$$|\psi(x)| \le \frac{C}{|x|^{N-2}+1}\big(\|\varphi\|_X + \|h\|_{L^\infty(B_{R_1+1})}\big)$$
$$\le \frac{C}{|x|^{N-2}+1}(\|\varphi\|_X + \|h\|_{**,\lambda}) \quad \text{for all } x \in D. \tag{3.5}$$

By the definition of w_λ

$$\sup_{R_1 \le |x| \le \frac{1}{\lambda}} w_\lambda(x) = \lambda^{\frac{2}{p-1}} \sup_{|x|\le\frac{1}{\lambda}} w(\lambda x) \le C\lambda^{\frac{2}{p-1}}. \tag{3.6}$$

Hence, by (3.5), (3.6)

$$\lambda^\sigma \sup_{|x|\le\frac{1}{\lambda}} \big(|x|^{2+\sigma} w_\lambda(x)^{p-1}|\psi(x)|\big) \le C\lambda^{2+\sigma}(\|\varphi\|_X + \|h\|_{**,\lambda}) \sup_{|x|\le\frac{1}{\lambda}} |x|^{4+\sigma-N}.$$

If $4+\sigma-N \ge 0$ (so $N=3$ or $N=4$) the supremum on the right hand side is bounded by $\lambda^{N-4-\sigma}$. On the other hand, if $4+\sigma-N<0$ the supremum is bounded by a constant. Thus

$$\lambda^\sigma \sup_{|x|\le\frac{1}{\lambda}} \big(|x|^{2+\sigma} w_\lambda(x)^{p-1}|\psi(x)|\big) \le C\lambda^\gamma(\|\varphi\|_X + \|h\|_{**,\lambda}) \tag{3.7}$$

where $\gamma = \min(2+\sigma, N-2)$.

On the other hand $w_\lambda(x) \le C(1+|x|)^{-\frac{2}{p-1}}$ for all $x \in \mathbb{R}^N$ with C independent of λ and hence using (3.5)

$$\lambda^{\frac{2}{p-1}} \sup_{|x|>\frac{1}{\lambda}} \Big(|x|^{2+\frac{2}{p-1}} w_\lambda(x)^{p-1}|\psi(x)|\Big) \le C\lambda^{N-2}(\|\varphi\|_X + \|h\|_{**,\lambda}). \tag{3.8}$$

Combining (3.7) and (3.8) we find

$$\|\zeta w_\lambda^{p-1}\psi\|_{**,\lambda} \le C\lambda^\gamma(\|\varphi\|_X + \|h\|_{**,\lambda})$$

and this together with (3.4) yields

$$\|F(\varphi)\|_{*,\lambda} \le C(\lambda^\gamma\|\varphi\|_X + \|h\|_{**,\lambda}). \tag{3.9}$$

By standard elliptic estimates and using (3.5)

$$\|\nabla F(\varphi)\|_{L^\infty(E)} \le C(\|F(\varphi)\|_{*,\lambda} + \|h\|_{**,\lambda} + \lambda^\gamma(\|\varphi\|_X + \|h\|_{**,\lambda})).$$

This and (3.9) imply

$$\|F(\varphi)\|_X \le C(\lambda^\gamma\|\varphi\|_X + \|h\|_{**,\lambda}).$$

Thus for λ small F defines a contraction mapping of the region $\{\varphi \in X \mid \|\varphi\|_X \le 2C\|h\|_{**,\lambda}\}$. This fixed point inherits the solution with the required properties.

In order to prove part (b) one argues exactly as before, except for stating the fixed point problem in the class of function respecting the symmetry with respect to the N axes and using Remark 2.1. □

4. Solving the Nonlinear Problem

Let $\eta \in C^\infty(\mathbb{R}^N)$, $0 \le \eta \le 1$, $\eta(x) = 0$ for $|x| \le R$, $\eta(x) = 1$ for $|x| \ge R+1$ and $R > 0$ is fixed. Then $u = \eta w_\lambda + \phi$ is a solution of (1.3)–(1.4) if ϕ solves

$$\begin{cases} \Delta\phi + p w_\lambda^{p-1}\phi = N(\phi) + E & \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}} \\ \phi = 0 & \text{on } \partial\mathcal{D} \\ \phi(x) \to 0 & \text{as } |x| \to +\infty \end{cases} \tag{4.1}$$

where

$$N(\phi) = -(\eta w_\lambda + \phi)^p + (\eta w_\lambda)^p + p(\eta w_\lambda)^{p-1}\phi + p(1 - \eta^{p-1}) w_\lambda^{p-1}\phi,$$

and

$$E = -\Delta(\eta w_\lambda) - (\eta w_\lambda)^p.$$

By Proposition 3.1 we thus have a solution to (4.1) if ϕ solves the fixed point problem

$$\phi = \mathcal{T}_\lambda(N(\phi) + E). \tag{4.2}$$

Let us estimate $\|N(\phi)\|_{**,\lambda}$. We observe that

$$\begin{aligned} \|p(1-\eta^{p-1}) w_\lambda^{p-1}\phi\|_{**,\lambda} &= C \sup_{|x| \le R+1} |x|^{2+\sigma} w_\lambda(x)^{p-1} |\phi(x)| \\ &\le C\lambda^2 \|\phi\|_{*,\lambda}. \end{aligned} \tag{4.3}$$

Let us now define

$$N_1(\phi) = N(\phi) - p(1 - \eta^{p-1}) w_\lambda^{p-1}\phi. \tag{4.4}$$

To estimate this quantity observe that

$$|N_1(\phi)| \le C\big(w_\lambda^{p-2}\phi^2 + |\phi|^p\big).$$

Let us work with $0 < \sigma < \min(2, 2/(p-1))$. Since

$$|\phi(x)| \le C\lambda^{-\frac{2}{p-1}} |x|^{-\frac{2}{p-1}} \|\phi\|_{*,\lambda}, \quad \text{for all } |x| \ge \frac{1}{\lambda}$$

and

$$w_\lambda(x) \le C(1 + |x|)^{-\frac{2}{p-1}} \quad \text{for all } x \in \mathbb{R}^N,$$

we have on one hand

$$\lambda^{\frac{2}{p-1}} \sup_{|x|\geq\frac{1}{\lambda}} |x|^{2+\frac{2}{p-1}} w_\lambda(x)^{p-2}|\phi(x)|^2 \leq C\lambda^{-\frac{2}{p-1}}\|\phi\|^2_{*,\lambda}. \tag{4.5}$$

On the other hand,

$$|\phi(x)| \leq C\lambda^{-\sigma}|x|^{-\sigma}\|\phi\|_{*,\lambda} \quad \text{for all } |x| \leq \frac{1}{\lambda}$$

and therefore, using now that

$$w_\lambda(x) \leq C\lambda^{\frac{2}{p-1}} \quad \text{for all } |x| \leq \frac{1}{\lambda}$$

we obtain

$$\begin{aligned}\lambda^{\sigma} \sup_{|x|\leq\frac{1}{\lambda}} |x|^{2+\sigma} w_\lambda(x)^{p-2}|\phi(x)|^2 &\leq C\lambda^{\frac{2(p-2)}{p-1}-\sigma}\|\phi\|^2_{*,\lambda} \sup_{|x|\leq\frac{1}{\lambda}}|x|^{2-\sigma} \\ &= C\lambda^{-\frac{2}{p-1}}\|\phi\|^2_{*,\lambda}. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6) it follows that

$$\left\| w_\lambda^{p-2}\phi^2 \right\|_{**,\lambda} \leq C\lambda^{-\frac{2}{p-1}}\|\phi\|^2_{*,\lambda}. \tag{4.7}$$

To estimate $\| \, |\phi|^p\|_{**,\lambda}$ we compute

$$\lambda^{\sigma} \sup_{|x|\leq\frac{1}{\lambda}} |x|^{2+\sigma}|\phi(x)|^p \leq C\lambda^{-2}\|\phi\|^p_{*,\lambda}. \tag{4.8}$$

Similarly

$$\lambda^{\frac{2}{p-1}} \sup_{|x|\geq\frac{1}{\lambda}} |x|^{2+\frac{2}{p-1}}|\phi(x)|^p \leq C\lambda^{-2}\|\phi\|^p_{*,\lambda}. \tag{4.9}$$

From (4.8) and (4.9) it follows that

$$\| \, |\phi|^p\|_{**,\lambda} \leq C\lambda^{-2}\|\phi\|^p_{*,\lambda}, \tag{4.10}$$

By (4.7) and (4.10) we have

$$\|N_1(\phi)\|_{**,\lambda} \leq C\Big(\lambda^{-\frac{2}{p-1}}\|\phi\|^2_{*,\lambda} + \lambda^{-2}\|\phi\|^p_{*,\lambda}\Big). \tag{4.11}$$

Thus from (4.3), (4.11) for any $p > 1$,

$$\|N(\phi)\|_{**,\lambda} \leq C\Big(\lambda^2\|\phi\|_{*,\lambda} + \lambda^{-\frac{2}{p-1}}\|\phi\|^2_{*,\lambda} + \lambda^{-2}\|\phi\|^p_{*,\lambda}\Big). \tag{4.12}$$

Next we estimate E. Explicitly we have

$$-E = \eta\Delta w_\lambda + 2\nabla\eta\nabla w_\lambda + w_\lambda\Delta\eta + \eta^p w_\lambda^p = \eta(\eta^{p-1}-1)w_\lambda^p + 2\nabla\eta\nabla w_\lambda + w_\lambda\Delta\eta$$

and we see that since E has compact support

$$\|E\|_{**,\lambda} \leq \lambda^{\sigma} \sup_{x\in \mathrm{supp}(E)} |x|^{2+\sigma}|E(x)| \leq C\lambda^{\frac{2}{p-1}+\sigma} \tag{4.13}$$

for some constant C independent of λ.

Proof of Theorem 1. We have already observed that $u = \eta w_{\lambda} + \phi$ is a solution of (1.3)–(1.4) if ϕ satisfies equation (4.2). Define $\phi_0 = \mathcal{T}_{\lambda}(E)$. From Proposition 3.1, part (a), and (4.13), we get $\|\phi_0\|_{*,\lambda} \leq C\lambda^{\frac{2}{p-1}+\sigma}$. Let us write $\phi = \phi_0 + \phi_1$. Then solving equation (4.2) is equivalent to solving the fix point problem $\phi_1 = \mathcal{T}_{\lambda}(N(\phi_0 + \phi_1))$. Consider the set

$$\mathcal{F} = \left\{\phi \in L^{\infty}(D)/\|\phi\|_{*,\lambda} \leq \rho\lambda^{\frac{2}{p-1}}\right\}$$

where $\rho > 0$ is going to be fixed independently of λ, and the operator

$$\mathcal{A}(\phi_1) = \mathcal{T}_{\lambda}(N(\phi_0 + \phi_1)).$$

We prove that $\mathcal{A}$ has a fixed point in $\mathcal{F}$. For $\phi_1 \in \mathcal{A}$ we have

$$\begin{aligned}\|\mathcal{A}(\phi_1)\|_{*,\lambda} &\leq C\|N(\phi_0 + \phi_1)\|_{**,\lambda} \\ &\leq C\big(\lambda^2\|\phi_0 + \phi_1\|_{*,\lambda} + \lambda^{-\frac{2}{p-1}}\|\phi_0 + \phi_1\|^2_{*,\lambda} + \lambda^{-2}\|\phi_0 + \phi_1\|^p_{*,\lambda}\big)\end{aligned}$$

by (4.12). Thus for small λ

$$\|\mathcal{A}(\phi_1)\|_{*,\lambda} \leq C\lambda^{\frac{2}{p-1}}(\rho\lambda^2 + \lambda^{2\sigma} + \lambda^{p\sigma} + \rho^2 + \rho^p) \leq \rho\lambda^{\frac{2}{p-1}}$$

if we fix $\rho > 0$ small and then let $\lambda \to 0$. Hence $\mathcal{A}(\mathcal{F}) \subset \mathcal{F}$ for small λ.

Now we show that $\mathcal{A}$ is a contraction mapping in $\mathcal{F}$. Let us take ϕ_1, ϕ_2 in $\mathcal{F}$. Then

$$\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_{*,\lambda} \leq C\|N(\phi_0 + \phi_1) - N(\phi_0 + \phi_2)\|_{**,\lambda}. \tag{4.14}$$

Write

$$N(\phi_0 + \phi_1) - N(\phi_0 + \phi_2) = D_{\bar{\phi}}N(\bar{\phi})(\phi_1 - \phi_2)$$

where $\bar{\phi}$ lies in the segment joining $\phi_0 + \phi_1$ and $\phi_0 + \phi_2$.

For $|x| \leq \frac{1}{\lambda}$,

$$\lambda^{\sigma}|x|^{2+\sigma}|N(\phi_0 + \phi_1) - N(\phi_0 + \phi_2)| \leq |x|^2|D_{\bar{\phi}}N(\bar{\phi})|\|\phi_1 - \phi_2\|_{*,\lambda}$$

while, for $|x| \geq \frac{1}{\lambda}$,

$$\lambda^{\frac{2}{p-1}}|x|^{2+\frac{2}{p-1}}|N(\phi_0 + \phi_1) - N(\phi_0 + \phi_2)| \leq |x|^2|D_{\bar{\phi}}N(\bar{\phi})|\|\phi_1 - \phi_2\|_{*,\lambda}.$$

Then we have

$$\|N(\phi_1) - N(\phi_2)\|_{**,\lambda} \le C \sup_x \big(|x|^2 |D_{\bar\phi} N(\bar\phi)|\big) \|\phi_1 - \phi_2\|_{*,\lambda}. \tag{4.15}$$

Directly from the definition of N, we compute

$$D_{\bar\phi} N(\bar\phi) = DN_1(\bar\phi) - p\big(1 - \eta^{p-1}\big)\omega_\lambda^{p-1} \tag{4.16}$$

where

$$DN_1(\bar\phi) = p\big[(\eta\omega_\lambda + \bar\phi)^{p-1} - (\eta\omega_\lambda)^{p-1}\big].$$

Thus

$$|DN_1(\bar\phi)| \le C\big(w_\lambda^{p-2}|\bar\phi| + |\bar\phi|^{p-1}\big).$$

For all x we have

$$|x|^2 w_\lambda^{p-2} |\bar\phi(x)| \le C\lambda^{-\frac{2}{p-1}} (\|\phi_1\|_{*,\lambda} + \|\phi_2\|_{*,\lambda}) \le C\rho. \tag{4.17}$$

Similarly, for all x

$$|x|^2 |\bar\phi(x)|^{p-1} \le C\lambda^{-2}\big(\|\phi_1\|_{*,\lambda}^{p-1} + \|\phi_2\|_{*,\lambda}^{p-1}\big) \le C\rho^{p-1}. \tag{4.18}$$

Estimates (4.16)–(4.18) show that

$$\sup_x \big(|x|^2 |D_{\bar\phi} N(\bar\phi)|\big) \le C\big(\rho + \rho^{p-1} + \lambda^2\big). \tag{4.19}$$

Gathering relations (4.14), (4.15) and (4.19) we conclude that $\mathscr{A}$ is a contraction mapping in $\mathscr{F}$, and hence a fixed point in this region indeed exists. This finishes the proof of the theorem. □

Proof of Theorem 2. Let us observe that $N(\phi)$ and E are even with respect to each of their coordinates if so is ϕ. One can then formulate the fixed point problem (4.2) restricting it to the space of functions respecting these symmetries, using the linear operator predicted by Proposition 3.1, part (b). The rest of the proof continues in identical way as that of Theorem 1. □

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