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Comparison results for PDEs with a singular potential

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Several comparison results are obtained for solutions to linear elliptic and parabolic equations with a singular potential. Solutions to these equations are singular in many cases, and our results roughly say that they all have comparable singularities, provided that they belong to an appropriate space. We formulate the hypothesis on the potential in terms of an inequality, which in the case of the well-known inverse-square potential, is a consequence of an improvement of Hardy's inequality due to Vázquez and Zuazua.

1. Introduction

Here we consider comparison results for linear elliptic and parabolic equations with singular potentials. Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain and let $a \in L^1_{\mathrm{loc}}(\Omega)$, $a \geqslant 0$. To motivate the discussion, assume initially that $a(x)$ is smooth and bounded, and suppose that

$$\lambda_1 = \inf_{\varphi \in C^1_c(\Omega)} \frac{\int_\Omega (|\nabla\varphi|^2 - a(x)\varphi^2)}{\int_\Omega \varphi^2} > 0, \tag{1.1}$$

i.e. the first eigenvalue for the problem

$$\left.\begin{aligned} -\Delta\varphi_1 - a(x)\varphi_1 &= \lambda_1\varphi_1 \quad \text{in } \Omega, \\ \varphi_1 &= 0 \qquad\quad \text{on } \partial\Omega \end{aligned}\right\} \tag{1.2}$$

is positive. Since a is smooth, it is well known that

$$C^{-1}\zeta_0 \leqslant \varphi_1 \leqslant C\zeta_0 \tag{1.3}$$

for some positive constant C, where ζ_0 is the solution of

$$\left.\begin{aligned} -\Delta\zeta_0 - a(x)\zeta_0 = 1 \quad &\text{in } \Omega, \\ \zeta_0 = 0 \quad &\text{on } \partial\Omega. \end{aligned}\right\} \tag{1.4}$$

Note that this problem is well posed and that $\zeta_0 > 0$, since $\lambda_1 > 0$.

We can formulate condition (1.1) without any assumption on the smoothness of a. An interesting example is the so-called inverse-square potential

$$a(x) = \frac{c}{|x|^2}, \tag{1.5}$$

where $n \geqslant 3$ and $0 < c \leqslant \frac{1}{4}(n-2)^2$. An improved version of Hardy's inequality (see [7,18]) shows that it satisfies (1.1). On the other hand, it just fails to belong to $L^{n/2}(\Omega)$ if $0 \in \Omega$, and therefore the standard elliptic regularity theory is not sufficient to conclude an estimate like (1.3). In fact, for this potential, there exists a constant $\alpha > 0$, more precisely,

$$\alpha = \tfrac{1}{2}(n-2) - \sqrt{\tfrac{1}{4}(n-2)^2 - c},$$

such that ζ_0 and φ_1 behave like $|x|^{-\alpha}$ near the origin (see [11]), so that (1.3) can be interpreted as 'φ_1 cannot have worse singularities than ζ_0, and vice versa'.

In this paper we prove (1.3) under a slightly stronger condition than (1.1).

We also want to extend the following version of the strong maximum principle for the heat equation (see, for example, [5,15]). Let $T > 0$ and $u = u(x,t) \geqslant 0$ be a solution of

$$\begin{aligned} u_t - \Delta u = 0 \quad &\text{in } \Omega \times (0,T), \\ u = 0 \quad &\text{on } \partial\Omega \times (0,T). \end{aligned}$$

Then either $u \equiv 0$ or

$$u(x,t) \geqslant c(t)\delta(x), \tag{1.6}$$

where c is a positive function of $t \in (0,T)$ and $\delta(x) = \operatorname{dist}(x, \partial\Omega)$.

Using Hopf's boundary lemma on one hand, and elliptic regularity on the other, observe that, for some $C > 0$,

$$C^{-1}\delta \leqslant \tilde{\zeta}_0 \leqslant C\delta,$$

where $\tilde{\zeta}_0$ is the solution of

$$\begin{aligned} -\Delta\tilde{\zeta}_0 = 1 \quad &\text{in } \Omega, \\ \tilde{\zeta}_0 = 0 \quad &\text{on } \partial\Omega. \end{aligned}$$

Hence (1.6) is equivalent to

$$u(x,t) \geqslant c(t)\tilde{\zeta}_0(x). \tag{1.7}$$

We would like to extend (1.7) to the case where $\tilde{\zeta}_0$ is replaced by ζ_0 solving (1.4) and $u > 0$ solves

$$\left.\begin{aligned} u_t - \Delta u - a(x)u = 0 \quad &\text{in } \Omega \times (0,T), \\ u = 0 \quad &\text{on } \partial\Omega \times (0,T). \end{aligned}\right\} \tag{1.8}$$

Inequality (1.7) was already proven for the inverse-square potential in [1] and the authors mentioned (see remark 7.1 in [1]) that their methods applied to potentials of the form $a(x) = -\Delta\phi/\phi$, where ϕ satisfies a certain weighted Sobolev inequality. In our proof, we derive a similar Sobolev inequality (see lemma 4.1) under an almost optimal assumption on the potential $a(x)$ (see (2.1)). As in [1], we also make use of Moser iteration-type arguments, but our approach is, we believe, simpler.

The comparison results obtained in this paper are motivated by, and apply to, some semilinear parabolic equations studied in [12]. As we shall see, they also generalize to problems involving other boundary conditions and complement the results obtained in [10].

2. Main results

The assumption on the potential $a(x)$ is the following: $a \in L^1_{\mathrm{loc}}(\Omega)$, $a \geqslant 0$ and there exists $r > 2$ such that

$$\gamma(a) := \inf_{\varphi \in C^1_c(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 - \int_\Omega a(x)\varphi^2}{(\int_\Omega |\varphi|^r)^{2/r}} > 0. \tag{2.1}$$

Remark 2.1. Observe that if a satisfies equation (1.1), then, for any small $\epsilon > 0$, $a_\epsilon := (1-\epsilon)a$ satisfies (2.1) with $r = 2^* = 2n/(n-2)$ (when $n = 2$, pick any $r \in (2, \infty)$), by Sobolev's embedding. In particular, equation (1.1) can be seen as a limiting case of (2.1).

We also observe that if $n \geqslant 3$, the inverse square potential (1.5) satisfies (2.1), with $r = 2^*$ if $0 \leqslant c < \frac{1}{4}(n-2)^2$ and with any $2 < r < 2^*$ for $c = \frac{1}{4}(n-2)^2$ (see [7, 18]).

Before stating our results, we clarify in what sense we consider the solutions to (1.2) and (1.4). This is necessary because, in the context of weak solutions, or solutions in the sense of distributions, uniqueness may not hold in general, and (1.3) can fail. For example, in the case of the inverse square potential (1.5), when Ω is the unit ball $B_1(0)$ and $0 < c < \frac{1}{4}(n-2)^2$, $n \geqslant 3$, there is a positive solution u to

$$\left.\begin{aligned} -\Delta u - \frac{c}{|x|^2}u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.2}$$

which is smooth except at the origin and belongs to $W^{1,1}(\Omega)$. This shows that uniqueness, in general, does not hold.

Furthermore, there exists a solution ζ_0 of (1.4), smooth in $\Omega \setminus \{0\}$, behaving like $|x|^{-\alpha'}$ near the origin, where

$$\alpha' = \tfrac{1}{2}(n-2) + \sqrt{\tfrac{1}{4}(n-2)^2 - c},$$

and a solution φ_1 of (1.2), which behaves like $|x|^{-\alpha}$ where

$$\alpha = \tfrac{1}{2}(n-2) - \sqrt{\tfrac{1}{4}(n-2)^2 - c} < \alpha'.$$

But then (1.3) would fail. For details, see [11].

Hence we only consider solutions that belong to the Hilbert space H, defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_H^2 = \int_\Omega |\nabla u|^2 - \int_\Omega a(x)u^2. \tag{2.3}$$

This norm comes from an inner product $(\cdot|\cdot)_H$ in H, and, with some abuse of notation, we can write

$$(u|v)_H = \int_\Omega \nabla u \cdot \nabla v - \int_\Omega a(x)uv.$$

We denote by H^* the dual of H. Observe that $H_0^1(\Omega) \subset H \subset L^2(\Omega)$, and therefore $L^2(\Omega) \subset H^* \subset H^{-1}(\Omega)$.

Definition 2.2. If $f \in H^*$, we say that $u \in H$ is an H-solution of

$$\left.\begin{aligned} -\Delta u - a(x)u = f \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega \end{aligned}\right\} \tag{2.4}$$

if

$$(u|v)_H = \langle f, v\rangle_{H^*,H}$$

for all $v \in H$. With the obvious abuse of notation, this is equivalent to

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Omega a(x)uv = \int_\Omega fv \quad \text{for all } v \in H.$$

From now on, we only deal with solutions in this sense, i.e. H-solutions.

Lemma 2.3. *Suppose (1.1) holds and let $f \in H^*$. Then there exists a unique H-solution u of (2.2). Furthermore,*

$$\|u\|_H = \|f\|_{H^*},$$

and if $f \geqslant 0$ in the sense of distributions, then $u \geqslant 0$ a.e.

For a proof, see [12].

We also have to mention how to obtain a first eigenfunction for the operator $-\Delta - a(x)$ with zero Dirichlet boundary data.

Lemma 2.4. *Suppose $a(x) \geqslant 0$ satisfies (2.1). Then H embeds compactly in $L^2(\Omega)$. In particular, the operator $L := -\Delta - a(x) : D(L) \subset L^2(\Omega) \to L^2(\Omega)$, where $D(L) = \{u \in H \mid -\Delta u - a(x)u \in L^2(\Omega)\}$ has a positive first eigenvalue*

$$\lambda_1 = \inf_{\varphi \in H\setminus\{0\}} \frac{\int_\Omega |\nabla\varphi|^2 - \int_\Omega a(x)\varphi^2}{\int_\Omega \varphi^2}.$$

This infimum is attained at a positive $\varphi_1 \in H$ that satisfies (1.2). Moreover, λ_1 is a simple eigenvalue for $-\Delta - a(x)$ and, if φ is a non-negative non-trivial H-solution of

$$\begin{aligned} -\Delta\varphi - a(x)\varphi = \lambda\varphi \quad &\text{in } \Omega, \\ \varphi = 0 \quad &\text{on } \partial\Omega \end{aligned}$$

for some $\lambda \in \mathbb{R}$, then $\lambda = \lambda_1$.

Similarly, we can define H-solutions of the evolution equation (1.8) with initial condition $u(0) = u_0 \in L^2(\Omega)$.

Definition 2.5. The operator L defined in lemma 2.4 is a bounded-below self-adjoint operator with dense domain and generates an analytic semigroup $(S(t))_{t\geqslant 0}$ in L^2.

Hence, for $u_0 \in L^2(\Omega)$, there exists a unique

$$u := S(t)u_0 \in C([0,\infty), L^2) \cap C^1((0,\infty), L^2) \cap C((0,\infty); H)$$

solving

$$\begin{aligned} u_t + Lu &= 0 \quad \text{for } t > 0, \\ u(0) &= u_0, \end{aligned}$$

which we call the H-solution (or simply the solution) of (1.8) with initial condition $u(0) = u_0 \in L^2(\Omega)$.

The main results of this paper are the following.

Theorem 2.6. *Assume $a : \Omega \to [0,\infty)$ satisfies (2.1). Let $\varphi_1 > 0$ denote the first eigenfunction for the operator $-\Delta - a(x)$ with zero Dirichlet boundary condition, normalized by $\|\varphi_1\|_{L^2(\Omega)} = 1$, and ζ_0 denote the solution of (1.4). Then there exists $C = C(\Omega, \gamma(a), r) > 0$ such that*

$$C^{-1}\zeta_0 \leqslant \varphi_1 \leqslant C\zeta_0.$$

Theorem 2.7. *Assume that $a : \Omega \to [0,\infty)$ satisfies (2.1). Let $u_0 \in L^2(\Omega)$, $u_0 \geqslant 0$, $u_0 \not\equiv 0$, and let u denote the solution of (1.8) with initial condition u_0. Let ζ_0 denote again the solution of (1.4). Then*

$$u(t) \geqslant c(t)\zeta_0$$

for some $c(t) > 0$ depending on u_0, Ω, $\gamma(a)$, r and t.

Corollary 2.8. *Under the assumptions of theorem 2.7, we have, more precisely,*

$$u(t) \geqslant c(t)\biggl(\int_\Omega u_0\zeta_0\biggr)\zeta_0,$$

where one can choose $c(t) = \mathrm{e}^{-K(t+1/t)}$ for some $K = K(\Omega, \gamma(a), r) > 0$.

Corollary 2.9. *Assume $a : \Omega \to [0,\infty)$ satisfies (2.1) and let u solve (2.4) for some $f \geqslant 0$. Then*

$$u \geqslant c\biggl(\int_\Omega f\zeta_0\biggr)\zeta_0,$$

where $c = c(\Omega, \gamma(a), r)$.

Remark 2.10. All of the previous results still hold for a sign-changing potential $a(x)$ under the following additional hypothesis.

Theorem 2.11. *Suppose that $a : \Omega \to \mathbb{R}$ satisfies (2.1) and that*

$$\left.\begin{aligned} a(x) = a^+(x) - a^-(x), \quad a^+, a^- \geqslant 0, \\ a^+ \in L^1_{\mathrm{loc}}(\Omega) \quad \text{and} \quad a^-(x) \in L^\infty(\Omega). \end{aligned}\right\} \tag{2.5}$$

Then theorems 2.6 and 2.7 and corollaries 2.8 and 2.9 still hold if the constants are allowed to also depend on a^-.

Remark 2.12. Theorem 2.7 and corollary 2.8 can be extended under an even less restrictive hypothesis. Suppose that, for some $M = M(a) > 0$, $\gamma = \gamma(a) > 0$ and $r > 2$,

$$\gamma\left(\int_\Omega |\varphi|^r\right)^{2/r} \leqslant \int_\Omega (|\nabla\varphi|^2 - a(x)\varphi^2 + M\varphi^2) \tag{2.6}$$

for all $\varphi \in C_c^\infty(\Omega)$, and define H in this case as the completion of $C_c^\infty(\Omega)$ under the norm

$$\|u\|_H^2 = \int_\Omega (|\nabla u|^2 - a(x)u^2 + Mu^2).$$

Theorem 2.13. *Suppose that $a(x)$ satisfies conditions (2.5) and (2.6). Let $u_0 \in L^2(\Omega)$, $u_0 \geqslant 0$, $u_0 \not\equiv 0$, and let u denote the H-solution of (1.8) with initial condition u_0. Then*

$$u(t) \geqslant c(t)\left(\int_\Omega u_0\varphi_1\right)\varphi_1,$$

where one can choose $c(t) = \mathrm{e}^{-K(t+1/t)}$ for some K depending on Ω, $\gamma(a)$, r, M and p, and where $0 < \varphi_1 \in H$ is the first eigenfunction for $-\Delta - a(x)$ normalized by $\|\varphi_1\|_{L^2} = 1$.

In § 7 we outline the proofs of these theorems and mention some examples of potentials satisfying (2.6) for which the stronger condition (2.1) may fail.

Observe that condition (2.6) implies the more standard inequality

$$\inf_{\varphi \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 - a(x)\varphi^2}{\int_\Omega \varphi^2} > -\infty,$$

which is a necessary condition for the existence of global non-negative solutions with exponential growth to the linear parabolic equation (1.8) (see [8]).

Remark 2.14. The method presented here for the parabolic problem also applies to equations with mixed boundary condition, extending a result of [10] to the parabolic case. Let Γ_1, Γ_2 be a partition of $\partial\Omega$, with $\Gamma_1 \neq \emptyset$. For simplicity, we can assume that Γ_1, Γ_2 are smooth, but this is not important.

In this context, let $\bar\zeta$ denote the solution of

$$\begin{aligned} -\Delta\bar\zeta &= 1 \quad \text{in } \Omega, \\ \bar\zeta &= 0 \quad \text{on } \Gamma_1, \\ \frac{\partial\bar\zeta}{\partial\nu} &= 0 \quad \text{on } \Gamma_2, \end{aligned}$$

where ν denotes the unit outward normal vector to $\partial\Omega$.

THEOREM 2.15. *Let $u_0 \in L^2(\Omega)$, $u_0 \geqslant 0$, and let u denote the solution to*

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } \Omega \times (0,\infty), \\ u &= 0 \quad \text{on } \Gamma_1 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_2 \times (0,\infty), \\ u(0) &= u_0 \quad \text{in } \Omega. \end{aligned}$$

Then

$$u(t) \geqslant c(t)\Big(\int_\Omega u_0 \bar{\zeta}\Big)\bar{\zeta},$$

where $c(t) = \mathrm{e}^{-K(t+1/t)}$ for some $K = K(\Omega, \Gamma_1, \Gamma_2)$.

We omit the proof, which is a slight modification of the one given for theorem 2.7.

3. Some preliminaries

We start this section with some preliminary results on the linear equation

$$\left.\begin{aligned} -\Delta u - a(x)u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.1}$$

when the potential $a(x)$ satisfies (2.1). As mentioned before, all solutions to (3.1) are assumed to be in H.

LEMMA 3.1. *Assume that $a(x)$ satisfies (2.1) and that $f \in L^2(\Omega)$. Then the solution u to (3.1) satisfies*

$$\int_\Omega u(-\Delta\zeta) = \int_\Omega a(x)u\zeta + \int_\Omega f\zeta \tag{3.2}$$

for all $\zeta \in C^2(\bar{\Omega})$, $\zeta = 0$ on $\partial\Omega$, and all integrals in (3.2) exist and are finite. In particular, by taking $\zeta = \tilde{\zeta}$ to be the solution of

$$\left.\begin{aligned} -\Delta\tilde{\zeta}_0 &= 1 \quad \text{in } \Omega, \\ \tilde{\zeta}_0 &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.3}$$

we conclude that $a(x)u + f \in L^1_{\mathrm{loc}}(\Omega)$.

Proof. By working with f^+, f^-, we can assume that $f \geqslant 0$. Let

$$a_k(x) = \min(a(x), k), \quad k > 0,$$

and let u_k be the solution to (3.1), with the potential $a(x)$ replaced by the potential $a_k(x)$. Then it is easy to check that u_k is non-decreasing in k and converges to u in $L^2(\Omega)$. Now take $\zeta \in C^2(\bar{\Omega})$, $\zeta = 0$ on $\partial\Omega$. Then

$$\int_\Omega u_k(-\Delta\zeta) = \int_\Omega a_k(x)u_k\zeta + f\zeta, \tag{3.4}$$

and note that here all the integrals are finite. By taking, in particular, $\zeta = \tilde{\zeta}_0$ (where $\tilde{\zeta}_0$ is the solution of (3.3)), and using Fatou's lemma, we see that $\int_\Omega a(x)u\tilde{\zeta}_0$

exists and is finite. Given any $\zeta \in C^2(\bar{\Omega})$, $\zeta = 0$ on $\partial\Omega$, we can find $C > 0$ so that $|\zeta| \leqslant C\tilde{\zeta}_0$. It follows that we can pass now to the limit in (3.4) and conclude that (3.2) holds. □

Lemma 3.2. *Assume that $a(x)$ satisfies (2.1) and let $T : L^2(\Omega) \to L^2(\Omega)$ be the operator defined by $Tf = u$, where u is the H-solution to (3.1) (i.e. $T = L^{-1}$, where L was defined in lemma 2.4). Then T is compact.*

Proof. Let (f_j) be a bounded sequence in $L^2(\Omega)$, and $u_j = Tf_j$. Then u_j is bounded in $L^r(\Omega)$ by (2.1). Let $\tilde{\zeta}_0$ be the solution to (3.3). Then, by (3.2), we have

$$\int_\Omega a(x) u_j \tilde{\zeta}_0 \leqslant \|\tilde{\zeta}_0\|_{L^\infty} \int_\Omega |f_j| + \|\tilde{\zeta}_0\|_{C^2} \int_\Omega |u_j|.$$

Therefore, $-\Delta u_j = a(x)u_j + f_j$ is bounded in $L^1_{\text{loc}}(\Omega)$ and, by the Gagliardo–Nirenberg inequality, u_j is bounded in $W^{1,1}_{\text{loc}}(\Omega)$. We conclude that, for a subsequence (denoted the same), $u_j \to u$ in $L^q(\Omega)$ for some fixed $1 \leqslant q < n/(n-1)$, and a.e. To conclude that u_j converges strongly in $L^2(\Omega)$, let $\varepsilon > 0$ be given. Then, by Egorov's theorem, there exists $E \subset \Omega$, measurable with $|E| \leqslant \varepsilon$, so that $u_j \to u$ uniformly in $\Omega \setminus E$. Hence

$$\begin{aligned}\limsup \int_\Omega |u_j - u|^2 &\leqslant \limsup \int_{\Omega\setminus E} |u_j - u|^2 + \limsup \int_E |u_j - u|^2 \\ &\leqslant \|u_j - u\|^2_{L^r} |E|^{1-2/r} \\ &\leqslant C\varepsilon^{1-2/r}\end{aligned}$$

by the uniform bound of u_j in $L^r(\Omega)$. □

To prove that the embedding $H \subset L^2(\Omega)$ is compact, we use the following result combined with the previous lemma.

Lemma 3.3. *Let $\mathcal{H}$, V be real Hilbert spaces and $J : \mathcal{H} \to V$ be a bounded linear map. Then J is compact if and only if JJ^* is compact.*

Proof. Clearly, if J is compact, then JJ^* is compact.

Let $\varepsilon > 0$. Then the map $S_\varepsilon := JJ^* + \varepsilon I : V \to V$ is self-adjoint and coercive, in the sense that $\|S_\varepsilon y\|_V \geqslant \varepsilon \|y\|_V$. It follows that S_ε is invertible. Therefore, given $x \in \mathcal{H}$, there is $y \in V$ such that

$$JJ^* y + \varepsilon y = Jx. \tag{3.5}$$

But

$$(J^* y \mid x)_{\mathcal{H}} \leqslant \tfrac{1}{2}\|x\|^2_{\mathcal{H}} + \tfrac{1}{2}\|J^* y\|^2_{\mathcal{H}},$$

and so

$$(J^* y \mid x - J^* y)_{\mathcal{H}} \leqslant \tfrac{1}{2}\|x\|^2_{\mathcal{H}} - \tfrac{1}{2}\|J^* y\|^2_{\mathcal{H}} \leqslant \tfrac{1}{2}\|x\|^2_{\mathcal{H}}. \tag{3.6}$$

From (3.5), it follows that

$$y = \epsilon^{-1}(Jx - JJ^* y).$$

Plugging this in (3.6), we obtain

$$(J^*(Jx - JJ^* y) \mid x - J^* y)_{\mathcal{H}} = \|JJ^* y - Jx\|^2_V \leqslant \tfrac{1}{2}\varepsilon \|x\|^2_{\mathcal{H}}. \tag{3.7}$$

Now assume that JJ^* is compact and let x_j be a bounded sequence in $\mathcal{H}$. Let $M = \sup_j \|x_j\|_{\mathcal{H}}$ and set $\varepsilon_k = 2^{-2k}$ for $k = 1, 2, \dots$. Take $k = 1$ and let $y_j = S_{\varepsilon_1}^{-1}(Jx_j)$. Then y_j is a bounded sequence and, since JJ^* is compact, there is a subsequence and some $z_1 \in V$ such that $JJ^* y_j \to z_1$. Therefore, using (3.7), we see that there is some j_1 such that $\|Jx_{j_1} - z_1\|_V \leqslant 2M\sqrt{\varepsilon_1} = 2M$. Using a diagonal argument, one can find a subsequence j_k and $z_k \in V$ such that $\|Jx_{j_l} - z_k\|_V \leqslant 2^{-k+1}M$ for all $l \geqslant k$. This implies that $\|z_{k+1} - z_k\|_V \leqslant 2^{-k+2}$, and therefore z_k is a Cauchy sequence in V. Thus z_k converges, and so Jx_{j_k} is also convergent. □

We are now in a position to prove lemma 2.4.

Proof of lemma 2.4. Take $V = L^2(\Omega)$, $\mathcal{H} = H$ and denote by $J : H \to L^2(\Omega)$ the canonical injection. We see that $T = JJ^*$, where $Tf = u$, and u is the H-solution to (3.1). By lemma 3.2, JJ^* is compact and hence, by lemma 3.3, the embedding $H \subset L^2(\Omega)$ is compact.

Since T is self-adjoint and compact, $L = T^{-1}$ has a smallest eigenvalue. This eigenvalue is simple, which can be proved in the same way as for smooth elliptic operators.

In fact, let $\varphi_1 \not\equiv 0$ be a non-negative minimizer of

$$\lambda_1 = \inf_{\varphi \in C_c^1(\Omega)} \frac{\int_\Omega (|\nabla\varphi|^2 - a(x)\varphi^2)}{\int_\Omega \varphi^2}, \tag{3.8}$$

which exists by the compactness of T. By the standard arguments of the calculus of variations, φ_1 satisfies (1.2) and, by a version of the strong maximum principle (see, for example [3]), $\varphi_1 > 0$. Now let φ denote another eigenfunction for λ_1. Then, for any $\mu \in \mathbb{R}$, we have that $\psi = \varphi_1 - \mu\varphi$ satisfies the equation

$$\left.\begin{aligned} -\Delta\psi - a(x)\psi &= \lambda_1\psi \quad \text{in } \Omega, \\ \psi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.9}$$

Now, because ψ satisfies (3.9), if $\psi \not\equiv 0$, then it minimizes (3.8). Then $|\psi|$ also minimizes (3.8), and therefore satisfies (3.9). Since

$$\begin{aligned} -\Delta|\psi| &= a(x)|\psi| + \lambda_1|\psi| \geqslant 0 && \text{in } \Omega, \\ |\psi| &= 0 && \text{on } \partial\Omega, \end{aligned}$$

by the strong maximum principle (see [3]), we conclude that if $\psi \not\equiv 0$, then $|\psi| > c\delta$ a.e. in Ω, where $c > 0$. This, combined with the fact that $\psi \in W^{1,1}_{\text{loc}}(\Omega)$ (by lemma 3.1), shows that either $\psi > 0$ or $\psi < 0$ in Ω (assuming Ω is connected (see, for example, [9])). That is, for any $\mu \in \mathbb{R}$, either $\varphi \geqslant \mu\varphi_1$ or $\varphi \leqslant \mu\varphi_1$. Setting $\mu_0 = \sup\{\mu : \varphi \geqslant \mu\varphi_1\}$, we see that $\varphi = \mu_0\varphi_1$. □

The last two lemmas of this section will allow us to reduce the proofs of the main results of this paper to the case of a bounded potential. Define

$$a_k = \min(a, k), \quad k > 0. \tag{3.10}$$

We denote by λ_1^k, φ_1^k, ζ_0^k the first eigenvalue, first eigenfunction and solution of (1.4) associated with the potential a_k, which are all defined in the usual sense, since a_k

is bounded. Let ζ_0 be the solution to (1.4) in the sense of lemma 2.3. Since a satisfies (2.1) (hence (1.1)), it is easy to check that $\zeta_0^k \to \zeta_0$ in $L^2(\Omega)$.

LEMMA 3.4. *Normalize* φ_1^k *by* $\|\varphi_1^k\|_{L^2(\Omega)} = 1$. *Then*

$$\lambda_1^k \to \lambda_1 \quad \text{and} \quad \varphi_1^k \to \varphi_1 \quad \text{in } H$$

as $k \to \infty$, *where* λ_1 *is given by (1.1) and* φ_1 *is given by lemma 2.4, normalized so that* $\|\varphi_1\|_{L^2(\Omega)} = 1$.

Proof. Observe that

$$\lambda_1^k = \inf_{\varphi \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 - \int_\Omega a_k(x)\varphi^2}{\int_\Omega \varphi^2} \tag{3.11}$$

is non-increasing as k increases. Therefore, the limit $\lim_{k\to\infty} \lambda_1^k$ exists. We claim that

$$\lim_{k\to\infty} \lambda_1^k = \lambda_1.$$

Indeed, note that $\lambda_1 \leqslant \lambda_1^k$ for all k, and also that, for any $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega a_k(x)\varphi^2 \to \int_\Omega a(x)\varphi^2, \tag{3.12}$$

by monotone convergence. Now take $\varphi \in C_c^\infty(\Omega)$, with $\|\varphi\|_{L^2} = 1$. Then

$$\lambda_1^k \leqslant \int_\Omega |\nabla\varphi|^2 - a_k(x)\varphi^2$$

and, using (3.12), we see that

$$\limsup \lambda_1^k \leqslant \int_\Omega |\nabla\varphi|^2 - a(x)\varphi^2.$$

Taking the infimum over φ, we obtain

$$\limsup \lambda_1^k \leqslant \lambda_1.$$

Recall that we normalize φ_1^k by $\|\varphi_1^k\|_{L^2} = 1$, and so

$$\int_\Omega |\nabla\varphi_1^k|^2 - \int_\Omega a_k(x)|\varphi_1^k|^2 = \lambda_1^k \to \lambda_1 \quad \text{as } k \to \infty. \tag{3.13}$$

In particular, φ_1^k is bounded in H and, by lemma 2.4, we can find a subsequence such that $\varphi_1^k \to \varphi_1$ in $L^2(\Omega)$. We observe that $\varphi_1 \geqslant 0$ and $\|\varphi_1\|_{L^2} = 1$.

CLAIM 3.5. φ_1 *minimizes*

$$\lambda_1 = \frac{\int_\Omega |\nabla\varphi|^2 - \int_\Omega a(x)\varphi^2}{\int_\Omega \varphi^2}.$$

Indeed, testing the equation for φ_1^k with $\varphi \in C_c^\infty(\Omega)$, $\varphi \geqslant 0$, we find

$$\int_\Omega \nabla\varphi_1^k \cdot \nabla\varphi - \int_\Omega a_k(x)\varphi_1^k\varphi = \lambda_1^k \int_\Omega \varphi_1^k\varphi,$$

and therefore

$$\int_\Omega \nabla\varphi_1^k \cdot \nabla\varphi - \int_\Omega a(x)\varphi_1^k\varphi \leqslant \lambda_1^k \int_\Omega \varphi_1^k\varphi.$$

Taking limits on both sides, we obtain

$$\int_\Omega \nabla\varphi_1 \cdot \nabla\varphi - \int_\Omega a(x)\varphi_1\varphi \leqslant \lambda_1 \int_\Omega \varphi_1\varphi.$$

By density, this is true for all $\varphi \in H$, $\varphi \geqslant 0$, and, taking $\varphi = \varphi_1$, we find that

$$\frac{\int_\Omega |\nabla\varphi_1|^2 - \int_\Omega a(x)\varphi_1^2}{\int_\Omega \varphi_1^2} \leqslant \lambda_1,$$

and the claim is proved.

Then the standard arguments of the calculus of variations show that φ_1 satisfies (1.2), and hence φ_1 is indeed the first eigenfunction of $-\Delta - a(x)$. The strong convergence $\varphi_1^k \to \varphi_1$ in H is a consequence of

$$\|\varphi\|_H = \lambda_1 \leqslant \|\varphi_1^k\|_H \leqslant \lambda_1^k.$$

The first inequality follows from the definition of λ_1 and the second from the fact that $a_k \leqslant a$. This implies that $\|\varphi_1^k\|_H \to \|\varphi\|_H$. ☐

Lemma 3.6. *It suffices to prove theorems 2.6 and 2.7 and corollaries 2.8 and 2.9 in the case where the potential $a(x)$ is bounded.*

Proof. We only give the argument for theorem 2.6, which can easily be carried out for the other results. Let $a \geqslant 0$ be any potential satisfying (2.1) and a_k its truncation defined by (3.10). Observe that

$$\inf_{\varphi \in C_c^1(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 - \int_\Omega a_k(x)\varphi^2}{(\int_\Omega |\varphi|^r)^{2/r}} \geqslant \gamma(a).$$

So, if theorem 2.6 holds for bounded potentials, we must have

$$C^{-1}\zeta_0^k \leqslant \varphi_1^k \leqslant C\zeta_0^k, \tag{3.14}$$

where ζ_0^k, φ_1^k were defined at the beginning of this section and $C = C(\Omega, \gamma(a)) > 0$ is independent of k. Since $\zeta_0^k \to \zeta_0$ in L^2 and lemma 3.4 holds, we can pass to the limit in (3.14). ☐

4. Proof of theorem 2.6

By lemma 3.6 in the previous section, it is enough to establish the result in the case that $a(x)$ is bounded.

The main idea is to consider the function

$$w = \frac{\varphi_1}{\zeta_0}$$

and notice that it satisfies (formally) an elliptic equation

$$\left.\begin{aligned} -\nabla\cdot(\zeta_0^2\nabla w) &= \lambda_1\varphi_1\zeta_0-\varphi_1 && \text{in } \Omega,\\ \zeta_0^2\nabla w\cdot\nu &= 0 && \text{on } \partial\Omega,\end{aligned}\right\} \tag{4.1}$$

where ν denotes the outer unit normal to the boundary $\partial\Omega$. Then we will use Moser's iteration argument, combined with a Sobolev inequality, to prove that w is bounded.

STEP 1. *Formal derivation of an iteration formula.* There exists $q > 2$ and $C > 0$ such that, for all $j \geqslant 1$,

$$\left(\int_\Omega \zeta_0^2 w^{qj}\right)^{2/q} \leqslant Cj\int_\Omega \zeta_0^2 w^{2j}. \tag{4.2}$$

Proof. Multiplying (4.1) by w^{2j-1}, where $j \geqslant 1$, and integrating by parts, we obtain

$$\frac{2j-1}{j^2}\int_\Omega \zeta_0^2|\nabla w^j|^2 = \int_\Omega(\lambda_1\varphi_1\zeta_0-\varphi_1)w^{2j-1} \leqslant \lambda_1\int_\Omega \zeta_0^2 w^{2j}. \tag{4.3}$$

Now we use the next lemma, which is a kind of Sobolev inequality.

LEMMA 4.1. *Assume u satisfies*

$$\left.\begin{aligned} -\Delta u - a(x)u &= c(x)u + f && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega,\end{aligned}\right\} \tag{4.4}$$

where $c, f \in L^\infty(\Omega)$, $f \geqslant 0$, $f \not\equiv 0$. Assume also that a satisfies (2.1). Then, for any $2 \leqslant q \leqslant r$, there is a constant $C > 0$ depending only Ω, r, $\gamma(a)$, $\|c\|_{L^\infty}$ and f such that

$$\left(\int_\Omega u^s|\varphi|^q\right)^{2/q} \leqslant C\int_\Omega u^2(|\nabla\varphi|^2+\varphi^2)$$

for all $\varphi \in C^1(\bar\Omega)$, where s is given by the relation

$$\frac{s}{r} = \frac{q-2}{r-2}. \tag{4.5}$$

(A proof of this lemma is given in step 4.)

We continue the proof of step 1. Taking $u = \zeta_0$, $f \equiv 1$, $c \equiv 0$ and $s = 2$, by lemma 4.1, there exist $q = 2(1+(r-2)/r) > 2$ and $C > 0$ such that

$$\left(\int_\Omega \zeta_0^2|\varphi|^q\right)^{2/q} \leqslant C\int_\Omega \zeta_0^2(|\nabla\varphi|^2+\varphi^2) \tag{4.6}$$

for all $\varphi \in C^1(\bar\Omega)$. This applied to $\varphi = w^j$ and combined with (4.3) yields (4.2). □

STEP 2. We have

$$\varphi_1 \leqslant C\zeta_0. \tag{4.7}$$

Proof. We iterate (4.2). Define $\mu = \frac{1}{2}q > 1$ and $j_k = 2\mu^k$, for $k = 0, 1, \dots$. Let

$$\theta_k = \left(\int_\Omega \zeta_0^2 w^{j_k}\right)^{1/j_k}.$$

Then (4.2) can be rewritten as

$$\theta_{k+1} \leqslant (C\mu^k)^{1/\mu^k}\theta_k.$$

Using this recursively yields

$$\theta_k \leqslant C\theta_0 = C\left(\int_\Omega \zeta_0^2\right)^{1/2} < \infty$$

for all $k = 0, 1, 2, \dots$, with C independent of k. But

$$\lim_{k\to\infty} \theta_k = \sup_\Omega w$$

(because $\zeta_0 > 0$ in Ω) and this shows that $w \leqslant C$. □

STEP 3. *Justification of step 1.* To be rigorous, we need to justify the derivation of (4.2), which has been formal only. One possible approach is the following.

Proof of (4.2). Consider the family of smooth domains

$$\Omega_\varepsilon = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, \Omega) < \varepsilon\},$$

where $\varepsilon > 0$ is small. Let ζ_0^ε be the solution to

$$\left.\begin{aligned} -\Delta\zeta_0^\varepsilon - a(x)\zeta_0^\varepsilon &= 1 \quad \text{in } \Omega_\varepsilon, \\ \zeta_0^\varepsilon &= 0 \quad \text{on } \partial\Omega_\varepsilon, \end{aligned}\right\} \tag{4.8}$$

where a is extended by 0 outside Ω. Then $\zeta_0^\varepsilon \searrow \zeta_0$ as $\varepsilon \to$ uniformly in $\bar{\Omega}$ (because we have a uniform bound in $C^{1,\alpha}(\bar{\Omega})$). Furthermore, $\zeta_0^\varepsilon \geqslant c_\varepsilon > 0$ in Ω, by the strong maximum principle. Letting

$$w_\varepsilon = \frac{\varphi_1}{\zeta_0^\varepsilon},$$

it follows that $w_\varepsilon \in C^{1,\alpha}(\bar{\Omega})$, $w_\varepsilon = 0$ on $\partial\Omega$, and all the formal computations done with w apply rigorously to w_ε, so that (4.2) holds for w_ε in place of w and ζ_0^ε in place of ζ_0. It is then easy to pass to the limit as $\varepsilon \to 0$, using, for example, monotone convergence. □

STEP 4.

Proof of lemma 4.1. First observe that $u \geqslant c\delta$ for some $c > 0$ (see, for example [3]), and recall Hardy's inequality

$$\int_\Omega \frac{\psi^2}{\delta^2} \leqslant C\int_\Omega |\nabla\psi|^2 \quad \text{for all } \psi \in C_c^1(\Omega),$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. Using this with $\psi = \delta\varphi$ as in [10], it is easy to check that

$$\int_\Omega \varphi^2 \leqslant C \int_\Omega \delta^2(|\nabla\varphi|^2 + \varphi^2) \tag{4.9}$$

for all $\varphi \in C^1(\bar{\Omega})$. This shows that

$$\int_\Omega \varphi^2 \leqslant C \int_\Omega u^2(|\nabla\varphi|^2 + \varphi^2). \tag{4.10}$$

The next step consists in proving

$$\left(\int_\Omega |u\varphi|^r\right)^{2/r} \leqslant C \int_\Omega u^2(|\nabla\varphi|^2 + \varphi^2) \quad \text{for all } \varphi \in C^1(\bar{\Omega}). \tag{4.11}$$

To achieve this, note that, by (2.1), we have

$$\left(\int_\Omega |u\varphi|^r\right)^{2/r} \leqslant C \int_\Omega |\nabla(u\varphi)|^2 - \int_\Omega a(x)(u\varphi)^2. \tag{4.12}$$

But

$$\int_\Omega |\nabla(u\varphi)|^2 = \int_\Omega u^2|\nabla\varphi|^2 + \int_\Omega \nabla u \nabla(u\varphi^2), \tag{4.13}$$

and, multiplying (4.4) by $u\varphi^2$ and integrating, we get

$$\int_\Omega \nabla u \nabla(u\varphi^2) - \int_\Omega a(x)(u\varphi)^2 = \int_\Omega c(x)u^2\varphi^2 + \int_\Omega f u\varphi^2. \tag{4.14}$$

Combining (4.12), (4.13) and (4.14), we find

$$\left(\int_\Omega |u\varphi|^r\right)^{2/r} \leqslant C \int_\Omega u^2|\nabla\varphi|^2 + \int_\Omega c(x)u^2\varphi^2 + \int_\Omega f u\varphi^2.$$

The last two terms in the right-hand side can be estimated by

$$\begin{aligned}\int_\Omega c(x)u^2\varphi^2 + \int_\Omega f u\varphi^2 &\leqslant \|c\|_{L^\infty} \int_\Omega u^2\varphi^2 + \|f\|_{L^\infty}\left(\int_\Omega u^2\varphi^2\right)^{1/2}\left(\int_\Omega \varphi^2\right)^{1/2} \\ &\leqslant C \int_\Omega u^2(|\nabla\varphi|^2 + \varphi^2)\end{aligned}$$

by (4.10). This proves (4.11).

Finally, we interpolate (4.10) and (4.11). By Hölder's inequality,

$$\int_\Omega u^s|\varphi|^q \leqslant \left(\int_\Omega u^r|\varphi|^r\right)^\lambda \left(\int_\Omega \varphi^2\right)^{1-\lambda}$$

if λ and s are chosen so that

$$s = \lambda r \quad \text{and} \quad r\lambda + 2(1-\lambda) = q.$$

This gives the relation (4.5) and proves the lemma. □

STEP 5. We have

$$\zeta_0 \leqslant C\varphi_1.$$

Proof. This time, we consider the quotient

$$w = \frac{\zeta_0}{\varphi_1},$$

which satisfies

$$\begin{aligned} -\nabla \cdot (\varphi_1^2 \nabla w) &= \varphi_1 - \lambda_1 \varphi_1 \zeta_0 && \text{in } \Omega, \\ \varphi_1^2 \nabla w \cdot \nu &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Again, we multiply this equation by $\varphi = w^{2j-1}$ to find

$$\frac{2j-1}{j^2} \int_\Omega \varphi_1^2 |\nabla w^j|^2 = \int_\Omega (\varphi_1 - \lambda_1 \varphi_1 \zeta_0) w^{2j-1} \leqslant \int_\Omega \varphi_1 w^{2j-1}.$$

Here, we use (4.7) to conclude that

$$\varphi_1 w^{2j-1} \leqslant C \zeta_0 w^{2j-1} = C \varphi_1 w^{2j},$$

and so

$$\int_\Omega \varphi_1^2 |\nabla w^j|^2 \leqslant Cj \int_\Omega \varphi_1 w^{2j}. \tag{4.15}$$

Letting $\varphi = w^j$ and using consecutively Hölder's inequality and lemma 4.1 (with $u = \varphi_1$, $f \equiv 0$, $c = \lambda_1$, $s = 0$ and $q = 2$), it follows from (4.15) that

$$\begin{aligned} \int_\Omega \varphi_1^2 |\nabla \varphi|^2 &\leqslant Cj \Big(\int_\Omega \varphi_1^2 \varphi^2 \Big)^{1/2} \Big(\int_\Omega \varphi^2 \Big)^{1/2} \\ &\leqslant Cj \Big(\int_\Omega \varphi_1^2 \varphi^2 \Big)^{1/2} \Big(\int_\Omega \varphi_1^2 (\varphi^2 + |\nabla \varphi|^2) \Big)^{1/2}. \end{aligned}$$

And, by Young's inequality,

$$\int_\Omega \varphi_1^2 |\nabla \varphi|^2 \leqslant Cj^2 \int_\Omega \varphi_1^2 \varphi^2 + \frac{1}{2} \Big(\int_\Omega \varphi_1^2 (\varphi^2 + |\nabla \varphi|^2) \Big),$$

so that

$$\int_\Omega \varphi_1^2 |\nabla \varphi|^2 \leqslant Cj^2 \int_\Omega \varphi_1^2 \varphi^2. \tag{4.16}$$

Using lemma 4.1 with $u = \varphi_1$, $f \equiv 0$, $c = \lambda_1$ and $s = 2$, we obtain a constant $q = 2(1 + (r-2)/r) > 2$ and $C > 0$ such that

$$\Big(\int_\Omega \varphi_0^2 w^{qj} \Big)^{2/q} \leqslant C \int_\Omega \varphi_1^2 (|\nabla w^j|^2 + w^{2j}).$$

Combining with (4.16), we obtain

$$\Big(\int_\Omega \varphi_1^2 w^{qj} \Big)^{2/q} \leqslant Cj^2 \int_\Omega \varphi_1^2 w^{2j}. \tag{4.17}$$

An iteration argument as in step 2 then shows that

$$\sup_{\Omega} w \leqslant C.$$

As in step 3, we need to justify the derivation of (4.17) by an approximation argument. This time, however, it is more convenient to consider

$$\Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\},$$

let ζ_0^ε solve (4.8) and do all of the above computations in Ω_ε in place of Ω. We leave the details to the reader. □

5. Proof of theorem 2.7

As in the elliptic case, using lemma 3.6, it is enough to establish the result for bounded $a(x)$.

Let u be the solution of (1.8) and ζ_0 be the solution of (1.4). We note that $u(t) \geqslant c(t)\delta$ for some positive function $c(t)$ (see [5]). We will replace $u(t)$ with $u(t-\tau)$ where $\tau > 0$ is fixed, and so we can assume

$$u(t) \geqslant c\delta \quad \text{for } t \in [0, T],$$

where $T > 0$ is fixed and $c > 0$ is independent of t for $t \in [0, T]$. By (4.9), we then have

$$\int_\Omega \varphi^2 \leqslant C \int_\Omega u(t)^2 (|\nabla\varphi|^2 + \varphi^2) \tag{5.1}$$

for $t \in [0, T]$, with C independent of t. Since, by theorem 2.6,

$$\zeta_0 \leqslant C\varphi_1,$$

where φ_1 denotes the first eigenfunction for $-\Delta - a(x)$, it is enough to show that, for some constant C, we have

$$\varphi_1 \leqslant Cu(t).$$

We will work with

$$v = \mathrm{e}^{-\lambda_1 t}\varphi_1,$$

which satisfies

$$\begin{aligned} \partial_t v - \Delta v - a(x)v &= 0 \quad \text{in } \Omega \times (0, \infty), \\ v &= 0 \quad \text{on } \partial\Omega \times (0, \infty). \end{aligned}$$

Set

$$w = \frac{v}{u},$$

and note that it formally satisfies

$$\left.\begin{aligned} u^2 w_t - \nabla \cdot (u^2 \nabla w) &= 0 \quad \text{in } \Omega \times (0, T), \\ u^2 \nabla w \cdot \nu &= 0 \quad \text{on } \partial\Omega \times (0, T). \end{aligned}\right\} \tag{5.2}$$

We claim that

$$w(t) \leqslant Ct^{-\beta} \quad \text{for } t \in [0, T],$$

where $\beta, C > 0$ are independent of t.

To accomplish this, we follow the idea in the paper by Brezis and Cazenave [4], which is inspired by a work of Fabes and Stroock [13]. To simplify the exposition, we first work formally with (5.2).

First, for $j \geqslant 1$ and $t \in [0, T]$, we define the quantity

$$\theta_j(t) = \int_\Omega u(t)^2 w(t)^j.$$

We also use the notation

$$\varphi = w^j.$$

Our first step is to derive the following.

CLAIM 5.1. *We have*

$$\theta'_{2j}(t) + 2\|u\varphi\|_H^2 + \frac{2(j-1)}{j} \int_\Omega u^2 |\nabla \varphi|^2 = 0, \tag{5.3}$$

where $\|\cdot\|_H$ was defined in (2.3).

Proof of (5.3). Multiplying (5.2) by w^{2j-1}, we find

$$\frac{1}{2j} \int_\Omega u^2 (w^{2j})_t + \frac{2j-1}{j^2} \int_\Omega u^2 |\nabla w^j|^2 = 0. \tag{5.4}$$

Then observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_{2j}(t) = \theta_{2j}(t)' = 2\int_\Omega u u_t \varphi^2 + \int_\Omega u^2 (\varphi^2)_t. \tag{5.5}$$

Hence, by (5.4) and using (5.5), we obtain

$$\frac{1}{2j}\left(\theta'_{2j} - 2\int_\Omega u u_t \varphi^2\right) + \frac{2j-1}{j^2} \int_\Omega u^2 |\nabla w^j|^2 = 0. \tag{5.6}$$

Now we multiply (1.8) by $u\varphi^2$ and integrate on Ω. This gives the relation

$$\int_\Omega u u_t \varphi^2 + \int_\Omega \nabla u \nabla (u\varphi^2) - \int_\Omega a u^2 \varphi^2 = 0.$$

Therefore,

$$\begin{aligned}\int_\Omega u u_t \varphi^2 &= \int_\Omega a u^2 \varphi^2 - \int_\Omega \nabla u \nabla (u\varphi^2) \\ &= \int_\Omega a u^2 \varphi^2 - \int_\Omega |\nabla (u\varphi)|^2 + \int_\Omega u^2 |\nabla \varphi|^2.\end{aligned}$$

Substituting the expression $\int u u_t \varphi^2$ from the previous equation in (5.6) yields (5.3). □

CLAIM 5.2. *From (5.3), it immediately follows that $\theta'_{2j}(t) \leqslant 0$ and therefore*

$$\theta_j(t) \leqslant \theta_j(0) \quad \textit{for all } t \in [0, T] \textit{ and } j \geqslant 2. \tag{5.7}$$

Claim 5.3. *There is a constant C such that*

$$\theta_{2j}'(t) + \frac{1}{C}\frac{\theta_{2j}(t)^{1+\gamma}}{\theta_j(0)^{2\gamma}} \leqslant \theta_{2j}(t) \quad \text{for } t \in [0,T], \tag{5.8}$$

where $\gamma > 0$ depends only on r.

Proof. By Hölder's inequality,

$$\theta_{2j}(t) = \int_\Omega u^2\varphi^2 \leqslant \left(\int_\Omega (u\varphi)^r\right)^{2/(3r-4)} \left(\int_\Omega \varphi^2\right)^{(r-2)/(3r-4)} \left(\int_\Omega u^2\varphi\right)^{(2r-4)/(3r-4)}.$$

Now we use assumption (2.1) and (5.1) to get

$$\begin{aligned}\theta_{2j}(t) &\leqslant C\|u\varphi\|_H^{r/(3r-4)} \left(\int_\Omega u^2(|\nabla\varphi|^2 + \varphi^2)\right)^{(r-2)/(3r-4)} \left(\int_\Omega u^2\varphi\right)^{(2r-4)/(3r-4)} \\ &= C\|u\varphi\|_H^{r/(3r-4)} \left(\int_\Omega u^2|\nabla\varphi|^2 + \theta_{2j}(t)\right)^{(r-2)/(3r-4)} \theta_j(t)^{(2r-4)/(3r-4)}.\end{aligned}$$

And, by Young's inequality,

$$\theta_{2j}(t) \leqslant C\left(\|u\varphi\|_H^2 + \int_\Omega u^2|\nabla\varphi|^2 + \theta_{2j}(t)\right)^{(2r-2)/(3r-4)} \theta_j(t)^{(2r-4)/(3r-4)}. \tag{5.9}$$

Let

$$\gamma = \frac{3r-4}{2r-2} - 1 > 0,$$

so that, by (5.9) and (5.7),

$$\theta_{2j}(t)^{1+\gamma} \leqslant C\left(\|u\varphi\|_H^2 + \int_\Omega u^2|\nabla\varphi|^2 + \theta_{2j}(t)\right)\theta_j(0)^{2\gamma}. \tag{5.10}$$

Rearranging (5.10) yields

$$\frac{1}{C}\frac{\theta_{2j}(t)^{1+\gamma}}{\theta_j(0)^{2\gamma}} - \theta_{2j}(t) \leqslant \|u\varphi\|_H^2 + \int_\Omega u^2|\nabla\varphi|^2$$

and, combining the last expression with (5.3), we obtain (5.8). □

Claim 5.4. *Using (5.8), we have*

$$\theta_{2j}(t) \leqslant Ct^{-1/\gamma}\theta_j(0)^2, \quad t \in [0,T]. \tag{5.11}$$

The derivation of this estimate has been formal only, but, as in step 3 of § 4, we can make it rigorous using the same approximation argument on Ω.

Claim 5.5. *Iterating (5.11), we find*

$$\|w(t)\|_{L^\infty} \leqslant Ct^{-1/2\gamma} \quad \text{for } t \in [0,T].$$

Indeed, for $k = 1, 2, \dots$, set $t_k = t(1-2^{-k+1})$ and $j_k = 2^k$. Then $t_{k+1} - t_k = 2^{-k}t$. So, from (5.11), we have

$$\theta_{j_{k+1}}(t_{k+1}) = \theta_{2j_k}(t_k + 2^{-k}t) \leqslant C2^{k/\gamma}t^{-1/\gamma}\theta_{j_k}(t_k)^2. \tag{5.12}$$

But recall that

$$\theta_j(t) = \int_\Omega u(t)^2 w(t)^j,$$

so from (5.12) we have

$$\begin{aligned}\left(\int_\Omega u(t_{k+1})^2 w(t_{k+1})^{2^{k+1}}\right)^{1/2^{k+1}} &\leqslant (C2^{k/\gamma}t^{-1/\gamma})^{1/2^{k+1}}\left(\int_\Omega u(t_k)^2 w(t_k)^{2^k}\right)^{1/2^k}\\ &\leqslant C' t^{-(1/\gamma)\sum_{j=2}^{k+1}2^{-j}}\left(\int_\Omega u(0)^2 w(0)^2\right)^{1/2}.\end{aligned}$$

Letting $k \to \infty$, we find that

$$\sup_\Omega w(t) \leqslant C' t^{-1/2\gamma}\|\varphi_1\|_{L^2}.$$

6. Proof of corollaries 2.8 and 2.9

Again, it is enough to reduce to the case where $a(x)$ is bounded.

STEP 1. *A first estimate involving* $\delta(x) = \operatorname{dist}(x, \partial\Omega)$.

Using a fine version of the maximum principle for the heat equation (see [5] for the time dependence of the constant and [15] for the dependence to the initial condition), we have that

$$u(t) \geqslant \mathrm{e}^{-K/t}\left(\int_\Omega u_0\delta\right)\delta(x) \quad \text{for } t \in [0, T],$$

where $K = K(\Omega, T) > 0$. Letting $\mu_1 > 0$ and $\psi_1 > 0$ be the first eigenvalue and eigenfunction of the Laplace operator (with zero boundary condition) and possibly increasing the constant K, it follows that

$$u(t) \geqslant \mathrm{e}^{-K/t}\left(\int_\Omega u_0\delta\right)\psi_1(x) \quad \text{for } t \in [0, 1],$$

where $K = K(\Omega)$. Now let

$$v(t) = \mathrm{e}^{\mu_1 - K}\left(\int_\Omega u_0\delta\right)\mathrm{e}^{-\mu_1 t}\psi_1(x).$$

Then

$$v_t - \Delta v = 0, \qquad v(1) \leqslant u(1).$$

So, by the maximum principle, $u(t) \geqslant v(t)$ for $t \in (1, \infty)$ and we finally obtain

$$u(t) \geqslant \mathrm{e}^{-K(t+1/t)}\left(\int_\Omega u_0\delta\right)\delta(x) \quad \text{for } t \in [0, \infty), \tag{6.1}$$

where $K = K(\Omega)$.

STEP 2. *An estimate for* $u^{x_0} = S(t)\delta_{x_0}$.

First, looking carefully at the previous section, we see that if $u \geqslant 0$ solves (1.8) and

$$u(t) \geqslant \delta(x) \quad \text{for } t \in [0, T],$$

then

$$u(t) \geqslant Ct^{\beta}\mathrm{e}^{-\lambda_1 t}\zeta_0 \quad \text{for } t \in [0,T], \tag{6.2}$$

where C and β depend only on Ω and $\gamma(a)$.

Next, fix a ball $B \subset\subset \Omega$ and for $x_0 \in B$, let δ_{x_0} denote the Dirac mass supported by $\{x_0\}$ and u^{x_0} the solution of (1.8) with initial condition $u_0 = \delta_{x_0}$. Given $t_0 > 0$, we have, by (6.1),

$$u^{x_0}(t_0) \geqslant \delta(x_0)\mathrm{e}^{-K(t_0+1/t_0)}\delta(x) \geqslant \mathrm{e}^{-K'(t_0+1/t_0)}\delta(x),$$

where K' depends only on Ω. Hence, for $t \in [0,T]$,

$$\begin{aligned} u^{x_0}(t+t_0) &\geqslant \mathrm{e}^{-K'(t_0+1/t_0)}S(t)\delta(x) \\ &\geqslant c\mathrm{e}^{-K'(t_0+1/t_0)}S(t)\psi_1(x) \\ &\geqslant c\mathrm{e}^{-K'(t_0+1/t_0)}\mathrm{e}^{-\mu_1 t}\psi_1(x) \\ &\geqslant c\mathrm{e}^{-K(t_0+1/t_0+T)}\delta(x), \end{aligned}$$

where $K = K(\Omega)$. Using (6.2), we obtain, for $t \in [0,T]$,

$$u^{x_0}(t+t_0) \geqslant Ct^{\beta}\mathrm{e}^{-\lambda_1 t}\mathrm{e}^{-K(t_0+1/t_0+T)}\zeta_0,$$

so that, choosing $t = T = t_0$,

$$u^{x_0}(2t_0) \geqslant \mathrm{e}^{-K''(t_0+1/t_0)}\zeta_0,$$

where K'' depends solely on Ω and $\gamma(a)$. Since $t_0 > 0$ was chosen arbitrarily, we finally obtain, for all $t > 0$,

$$u^{x_0}(t) \geqslant \mathrm{e}^{-K''(t+1/t)}\zeta_0. \tag{6.3}$$

STEP 3. Let u^B be the solution of (1.8) with initial condition $u_0 = \chi_B$. Proceeding as in the previous step, we can show that

$$u^B \geqslant \mathrm{e}^{-K(t+1/t)}\zeta_0. \tag{6.4}$$

Now, let u be the solution of (1.8) with arbitrary initial condition $u_0 \in L^2(\Omega)$, $u_0 \geqslant 0$. Using (6.3), we then have, for $x \in B$,

$$u(t,x) = \langle \delta_x, S(t)u_0 \rangle = \int_\Omega u_0 u^x \geqslant \mathrm{e}^{-K''(t+1/t)} \int_\Omega u_0\zeta_0.$$

In other words,

$$u(t) \geqslant \mathrm{e}^{-K''(t+1/t)}\left(\int_\Omega u_0\zeta_0\right)\chi_B.$$

Hence, using (6.4), it follows that

$$u(2t) \geqslant \mathrm{e}^{-K(t+1/t)}\left(\int_\Omega u_0\zeta_0\right)\zeta_0,$$

with $K = K(\Omega, \gamma(a))$, which completes the proof of corollary 2.8.

For corollary 2.9, one just needs to apply corollary 2.8 and Duhamel's principle: if u solves (2.4), then

$$u = S(1)u + \int_0^1 S(s)f \,\mathrm{d}s \geqslant \left(\int_0^1 \mathrm{e}^{-K(s+1/s)} \,\mathrm{d}s\right)\left(\int_\Omega f\zeta_0\right)\zeta_0.$$

7. Further results and open problems

In this section, we question the optimality of our assumption (2.1) on the potential $a(x)$. As we shall see, potentials of the form $a(x) = c/d(x)^2$, where

$$d(x) = \operatorname{dist}(x, \Sigma)$$

is the distance function to an embedded manifold $\Sigma \subset \mathbb{R}^n$, do not necessarily satisfy our assumption (2.1), but instead its weaker version (2.6). As stated in theorem 2.13, *some* comparison results can still be obtained. The outline of the proofs of theorems 2.11 and 2.13 is then given. Finally, we ask whether pointwise estimates for the Green function of the operator $-\Delta - a(x)$ can be obtained.

First, we state the following generalized Hardy inequalities.

THEOREM 7.1. *Let Σ be a smooth manifold of codimension $k \neq 2$ embedded in $\mathbb{R}^n$ and $d(x) = \operatorname{dist}(x, \Sigma)$.*

(i) *If Σ is compact, then, for any $\epsilon > 0$ and $2 < r < 2n/(n-2)$, there exist $C(\epsilon) > 0$, $\gamma > 0$ such that*

$$C(\epsilon)\int_\Omega \varphi^2 + \int_\Omega |\nabla\varphi|^2 - \tfrac{1}{4}(k-2-\epsilon)^2 \int_\Omega \frac{\varphi^2}{d^2} \geqslant \gamma\left(\int_\Omega |\varphi|^r\right)^{2/r}$$

for all $\varphi \in C_c^\infty(\Omega \setminus \Sigma)$.

(ii) *If Σ is oriented then for some $r > 2$, there exist $C, \gamma > 0$ such that*

$$C\int_\Omega \varphi^2 + \int_\Omega |\nabla\varphi|^2 - \tfrac{1}{4}(k-2)^2 \int_\Omega \frac{\varphi^2}{d^2} \geqslant \gamma\left(\int_\Omega |\varphi|^r\right)^{2/r}$$

for all $\varphi \in C_c^\infty(\Omega \setminus \Sigma)$.

(iii) *If Σ is such that $\Delta d^{k-2} \leqslant 0$ in $\mathcal{D}'(\Omega \setminus \Sigma)$, then, for any $2 < r < 2n/(n-2)$, there exists $\gamma > 0$ such that*

$$\int_\Omega |\nabla\varphi|^2 - \tfrac{1}{4}(k-2)^2 \int_\Omega \frac{\varphi^2}{d^2} \geqslant \gamma\left(\int_\Omega |\varphi|^r\right)^{2/r}$$

for all $\varphi \in C_c^\infty(\Omega \setminus \Sigma)$.

(iv) *In particular, if $\Sigma = \partial\Omega$ and Ω is convex, then, for any $2 < r < 2n/(n-2)$, there exists $\gamma > 0$ such that*

$$\int_\Omega |\nabla\varphi|^2 - \frac{1}{4}\int_\Omega \frac{\varphi^2}{d^2} \geqslant \gamma\left(\int_\Omega |\varphi|^r\right)^{2/r}$$

for all $\varphi \in C_c^\infty(\Omega)$.

The fourth inequality was discovered with $\gamma = 0$ by Marcus *et al.* [14] and Matskevich and Sobolevskii [16]. It was then improved by Brezis and Marcus [6] to the case $\gamma > 0$ and $r = 2$. The general case for the third and fourth inequalities is due to Barbatis *et al.* [2]. We will prove the two others in a forthcoming publication.

Observe that the first two inequalities in theorem 7.1 provide examples of potentials for which (2.6) holds whereas (2.1) may fail. We now describe how to adapt the methods of this paper to prove theorems 2.11 and 2.13.

7.1. Outline of the proof of theorem 2.11

Proof of theorem 2.13. Pick $M > 0$ so that (2.6) holds and let $\tilde{a}(x) = a(x) - M$. Since a satisfies (2.6), $\tilde{a}$ satisfies (2.1). Set $\tilde{u} = \mathrm{e}^{-Mt}u$, where u solves (1.8), with the initial condition $u_0 \geqslant 0$, and observe that $\tilde{u}$ satisfies (1.8), with a replaced by $\tilde{a}$.

Combining theorem 2.6 and corollary 2.9 (which hold for $\tilde{a}$ by theorem 2.11) and observing that $-\Delta - a(x)$ and $-\Delta - \tilde{a}(x)$ have the same first eigenfunction, it follows that

$$\tilde{u}(t) \geqslant c(t)\biggl(\int_{\Omega} u_0\varphi_1\biggr)\varphi_1$$

and the estimate follows for u. □

7.2. Green's function

Another interesting direction to pursue concerns the Green function for the operator $-\Delta - a(x)$. We assume here that $a(x)$ satisfies (2.1). Let G_k be the Green function for the operator $-\Delta - a_k(x)$, where $a_k(x) = \min(a(x), k)$, that is,

$$\begin{aligned} -\Delta_y G_k(x,\cdot) - a_k(y)G_k(x,\cdot) &= \delta_x \quad \text{in } \Omega, \\ G_k(x,\cdot) &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where δ_x denotes the Dirac measure at some $x \in \Omega$. Then one can prove the following.

LEMMA 7.2. *We have $G_k \geqslant 0$ and the sequence G_k is non-decreasing and bounded in $L^1(\Omega \times \Omega)$. Therefore, it converges to a function $G \in L^1(\Omega \times \Omega)$. Moreover, for any $f \in L^\infty(\Omega)$, the solution u to*

$$\begin{aligned} -\Delta u - a(x)u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

can be represented as

$$u(x) = \int_{\Omega} G(x,y)f(y)\,\mathrm{d}y \quad \text{a.e. in } \Omega.$$

Then, as a consequence of the comparison result in corollary 2.9, we have the following.

COROLLARY 7.3. *There exists a constant $c > 0$, depending on Ω, r, $\gamma(a)$, such that*

$$G(x,y) \geqslant c\zeta_0(x)\zeta_0(y) \quad \text{a.e. in } \Omega \times \Omega.$$

We have not investigated the possibility of establishing pointwise upper bounds for G. For the special case of the inverse square potential $a(x) = c/|x|^2$, in dimension $n \geqslant 3$ and with $0 < c < \frac{1}{4}(n-2)^2$, Milman and Semenov [17] established upper and lower bounds for the heat kernel associated to the operator $-\Delta - a(x)$, from which upper bounds for Green's function can be derived.

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