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On finite Morse index solutions of two equations with negative exponent

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We consider the following equations involving negative exponent:

$$\Delta u = |x|^{\alpha} u^{-p}, \quad u > 0 \text{ in } \Omega \subset \mathbb{R}^n,$$

$$\Delta u = u^{-p} - 1, \quad u > 0 \text{ in } \Omega \subset \mathbb{R}^n,$$

where $p > 0$. Under optimal conditions on the parameters $\alpha > -2$ and $p > 0$, we prove the non-existence of finite Morse index solution on exterior domains or near the origin. We also prove an optimal regularity result for solutions with finite Morse index and isolated rupture at 0.

1. Introduction

Recently, many authors have studied solutions with finite Morse index for elliptic equations. For example, Farina [3] classified all finite Morse index classical solutions of $-\Delta u = |u|^{p-1}u$ in $\mathbb{R}^n$ for $1 < p < p_{\mathrm{JL}}$, where p_{JL} is the Joseph–Lundgren exponent. Motivated by models arising in engineering and physics, such as micro-electromechanical systems or thin films, elliptic equations with nonlinearities of negative exponent (for example, $f(x)u^{-p}$, $p > 0$) have also received a large amount of research attention (see, for example, [1, 2, 4] and the references therein).

We improve some results in [1,4] using simple arguments that can also be applied to similar problems with negative exponent.

In [1, theorem 1.2], it was proved that there are no solutions with finite Morse index of

$$\Delta u = |x|^{\alpha} u^{-p}, \quad u > 0 \text{ in } \Omega = \mathbb{R}^n \setminus B(0, R), \tag{1.1}$$

for any $n \geqslant 2$, $\alpha > -2$, $p > p_c(\alpha^-)$ and $R > 0$. Here, $\alpha^- = \min(\alpha, 0)$ and $B(x, r)$ denotes the ball of radius $r > 0$ centred at x. Moreover, for any $\alpha > -2$, the exponent $p_c(\alpha)$ is given by

$$p_c(\alpha) = \begin{cases} \dfrac{\alpha + n - \sqrt{(\alpha+2)(\alpha+2n-2)}}{\alpha - n + 4 + \sqrt{(\alpha+2)(\alpha+2n-2)}} & \text{if } 2 \leqslant n < 10 + 4\alpha, \\ +\infty & \text{if } n \geqslant 10 + 4\alpha. \end{cases}$$

For simplicity, we always consider classical solutions, i.e. $u \in C^2$. Let us recall that the Morse index of a solution u to (1.1) is defined as the maximal dimension of all subspaces X of $C_c^1(\Omega)$ such that

$$\int_\Omega |\nabla\varphi|^2 \, dx - p \int_\Omega |x|^\alpha u^{-p-1}\varphi^2 \, dx < 0 \quad \text{for all } \varphi \in X \setminus \{0\}. \tag{1.2}$$

We say that u is a stable solution to (1.1) if the Morse index is just 0.

Returning to [1, theorem 1.2], it is well known that $u_0(x) = \Lambda|x|^{(2+\alpha)/(p+1)}$ with

$$\Lambda = \left[\frac{2+\alpha}{p+1}\left(n - 2 + \frac{2+\alpha}{p+1}\right)\right]^{-1/(p+1)}$$

is a stable solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$, if $\alpha > -2$ and $0 < p \leqslant p_c(\alpha)$. So the situation for $\alpha > 0$ and $p_c(\alpha) < p \leqslant p_c(0)$ was left open in [1]. Our first result gives an answer for this.

Theorem 1.1. *Assume $n \geqslant 2$, $\alpha > -2$, $p > p_c(\alpha)$ and $R > 0$. Then there is no solution of (1.1) with finite Morse index.*

Theorem 1.1 here completes theorems 1.1 and 1.2 of [1]. Using the same idea, we also obtain the optimal non-existence result for finite Morse solution of (1.1) near the origin.

Theorem 1.2. *Assume $n \geqslant 2$, $\alpha > -2$, $p > p_c(\alpha)$ and $R > 0$. Then there is no solution of $\Delta u = |x|^\alpha u^{-p}$, $u > 0$ in $B(0, R) \setminus \{0\}$ with finite Morse index that has an isolated rupture at* 0.

This optimal result completes theorem 1.3 of [1], again for the case $\alpha > 0$ and $p_c(\alpha) < p \leqslant p_c(0)$. The fact that the solution u has isolated rupture at the origin means that $\lim_{x\to 0} u(x) = 0$. We define the Morse index in the same way as for (1.1), just replacing Ω by $\{0 < |x| < R\}$.

As a corollary, we also obtain the following regularity result, which generalizes theorem 1.4 of [1].

Theorem 1.3. *Assume $n \geqslant 2$, $\alpha > -2$ and $p > 0$. If u is a classical solution of $\Delta u = |x|^\alpha u^{-p}$ in $B(0, R) \setminus \{0\}$ with finite Morse index and an isolated rupture at 0, then u is Hölder continuous at 0. More precisely, defining $u(0) = 0$, we have*

$$u \in C^{(2+\alpha)/(p+1)}(B(0, R)).$$

In [4], the authors considered the equation

$$\Delta u = u^{-p} - 1, \quad u > 0 \text{ in } \mathbb{R}^n \setminus B(0, R), \tag{1.3}$$

with $p > 0$. In particular, they proved that, when $p > p_c(0)$, no solution with finite Morse index of (1.3) exists. More precisely, $p > \max(p_c(0), (n-2)^2/8n)$ was required, but we can easily check that the maximum is just $p_c(0)$. Consequently, theorem 1.3 of [4] was significant only for $n < 10$, since $p_c(0) = \infty$ if $n \geqslant 10$.

Here we wish to point out that problem (1.3) is of a very different nature from (1.1). It was proved (see [5]) that any non-trivial radial solution to $\Delta u = u^{-p} - 1$ in $\mathbb{R}^n$ oscillates infinitely many times around the value 1 as $r \to \infty$ whenever $p > 0$ and $n \geqslant 2$. This suggests that all solutions to (1.3) have infinite Morse index, which is confirmed as follows.

THEOREM 1.4. *For any $n \geqslant 2$, $p > 0$ and $R > 0$, there are no solutions of (1.3) with finite Morse index.*

The notion of finite Morse index for a solution u of (1.3) is similar to (1.1). More precisely, it is required that there is only a finite-dimensional vector space $X \subset C_c^1(\mathbb{R}^n \setminus B(0,R))$ such that

$$\int_{\mathbb{R}^n \setminus B(0,R)} |\nabla\varphi|^2 \, \mathrm{d}x - p \int_{\mathbb{R}^n \setminus B(0,R)} u^{-p-1}\varphi^2 \, \mathrm{d}x < 0 \quad \text{for all } \varphi \in X \setminus \{0\}.$$

Theorem 1.4 here generalizes theorems 1.1 and 1.3 in [4].

In what follows, the symbol C or C_i, C' always means a generic positive constant.

2. Proof of theorem 1.1

Suppose that u, a solution with finite Morse index to (1.1) exists with some $R > 0$. So it is stable outside a compact set, from the stability and (1.1), it is known by proposition 1 of [1] that

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B(0,R_0)} |x|^\alpha u^{\gamma-p}\psi^{2m} \\ &\qquad \leqslant C \int_{\mathbb{R}^n \setminus B(0,R_0)} |x|^{(\gamma+1)\alpha/(p+1)} (|\nabla\psi|^2 + |\psi\Delta\psi|)^{(p-\gamma)/(p+1)} \end{aligned} \tag{2.1}$$

for all $\psi \in C_c^\infty(B(0,R_0)^c)$ verifying $|\psi| \leqslant 1$. Here, $m \geqslant \max(2, (p-\gamma)/(p+1))$, $\gamma \in (\gamma_p, -1]$ and

$$\gamma_p = -1 - 2p - 2\sqrt{p(p+1)},$$

the radius $R_0 > R$ is chosen such that the solution u is stable outside $B(0,R_0)$.

Let $|y| \geqslant 4R_0$ and $R_1 = \frac{1}{4}|y|$, as $B(y, 2R_1) \subset B(0,R_0)^c$, using (2.1) with standard cut-off function, we have

$$|y|^\alpha \int_{B(y,R_1)} u^{\gamma-p} \leqslant C|y|^{(\gamma+1)\alpha/(p+1)} R_1^{n-2((p-\gamma)/(p+1))}$$

hence

$$\int_{B(y,|y|/4)} u^{\gamma-p} \leqslant C|y|^{n-((2+\alpha)(p-\gamma)/(p+1))}, \quad \forall |y| \geqslant 4R_0, \ \gamma \in (\gamma_p, -1]. \tag{2.2}$$

We now write equation (1.1) in polar variables:

$$u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2} \Delta_{S^{n-1}} u = r^\alpha u^{-p}$$

and integrate on S^{n-1} to obtain

$$r^{1-n}(r^{n-1}\bar{u}')' = r^\alpha g(r),$$

where

$$\bar{u}(r) = \int_{S^{n-1}} u(r,\sigma)\,\mathrm{d}\sigma,$$
$$g(r) = \int_{S^{n-1}} u(r,\sigma)^{-p}\,\mathrm{d}\sigma.$$

Integration yields, for all $r > r_1 > R$,

$$\bar{u}(r) = \bar{u}(r_1) + r_1^{n-1}\bar{u}'(r_1) \int_{r_1}^{r} t^{1-n}\,\mathrm{d}t + \int_{r_1}^{r} t^{1-n} \int_{r_1}^{t} s^{n-1+\alpha} g(s)\,\mathrm{d}s\,\mathrm{d}t. \tag{2.3}$$

From (2.2) and the Hölder inequality, it holds that, for $|y| \geqslant 4R_0$,

$$\int_{B(y,|y|/4)} u^{-p} \leqslant C|y|^{n-((2+\alpha)p/(p+1))}.$$

Using a covering argument, this implies that

$$\int_{B(0,2r)\setminus B(0,r)} u^{-p} \leqslant Cr^{n-(2+\alpha)p/(p+1)} \quad \text{for } r \geqslant 4R_0, \tag{2.4}$$

or, equivalently,

$$\int_{r}^{2r} s^{n-1} g(s)\,\mathrm{d}s \leqslant Cr^{n-((2+\alpha)p/(p+1))}, \quad \forall r \geqslant 4R_0.$$

The dyadic decomposition of the interval $[4R_0, r)$ gives the following estimate:

$$\int_{4R_0}^{r} s^{n-1+\alpha} g(s)\,\mathrm{d}s \leqslant Cr^{n-((2+\alpha)p/(p+1))+\alpha} \quad \text{for all } r \geqslant 4R_0, \tag{2.5}$$

where we used

$$n - \frac{(2+\alpha)p}{p+1} + \alpha = n - 2 + \frac{2+\alpha}{p+1} > 0.$$

Note that, for $n \geqslant 2$,

$$\int_{r_1}^{r} t^{1-n}\,\mathrm{d}t = o(r^{(2+\alpha)/(p+1)}) \quad \text{as } r \to \infty.$$

Combining (2.5) with (2.3), we have

$$\bar{u}(r) \leqslant Cr^{(2+\alpha)/(p+1)} \quad \text{for all } r \geqslant 4R_0,$$

which leads to

$$\int_{B(y,|y|/4)} u \leqslant \int_{B(0,2|y|)\setminus B(0,|y|/2)} u = \int_{|y|/2}^{2|y|} s^{n-1}\bar{u}(s)\,\mathrm{d}s \leqslant C|y|^{n+(2+\alpha)/(p+1)}$$

for all $|y| \geqslant 8R_0$. Since u is subharmonic, we directly obtain

$$u(y) \leqslant C|y|^{(2+\alpha)/(p+1)}, \quad \forall |y| \geqslant 8R_0.$$

This implies, for any $\gamma < 0$, that

$$u^{\gamma-p}(y) \geqslant C_1|y|^{(2+\alpha)/(p+1)(\gamma-p)}, \quad \forall |y| \geqslant 8R_0. \tag{2.6}$$

where $C_1 > 0$ is a fixed constant depending on γ.

Furthermore, we know that (see [1]) the unique solution γ to

$$n + \alpha + \frac{2+\alpha}{p+1}(\gamma - p) = n + \frac{(\gamma+1)\alpha}{p+1} - 2\frac{p-\gamma}{p+1} = 0 \tag{2.7}$$

belongs to $(\gamma_p, -1]$ if and only if $p > p_c(\alpha)$. Let γ satisfy (2.7). Using (2.6), we deduce that

$$\int_{B(0,r)\setminus B(0,8R_0)} |x|^\alpha u^{\gamma-p} \geqslant C\int_{8R_0}^{r} \frac{\mathrm{d}s}{s} \geqslant C\ln r - C_2 \quad \text{for all } r \geqslant 8R_0. \tag{2.8}$$

However, (2.1) with an appropriate test function (see [1,3]) gives that, for γ verifying (2.7),

$$\begin{aligned}\int_{B(0,r)\setminus B(0,8R_0)} |x|^\alpha u^{\gamma-p} &\leqslant \int_{\mathbb{R}^n\setminus B(0,R_0)} |x|^\alpha u^{\gamma-p}\psi^{2m}\\ &\leqslant C_3(1 + r^{n+((\gamma+1)\alpha/(p+1))-(2(p-\gamma)/(p+1))})\\ &= 2C_3 < \infty \end{aligned} \tag{2.9}$$

with C_3 independent of r. The estimates (2.8) and (2.9) are clearly in contradiction and show that u cannot be stable outside any compact set.

3. Proof of theorems 1.2 and 1.3

The main idea is very similar to the previous proof, so we just show the essential arguments and omit some details. Suppose that $u > 0$ satisfying $\Delta u = |x|^\alpha u^{-p}$ in $B(0,R) \setminus \{0\}$ has finite Morse index and a rupture at the origin. Then there exists $R_0 > 0$ small such that u is stable in $B(0,4R_0) \setminus \{0\}$. We can claim

$$\begin{aligned}&\int_{\{0<|x|<4R_0\}} |x|^\alpha u^{\gamma-p}\psi^{2m}\\ &\qquad \leqslant C\int_{\{0<|x|<4R_0\}} |x|^{(\gamma+1)\alpha/(p+1)}(|\nabla\psi|^2 + |\psi\Delta\psi|)^{(p-\gamma)/(p+1)}\end{aligned} \tag{3.1}$$

for all $\psi \in C_c^\infty(B(0,4R_0) \setminus \{0\})$ verifying $|\psi| \leqslant 1$, $\gamma \in (\gamma_p, -1]$, where

$$\gamma_p = -1 - 2p - 2\sqrt{p(p+1)}$$

and m sufficiently large.

Taking a suitable cut-off function, it holds, by the estimates [1, (2.2) and (2.3)], that

$$\int_{\{r\leqslant|x|\leqslant 2R_0\}}|x|^{\alpha}u^{\gamma-p} \leqslant C(1+r^{n+\alpha+((2+\alpha)/(p+1))(\gamma-p)}),\quad \forall r\in(0,2R_0),\ \gamma\in(\gamma_p,-1], \tag{3.2}$$

and

$$\int_{B(y,|y|/4)}u^{\gamma-p}\leqslant C|y|^{n-(2+\alpha)(p-\gamma)/(p+1)},\quad \forall 0<|y|\leqslant 2R_0,\ \gamma\in(\gamma_p,-1]. \tag{3.3}$$

Define $\bar u$ and $g(r)$ as above. The estimate (3.3) associated to the covering argument gives

$$\int_r^{2r}s^{n-1}g(s)\,\mathrm{d}s=\int_{B(0,2r)\setminus B(0,r)}u^{-p}\leqslant Cr^{n-(2+\alpha)p/(p+1)}\quad\text{for } 0<r\leqslant R_0.$$

Using a dyadic decomposition of $(0,r)$, we have

$$\int_0^r s^{n-1+\alpha}g(s)\,\mathrm{d}s\leqslant Cr^{n-((2+\alpha)p/(p+1))+\alpha}\quad\text{for all } r\leqslant 2R_0. \tag{3.4}$$

As $r^{1-n}(r^{n-1}\bar u')'=r^{\alpha}g(r)\geqslant 0$, $r^{n-1}\bar u'(r)$ is non-decreasing in r, we claim that

$$\ell=\lim_{r\to 0}r^{n-1}\bar u'(r)=0. \tag{3.5}$$

Indeed, $\ell\in[-\infty,\infty)$ exists by monotonicity of $r^{n-1}\bar u'$. As

$$\lim_{s\to 0}\bar u(s)=0$$

by the rupture assumption on u, $\ell\neq 0$ will lead to a contradiction, since r^{1-n} is not integrable at 0 for $n\geqslant 2$.

Integrating $(r^{n-1}\bar u')'=r^{n-1+\alpha}g(r)$, by (3.4) and (3.5), it holds that

$$r^{n-1}\bar u'(r)\leqslant Cr^{n-((2+\alpha)p/(p+1))+\alpha}\quad\text{if } 0<r\leqslant 2R_0.$$

Then, for any $r>0$, we see that

$$\lim_{s\to 0}s^{n-1}\bar u'(s)\int_s^r t^{1-n}\,\mathrm{d}t=0\quad\text{because } 2+\alpha-\frac{(2+\alpha)p}{p+1}=\frac{2+\alpha}{p+1}>0.$$

Combining with $\lim_{s\to 0}\bar u(s)=0$ and (3.4), tending r_1 to 0 in (2.3), we get

$$\bar u(r)\leqslant Cr^{(2+\alpha)/(p+1)}\quad\text{for all } 0<r<2R_0.$$

Using the fact that u is subharmonic, we can then conclude that

$$u(y)\leqslant C|y|^{(2+\alpha)/(p+1)}\quad\text{for all } 0<|y|<R_0. \tag{3.6}$$

If $p>p_c(\alpha)$, fix $\gamma\in(\gamma_p,-1]$ and verify (2.7). The inequality (3.2) (tending r to 0) implies

$$\int_{\{0<|x|\leqslant 2R_0\}}|x|^{\alpha}u^{\gamma-p}=\int_0^{2R_0}s^{n-1+\alpha}g(s)\,\mathrm{d}s<\infty.$$

However, by estimate (3.6), we see that

$$\int_{\{0<|x|\leqslant 2R_0\}} |x|^\alpha u^{\gamma-p} = \int_0^{2R_0} s^{n-1+\alpha} g(s)\,\mathrm{d}s \geqslant C_1 \int_0^{2R_0} \frac{\mathrm{d}s}{s} = \infty,$$

which is absurd. So such a solution with the rupture at zero cannot exist whenever $p > p_c(\alpha)$. The proof of theorem 1.2 is complete.

Finally, theorem 1.3 is just a direct consequence of the estimate (3.6) to the finite Morse index solution with isolated rupture at 0, which is valid for any $p > 0$ and $\alpha > -2$.

4. Proof of theorem 1.4

We argue by contradiction. Suppose that a solution u with finite Morse index to (1.3) exists. Using a very similar argument for (2.1), it is showed that (see estimate (2.1) in [4])

$$\int_{\mathbb{R}^n\setminus B(0,R_0)} u^{\gamma-p}\psi^{2m} \leqslant C \int_{\mathbb{R}^n\setminus B(0,R_0)} (|\nabla\psi|^2 + |\psi\Delta\psi|)^{(p-\gamma)/(p+1)} \tag{4.1}$$

for all $\psi \in C_c^\infty(B(0,R_0)^c)$ satisfying $|\psi| \leqslant 1$, $\gamma \in (\gamma_p, -1]$. Here again,

$$\gamma_p = -1 - 2p - 2\sqrt{p(p+1)}$$

and $R_0 > R$ is chosen such that the solution u is stable outside $B(0,R_0)$. We can proceed as above to get the corresponding estimates of (2.1), (2.4) with $\alpha = 0$, that is,

$$\int_{B(0,2r)\setminus B(0,r)} u^{-p} \leqslant Cr^{n-2p/(p+1)} \quad \text{for } r \geqslant 4R_0.$$

As $n - (2p/(p+1)) > 0$ for $n \geqslant 2$ and $p > 0$, the dyadic decomposition argument leads to the following estimate:

$$\int_{B(0,r)\setminus B(0,4R_0)} u^{-p} \leqslant C(1 + r^{n-2p/(p+1)}) \leqslant C' r^{n-2p/(p+1)} \quad \text{for any } r \geqslant 4R_0. \tag{4.2}$$

On the other hand, integrating equation (1.3) over $B(0,r) \setminus B(0,4R_0)$, we have

$$\int_{\partial B(0,r)} \frac{\partial u}{\partial \nu}\,\mathrm{d}\sigma - C = \int_{B(0,r)\setminus B(0,4R_0)} (u^{-p} - 1), \quad \forall r \geqslant 4R_0.$$

Applying (4.2), it holds that

$$\int_{\partial B(0,r)} \frac{\partial u}{\partial \nu}\,\mathrm{d}\sigma \leqslant Cr^n(r^{-n} + r^{-2p/(p+1)} - C_4), \quad \forall r \geqslant 4R_0.$$

Define $\bar{u}$ as before. Then

$$\bar{u}'(s) = \int_{S^{n-1}} \frac{\partial u}{\partial r}(s,\sigma)\,\mathrm{d}\sigma = r^{1-n}\int_{\partial B(0,r)} \frac{\partial u}{\partial \nu}\,\mathrm{d}\sigma.$$

Combining the above two formulae, for sufficiently large s,

$$\bar{u}'(s) \leqslant Cs(s^{-n} + s^{-2p/(p+1)} - C_4) \leqslant -C_5 s,$$

which then implies $\lim_{r\to\infty}\bar{u}(r) = -\infty$. This is impossible since $\bar{u} > 0$.

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References

1 Y. Du and Z. Guo. Positive solutions of an elliptic equation with negative exponent: stability and critical power. *J. Diff. Eqns* **246** (2009), 2387–2414.

2 P. Esposito, N. Ghoussoub and Y. Guo. *Mathematical analysis of partial differential equations modeling electrostatic MEMS*, Courant Lect. Not. Math., vol. 20 (Providence, RI: American Mathematical Society, 2010).

3 A. Farina. On the classification of solutions of the Lane–Emden equation on unbounded domains of $\mathbb{R}^N$. *J. Math. Pures Appl.* **87** (2007), 537–561.

4 Z. Guo and L. Ma. Finite Morse index steady states of van der Waals force driven thin film equations. *J. Math. Analysis Applic.* **368** (2010), 559–572.

5 Z. Guo, D. Ye and F. Zhou. Existence of singular positive solutions for some semilinear elliptic equations. *Pac. J. Math.* **236** (2008), 57–71.

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