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Concentration for an elliptic equation with singular nonlinearity

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Abstract

We are interested in nontrivial solutions of the equation:

$$-\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u^p, \quad u \geqslant 0 \quad \text{in } \Omega,$$

with $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is a bounded domain with smooth boundary, $0 < \beta < 1$, $1 \leqslant p < \frac{N+2}{N-2}$ if $N \geqslant 3$ ($p \geqslant 1$ if $N = 2$) and $\lambda > 0$. If $p > 1$ we prove existence of nontrivial solutions for every $\lambda > 0$. As $\lambda \to +\infty$ we find that the *least energy* solutions concentrate around a point that maximizes the distance to the boundary. We also study the behavior as $\lambda \to 0$. When $p = 1$ we have similar results, extending previous works for radial solutions in a ball.

Résumé

Dans cet article on étudie les solutions non triviales de l'équation :

$$-\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u^p, \quad u \geqslant 0 \quad \text{dans } \Omega,$$

avec $u = 0$ sur $\partial\Omega$, où $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, est un domaine borné de frontière réguliere, $0 < \beta < 1$, $1 \leqslant p < \frac{N+2}{N-2}$ si $N \geqslant 3$ ($p \geqslant 1$ si $N = 2$) avec $\lambda > 0$. Si $p > 1$ on démontre l'existence de solutions non triviales pour $\lambda > 0$. Quand $\lambda \to +\infty$ on obtient les solutions *d'energie minimale* concentrées autour d'un point qui maximise la distance à la frontière. On étudie égalment le comportement des solutions lorsque $\lambda \to 0$. Si $p = 1$ on obtient des résultats similaires qui étendent ainsi les résultats précédents au cas des solutions radiales dans une boule.

Keywords: Elliptic equations; Singular nonlinearity; Asymptotic behavior

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1. Introduction

We are interested in nontrivial solutions of the equation:

$$\begin{cases} -\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u^p & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is a bounded domain with smooth boundary, $0 < \beta < 1$, $1 \leqslant p < \frac{N+2}{N-2}$ if $N \geqslant 3$ ($p \geqslant 1$ if $N = 2$) and $\lambda > 0$. By a solution of (1.1) we mean a function $u \in H_0^1(\Omega)$, $u \geqslant 0$ such that $\int_{[u>0]} u^{-\beta} < \infty$, and

$$\int_\Omega \nabla u \nabla \varphi + \int_{[u>0]} u^{-\beta} \varphi = \lambda \int_\Omega u^p \varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

We use the notation $[u > 0] = \{x \in \Omega\colon\ u(x) > 0\}$.

1.1. *The case* $1 < p < \frac{N+2}{N-2}$ ($p > 1$ *if* $N = 2$)

The radial problem:

$$\begin{cases} -\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u^p, \quad u \geqslant 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.2}$$

where B_1 is the unit ball in $\mathbb{R}^N$ has been studied by several authors [2,7,13,20,27,28]. In [7,28] it is proved that there exists $\bar\lambda > 0$ such that (1.2) has a positive radial solution if and only if $0 < \lambda \leqslant \bar\lambda$, and this solution belongs to $C^2(B_1) \cap C^1(\overline{B}_1)$. Moreover the radial solution u_λ is unique, $u_\lambda'(1) < 0$ if $0 < \lambda < \bar\lambda$ and $u_\lambda'(1) = 0$ if $\lambda = \bar\lambda$. For $\lambda > \bar\lambda$ (1.2) still possesses a radial solution, but it has compact support.

In this work we address the existence question for (1.1) in general smooth bounded domains. We prove:

Theorem 1.1. *For all* $\lambda > 0$ *problem* (1.1) *has a nontrivial solution* $u_\lambda \in C^1(\overline{\Omega}) \cap C^\infty(\Omega)$.

We prove this result constructing solutions $u_{\lambda,\varepsilon}$ to

$$\begin{cases} -\Delta u + \dfrac{u}{(u+\varepsilon)^{1+\beta}} = \lambda u^p, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\varepsilon > 0$, through the mountain pass theorem. Then we prove that $u_{\lambda,\varepsilon}$ is bounded in $L^\infty(\Omega)$ and then in $C^{1,\mu}(\overline{\Omega})$ where $\mu = \frac{1-\beta}{1+\beta}$, uniformly as $\varepsilon \to 0$. Finally we show that for a fixed λ, $\lim_{\varepsilon\to 0} u_{\lambda,\varepsilon}$ is a nontrivial solution of (1.1).

We then study the asymptotic behavior of the solutions u_λ as $\lambda \to +\infty$. For this it convenient to remark the following. Let $\bar u$ be the radial solution of (1.2) corresponding to $\lambda = \bar\lambda$ and extended by zero outside B_1. Set

$$w(x) = \bar\lambda^{\frac{1}{p+\beta}} \bar u\big(\bar\lambda^{-\frac{1+\beta}{2(p+\beta)}} x\big), \quad x \in \mathbb{R}^N.$$

Then, since $\bar u$ has vanishing gradient on ∂B_1 we see that w satisfies:

$$-\Delta w + w^{-\beta} \chi_{[w>0]} = w^p \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

We will call w the radial ground state of (1.4).

Theorem 1.2. *For* $\lambda > 0$ *sufficiently large, the solution* u_λ *of* (1.1) *obtained as the limit* $u_\lambda = \lim_{\varepsilon\to 0} u_{\lambda,\varepsilon}$ *in Theorem* 1.1, *has the form*:

$$u_\lambda(x) = \lambda^{-\frac{1}{p+\beta}} w\big(\lambda^{\frac{1+\beta}{2(p+\beta)}}(x - x_\lambda)\big),$$

for some point $x_\lambda \in \Omega$. *Moreover*

$$\operatorname{dist}(x_\lambda, \partial\Omega) \to \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$$

as $\lambda \to \infty$.

We note that given a point $x_0 \in \Omega$, when $\lambda > 0$ is sufficiently large the function

$$u(x) = \lambda^{-\frac{1}{p+\beta}} w\big(\lambda^{\frac{1+\beta}{2(p+\beta)}}(x - x_0)\big)$$

is a solution of (1.1). However, Theorem 1.2 asserts that the solution constructed as the limit of least energy solutions of the approximation (1.3) selects the position of its maximum as the one that maximizes the distance to the boundary. This is because the energy functional associated to (1.3) has an expansion in terms of ε and λ which in the leading order has a term that penalizes the distance to the boundary.

The above phenomenon is similar to what happens in other equations. For instance Flucher and Wei [11] studied:

$$\begin{cases} -\varepsilon^2\Delta u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where

$$g(u) = \begin{cases} (u-1)^p & \text{for } u \geqslant 1, \\ 0 & \text{for } u < 1, \end{cases} \tag{1.6}$$

and $1 < p < \frac{N+2}{N-2}$ and $N > 2$. For every $\varepsilon > 0$ small, they proved existence of a positive solution u_ε, which is of mountain pass type. The boundary of the core $A_\varepsilon = [u_\varepsilon > 1]$ is a free boundary. They proved also that as $\varepsilon \to 0$ the mountain pass solution u_ε has a unique maximum point x_ε which converges to a harmonic center of Ω and A_ε is asymptotically spherical. To achieve these results they established energy estimates, for which they needed to characterize the kernel of the linearized operator $Lv := \Delta v + g'(w)v$ where w is the ground state solving (1.5) in $\mathbb{R}^N$.

Concentration at a point that maximizes the distance to the boundary was proved by Ni and Wei [24] for the least energy solution u_ε of

$$\begin{cases} -\varepsilon^2\Delta u + u = f(u), \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

for a nonlinearity f that includes $f(u) = u^p$ with $1 < p < \frac{N+2}{N-2}$. They have used similar techniques used by Ni and Takagi [22,23] for the same equation with boundary condition $\partial u/\partial\nu = 0$ on Ω. The argument in [24] is based on precise energy estimates of u_ε and requires to know uniqueness and nondegeneracy of the ground state:

$$\begin{cases} -\Delta w + w = f(w), \quad w > 0 & \text{in } \mathbb{R}^N, \\ w(x) \to 0 & \text{as } |x| \to 0. \end{cases} \tag{1.8}$$

Here we adopt the strategy presented in the work of del Pino and Felmer [9], which does not require uniqueness nor nondegeneracy of w. The estimates of the least energy of the associated functional are obtained by comparison to the least energy of solutions in balls.

A problem related to ours is Eq. (1.1) with $-1 < \beta < 0$ and $p \geqslant 1$. In [4] and [19] the authors proved that if u is a solution in $\mathbb{R}^N$ with compact support, then any connected component of the set $[u > 0]$ is a ball and u restricted to that ball is radially symmetric with respect to its center. This phenomenon is reminiscent to the one described in Theorem 1.2. Actually in the proof of our result it would be useful to have such a symmetry result, but unfortunately for the case $0 < \beta < 1$ it is not known. We overcome this difficulty because we work with least energy solutions. We believe that a result similar to Theorem 1.2 should be valid if $-1 < \beta < 0$ and $1 < p < \frac{N+2}{N-2}$. The motivation for this type of classification questions arises in the study of the blow-up set for the porous medium equation:

$$v_t = \Delta v^m + v^m \quad \text{in } \mathbb{R}^N,\ t > 0, \tag{1.9}$$

where $m > 1$. Regional blow-up was observed in the one-dimensional situation in [12], and it was proved in the higher dimensions in [3]. We note that if we consider (1.9) with $m < 0$, which corresponds to the very fast diffusion equation, we are lead to an equation with negative exponent of the form

$$\Delta u + u + u^{-\beta}\chi_{[u>0]} = 0 \quad \text{in } \mathbb{R}^N,$$

for some $\beta > 0$.

Finally, we prove:

Theorem 1.3. *Let u_λ denote the solution obtained as the limit $u_\lambda = \lim_{\varepsilon\to 0} u_{\lambda,\varepsilon}$ in Theorem* 1.1. *Then for $\lambda > 0$ small enough the solution u_λ is positive in Ω.*

1.2. The case $p = 1$

We consider now,

$$\begin{cases} -\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u, & u \geqslant 0 \quad \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

It is readily seen that if $\lambda \leqslant \lambda_1$, where λ_1 denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary condition, then a solution is identically 0. Indeed, let $\varphi_1 > 0$ be an eigenfunction of $-\Delta$ associated to λ_1. Then multiplying the equation by φ_1 yields:

$$\lambda_1 \int_\Omega u\varphi_1 + \int_{[u>0]} u^{-\beta}\varphi_1 = \lambda \int_\Omega u\varphi_1.$$

If $u \not\equiv 0$ then $\lambda > \lambda_1$.

The radial case in a ball is studied in [2], where it is shown that there exist λ^* and $\bar\lambda$ with $\lambda_1 \leqslant \lambda^* < \bar\lambda$ such that there exists a positive radial solution if and only if $\lambda \in (\lambda^*, \bar\lambda]$. In the case of dimension $N = 1$ and $\Omega = (-1, 1)$ the results of [2] imply that $\bar\lambda = (\frac{\pi}{1+\beta})^2$ and $\lambda^* \geqslant (\frac{\pi}{2})^2$. He also shows that the solution corresponding to $\bar\lambda$ has vanishing gradient on the boundary of the ball. Similar results for radial solutions in a ball are obtained in [20]. They prove that there is a positive radial solution, provided $\lambda_1 < \lambda < (1 + \frac{2(1-\beta)}{N(1+\beta)})\lambda_1(B)$, where $\lambda_1(B)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition on the unit ball B.

We have:

Theorem 1.4. *If $\lambda > \lambda_1$, then* (1.10) *has a nontrivial solution.*

We prove this theorem by letting $p \to 1$ in the solution $u_{\varepsilon,\lambda,p}$ of (1.3) and then $\varepsilon \to 0$.

Positive solutions as in [2,20] can be obtained in a general domain for $\lambda > \lambda_1$ and λ close to λ_1 through the implicit function theorem.

Theorem 1.5. *There exists $\bar\lambda > \lambda_1$ such that for $\lambda_1 < \lambda < \bar\lambda$ there exists a unique nontrivial solution to* (1.10)*, which is positive.*

In the above theorem the solution has the behavior $u_\lambda(x) = c(\lambda - \lambda_1)^{-\frac{1}{1+\beta}}\varphi_1(x)(1 + o(1))$ as $\lambda \to \lambda_1$ where φ_1 is the positive eigenfunction of $-\Delta$ with Dirichlet boundary condition normalized such that $\int_\Omega \varphi_1^2 = 1$, and c is given by,

$$c^{1+\beta} = \int_\Omega \varphi_1^{1-\beta}.$$

We conjecture that when λ is sufficiently large a concentration phenomenon similar to the one for the case $p > 1$ described before takes place. Namely we believe that the solution u_λ obtained through the approximation (1.3) takes the form of a rescaled version of a radial solution, centered at a point, which asymptotically maximizes the distance to the boundary.

The organization of the paper is the following. In Section 2 we present local and up to the boundary estimates in $C^{1,\mu}$ for solutions of (1.3), which are uniform in ε. Using these estimates we prove Theorems 1.1 and 1.4 in Section 3. We need to recall a few properties of the ground state w of (1.4) which we do in Section 4. In Section 5 we obtain some preliminary results on the asymptotic behavior as $\lambda \to \infty$ of the solution u_λ of (1.1), constructed in Theorem 1.1. Then Section 6 contains estimates of the energy of mountain pass solutions of (1.3) when the domain is a ball. Then in Section 7 we perform the proof of Theorem 1.2. In Section 8 we prove Theorem 1.3. Finally we prove Theorem 1.5 in Section 9.

2. Estimates in $C^{1,\mu}$

In this section we shall obtain a local estimate for solutions u to the perturbed equation (1.3). Actually we will obtain estimates for a slightly more general situation, useful in later sections.

Let $D \subseteq \mathbb{R}^N$ be an open set with smooth boundary and consider the equation

$$\begin{cases} -\Delta u + a(x)\dfrac{u}{(u+\varepsilon)^{1+\beta}} = f(x) & \text{in } D \cap B_2(0), \\ u = 0 & \text{on } \partial D \cap B_2(0), \end{cases} \tag{2.1}$$

where $0 < \beta < 1$, $a, f \in L^\infty(D)$, $a \geqslant 0$, $\varepsilon > 0$.

Proposition 2.1. *Let $M > 0$, $a \geqslant 0$ and $a, f \in L^\infty(D)$. Then there exists a constant $C > 0$ such that for every $u \in H^1(D \cap B_2(0)) \cap C(\overline{D} \cap B_2(0))$, $u \geqslant 0$ satisfying* (2.1)*, and*

$$\|u\|_{L^\infty(D\cap B_2(0))} \leqslant M, \tag{2.2}$$

we have

$$\left\|\nabla u(x)\right\|_{C^\mu(\overline{D\cap B_1(0)})} \leqslant C,$$

where $\mu = \frac{1-\beta}{1+\beta}$.

The constant C depends on M, $\|f\|_{L^\infty}$, $\|a\|_{L^\infty}$, N, β and the smoothness of D, but is independent of the solution and ε.

Another related estimate that we will need is the following:

Proposition 2.2. *Suppose that $u \in H^1(B_{2R}(0))$ satisfies:*

$$-\Delta u + a(x)\frac{u}{(u+\varepsilon)^{1+\beta}} = f(x) \quad \text{in } B_{2R}(0), \tag{2.3}$$

where $0 < \beta < 1$, $a, f \in L^\infty(B_{2R}(0))$, $a \geqslant 0$, $\varepsilon > 0$. Then

$$|\nabla u|^2 \leqslant C u^{1-\beta} \quad \text{in } B_R(0),$$

where C depends only on R, $\|u\|_{L^\infty(B_{2R}(0))}$, $\|f\|_{L^\infty(B_{2R}(0))}$, $\|a\|_{L^\infty(B_{2R}(0))}$.

We emphasize here that the estimates above do not depend on $\varepsilon > 0$, which will be kept fixed in all this section.

The arguments are based on the work of Phillips [25], where optimal interior regularity is obtained for minimizers of a certain functional. The difference with that work is that here we do not assume that solutions are minimizers, and we deal also with regularity up to the boundary, following techniques introduced in [5,6].

First, we introduce some notation. For $\varepsilon > 0$ we define:

$$g_\varepsilon(u) = \begin{cases} \frac{u}{(u+\varepsilon)^{1+\beta}} & \text{if } u \geqslant 0, \\ 0 & \text{if } u \leqslant 0, \end{cases} \tag{2.4}$$

and

$$G_\varepsilon(u) = \int_0^u g_\varepsilon(t)\,dt, \tag{2.5}$$

so that

$$G_\varepsilon(u) = \frac{\beta u + \varepsilon}{\beta(1-\beta)(u+\varepsilon)^\beta} - \frac{\varepsilon^{1-\beta}}{\beta(1-\beta)} \quad \text{for } u \geqslant 0.$$

Let

$$x_0 \in D \cap B_1(0) \quad \text{and} \quad \alpha = \frac{2}{\beta+1}.$$

Without loss of generality we may assume in what follows that $x_0 = 0$. Given $r > 0$ let

$$u_r(x) = r^{-\alpha} u(rx),$$

which is defined for

$$x \in D_r = \frac{1}{r} D.$$

By the smoothness of D for a small $r > 0$ one can construct a smooth domain V_r such that

$$D_r \cap B_{3/4}(0) \subseteq V_r \subseteq D_r \cap B_1(0).$$

The smoothness of V_r can be made independent of r if we restrict $0 < r < R$ for some $R > 0$.

The proof of Proposition 2.1 involves the construction of a suitable lower barrier.

Lemma 2.3. *There exists $r_1 > 0$, $m_0 > 0$ such that if $0 < r < r_1$ and $h \in C(\partial V_r)$, $h \geqslant 0$ satisfies,*

$$\int_{\partial V_r} h \geqslant m_0,$$

then there exists $w > 0$ satisfying,

$$-\Delta w + w^{-\beta} \leqslant 0 \quad \text{in } V_r, \qquad w = h \quad \text{on } \partial V_r,$$

and

$$w(x) \geqslant c_1 \left(\int_{\partial V_r} h \right) \operatorname{dist}(x, \partial V_r) \quad \text{for all } x \in V_r. \tag{2.6}$$

Here r_1, m_0, c_1 are fixed positive constants depending only on β, N and the smoothness of D.

The function w depends on x_0 and r, but for simplicity we will write just w. For the construction of w see [5]. The scaling r is going to be in the range $0 < r \leqslant r_0$ where r_0 is given by:

$$r_0 = \max\left\{ \left(\frac{u(x_0)}{C_1} \right)^{1/\alpha}, \left(\frac{u(x_0)}{C_1 \operatorname{dist}(x_0, \partial D)} \right)^{1/(\alpha-1)} \right\}, \tag{2.7}$$

where $C_1 > 0$ is a universal constant to be specified later. It will be convenient to note that

$$\left(\frac{u(x_0)}{C_1} \right)^{1/\alpha} \geqslant \left(\frac{u(x_0)}{C_1 \operatorname{dist}(x_0, \partial D)} \right)^{1/(\alpha-1)}$$

that is, $r_0 = (\frac{u(x_0)}{C_1})^{1/\alpha}$, if and only if

$$C_1 \operatorname{dist}(x_0, \partial D)^\alpha \geqslant u(x_0).$$

To proceed further we need the following consequence of standard elliptic estimates.

Lemma 2.4. *Suppose $v \in H^1(V_r)$ satisfies:*

$$-\Delta v \leqslant h \quad \text{in } V_r, \qquad v = 0 \quad \text{on } \partial V_r \cap B_{3/4}(0),$$

where $h \in L^\infty(V_r)$. Then

$$v(y) \leqslant \overline{C} \operatorname{dist}(y, \partial V_r) \left(\|h\|_{L^\infty(V_r)} + \int_{\partial V_r} |v| \right) \quad \text{for all } y \in B_{1/2}(0),$$

the constant $\overline{C}$ depends only on r.

Note that u_r satisfies,

$$-\Delta u_r + a_r(x) g_{\varepsilon,r}(u_r) = f_r(x) \quad \text{in } D_r,$$

where

$$a_r(x) = a\bigl(r(x+x_0)\bigr), \qquad f_r(x) = r^{2-\alpha} f\bigl(r(x+x_0)\bigr),$$
$$g_{\varepsilon,r}(v) = r^{2-\alpha} g_\varepsilon\bigl(r^\alpha v\bigr).$$

Lemma 2.5. *Assume* $0 < r \leqslant r_0$. *Then*

$$\int_{\partial V_r} u_r \geqslant m_0.$$

Proof. Using Lemma 2.4 we have:

$$\int_{V_r} u_r \geqslant \frac{u_r(0)}{\overline{C}\,\mathrm{dist}(0, \partial V_r)} - r^{2-\alpha} \|f\|_{L^\infty}.$$

On the other hand from the definitions,

$$\frac{1}{C_0} \min\bigl(\mathrm{dist}(x_0, \partial D), r\bigr) \leqslant r\,\mathrm{dist}(0, \partial V_r) \leqslant C_0 \min\bigl(\mathrm{dist}(x_0, \partial D), r\bigr),$$

for some constant C_0 depending only on the geometry of D. This and $u_r(0) = r^{-\alpha} u(x_0)$ yield

$$\int_{\partial V_r} u_r \geqslant \frac{u(x_0)}{C_0 \overline{C} r^{\alpha-1} \min(\mathrm{dist}(x_0, \partial D), r)} - r^{2-\alpha} \|f\|_{L^\infty}$$
$$\geqslant \frac{u(x_0)}{C_0 \overline{C} r_0^{\alpha-1} \min(\mathrm{dist}(x_0, \partial D), r_0)} - r_0^{2-\alpha} \|f\|_{L^\infty}.$$

Suppose that $C_1\,\mathrm{dist}(x_0, \partial D)^\alpha \leqslant u(x_0)$. Then $r_0 = (\frac{u(x_0)}{C_1\,\mathrm{dist}(x_0, \partial D)})^{1/(\alpha-1)}$ and we deduce:

$$r_0^{\alpha-1} = \frac{u(x_0)}{C_1\,\mathrm{dist}(x_0, \partial D)} \geqslant \mathrm{dist}(x_0, \partial D)^{\alpha-1},$$

so that $r_0 \geqslant \mathrm{dist}(x_0, \partial D)$. It follows that

$$\int_{\partial V_r} u_r \geqslant \frac{C_1}{C_0 \overline{C}} - r_0^{2-\alpha} \|f\|_{L^\infty} \geqslant \frac{C_1}{C_0 \overline{C}} - r_0^{2-\alpha} \|f\|_{L^\infty}. \tag{2.8}$$

Suppose now that $C_1\,\mathrm{dist}(x_0, \partial D)^\alpha \geqslant u(x_0)$. Then $r_0 = (\frac{u(x_0)}{C_1})^{1/\alpha}$ and we deduce $r_0 \leqslant \mathrm{dist}(x_0, \partial D)$. Hence (2.8) is still valid in this case.

To conclude we need to verify that r_0 has an upper bound. If $r_0 = (\frac{u(x_0)}{C_1})^{1/\alpha}$ from (2.2) it follows that $r_0 \leqslant (M/C_1)^{1/\alpha}$. If $r_0 = (\frac{u(x_0)}{C_1\,\mathrm{dist}(x_0, \partial D)})^{1/(\alpha-1)}$ an upper bound for $u(x_0)/\mathrm{dist}(x_0, \partial D)$ follows from Lemma 2.4. By choosing C_1 large we see from (2.8) that

$$\int_{\partial V_r} u_r \geqslant m_0, \quad \text{for all } 0 < r \leqslant r_0. \qquad \square$$

The main point in the proof of Proposition 2.1 is the following:

Proposition 2.6. *Assume* $0 < r \leqslant r_0$ *and let* w *be the function constructed in Lemma* 2.3 *with* $w = u_r$ *on* ∂V_r. *Then*

$$u_r \geqslant w \quad \text{in } V_r.$$

To prove the above result consider the problem:

$$\begin{cases} -\Delta v + a_r(x) g_{\varepsilon,r}(v) = f_r(x) & \text{in } V_r, \\ v = u_r & \text{on } \partial V_r, \end{cases} \tag{2.9}$$

where we regard u_r as a given boundary data. Notice that $v = u_r$ is a solution to (2.9) and that the solutions of (2.9) are critical points of the functional,

$$J_r(v) = \int_{V_r} \left(\frac{1}{2}|\nabla v|^2 + a_r(x) G_{\varepsilon,r}(v) - f_r(x) v \right) dx, \quad v \in H_0^1(V_r) + u_r,$$

where

$$G_{\varepsilon,r}(v) = \int_0^v g_{\varepsilon,r}(s)\, ds.$$

Lemma 2.7. *Suppose u_1, u_2 are subsolutions of* (2.9) *such that*

$$u_1 \leqslant u_2 \quad \text{on } \partial V_r$$

and

$$\int_{\partial V_r} u_1 \geqslant m_0.$$

Let w be the function constructed in Lemma 2.3 *with $w = u_1$ on ∂V_r and assume that*

$$u_1 \geqslant w \quad \text{in } V_r.$$

Then

$$J_r\big(\max(u_1, u_2)\big) \leqslant J_r(u_2) + \left(-\frac{1}{2} + \frac{C}{m_0^{1+\beta}} \right) \int_{V_r} \big|\nabla\big(\max(u_1, u_2) - u_2\big)\big|^2.$$

Proof. Let $u = \max(u_1, u_2)$, which satisfies:

$$\begin{cases} -\Delta u + a_r(x) g_{\varepsilon,r}(u) \leqslant f_r(x) & \text{in } V_r, \\ u \leqslant u_r & \text{on } \partial V_r. \end{cases} \tag{2.10}$$

Multiplying (2.10) by $u - u_2 \geqslant 0$ and integrating by parts we obtain:

$$\int_{V_r} \nabla u \cdot \nabla(u - u_2)\, dx + \int_{V_r} a_r(x) g_{\varepsilon,r}(u)(u - u_2)\, dx \leqslant \int_{V_r} f_r(x)(u - u_2)\, dx. \tag{2.11}$$

On the other hand

$$J_r(u) - J_r(u_2) = -\frac{1}{2} \int_{V_r} \big|\nabla(u - u_2)\big|^2 dx + \int_{V_r} \nabla u \cdot \nabla(u - u_2)\, dx$$
$$+ \int_{V_r} a_r(x)\big(G_{\varepsilon,r}(u) - G_{\varepsilon,r}(u_2)\big)\, dx - \int_{V_r} f_r(x)(u - u_2)\, dx.$$

Combining with (2.11) we find:

$$J_r(u) - J_r(u_2) = -\frac{1}{2} \int_{V_r} \big|\nabla(u - u_2)\big|^2 dx + \int_{V_r} a_r(x)\big[G_{\varepsilon,r}(u) - G_{\varepsilon,r}(u_2) - g_{\varepsilon,r}(u)(u - u_2)\big]\, dx. \tag{2.12}$$

But we have,

$$G_{\varepsilon,r}(u) - G_{\varepsilon,r}(u_2) - g_{\varepsilon,r}(u)(u-u_2) \leqslant C\big(r^2 + u^{-1-\beta}\big)(u-u_2)^2. \tag{2.13}$$

Before proving this we note here that $G_{\varepsilon,r}(v) = r^{2-2\alpha} G_\varepsilon(r^\alpha v)$. Since $0 \leqslant g_\varepsilon(u) \leqslant u^{-\beta}$ we see that

$$0 \leqslant G_{\varepsilon,r}(v) \leqslant \frac{v^{1-\beta}}{1-\beta} \quad \text{for } v \geqslant 0. \tag{2.14}$$

Now,

$$G_{\varepsilon,r}(u) - G_{\varepsilon,r}(u_2) - g_{\varepsilon,r}(u)(u-u_2) = \frac{1}{2}\frac{\partial g_{\varepsilon,r}}{\partial v}(\xi)(u-u_2)^2,$$

for some $\xi \in [u_2, u]$. But from $\frac{\partial g_{\varepsilon,r}}{\partial v}(\xi) = r^2 \frac{\partial g_\varepsilon}{\partial u}(r^\alpha \xi)$ and the inequality,

$$\left|\frac{\partial g_\varepsilon}{\partial u}(u)\right| = \left|\frac{\varepsilon - \beta u}{(u+\varepsilon)^{2+\beta}}\right| \leqslant C u^{-1-\beta} \quad \text{for all } u \geqslant 0,\ \varepsilon > 0,$$

we have:

$$\left|\frac{\partial g_{\varepsilon,r}}{\partial v}(\xi)\right| \leqslant C\xi^{-1-\beta}.$$

If $u \leqslant 2u_2$ then $\xi \geqslant u_2 \geqslant u/2$, and

$$\left|\frac{\partial g_{\varepsilon,r}}{\partial v}(\xi)\right| \leqslant C u^{-1-\beta}.$$

If, on the other hand, $u > 2u_2$ then $u \leqslant 2(u-u_2)$ and hence using (2.14) and $g_{\varepsilon,r} \geqslant 0$ we find:

$$G_{\varepsilon,r}(u) - G_{\varepsilon,r}(u_2) - g_{\varepsilon,r}(u)(u-u_2) \leqslant \frac{u^{1-\beta}}{1-\beta} \leqslant C u^{-1-\beta} u^2 \leqslant C u^{-1-\beta}(u-u_2)^2.$$

This proves (2.13) and combining with (2.12) we obtain:

$$J_r(u) - J_r(u_2) \leqslant -\frac{1}{2}\int_{V_r} \big|\nabla(u-u_2)\big|^2\,dx + C\|a\|_{L^\infty}\int_{V_r} u^{-1-\beta}(u-u_2)^2\,dx.$$

But $u \geqslant u_1 \geqslant w$ and w satisfies (2.6) we have:

$$\int_{V_r} u^{-1-\beta}(u-u_2)^2\,dx \leqslant C m_0^{-1-\beta}\int_{V_r} \operatorname{dist}(x,\partial V_r)^{-1-\beta}(u-u_2)^2\,dx.$$

By Hölder's and Hardy's inequality,

$$\int_{V_r} u^{-1-\beta}(u-u_2)^2\,dx \leqslant C m_0^{-1-\beta}\int_{V_r} \big|\nabla(u-u_2)\big|^2.$$

Hence

$$J_r(u) \leqslant J_r(u_2) + \left(-\frac{1}{2} + \frac{C}{m_0^{1+\beta}}\right)\int_{V_r} \big|\nabla(u-u_2)\big|^2. \qquad \square$$

Proof of Proposition 2.6. Throughout this proof we assume $0 < r \leqslant r_0$. By Lemma 2.5 we can apply Lemma 2.3. Henceforth we let w be the function constructed in Lemma 2.3 with $w = u_r$ on ∂V_r.

Step 1. If $r > 0$ is small enough then (2.9) has a unique solution.

Suppose that there exists a sequence $r_j \to 0$ and different solutions v_j^1, v_j^2 to the problem (2.9). Let $w_j = v_j^1 - v_j^2$. Then $w_j \not\equiv 0$ and satisfies:

$$\begin{cases} -\Delta w_j + b_j(x) w_j = 0 & \text{in } V_r, \\ w_j = 0 & \text{on } \partial V_r, \end{cases} \tag{2.15}$$

where b_j is given by,

$$b_j(x) = a_{r_j}(x) \frac{\partial g_{\varepsilon, r_j}}{\partial v}\big(\xi_j(x)\big),$$

and $\xi_j(x)$ is in between $v_j^1(x)$ and $v_j^2(x)$. But

$$\frac{\partial g_{\varepsilon, r_j}}{\partial v}(v) = r_j^2 \frac{\varepsilon - \beta r_j^\alpha v}{(r_j^\alpha v + \varepsilon)^{2+\beta}}.$$

Since $\varepsilon > 0$ is fixed and a_j is uniformly bounded in L^∞ we have:

$$b_j \to 0 \text{ uniformly in } V_r \quad \text{as } j \to \infty.$$

Hence the operator in (2.15) becomes coercive as $j \to \infty$ and therefore $w_j \equiv 0$ for large j.

Step 2. Suppose $u \in H_0^1(V_r) + u_r$ is a minimizer of J_r on this set. Then

$$u \geqslant w \quad \text{in } V_r.$$

To prove this, we apply Lemma 2.7 with $u_1 = w$ and $u_2 = u$. Taking m_0 large we see that if u is a minimizer then $\max(u, w) \equiv u$, that is, $u \geqslant w$ in V_r.

Step 3. The functional J_r has unique minimum in $H_0^1(V_r) + u_r$, which we will write as $\tilde{u}_r$. By the previous step,

$$\tilde{u}_r \geqslant w \quad \text{in } V_r.$$

Indeed, the existence of at least one minimizer of J_r follows from the lower semi-continuity and coercivity of the functional in $H_0^1(V_r) + u_r$. If u_1, u_2 are minimizers, then $u_1 \geqslant w$ and $u_2 \geqslant w$ by the previous step. Taking m_0 large we deduce, applying Lemma 2.7, that $u_1 \leqslant u_2$ and $u_2 \leqslant u_1$.

Step 4. The function u_r is the minimizer of J_r on the set $H_0^1(V_r) + u_r$. By the previous steps we conclude that

$$u_r \geqslant w \quad \text{in } V_r.$$

Indeed, problem (2.9) has a unique solution for small $r > 0$. Thus u_r is the minimizer of J_r for $r > 0$ small. If $0 < r \leqslant r_0$ the linear operator $DJ_r(\tilde{u}_r)$ is coercive on $H_0^1(V_r)$. Indeed

$$\big(DJ_r(\tilde{u}_r)\varphi, \varphi\big) = \int_{V_r} \big(|\nabla \varphi|^2 + a_r(x) g'_{\varepsilon,r}(\tilde{u}_r)\varphi^2\big)\, dx, \quad \text{for all } \varphi \in H_0^1(V_r).$$

But $g_{\varepsilon,r}(v) \geqslant -\beta v^{-1-\beta}$, and therefore

$$\big(DJ_r(\tilde{u}_r)\varphi, \varphi\big) \geqslant \int_{V_r} \big(|\nabla \varphi|^2 - \beta \|a_r\|_{L^\infty} \tilde{u}_r^{-1-\beta} \varphi^2\big)\, dx.$$

From $\tilde{u}_r \geqslant w$, (2.6) and Lemma 2.5

$$\tilde{u}_r \geqslant c_1 m_0 \operatorname{dist}(x, \partial V_r) \quad \text{for all } x \in V_r.$$

Thus

$$\big(DJ_r(\tilde{u}_r)\varphi, \varphi\big) \geqslant \int_{V_r} \big(|\nabla \varphi|^2 - \beta \|a_r\|_{L^\infty} (c_1 m_0)^{-1-\beta} \operatorname{dist}(x, \partial V_r)^{-1-\beta} \varphi^2\big)\, dx.$$

If we take m_0 large enough we can apply Hardy's inequality and obtain the coercivity of $DJ_r(\tilde{u}_r)$ on $H_0^1(V_r)$. Hence the branch of minimizers cannot bifurcate if $0 < r \leqslant r_0$, and since the branches u_r and $\tilde{u}_r$ coincide for small r we must have $u_r = \tilde{u}_r$ for $0 < r \leqslant r_0$. □

Proof of Propositions 2.1 and 2.2. By Proposition 2.6 and (2.6) we have established

$$u_r \geqslant c_1\Big(\int_{\partial V_r} u_r\Big)\operatorname{dist}(x,\partial V_r) \quad \text{for all } x\in V_r,$$

and for all $0<r\leqslant r_0$, where r_0 is given by (2.7). The estimates for ∇u can then be proved exactly in the same way as in Lemmas 10 and 11 and Theorem 3 of [5]. □

3. Existence of solutions: proof of Theorems 1.1 and 1.4

Given $1<p<\frac{N+2}{N-2}$ we define the following functionals in $H_0^1(\Omega)$:

$$J_\lambda(u)=\frac{1}{2}\int_\Omega |\nabla u|^2+\frac{1}{1-\beta}\int_\Omega u_+^{1-\beta}-\frac{\lambda}{p+1}\int_\Omega |u|^{p+1}, \tag{3.1}$$

and

$$J_{\lambda,\varepsilon}(u)=\frac{1}{2}\int_\Omega |\nabla u|^2+\int_\Omega G_\varepsilon(u)-\frac{\lambda}{p+1}\int_\Omega |u|^{p+1}, \tag{3.2}$$

where G_ε is defined in (2.5). Note that

$$G_\varepsilon(u)=\int_0^u g_\varepsilon(s)\,ds=\frac{\beta u+\varepsilon}{\beta(1-\beta)(u+\varepsilon)^\beta}-\frac{\varepsilon^{1-\beta}}{\beta(1-\beta)} \quad \text{for all } u\geqslant 0.$$

Let $A>0$ be sufficiently large and fixed to ensure,

$$\frac{1}{2}\int_\Omega \big|\nabla(A\varphi_1)\big|^2-\frac{\lambda}{p+1}\int_\Omega (A\varphi_1)^{p+1}<0,$$

for all $\lambda>\lambda_1$. Then

$$J_{\lambda,\varepsilon}(A\varphi_1)<0, \tag{3.3}$$

for all $\varepsilon>0$ and $\lambda>\lambda_1$.

Lemma 3.1. *Let $1<p<\frac{N+2}{N-2}$ and $\lambda>0$. Then*

$$\begin{cases} -\Delta u+g_\varepsilon(u)=\lambda u^p & \text{in } \Omega,\\ u\geqslant 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

has a positive solution $u_{\varepsilon,\lambda,p}$.

Proof. We solve (3.4) using the mountain pass theorem for the functional $J_{\lambda,\varepsilon}$ defined in (3.2). Note that since $G_\varepsilon\geqslant 0$ this functional satisfies:

$$\text{there exist } \rho>0, c>0 \quad \text{such that} \quad J_{\lambda,\varepsilon}(u)\geqslant c \quad \forall \|u\|_{H_0^1(\Omega)}=\rho.$$

This and (3.3) give the geometric condition for the mountain pass theorem, and the Ambrosetti–Rabinowitz condition, namely

$$\begin{aligned}&\exists\theta>2 \quad \text{such that}\\ &\theta\Big(\frac{\lambda}{p+1}u^{p+1}-G_\varepsilon(u)\Big)\leqslant \lambda u^p-g_\varepsilon(u) \quad \text{for sufficiently large } |u|\end{aligned}$$

is satisfied since the term that dominates in the nonlinearity for large u is u^p. Therefore there exists a critical point u of $J_{\lambda,\varepsilon}$ in $H_0^1(\Omega)$. By standard regularity theory u is $C^2(\overline{\Omega})$ and satisfies:

$$-\Delta u + g_\varepsilon(u) = \lambda |u|^p \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

We claim that $u > 0$ in Ω. To prove this it suffices to verify that $u \geqslant 0$ in Ω. Suppose on the contrary that $\omega = \{x \in \Omega\colon u(x) < 0\}$ is non-empty. Then

$$-\Delta u = \lambda |u|^p > 0 \quad \text{in } \omega, \qquad u = 0 \quad \text{on } \partial\omega,$$

and we deduce $u > 0$ in ω, a contradiction. Thus we have produced a positive solution u of (3.4) for every $\lambda > 0$. □

Proof of Theorem 1.1. By Lemma 3.1 we know that (3.4) has a solution $u_{\lambda,\varepsilon} > 0$, which we write for short u_ε. We claim that there exists a constant C independent of ε such that

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant C. \tag{3.5}$$

To prove this, we use a blow-up argument. Assuming that $m_\varepsilon \equiv \max_{\overline{\Omega}} u_\varepsilon \to +\infty$ as $\varepsilon \to 0$, define:

$$v_\varepsilon(x) = \frac{u_\varepsilon(m_\varepsilon^{-\frac{p-1}{2}} x)}{m_\varepsilon}, \quad x \in \Omega_\varepsilon = m_\varepsilon^{\frac{p-1}{2}} \Omega.$$

Then $0 \leqslant v_\varepsilon \leqslant 1$ and satisfies

$$\begin{cases} -\Delta v_\varepsilon + \dfrac{1}{m_\varepsilon^{1+\beta}} \dfrac{v_\varepsilon}{(v_\varepsilon + \varepsilon/m_\varepsilon)^{p+\beta}} = \lambda v_\varepsilon^p & \text{in } \Omega_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{3.6}$$

For a subsequence we may assume that $\Omega_\varepsilon \to U$ where $U = \mathbb{R}^N$ or U is a half space. By Proposition 2.1, v_ε is bounded in $C^{1,\mu}(B_R(0) \cap \overline{\Omega}_\varepsilon)$ norm for every $R > 0$, where $\mu = \frac{1-\beta}{1+\beta}$. It follows that v_ε converges uniformly as $\varepsilon \to 0$ on compact sets of $\overline{U}$ to some function v which is $C^1(\overline{U})$ and satisfies $\|v\|_{L^\infty(U)} = 1$ and $v(0) = 1$. Using test functions with support in the open set $[v > 0]$ we see that $-\Delta v = v^p$ in the set $[v > 0]$. Using the strong maximum principle we deduce that actually $v > 0$ in all U and hence

$$\begin{cases} -\Delta v = v^p & \text{in } U, \\ v = 0 & \text{on } \partial U \text{ if } \partial U \neq \emptyset, \\ 0 \leqslant v \leqslant 1. \end{cases}$$

By the results due to Gidas and Spruck [15,16] we conclude $v \equiv 0$, which contradicts $v(0) = 1$. This proves (3.5).

In addition we also have a fixed lower bound for $\|u_\varepsilon\|_{L^\infty(\Omega)}$. Indeed, if x_0 is a maximum of u_ε, then

$$\frac{u_\varepsilon(x_0)}{(u_\varepsilon(x_0) + \varepsilon)^{1+\beta}} \leqslant \lambda u_\varepsilon(x_0)^p.$$

This implies that

$$u_\varepsilon(x_0) \geqslant c > 0.$$

Since u_ε is uniformly bounded in $L^\infty(\Omega)$, applying Proposition 2.1 we have that u_ε is uniformly bounded in $C^{1,\mu}(\overline{\Omega})$. Therefore up to a subsequence u_ε converges in $C^1(\overline{\Omega})$ to a non-zero function $u \in C^{1,\mu}(\overline{\Omega})$. We need to show that u is a solution to (1.1). Since $\nabla u = 0$ on the set $\Omega \setminus [u > 0]$ we then need to prove that

$$\int_{[u>0]} \nabla u \nabla \varphi = \int_{[u>0]} \left(-u^{-\beta} + \lambda u^p\right)\varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{3.7}$$

First we remark that using test functions supported in the open set $[u > 0]$ we find that u satisfies:

$$-\Delta u + u^{-\beta} = \lambda u^p \quad \text{in } [u > 0]. \tag{3.8}$$

Then observe that integrating (3.4) one obtains,

$$\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon + \varepsilon)^{1+\beta}} \leqslant \lambda \int_\Omega u_\varepsilon^p \leqslant C,$$

since u_ε is bounded in $L^\infty(\Omega)$. By Fatou's lemma we find:

$$\chi_{[u>0]}u^{-\beta} \in L^1(\Omega).$$

Take $\varphi \in C_0^\infty(\Omega)$ and $\eta \in C_0^\infty([u>0])$. Then, using (3.8) we have:

$$\int_\Omega \eta \nabla u \nabla \varphi + \int_\Omega \varphi \nabla u \nabla \eta = \int_\Omega \left(-u^{-\beta} + \lambda u^p\right)\varphi \eta.$$

Let $h \in C^\infty(\mathbb{R})$ be such that $h(t) = 0$ for $t \leqslant 1$ and $h(t) = 1$ for $t \geqslant 2$. Given $\varepsilon > 0$ we take $\eta_\varepsilon = h(u/\varepsilon)$. By dominated convergence,

$$\lim_{\varepsilon \to 0} \int_\Omega \eta_\varepsilon \nabla u \nabla \varphi = \int_{[u>0]} \nabla u \nabla \varphi,$$

and

$$\lim_{\varepsilon \to 0} \int_\Omega \left(-u^{-\beta} + \lambda u\right)\varphi \eta_\varepsilon = \int_{[u>0]} \left(-u^{-\beta} + \lambda u^p\right)\varphi.$$

Let S be the support of φ. Then by Proposition 2.2 there exists some constant C such that

$$|\nabla u|^2 \leqslant C u^{1-\beta} \quad \text{in } S.$$

Hence in S

$$|\nabla u \nabla \eta_\varepsilon| = \frac{1}{\varepsilon}\left|h'(u/\varepsilon)\right| |\nabla u|^2 \leqslant \frac{C}{\varepsilon} u^{1-\beta} \chi_{[\varepsilon<u<2\varepsilon]} \leqslant C u^{-\beta} \chi_{[\varepsilon<u<2\varepsilon]}$$

and it follows that

$$\left| \int_\Omega \varphi \nabla u \nabla \eta_\varepsilon \right| \leqslant C \|\varphi\|_{L^\infty(\Omega)} \int_{[u>0]} u^{-\beta} \chi_{[\varepsilon<u<2\varepsilon]} \to 0,$$

since $\chi_{[\varepsilon<u<2\varepsilon]} \to 0$ as $\varepsilon \to 0$ and $\chi_{[u>0]}u^{-\beta} \in L^1(\Omega)$. This establishes (3.7). □

Proof of Theorem 1.4. By Lemma 3.1 we know that (3.4) has a solution $u_{\lambda,\varepsilon,p} > 0$. We let first $p \to 1$. For this we claim that if $\lambda > \lambda_1$ then there exists $p_0 = p_0(\varepsilon, \lambda) > 1$ such that

$$\|u_{\varepsilon,\lambda,p}\|_{L^\infty(\Omega)} \leqslant C_\varepsilon \quad \text{for } 1 < p < p_0(\varepsilon, \lambda). \tag{3.9}$$

To prove (3.9), let us write $u = u_{\lambda,\varepsilon,p}$. Multiply the equation by φ_1 and integrate,

$$\lambda \int_\Omega u^p \varphi_1 = \int_\Omega \lambda_1 u \varphi_1 + g_\varepsilon(u)\varphi_1 \leqslant \frac{\lambda_1}{p} \int_\Omega u^p \varphi_1 + \frac{\lambda_1}{p'} + \max(g_\varepsilon)\|\varphi_1\|_{L^1},$$

which yields:

$$\left(\lambda - \frac{\lambda_1}{p}\right) \int_\Omega u^p \varphi_1 \leqslant \frac{\lambda_1}{p'} + C_\varepsilon.$$

This implies,

$$\int_\Omega u^p \varphi_1 \leqslant C_\varepsilon,$$

if we take $p > \lambda_1/\lambda$. By a bootstrap argument, if $p - 1$ is small, we can reach the estimate $\|u\|_{L^\infty} \leqslant C_\varepsilon$. The constant C_ε is independent of p.

As in the proof of Theorem 1.1 we also have:

$$\|u_{\lambda,\varepsilon,p}\|_{L^\infty(\Omega)} \geqslant c,$$

with $c > 0$ independent of p and ϵ.

By Proposition 2.1 $u_\varepsilon = \lim_{p\to 1} u_{\lambda,\varepsilon,p}$ exists and is a nontrivial solution of

$$\begin{cases} -\Delta u + g_\varepsilon(u) = \lambda u, \quad u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

By the strong maximum principle $u_\varepsilon > 0$ in Ω.

We claim that there exists a constant C independent of ε such that

$$\|u_\varepsilon\|_{L^\infty} \leqslant C. \tag{3.11}$$

To prove this, we use a similar blow-up argument as in the proof of Theorem 1.1, except that now the domain stays fixed. Assuming that $m_\varepsilon \equiv \max_{\overline{\Omega}} u_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Define:

$$v_\varepsilon(x) = \frac{u_\varepsilon(x)}{m_\varepsilon}, \quad x \in \Omega.$$

Then $0 \leqslant v_\varepsilon \leqslant 1$ and satisfies:

$$\begin{cases} -\Delta v_\varepsilon + \dfrac{1}{m_\varepsilon^{1+\beta}} \dfrac{v_\varepsilon}{(v_\varepsilon + \varepsilon/m_\varepsilon)^{1+\beta}} = \lambda v_\varepsilon & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

By Proposition 2.1, v_ε is bounded in $C^{1,\mu}(\overline{\Omega})$, $\mu = \frac{1-\beta}{1+\beta}$, and therefore along some subsequence it converges in $C^1(\overline{\Omega})$ to a function $v \in C^{1,\mu}(\overline{\Omega})$. Since $\|v_\varepsilon\|_{L^\infty(\Omega)} = 1$ we also have $\|v\|_{L^\infty(\Omega)} = 1$. Taking test functions with support in $[v > 0]$ it follows that v satisfies,

$$-\Delta v = \lambda v \quad \text{in } [v > 0].$$

Since $v \geqslant 0$ in Ω and $v \not\equiv 0$ by the strong maximum principle we deduce that $v > 0$ in Ω. This yields a contradiction with $\lambda > \lambda_1$ and establishes (3.11).

Since u_ε is uniformly bounded in $L^\infty(\Omega)$, applying Proposition 2.1 we have that u_ε is uniformly bounded in $C^{1,\mu}(\overline{\Omega})$. Therefore up to a subsequence u_ε converges in $C^1(\overline{\Omega})$ to a function $u \in C^{1,\mu}(\overline{\Omega})$. We may now prove that u is a solution to (1.10) following the same steps as in the proof of Theorem 1.1. □

4. Ground states

We recall that there exists a radial solution with compact support of the equation

$$-\Delta w + \chi_{[w>0]} w^{-\beta} = w^p \quad \text{in } \mathbb{R}^N. \tag{4.1}$$

More precisely, consider,

$$-\Delta u + u^{-\beta} = u^p, \qquad u > 0 \quad \text{in } B_R, \qquad u = 0 \quad \text{on } \partial B_R, \tag{4.2}$$

where $1 < p < \frac{N+2}{N-2}$ ($p > 1$ if $N = 2$) and $R > 0$. In [7, Corollary 1.2] (see also [28]) it is proved that there exists $\overline{R} > 0$ such that (4.2) has a radial solution u if and only if $0 < R \leqslant \overline{R}$, and it is unique in the class of radial solutions. Moreover the radial solution to (4.2) with $R = \overline{R}$ has vanishing gradient on the boundary and hence satisfies (4.1). For $0 < R < \overline{R}$ the radial solution u to (4.2) satisfies $u'(R) < 0$.

Let us write w the solution of (4.2) with $R = \overline{R}$. We call this function the radial ground state of (4.1).

For $u \in H^1(\mathbb{R}^N) \cap L^{1-\beta}(\mathbb{R}^N)$ define:

$$J(u) = \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla u|^2 + \frac{u^{1-\beta}}{1-\beta} - \frac{u^{p+1}}{p+1} \right), \qquad M(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^{1-\beta} - u^{p+1} \right). \tag{4.3}$$

Proposition 4.1. *Let $R_1 > \overline{R}$. Suppose $u \in H_0^1(B_{R_1}(0))$, $u \neq 0$ solves* (4.1) *in the sense $\chi_{[u>0]} u^{-\beta} \in L^1(\mathbb{R}^N)$, and*

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi + \chi_{[u>0]} u^{-\beta} \varphi = \int_{\mathbb{R}^N} u^p \varphi \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N), \tag{4.4}$$

and satisfies

$$J(u) \leqslant J(\varphi),$$

for all $\varphi \in H_0^1(B_{R_1}(0))$ such that $M(\varphi) = 0$. Then up to translation $u = w$.

Proof. Let u^* denote the Schwarz symmetrization of u. Then u^* is radially symmetric, radially nonincreasing and $u^* \in H_0^1(B_R(0))$. It also satisfies $\int_{\mathbb{R}^N} (u^*)^{p+1} = \int_{\mathbb{R}^N} u^{p+1}$, $\int_{\mathbb{R}^N} (u^*)^{1-\beta} = \int_{\mathbb{R}^N} u^{1-\beta}$, and

$$\int_{\mathbb{R}^N} \left|\nabla u^*\right|^2 \leqslant \int_{\mathbb{R}^N} |\nabla u|^2,$$

with equality if and only if $u = u^*$ (after translating). For these properties see for example [21]. Choose $t_0 > 0$ such that $M(t_0 u^*) = 0$. Since u solves (4.1) it satisfies $M(u) = 0$. Note that there is a unique $t > 0$ such that $M(tu) = 0$ and this number is the one that maximizes $t \mapsto J(tu)$. Therefore

$$J(u) = \sup_{t \geqslant 0} J(tu).$$

Thus

$$J\left(t_0 u^*\right) = \int_{B_R(0)} \left(\frac{t_0^2}{2} \left|\nabla u^*\right|^2 + \frac{t_0^{1-\beta}}{1-\beta} \left(u^*\right)^{1-\beta} - \frac{t_0^{p+1}}{p+1} \left(u^*\right)^{p+1} \right) \leqslant J(t_0 u) \leqslant J(u),$$

with strict inequality unless $\int_{\mathbb{R}^N} |\nabla u^*|^2 = \int_{\mathbb{R}^N} |\nabla u|^2$, that is, $u = u^*$ after translation. Since u minimizes J with respect to functions $\varphi \in H_0^1(B_{R_1}(0))$ with $M(\varphi) = 0$ we deduce that $J(t_0 u^*) = J(u)$. Therefore $u = u^*$ after translating, which means u is a radial solution of (4.1). Let $0 < R_0 \leqslant R_1$ be such that $u > 0$ in B_{R_0} and $u(r) = 0$ for $r \geqslant R_0$. Then $u'(R_0) = 0$. By the uniqueness of $\overline{R}$ and the solution we find $R_0 = \overline{R}$ and $u = w$. □

5. Asymptotic behavior as λ → +∞, Part 1

Proposition 5.1. *Let u_λ be the solution to*

$$\begin{cases} -\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

constructed in Theorem 1.1 *or Theorem* 1.4 *where $1 \leqslant p < \frac{N+2}{N-2}$. Then for some $C > 0$,*

$$\lambda^{-\frac{1}{p+\beta}} \leqslant \|u_\lambda\|_{L^\infty(\Omega)} \leqslant C \lambda^{-\frac{1}{p+\beta}} \quad \text{for all large } \lambda. \tag{5.1}$$

Proof. Let $x_\lambda \in \Omega$ be a point where u_λ attains its maximum. The first inequality in the statement follows from

$$u(x_\lambda)^{-\beta} - \lambda u(x_\lambda)^p = \Delta u(x_\lambda) \leqslant 0.$$

Let

$$m_\lambda = \|u_\lambda\|_{L^\infty(\Omega)},$$

and assume by contradiction that for some sequence $\lambda \to +\infty$ we have:

$$m_\lambda \lambda^{\frac{1}{p+\beta}} \to +\infty. \tag{5.2}$$

Define:

$$v_\lambda(x) = \frac{u_\lambda(x_\lambda + \lambda^{-1/2} m_\lambda^{\frac{1-p}{2}} x)}{m_\lambda}.$$

Then v_λ satisfies,

$$-\Delta v_\lambda + \frac{1}{\lambda m_\lambda^{p+\beta}} \chi_{[v_\lambda > 0]} v_\lambda^{-\beta} = v_\lambda^p \quad \text{in } \Omega_\lambda,$$

where $\Omega_\lambda = \lambda^{1/2} m_\lambda^{\frac{p-1}{2}} (\Omega - x_\lambda)$. If $m_\lambda \to +\infty$ as $\lambda \to +\infty$ then $\lambda^{1/2} m_\lambda^{\frac{p-1}{2}} \to +\infty$ as well. If m_λ is bounded, then $\lambda m_\lambda^{p-1} = \lambda m_\lambda^{p+\beta} m_\lambda^{-1-\beta} \to +\infty$ by (5.2). Thus in all cases $\lambda^{1/2} m_\lambda^{\frac{p-1}{2}} \to +\infty$ as $\lambda \to +\infty$. For a subsequence we may assume that $\Omega_\lambda \to U$ where either $U = \mathbb{R}^N$ or U is a half space.

Observe that $0 \leqslant v_\lambda \leqslant 1$ and $v_\lambda(0) = 1$. For a fixed $\lambda > 0$, v_λ is the limit of $v_{\lambda,\varepsilon}$ as $\varepsilon \to 0$ which are uniformly bounded solutions of

$$\begin{cases} -\Delta v + \dfrac{1}{\lambda m_\lambda^p} g_\varepsilon(m_\lambda v) = v^p & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

to which Proposition 2.1 can be applied. Therefore v_λ is bounded in the $C^{1,\mu}(B_R(0) \cap \overline{\Omega}_\lambda)$ norm for every $R > 0$. It follows that v_λ converges uniformly as $\lambda \to +\infty$ on a bounded set of U to some function v which is $C^1(\overline{U})$ and satisfies $\|v\|_{L^\infty(U)} = 1$ and $v(0) = 1$.

Using test functions with support in the open set $[v > 0]$ we see that $-\Delta v = v^p$ in the set $[v > 0]$. Using the strong maximum principle we deduce that actually $v > 0$ in all U and hence

$$\begin{cases} -\Delta v = v^p & \text{in } U, \\ v = 0 & \text{on } \partial U \text{ if } \partial U \neq \emptyset, \\ 0 \leqslant v \leqslant 1. \end{cases}$$

If $1 < p < \frac{N+2}{N-2}$ by the results Gidas and Spruck [15,16] we conclude $v \equiv 0$. If $p = 1$ then also $v \equiv 0$ because otherwise v is a positive supersolution for the operator $-\Delta - \lambda_1(R)$ in every large ball $B_R(x) \subseteq U$ where the first eigenvalue with Dirichlet boundary condition is $O(R^{-2})$. Thus $v \equiv 0$ which contradicts $v(0) = 1$. □

Define:

$$\mu_\lambda = \inf_{\gamma \in \Gamma_\lambda} \sup_{t \in [0,1]} J_\lambda(\gamma(t)),$$

where

$$\Gamma_\lambda = \{\gamma : [0,1] \to H_0^1(\Omega)\colon \gamma \text{ is continuous}, \gamma(0) = 0, \gamma(1) = A\varphi_1\},$$

and

$$\mathcal{N}_\lambda = \left\{u \in H_0^1(\Omega)\colon u \neq 0 \text{ and } \int_\Omega |\nabla u|^2 = \int_\Omega \left(\lambda \frac{|u|^{p+1}}{p+1} - \frac{u_+^{1-\beta}}{1-\beta}\right)\right\}.$$

In the above definition of Γ_λ the constant A is fixed such that $J_{\lambda,\varepsilon}(A\varphi_1) < 0$ for all $\varepsilon > 0$ and all $\lambda > \lambda_1$.

The following lemma is standard for continuous nonlinearities $f(u)$ satisfying the classical hypotheses for the existence of mountain pass solutions, see [1], and the condition $f(u)/u$ increasing, see [10,29]. The nonlinearity $f(u) = \lambda u^p - u^{-\beta}$ satisfies the last assumption, but since it is discontinuous we provide the proof.

Lemma 5.2. *Let J_λ be the functional defined in* (3.1). *We have*:

$$J_\lambda(u_\lambda) = \mu_\lambda = \inf_{\mathcal{N}_\lambda} J_\lambda.$$

Proof. Let u_ε denote the solution of (3.4) constructed in Lemma 3.1 with the mountain pass theorem. Multiplying Eq. (3.4) by u_ε and integrating we have:

$$\int_\Omega \left(|\nabla u_\varepsilon|^2 + g_\varepsilon(u_\varepsilon)u_\varepsilon - \lambda u_\varepsilon^{p+1}\right) = 0.$$

Since $g_\varepsilon(u)u \to u_+^{1-\beta}$ uniformly for u on compact sets of $\mathbb{R}$ we have:

$$\int_\Omega \left(|\nabla u_\varepsilon|^2 + u_\varepsilon^{1+\beta} - \lambda u_\varepsilon^{p+1}\right) = o(\varepsilon)$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus there exists $t(\varepsilon) = 1 + o(\varepsilon)$ with the property $t(\varepsilon)u_\varepsilon \in \mathcal{N}_\lambda$. Hence

$$\inf_{\mathcal{N}_\lambda} J_\lambda \leqslant J_\lambda\big(t(\varepsilon)\, u_\varepsilon\big).$$

But

$$J_\lambda\big(t(\varepsilon)\, u_\varepsilon\big) = J_\lambda(u_\varepsilon) + o(\varepsilon) = J_{\lambda,\varepsilon}(u_\varepsilon) + o(\varepsilon),$$

where $J_{\lambda,\varepsilon}$ is the functional defined in (3.2). By construction of u_ε

$$J_{\lambda,\varepsilon}(u_\varepsilon) = \mu_{\lambda,\varepsilon},$$

where

$$\mu_{\lambda,\varepsilon} = \inf_{\gamma\in\Gamma_\lambda} \sup_{t\in[0,1]} J_{\lambda,\varepsilon}\big(\gamma(t)\big).$$

Thus

$$\inf_{\mathcal{N}_\lambda} J_\lambda \leqslant \mu_{\lambda,\varepsilon} + o(\varepsilon).$$

For a fixed $\gamma \in \Gamma_\lambda$

$$\mu_{\lambda,\varepsilon} \leqslant \sup_{t\in[0,1]} J_{\lambda,\varepsilon}\big(\gamma(t)\big),$$

and letting $\varepsilon \to 0$

$$\limsup_{\varepsilon\to 0} \mu_{\lambda,\varepsilon} \leqslant \sup_{t\in[0,1]} J_\lambda\big(\gamma(t)\big).$$

Therefore

$$\limsup_{\varepsilon\to 0} \mu_{\lambda,\varepsilon} \leqslant \mu_\lambda,$$

and we deduce

$$\inf_{\mathcal{N}_\lambda} J_\lambda \leqslant \mu_\lambda.$$

To prove the converse let $u \in \mathcal{N}_\lambda$. Given $c_1 > 0$, $c_2 \geqslant 0$, $c_3 > 0$ we consider the function:

$$f(t) = c_1\frac{t^2}{2} + c_2\frac{t^{1-\beta}}{1-\beta} - c_3\frac{t^{p+1}}{p+1} \quad \text{for } t > 0.$$

Note that

$$\frac{f'(t)}{t} = c_1 + c_2 t^{-\beta-1} - c_3 t^{p-1}$$

is a decreasing function with limit $+\infty$ as $t \to 0$ and $-\infty$ as $t \to +\infty$. Thus f has a unique critical point, which corresponds to a maximum and is nondegenerate. Thus there is a unique $t^*(u) > 0$ which is critical point of

$$t \mapsto J_\lambda(tu),$$

and hence $t^*(u) = 1$. Thus $J_\lambda(tu) \leqslant J_\lambda(t^*(u)u)$ for all $t \geqslant 0$. Let $t_1 > t^*(u)$ be large such that $J_\lambda(t_1 u) < 0$. We take γ as the path that connects 0 with $t_1 u$ with a straight line and then $t_1 u$ with $A\varphi_1$ on the affine space $\{s_1(t_1 u) + s_2 A\varphi_1\colon s_1, s_2 \in \mathbb{R}\}$ along which J_λ is negative. Then $\max_{t\in[0,1]} J_\lambda(\gamma(t)) = J_\lambda(u)$. □

Lemma 5.3. *If* $1 < p < \frac{N+2}{N-2}$ *we have*:

$$\int_\Omega |\nabla u_\lambda|^2 \leqslant C\lambda^{-q}, \qquad \lambda \int_\Omega u_\lambda^{p+1} \leqslant C\lambda^{-q}, \qquad \int_\Omega u_\lambda^{1-\beta} \leqslant C\lambda^{-q}, \tag{5.3}$$

where $q = \frac{2+N+(N-2)\beta}{2(p+\beta)}$.

Proof. Let $\varphi = a\varphi_0$ where $\varphi_0 \in C_0^\infty(\mathbb{R}^N)$, $\varphi_0 \geqslant 0$ and $\varphi_0 \not\equiv 0$. We choose the constant $a > 0$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla\varphi|^2 + \varphi^{1-\beta} - \varphi^{p+1}\right) = 0.$$

For the next calculation we assume that $0 \in \Omega$, so that the support of $\varphi_\lambda(x) = \lambda^{-\frac{1}{p+\beta}} \varphi(\lambda^{\frac{1+\beta}{2(p+\beta)}} x)$ is contained in Ω for λ sufficiently large. Then

$$\begin{aligned}
\int_\Omega |\nabla\varphi_\lambda|^2 &= \lambda^{\frac{-2-N-(N-2)\beta}{2(p+\beta)}} \int_{\mathbb{R}^N} |\nabla\varphi|^2, \\
\int_\Omega \varphi_\lambda^{1-\beta} &= \lambda^{\frac{-2-N-(N-2)\beta}{2(p+\beta)}} \int_{\mathbb{R}^N} \varphi^{1-\beta}, \\
\int_\Omega \varphi_\lambda^{p+1} &= \lambda^{\frac{-2-N-(N-2)\beta}{2(p+\beta)}} \int_{\mathbb{R}^N} \varphi^{p+1}.
\end{aligned}$$

Hence $\varphi_\lambda \in \mathcal{N}_\lambda$ and since $J_\lambda(\varphi_\lambda) = C\lambda^{\frac{-2-N-(N-2)\beta}{2(p+\beta)}}$ we obtain:

$$J_\lambda(u_\lambda) \leqslant C\lambda^{\frac{-2-N-(N-2)\beta}{2(p+\beta)}}.$$

Therefore

$$\begin{aligned}
C\lambda^{\frac{-2-N-(N-2)\beta}{2(p+\beta)}} \geqslant J_\lambda(u_\lambda) &= \int_\Omega \frac{1}{2}|\nabla u_\lambda|^2 + \frac{u_\lambda^{1-\beta}}{1-\beta} - \lambda\frac{u^{p+1}}{p+1} \\
&= \int_\Omega \lambda\left(\frac{1}{2} - \frac{1}{p+1}\right)u_\lambda^{p+1} + \left(\frac{1}{1-\beta} - \frac{1}{2}\right)u_\lambda^{1-\beta} \\
&\geqslant c\lambda \int_\Omega u_\lambda^{p+1} \geqslant c\int_\Omega |\nabla u_\lambda|^2. \qquad \square
\end{aligned}$$

Let $x_\lambda \in \Omega$ be a maximum point of u_λ. Let us introduce the rescaled function

$$v_\lambda(x) = \lambda^{\frac{1}{p+\beta}} u_\lambda\left(x_\lambda + \lambda^{-\frac{1+\beta}{2(p+\beta)}} x\right),$$

so that v_λ solves,

$$-\Delta v_\lambda + \chi_{[v_\lambda>0]} v_\lambda^{-\beta} = v_\lambda^p \quad \text{in } \Omega_\lambda, \qquad v_\lambda = 0 \quad \text{on } \partial\Omega_\lambda, \tag{5.4}$$

where $\Omega_\lambda = \lambda^{\frac{1+\beta}{2(p+\beta)}}(\Omega - x_\lambda)$.

Lemma 5.4. *For a large $\lambda > 0$ the solution u_λ has support in a ball $B_{R_\lambda}(x_\lambda)$ with $R_\lambda = C\lambda^{-\frac{1+\beta}{2(p+\beta)}}$.*

Proof. By (5.1)

$$1 \leqslant \|v_\lambda\|_{L^\infty(\Omega_\lambda)} \leqslant C,$$

and in particular $1 \leqslant v_\lambda(0) \leqslant C$. By Proposition 2.1 for every $R > 0$ there exists $C_R > 0$ such that for every $x \in \Omega_\lambda$,

$$\|\nabla v_\lambda\|_{L^\infty(B_R(x)\cap\Omega_\lambda)} \leqslant C_R,$$

and therefore

$$\|\nabla v_\lambda\|_{L^\infty(\Omega_\lambda)} < +\infty. \tag{5.5}$$

On the other hand by (5.3) there is some constant such that

$$\int_{\Omega_\lambda} v_\lambda^{p+1} \leqslant C. \tag{5.6}$$

Define $FB_\lambda = \partial\{x \in \Omega_\lambda\colon\ v_\lambda(x) > 0\}$. There exists $c, c' > 0$ such that if $x_0 \in FB_\lambda$ then for every $0 < r \leqslant c$,

$$\text{there exists } x_r \in \Omega \text{ with } |x_r - x_0| = r \quad \text{such that} \quad v_\lambda(x_r) \geqslant c' r^\alpha. \tag{5.7}$$

This is standard, see e.g. [26], but to be self-contained we give the explanation. By (5.5) we can select $c > 0$ such that

$$v_\lambda(y)^{p+\beta} \leqslant \frac{1}{2} \quad \text{for all } y \in \Omega_\lambda, \qquad |y - x_0| \leqslant 2c.$$

Then on the set $\{v_\lambda > 0\} \cap B_{2c}(x)$

$$\Delta\big(v_\lambda^{1+\beta}\big) = (1+\beta)\big(1 - v_\lambda^{p+\beta}\big) + (1+\beta)\beta v_\lambda^{\beta-1}|\nabla v_\lambda|^2 \geqslant \frac{1+\beta}{2}.$$

Let $z(y) = \frac{1+\beta}{4N}|x - \bar{x}|^2$ where $\bar{x}$ is a point close to x_0, say $|\bar{x} - x_0| \leqslant c/2$, such that $v_\lambda(\bar{x}) > 0$. Let $0 < r \leqslant c$. If $v_\lambda^{1+\beta} \leqslant z$ for $y \in \partial B_r(\bar{x}) \cap \Omega_\lambda$ then by the maximum principle $v_\lambda(\bar{x}) \leqslant z(\bar{x}) = 0$, which is not possible. Hence there is a point $y \in \partial B_r(\bar{x}) \cap \Omega_\lambda$ such that

$$v_\lambda(y)^{1+\beta} \geqslant z(y).$$

Letting $\bar{x} \to x_0$ establishes the assertion.

We observe that $\{x \in \Omega_\lambda\colon\ v_\lambda(x) > 0\}$ is connected. Otherwise let ω denote the connected component containing the origin, and assume that $v_\lambda > 0$ for some point outside ω. Let $v_1 = v\chi_\omega$ and $v_2 = v\chi_{\Omega\setminus\bar{\omega}}$, which are nontrivial solutions to (5.4). Let

$$J(v) = \int_{\Omega_\lambda} \frac{1}{2}|\nabla v|^2 + \frac{v^{1-\beta}}{1-\beta} - \frac{v^{p+1}}{p+1}.$$

By the characterization of Lemma 5.2

$$J(v_\lambda) \leqslant J(v_1) \quad \text{and} \quad J(v_\lambda) \leqslant J(v_2).$$

Since $J(v_\lambda) = J(v_1) + J(v_2)$ we must have $J(v_1) \leqslant 0$ and $J(v_2) \leqslant 0$, which implies $J(v_\lambda) = 0$. But this and the fact that v_λ is a solution give,

$$\int_{\Omega_\lambda} \left(\frac{1}{2} - \frac{1}{p+1}\right) v_\lambda^{p+1} + \left(\frac{1}{1-\beta} - \frac{1}{2}\right) v_\lambda^{1-\beta} = 0,$$

which implies $v_\lambda = 0$, a contradiction.

Using the previous properties and (5.5) and (5.6) we can now show that the support of v_λ is contained in a ball of fixed radius. Suppose $x_k \in FB_\lambda$, $k = 1, \ldots, m$, are at distance at least $2c$ from each other. By (5.7) there exists $y_k \in \Omega$ with $|y_k - x_k| = c$ such that

$$v_\lambda(y_k) \geqslant \tilde{c} > 0.$$

By the uniform Lipschitz bound we deduce that

$$\int_{\Omega_\lambda} v_\lambda^{p+1} \geqslant mc''$$

for some $c'' > 0$. Thus from the upper bound (5.6) we deduce $m \leqslant C$ for some constant C independent of λ. This shows that the support of v_λ is contained in ball $B_M(0)$ with M bounded independently of λ and this establishes the lemma. □

6. Energy estimates in balls

We work here with $1 < p < \frac{N+2}{N-2}$ and $0 < \beta < 1$. Given $\rho > 0$ and $\varepsilon > 0$ we consider the equation,

$$\begin{cases} -\Delta w + \dfrac{w}{(w+\varepsilon)^{1+\beta}} = w^p, \quad w > 0 & \text{in } B_\rho, \\ w = 0 & \text{on } \partial B_\rho, \end{cases} \tag{6.1}$$

and its associated functional $J_{\rho,\varepsilon} : H_0^1(B_\rho) \to \mathbb{R}$ defined by:

$$J_{\rho,\varepsilon}(u) = \int_{B_\rho} \frac{1}{2}|\nabla u|^2 + G_\varepsilon(u) - \frac{(u^+)^{p+1}}{p+1}.$$

This functional has a least positive critical value $c_{\rho,\varepsilon}$. Since the nonlinearity $f(u) = u^p - \frac{u}{(u+\varepsilon)^{1+\beta}}$ satisfies that $f(u)/u$ is strictly increasing for $u > 0$, then $c_{\rho,\varepsilon}$ can be characterized by,

$$c_{\rho,\varepsilon} = \inf_{u \in H_0^1(B_\rho),\, u \neq 0} \sup_{t \geqslant 0} J_{\rho,\varepsilon}(tu), \tag{6.2}$$

see [10,29].

For $\varepsilon > 0$ we also consider the equation:

$$\begin{cases} -\Delta w + \dfrac{w}{(w+\varepsilon)^{1+\beta}} = w^p, \quad w > 0 & \text{in } \mathbb{R}^N, \\ w(x) \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{6.3}$$

Critical points of the functional,

$$J_{\mathbb{R}^N,\varepsilon}(u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 + G_\varepsilon(u) - \frac{(u^+)^{p+1}}{p+1}, \quad u \in H^1(\mathbb{R}^N),$$

give rise to solutions of (6.3). This functional has a least positive critical value c_ε characterized by:

$$c_\varepsilon = \inf_{u \in H^1(\mathbb{R}^N),\, u \neq 0} \sup_{t \geqslant 0} J_{\mathbb{R}^N,\varepsilon}(tu). \tag{6.4}$$

The main result in this section is

Proposition 6.1. *There are $R_1 > R_2 > 0$, $C > 0$ and $\varepsilon_0 > 0$ such that for all $\rho \geqslant R_1 + 2$ and $0 < \varepsilon \leqslant \varepsilon_0$: we have*

$$c_{\rho,\varepsilon} \geqslant c_\varepsilon + e^{-2(\varepsilon^{-(1+\beta)/2}+C)(\rho-R_2)-C\log\rho}, \tag{6.5}$$

$$c_{\rho,\varepsilon} \leqslant c_\varepsilon + e^{-2(\varepsilon^{-(1+\beta)/2}-C)(\rho-R_1)+C\log\rho}. \tag{6.6}$$

Solutions of (6.1) with energy $c_{\rho,\varepsilon}$ are called least energy solutions, and similarly for solutions of (6.3) with energy c_ε. Using the Schwarz symmetrization we can show that the existence of a radial, decreasing least energy solution of (6.1) that we call $w_{\rho,\varepsilon}$ and a radial, decreasing least energy solution w_ε of (6.3).

Lemma 6.2. *Let $R_0 > 0$, $\sigma > 0$ and $\bar{w}$ be the solution to*

$$\bar{w}'' = \frac{1}{2}\frac{\bar{w}}{(w+\varepsilon)^{1+\beta}}, \quad r > R_0, \tag{6.7}$$

$$\bar{w}(R_0) = \sigma, \qquad \lim_{r\to+\infty} \bar{w}(r) = 0. \tag{6.8}$$

If $\sigma > 0$ is sufficiently small, only depending on β, there is $c_0 > 0$ such that

$$\bar{w}(R_0+1) \leqslant e^{-c_0\varepsilon^{-(1+\beta)/2}}. \tag{6.9}$$

Proof. We write in this proof w instead of $\bar{w}$. The solution to (6.7), (6.8) is obtained by multiplying (6.7) by w' and integrating:

$$\frac{1}{2}\big(w'\big)^2 = \frac{1}{2}G_\varepsilon(w),$$

which gives the relation

$$\int_{w(r)}^{\sigma} \frac{1}{\sqrt{G_\varepsilon(s)}}\, ds = r - R_0. \tag{6.10}$$

Using the definition of g_ε we see that

$$g_\varepsilon(u) = \frac{u}{(u+\varepsilon)^{1+\beta}} \geqslant \frac{u}{(2\varepsilon)^{1+\beta}} \quad \text{for } 0 \leqslant u \leqslant \varepsilon,$$

and therefore

$$G_\varepsilon(u) \geqslant \frac{u^2}{2^{2+\beta}\varepsilon^{1+\beta}} \quad \text{for } 0 \leqslant u \leqslant \varepsilon.$$

Similarly

$$g_\varepsilon(u) \geqslant \frac{1}{2^{2+\beta}u^\beta} \quad \text{for } u \geqslant \varepsilon,$$

and

$$G_\varepsilon(u) \geqslant \frac{\varepsilon^{1-\beta}}{2^{2+\beta}} + \frac{u^{1-\beta} - \varepsilon^{1-\beta}}{2^{1+\beta}(1-\beta)} \quad \text{for } u \geqslant \varepsilon.$$

By choosing $\sigma > 0$ small we get $w(R_0+1) \leqslant \varepsilon$. Indeed, assume $w(R_0+1) \geqslant \varepsilon$. Then Eq. (6.10) and the estimates for G_ε imply that

$$\int_{\varepsilon}^{\sigma} \left(\frac{\varepsilon^{1-\beta}}{2^{2+\beta}} + \frac{s^{1-\beta} - \varepsilon^{1-\beta}}{2^{1+\beta}(1-\beta)}\right)^{-1/2} ds \geqslant 1.$$

But

$$\int_{\varepsilon}^{\sigma} \left(\frac{\varepsilon^{1-\beta}}{2^{2+\beta}} + \frac{s^{1-\beta} - \varepsilon^{1-\beta}}{2^{1+\beta}(1-\beta)}\right)^{-1/2} ds \leqslant C\int_{\varepsilon}^{\sigma} \frac{1}{s^{(1-\beta)/2}}\, ds \leqslant C\sigma^{(1+\beta)/2} \leqslant \frac{1}{2}, \tag{6.11}$$

if we choose $\sigma > 0$ small. This gives a contradiction.

Since $w(R_0+1) \leqslant \varepsilon$ we find from (6.10) and the estimates for G_ε that

$$C\varepsilon^{(1+\beta)/2} \int_{w(R_0+1)}^{\varepsilon} \frac{1}{s}\, ds + \int_{\varepsilon}^{\sigma} \left(\frac{\varepsilon^{1-\beta}}{2^{2+\beta}} + \frac{s^{1-\beta} - \varepsilon^{1-\beta}}{2^{1+\beta}(1-\beta)}\right)^{-1/2} ds \geqslant 1, \tag{6.12}$$

where $C > 0$. Then (6.11) and (6.12) yield:

$$C\varepsilon^{(1+\beta)/2}\log\left(\frac{\varepsilon}{w(R_0+1)}\right)\geqslant\frac{1}{2}.$$

This estimate implies the inequality (6.9). □

Lemma 6.3. *Let w be a radial, decreasing solution w of* (6.1) *or* (6.3)*. There is a fixed $R_1>0$ such that for every $\delta>0$ there is $\varepsilon_0>0$ such that*

$$w(r)\leqslant e^{-(\varepsilon^{-1-\beta}-\delta)^{1/2}(r-R_1)}\quad \textit{for all } r\geqslant R_1 \textit{ and all } 0<\varepsilon\leqslant\varepsilon_0.$$

Proof. We first remark that there is an a priori bound M for every radial, decreasing solution w of (6.3), that is, $w(r)\leqslant M$ for all $r\geqslant 0$. Moreover given a $\sigma>0$ there is some $R_0>0$ such that for every radial solution w of (6.3)

$$w(r)\leqslant\sigma\quad\text{for all } r\geqslant R_0. \tag{6.13}$$

We apply this property with $\sigma>0$ given in Lemma 6.2 and find R_0 such that (6.13) holds. Let $\bar w$ be the function constructed in Lemma 6.2. Then, taking $\sigma>0$ smaller if necessary, we see that $\bar w$ is a supersolution to (6.3) and we deduce $w(r)\leqslant\bar w(r)$ for all $r\geqslant R_0$. In particular

$$w(R_0+1)\leqslant e^{-c_0\varepsilon^{-(1+\beta)/2}},$$

where $c_0>0$. Set $R_1=R_0+1$. Let $\delta>0$ be given and define:

$$\bar w_2(r)=e^{-c_0\varepsilon^{-(1+\beta)/2}}e^{-(\varepsilon^{-1-\beta}-\delta)^{1/2}(r-R_1)}.$$

Then $\bar w_2$ satisfies,

$$\bar w_2''=\left(\varepsilon^{-1-\beta}-\delta\right)\bar w_2,$$

and we can arrange the constants so that $\bar w_2$ is a supersolution. Indeed,

$$\Delta\bar w_2+\bar w_2^p-\frac{\bar w_2}{(\bar w_2+\varepsilon)^{1+\beta}}\leqslant-\frac{\delta}{2}\bar w_2+\bar w_2^p+\bar w_2\left(\varepsilon^{-1-\beta}-\frac{\delta}{2}-\frac{1}{(\bar w_2+\varepsilon)^{1+\beta}}\right).$$

Provided $\bar w_2^{p-1}\leqslant\delta/2$ and $\bar w_2/\varepsilon\leqslant 1/2$ we have:

$$\Delta\bar w_2+\bar w_2^p-\frac{\bar w_2}{(\bar w_2+\varepsilon)^{1+\beta}}\leqslant\bar w_2\left(-\frac{\delta}{2}+O\left(\frac{\bar w_2}{\varepsilon^{2+\beta}}\right)\right)\leqslant 0,$$

if also $\bar w_2\leqslant C\delta\varepsilon^{2+\beta}$. But since $\bar w_2(r)\leqslant e^{-c_0\varepsilon^{-(1+\beta)/2}}$ for $r\geqslant R_1$ we can achieve the inequalities by taking $\varepsilon>0$ small, depending on δ. By comparison

$$w(r)\leqslant\bar w_2(r)\quad\forall r\geqslant R_1.\qquad\square$$

Lemma 6.4. *Let w be a radial, decreasing solution of* (6.1)*. There are fixed numbers $R_2>0$, $C>0$ and $\varepsilon_0>0$ such that*

$$w\left(\rho-\varepsilon^{-(1+\beta)/2}\right)\geqslant e^{-(\varepsilon^{-(1+\beta)/2}+C)(\rho-R_2)-C}\quad\textit{for all } r\geqslant R_2 \textit{ and all } 0<\varepsilon\leqslant\varepsilon_0.$$

Proof. There is a uniform lower bound for $w_{\rho,\varepsilon}(0)$. Indeed, at the origin $\Delta w_{\rho,\varepsilon}\leqslant 0$ and the equation yields $w_{\rho,\varepsilon}(0)^{p-1}(w_{\rho,\varepsilon}+\varepsilon)^{1+\beta}\geqslant 1$ and this implies $w_{\rho,\varepsilon}(0)\geqslant c$ for some $c>0$ independent of $\varepsilon\in(0,1]$ and of $\rho\geqslant 1$. The uniform estimate for the gradient Proposition 2.2 implies that if we fix $R_2>0$ small, then $w_{\rho,\varepsilon}(R_2)\geqslant c/2$ for all $0<\varepsilon\leqslant 1$ and all $\rho\geqslant 1$.

Let $\underline{w}(r)$ be the solution of

$$\underline{w}''+\frac{N-1}{R_2}\underline{w}'=\frac{1}{\varepsilon^{1+\beta}}\underline{w},\quad r\in(0,\rho),$$
$$\underline{w}(R_2)=c,\qquad \underline{w}(\rho)=0.$$

Then $\underline{w}$ is explicitly given by:

$$\underline{w}(r) = c\frac{e^{\lambda_+(\rho-r)} - e^{\lambda_-(\rho-r)}}{e^{\lambda_+(\rho-R_2)} - e^{\lambda_-(\rho-R_2)}}, \quad r \in [0,\rho],$$

where

$$\lambda_\pm = -\frac{N-1}{2R_2} \pm \sqrt{\left(\frac{N-1}{2R_2}\right)^2 + \varepsilon^{-1-\beta}}.$$

Moreover, $\underline{w}$ is a subsolution of (6.1). Therefore $w_{\rho,\varepsilon}(\rho - \varepsilon^{(1+\beta)/2}) \geqslant \underline{w}(\rho - \varepsilon^{(1+\beta)/2})$, so

$$w_{\rho,\varepsilon}\big(\rho - \varepsilon^{(1+\beta)/2}\big) \geqslant c\frac{e^{\lambda_+\varepsilon^{(1+\beta)/2}} - e^{\lambda_-\varepsilon^{(1+\beta)/2}}}{e^{\lambda_+(\rho-R_2)} - e^{\lambda_-(\rho-R_2)}}.$$

Since

$$\lambda_\pm = \pm\varepsilon^{-(1+\beta)/2} - \frac{N-1}{2R_2} + O\left(\frac{\varepsilon^{(1+\beta)/2}}{R_2^2}\right)$$

the conclusion follows. □

Proof of Proposition 6.1. We prove first the upper estimate. Let w_ε denote a radial, decreasing least energy solution of (6.3). Let $\rho > 0$ and v_ρ be the solution to

$$\Delta v_\rho = \frac{1}{\varepsilon^{1+\beta}} v_\rho \quad \text{in } B_\rho \setminus \overline{B}_{\rho-1}, \tag{6.14}$$

with $v_\rho(\rho-1) = w_\varepsilon(\rho-1)$ and $v_\rho(\rho) = 0$. Define:

$$\bar{w}_{\rho,\varepsilon}(r) = \begin{cases} w_\varepsilon(r) & \text{if } 0 \leqslant r \leqslant \rho-1, \\ v_\rho(r) & \text{if } \rho-1 \leqslant r \leqslant \rho. \end{cases}$$

Then

$$c_{\rho,\varepsilon} = J_{\rho,\varepsilon}(w_{\rho,\varepsilon}) = \max_{t\geqslant 0} J_{\rho,\varepsilon}(tw_{\rho,\varepsilon}) \leqslant \max_{t\geqslant 0} J_{\rho,\varepsilon}(t\bar{w}_{\rho,\varepsilon}) = J_{\rho,\varepsilon}(t_{\rho,\varepsilon}\bar{w}_{\rho,\varepsilon}),$$

where $t_{\rho,\varepsilon} > 0$ is the unique t where the last maximum is attained. Since $w_{\rho,\varepsilon} \to w_\varepsilon$ as $\rho \to +\infty$ uniformly for $\varepsilon > 0$ small, we have $t_{\rho,\varepsilon} \to 1$ as $\rho \to +\infty$, uniformly for $\varepsilon > 0$ small. Then

$$J_{\rho,\varepsilon}(t_{\rho,\varepsilon}\bar{w}_{\rho,\varepsilon}) \leqslant \int_{B_{\rho-1}} \frac{1}{2}t_{\rho,\varepsilon}^2|\nabla w_\varepsilon|^2 + G_\varepsilon(t_{\rho,\varepsilon}w_\varepsilon) - \frac{(t_{\rho,\varepsilon}w_\varepsilon)^{p+1}}{p+1} + \int_{B_\rho\setminus B_{\rho-1}} \frac{1}{2}t_{\rho,\varepsilon}^2|\nabla v_\rho|^2 + G_\varepsilon(t_{\rho,\varepsilon}v_\rho).$$

We have the following expansion for G_ε

$$G_\varepsilon(u) = \frac{u^2}{2\varepsilon^{1+\beta}} + O\left(\frac{u^3}{\varepsilon^{2+\beta}}\right), \tag{6.15}$$

where $O(\frac{u^3}{\varepsilon^{2+\beta}})$ is uniform for $u \leqslant \varepsilon/2$. Let $\delta > 0$ be given and $\varepsilon_0 > 0$, $R_1 > 0$ be as in Lemma 6.3. Let us work with $\rho \geqslant R_1 + 2$. Then we have $v_\rho(r) \leqslant e^{-(\varepsilon^{-1-\beta}-\delta)^{1/2}}$ for $r \in [\rho-1,\rho]$. Thus by taking $\varepsilon > 0$ small we have

$$\begin{aligned} J_{\rho,\varepsilon}(t_{\rho,\varepsilon}\bar{w}_{\rho,\varepsilon}) \leqslant & \int_{\mathbb{R}^N} \frac{1}{2}t_{\rho,\varepsilon}^2|\nabla w_\varepsilon|^2 + G_\varepsilon(t_{\rho,\varepsilon}w_\varepsilon) - \frac{(t_{\rho,\varepsilon}w_\varepsilon)^{p+1}}{p+1} \\ & + \int_{B_\rho\setminus B_{\rho-1}} \left(\frac{1}{2}t_{\rho,\varepsilon}^2|\nabla v_\rho|^2 + t_{\rho,\varepsilon}^2\frac{v_\rho^2}{2\varepsilon^{1+\beta}} + Ct_{\rho,\varepsilon}^3\frac{v_\rho^3}{\varepsilon^{2+\beta}}\right) \end{aligned}$$

since also $\frac{1}{2}t_{\rho,\varepsilon}^2|\nabla w_\varepsilon|^2 + G_\varepsilon(t_{\rho,\varepsilon}w_\varepsilon) - \frac{(t_{\rho,\varepsilon}w_\varepsilon)^{p+1}}{p+1} \geqslant 0$ for $r \geqslant \rho - 1$. Therefore

$$c_{\rho,\varepsilon} \leqslant J_{\mathbb{R}^N,\varepsilon}(t_{\rho,\varepsilon} w_\varepsilon) - \frac{1}{2} t_{\rho,\varepsilon}^2 a_N \rho^{N-1} v_\rho'(\rho-1) v_\rho(\rho-1) + C \frac{t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{B_\rho \setminus B_{\rho-1}} v_\rho^3$$

$$\leqslant c_\varepsilon - \frac{1}{2} t_{\rho,\varepsilon}^2 \int_{\partial B_{\rho-1}} v_\rho' v_\rho + C \frac{t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{B_\rho \setminus B_{\rho-1}} v_\rho^3.$$

To estimate the last integral note that $v_\rho \leqslant w_\varepsilon$ for $\rho - 1 \leqslant r \leqslant \rho$ because w_ε is a supersolution of (6.14). Hence

$$\int_{B_\rho \setminus B_{\rho-1}} v_\rho^3 \leqslant C \rho^{N-1} e^{-3(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho - R_1 - 1)},$$

so

$$C \frac{t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{B_\rho \setminus B_{\rho-1}} v_\rho^3 \leqslant e^{-3(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho - R_1 - 1) + C_1 \log\rho + C_2 \log(1/\varepsilon) + C_3}.$$

To estimate the boundary integral let z be the solution to

$$\begin{cases} \Delta z = 0 & \text{in } B_\rho \setminus \overline{B}_{\rho-1}, \\ z(\rho-1) = 1, & z(\rho) = 0. \end{cases}$$

Then

$$\int_{\partial B_{\rho-1}} v_\rho' = \int_{\partial B_{\rho-1}} v_\rho z' - \frac{1}{\varepsilon^{1+\beta}} \int_{B_\rho \setminus B_{\rho-1}} v_\rho z.$$

As before

$$\frac{1}{\varepsilon^{1+\beta}} \int_{B_\rho \setminus B_{\rho-1}} v_\rho z \leqslant e^{-(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho - R_1 - 1) + C_1 \log\rho + C_2 \log(1/\varepsilon) + C_3},$$

so

$$c_{\rho,\varepsilon} \leqslant c_\varepsilon + e^{-2(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho - R_1 - 1) + C_1 \log\rho + C_2 \log(1/\varepsilon) + C_3}.$$

Taking a fixed $\delta > 0$, we obtain for $\rho \geqslant R_1 + 2$ the inequality (6.6).

To prove the lower bound, let $w_{\rho,\varepsilon}$ be a radial, decreasing least energy solution of (6.1). We let also $\tilde{v}_\rho$ be the solution of

$$\Delta \tilde{v}_\rho = \frac{1}{\varepsilon^{1+\beta}} \tilde{v}_\rho \quad \text{in } \mathbb{R}^N \setminus \overline{B}_{\rho - \varepsilon^{(1+\beta)/2}},$$

with $\tilde{v}_\rho(\rho - \varepsilon^{(1+\beta)/2}) = w_{\rho,\varepsilon}(\rho - \varepsilon^{(1+\beta)/2})$ and $\lim_{r\to+\infty} \tilde{v}_\rho(r) = 0$. Define:

$$\bar{w}_{\rho,\varepsilon}(r) = \begin{cases} w_{\rho,\varepsilon}(r) & \text{if } 0 \leqslant r \leqslant \rho - \varepsilon^{(1+\beta)/2}, \\ \tilde{v}_\rho(r) & \text{if } r \geqslant \rho - \varepsilon^{(1+\beta)/2}. \end{cases}$$

The for all $t > 0$,

$$c_{\rho,\varepsilon} \geqslant J_{\rho,\varepsilon}(t w_{\rho,\varepsilon}) \geqslant J_{\mathbb{R}^N,\varepsilon}(t \bar{w}_{\rho,\varepsilon}) + \int_{B_\rho \setminus B_{\rho - \varepsilon^{(1+\beta)/2}}} \left(\frac{1}{2} t^2 |\nabla w_{\rho,\varepsilon}|^2 + G_\varepsilon(t w_{\rho,\varepsilon}) - \frac{(t w_{\rho,\varepsilon})^{p+1}}{p+1} \right)$$

$$- \int_{\mathbb{R}^N \setminus B_{\rho - \varepsilon^{(1+\beta)/2}}} \left(\frac{1}{2} t^2 |\nabla \tilde{v}_\rho|^2 + G_\varepsilon(t \tilde{v}_\rho) \right).$$

We take now $t = t_{\rho,\varepsilon}$ such that $J_{\mathbb{R}^N,\varepsilon}(t_{\rho,\varepsilon} \bar{w}_{\rho,\varepsilon}) \geqslant c_\varepsilon$. Since $w_{\rho,\varepsilon} \to w_\varepsilon$ as $\rho \to +\infty$ uniformly for $\varepsilon > 0$ small, we have $t_{\rho,\varepsilon} \to 1$ as $\rho \to +\infty$, uniformly for $\varepsilon > 0$ small.

We use the expansion (6.15). Let $\delta > 0$ be given and $\varepsilon_0 > 0$, $R_1 > 0$ be as in Lemma 6.3. Let us work with $\rho \geqslant R_1 + 2$. Then we have $w_{\rho,\varepsilon}(r) \leqslant e^{-(\varepsilon^{-1-\beta}-\delta)^{1/2}}$ for $\rho - 1 \leqslant r \leqslant \rho$. Thus by taking $\varepsilon > 0$ small we can estimate:

$$\begin{aligned}
&\int\limits_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} G_\varepsilon(t_{\rho,\varepsilon} w_{\rho,\varepsilon}) - \frac{(t_{\rho,\varepsilon} w_{\rho,\varepsilon})^{p+1}}{p+1} \\
&\quad \geqslant \int\limits_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} t_{\rho,\varepsilon}^2 \frac{w_{\rho,\varepsilon}^2}{2\varepsilon^{1+\beta}} - C t_{\rho,\varepsilon}^3 \frac{w_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} - \frac{(t_{\rho,\varepsilon} w_{\rho,\varepsilon})^{p+1}}{p+1} \\
&\quad = \frac{t_{\rho,\varepsilon}^2}{2\varepsilon^{1+\beta}} \int\limits_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} w_{\rho,\varepsilon}^2 \left(1 - \frac{2\varepsilon^{1+\beta}}{p+1} t_{\rho,\varepsilon}^{p-1} w_{\rho,\varepsilon}^{p-1}\right) - \frac{C t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int\limits_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} w_{\rho,\varepsilon}^3 \\
&\quad \geqslant \frac{t_{\rho,\varepsilon}^2}{2\varepsilon^{1+\beta}} m_{\rho,\varepsilon} \int\limits_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} w_{\rho,\varepsilon}^2 - \frac{C t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int\limits_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} w_{\rho,\varepsilon}^3,
\end{aligned}$$

where

$$m_{\rho,\varepsilon} = \max_{\rho-\varepsilon^{(1+\beta)/2} \leqslant r \leqslant \rho} \left(1 - \frac{2\varepsilon^{1+\beta}}{p+1} t_{\rho,\varepsilon}^{p-1} w_{\rho,\varepsilon}^{p-1}\right).$$

We also have $\tilde{v}_\rho(r) \leqslant e^{-(\varepsilon^{-1-\beta}-\delta)^{1/2}}$ for $r \geqslant \rho - 1$. Then for $\varepsilon > 0$ small we can estimate:

$$\int\limits_{\mathbb{R}^N \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} \left(\frac{1}{2} t^2 |\nabla \tilde{v}_\rho|^2 + G_\varepsilon(t \tilde{v}_\rho)\right) \leqslant \int\limits_{\mathbb{R}^N \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} \left(\frac{1}{2} t^2 |\nabla \tilde{v}_\rho|^2 + t_{\rho,\varepsilon}^2 \frac{\tilde{v}_\rho^2}{2\varepsilon^{1+\beta}} + C t_{\rho,\varepsilon}^3 \frac{\tilde{v}_\rho^3}{\varepsilon^{2+\beta}}\right).$$

Let z be the solution to

$$\Delta z = \frac{m_{\rho,\varepsilon}}{\varepsilon^{1+\beta}} z \quad \text{in } B_\rho \setminus \overline{B}_{\rho-\varepsilon^{(1+\beta)/2}},$$

with $z(\rho - \varepsilon^{(1+\beta)/2}) = w_{\rho,\varepsilon}(\rho - \varepsilon^{(1+\beta)/2})$ and $z(\rho) = 0$. Then,

$$\begin{aligned}
c_{\rho,\varepsilon} &\geqslant c_\varepsilon + \frac{t_{\rho,\varepsilon}^2}{2} \int\limits_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} \left(|\nabla z|^2 + \frac{m_{\rho,\varepsilon}}{\varepsilon^{1+\beta}} z^2\right) \\
&\quad - \frac{t_{\rho,\varepsilon}^2}{2} \int\limits_{\mathbb{R}^N \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} \left(|\nabla \tilde{v}_\rho|^2 + \frac{1}{\varepsilon^{1+\beta}} \tilde{v}_\rho^2\right) \\
&\quad - \frac{C t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int\limits_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} w_{\rho,\varepsilon}^3 - \frac{C t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int\limits_{\mathbb{R}^N \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} \tilde{v}_\rho^3 \\
&= c_\varepsilon - \frac{t_{\rho,\varepsilon}^2}{2} w_{\rho,\varepsilon}\big(\rho - \varepsilon^{(1+\beta)/2}\big) \int\limits_{\partial B_{\rho-\varepsilon^{(1+\beta)/2}}} \left(\frac{\partial z}{\partial \nu} - \frac{\partial \tilde{v}_\rho}{\partial \nu}\right) \\
&\quad - \frac{C t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int\limits_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} w_{\rho,\varepsilon}^3 - \frac{C t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int\limits_{\mathbb{R}^N \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} \tilde{v}_\rho^3.
\end{aligned}$$

By similar estimates as for the upper bound,

$$\frac{t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} w_{\rho,\varepsilon}^3 \leqslant e^{-3(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho-R_1-1)+C_1\log\rho+C_2\log(1/\varepsilon)+C_3},$$

and

$$\frac{Ct_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{\mathbb{R}^N \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} \tilde{v}_\rho^3 \leqslant e^{-3(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho-R_1-1)+C_1\log\rho+C_2\log(1/\varepsilon)+C_3}.$$

Using suitable barriers one can prove:

$$\tilde{v}_\rho'\big(\rho-\varepsilon^{(1+\beta)/2}\big) \geqslant w_{\rho,\varepsilon}\big(\rho-\varepsilon^{(1+\beta)/2}\big)\left[-\varepsilon^{-(1+\beta)/2} - \frac{N-1}{2(\rho-\varepsilon^{-(1+\beta)/2})} + O\left(\frac{\varepsilon^{(1+\beta)/2}}{\rho^2}\right)\right],$$

and

$$z'\big(\rho-\varepsilon^{(1+\beta)/2}\big) \leqslant -w_{\rho,\varepsilon}\big(\rho-\varepsilon^{(1+\beta)/2}\big)\frac{m_{\rho,\varepsilon}^{1/2}}{\varepsilon^{(1+\beta)/2}}\frac{e^{m_{\rho,\varepsilon}^{1/2}}+e^{-m_{\rho,\varepsilon}^{1/2}}}{e^{m_{\rho,\varepsilon}^{1/2}}-e^{-m_{\rho,\varepsilon}^{1/2}}}.$$

This implies

$$\tilde{v}_\rho'\big(\rho-\varepsilon^{(1+\beta)/2}\big) - z'\big(\rho-\varepsilon^{(1+\beta)/2}\big) \geqslant w_{\rho,\varepsilon}\big(\rho-\varepsilon^{(1+\beta)/2}\big)^2\left[\varepsilon^{-(1+\beta)/2}\left(m_{\rho,\varepsilon}^{1/2}\frac{e^{m_{\rho,\varepsilon}^{1/2}}+e^{-m_{\rho,\varepsilon}^{1/2}}}{e^{m_{\rho,\varepsilon}^{1/2}}-e^{-m_{\rho,\varepsilon}^{1/2}}}-1\right)-\frac{C}{\rho}\right].$$

Since $m_{\rho,\varepsilon} \to 1$ as $\rho \to +\infty$ uniformly for $\varepsilon > 0$, combining the previous inequalities with Lemma 6.4, we deduce:

$$c_{\rho,\varepsilon} \geqslant c_\varepsilon + e^{-2(\varepsilon^{-(1+\beta)/2}+C)(\rho-R_2)-C\log\rho-C\log(1/\varepsilon)-C},$$

where $R_2 > 0$ is a small constant. From this we obtain (6.5) for ρ large. □

7. Asymptotic behavior as $\lambda \to +\infty$, Part 2

In this section we prove Theorem 1.2 following the argument of [9]. For $\lambda > 0$ and $\varepsilon > 0$, let $u_{\lambda,\varepsilon}$ denote the solution of (1.3) obtained through the mountain pass theorem, as in the proof of Theorem 1.1. Let $x_{\lambda,\varepsilon} \in \Omega$ denote a point where $u_{\lambda,\varepsilon}$ attains its maximum. It will be convenient to introduce the rescaled functions,

$$v_{\lambda,\varepsilon}(x) = \lambda^{\frac{1}{p+\beta}} u_{\lambda,\varepsilon}\big(x_{\lambda,\varepsilon} + \lambda^{-\frac{1+\beta}{2(p+\beta)}}x\big), \quad x \in \Omega_{\lambda,\varepsilon},$$

where $\Omega_{\lambda,\varepsilon} = \lambda^{\frac{1+\beta}{2(p+\beta)}}(\Omega - x_{\lambda,\varepsilon})$. Then $v_{\lambda,\varepsilon}$ solves

$$-\Delta v + \frac{v}{(v+\varepsilon\lambda^{\frac{1}{p+\beta}})^{1+\beta}} = v^p \quad \text{in } \Omega_{\lambda,\varepsilon}, \qquad v = 0 \quad \text{on } \partial\Omega_{\lambda,\varepsilon}. \tag{7.1}$$

Associated to (7.1) we have the functional,

$$J_{\lambda,\varepsilon}(v) = \int_{\Omega_{\lambda,\varepsilon}} \left(\frac{1}{2}|\nabla v|^2 + G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}(v) - \frac{(v^+)^{p+1}}{p+1}\right), \quad v \in H_0^1(\Omega_{\lambda,\varepsilon}),$$

with least energy

$$c_{\lambda,\varepsilon} = \inf_{v\in H_0^1(\Omega_{\lambda,\varepsilon}), v\neq 0} \sup_{t\geqslant 0} J_{\lambda,\varepsilon}(tv).$$

Let c_ε be the value defined in (6.4) and $\bar{c}_{\lambda,\varepsilon} = c_{\varepsilon\lambda^{1/(p+\beta)}}$. Let

$$d_{\lambda,\varepsilon} = \operatorname{dist}(0, \partial\Omega_{\lambda,\varepsilon}) \quad \text{and} \quad d_{max} = \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$$

so that

$$\max_{x\in\Omega_{\lambda,\varepsilon}} \operatorname{dist}(x, \partial\Omega_{\lambda,\varepsilon}) = \lambda^{\frac{1+\beta}{2(p+\beta)}} d_{max}.$$

We will prove that

$$c_{\lambda,\varepsilon} \leqslant \bar{c}_{\lambda,\varepsilon} + \exp\bigl(-2\bigl(\varepsilon^{-\frac{1+\beta}{2}}\lambda^{-\frac{1+\beta}{2(p+\beta)}} - C\bigr)\bigl(\lambda^{\frac{1+\beta}{2(p+\beta)}} d_{max} - R_1\bigr) + C\log(\lambda)\bigr), \tag{7.2}$$

and

$$c_{\lambda,\varepsilon} \geqslant \bar{c}_{\lambda,\varepsilon} + \exp\bigl(-2\bigl(\varepsilon^{-\frac{1+\beta}{2}}\lambda^{-\frac{1+\beta}{2(p+\beta)}} + C\bigr)\bigl(d_{\lambda,\varepsilon} + o(1) - R_2\bigr) - C\log(\lambda)\bigr), \tag{7.3}$$

where for each fixed λ large, $o(1) \to 0$ as $\varepsilon \to 0$, and C, R_1, R_2 are constants. Inequalities (7.2) and (7.3) imply that

$$\liminf_{\varepsilon\to 0} \operatorname{dist}(x_{\lambda,\varepsilon}, \partial\Omega) \geqslant d_{max} - C\lambda^{-\frac{1+\beta}{2(p+\beta)}} \tag{7.4}$$

for some constant C.

We postpone the proof of (7.2), (7.3) and finish the proof of the theorem. As in the proof of Theorem 1.1 up to a subsequence $u_{\lambda,\varepsilon}$ converges in $C^1(\overline{\Omega})$ as $\varepsilon \to 0$ to a non-zero function $u_\lambda \in C^{1,\mu}(\overline{\Omega})$ which solves (1.1). Let $x_\lambda \in \Omega$ be a maximum point of u_λ. Then (7.4) implies that

$$\operatorname{dist}(x_\lambda, \partial\Omega) \geqslant d_{max} - C\lambda^{-\frac{1+\beta}{2(p+\beta)}}.$$

By Lemma 5.4, for a large $\lambda > 0$ the solution u_λ has support contained in a ball $B_{R_\lambda}(x_\lambda)$ with $R_\lambda = C\lambda^{-\frac{1+\beta}{2(p+\beta)}}$ for some constant C. Thus by taking $\lambda > 0$ large, u_λ has compact support in Ω and hence solves,

$$-\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u^p \quad \text{in } \mathbb{R}^N.$$

Let

$$v_\lambda(x) = \lambda^{\frac{1}{p+\beta}} u_\lambda\bigl(x_\lambda + \lambda^{-\frac{1+\beta}{2(p+\beta)}} x\bigr),$$

so that v_λ solves

$$-\Delta v_\lambda + \chi_{[v_\lambda>0]} v_\lambda^{-\beta} = v_\lambda^p \quad \text{in } \mathbb{R}^N.$$

Let J, M be defined by (4.3). By Lemma 5.2

$$J(v) \leqslant J(\varphi)$$

for all $\varphi \in H_0^1(\Omega_\lambda)$ where $\Omega_\lambda = \lambda^{\frac{1+\beta}{2(p+\beta)}}(\Omega - x_\lambda)$, which satisfies $M(\varphi) = 0$. By Proposition 4.1, $v = w$ because we have fixed the maximum of v at the origin. This concludes the proof of Theorem 1.2, up to (7.2) and (7.3).

The proof of (7.2) and (7.3) are similar to those in [9], except for one estimate of the energy. For completeness we give the details. To prove the upper bound (7.2) let $\bar{x}_{\lambda,\varepsilon} \in \Omega_{\lambda,\varepsilon}$ be a point that realizes the maximum distance to $\partial\Omega_{\lambda,\varepsilon}$. Since the least energy values for the functional $J_{\lambda,\varepsilon}$ in $\Omega_{\lambda,\varepsilon}$ and the ball with center $\bar{x}_{\lambda,\varepsilon}$ and radius $\lambda^{-\frac{1+\beta}{2(p+\beta)}} d_{max}$ are ordered we have:

$$c_{\lambda,\varepsilon} \leqslant c_{\rho,\varepsilon\lambda^{\frac{1}{p+\beta}}},$$

where $c_{\rho,\varepsilon}$ is defined in (6.2) and $\rho = \lambda^{\frac{1+\beta}{2(p+\beta)}} d_{max}$. By estimate (6.5) we deduce (7.2).

We now derive (7.3). For simplicity of the notation we write $v = v_{\lambda,\varepsilon}$. Up to subsequence we can assume that $x_{\lambda,\varepsilon} \to x_\lambda \in \overline{\Omega}$ as $\varepsilon \to 0$. Then

$$\operatorname{dist}(x_{\lambda,\varepsilon}, \partial\Omega) \to \operatorname{dist}(x_\lambda, \partial\Omega) \quad \text{as } \varepsilon \to 0.$$

Let $R_{\lambda,\varepsilon} = \operatorname{dist}(x_{\lambda,\varepsilon}, \partial\Omega)$ and $R_0 = \operatorname{dist}(x_\lambda, \partial\Omega)$. Let $\delta > 0$ be given and take $R_0' > 0$ such that

$$vol\bigl(B_{R_0'}(x_\lambda)\bigr) = vol\bigl(\Omega \cap B_{R_0+\delta}(x_\lambda)\bigr).$$

Now let $0 < \delta' < \delta$ be such that $R_0' < R_0 + \delta'$. Let η be a $C^\infty(\mathbb{R})$ cut-off function such that $\eta(s) = 1$ for $0 \leqslant s \leqslant R_{\lambda,\varepsilon} + \delta'$, $\eta(s) = 0$ for $s \geqslant R_{\lambda,\varepsilon} + \delta$, and with uniformly bounded gradient. We set $\eta_\lambda(s) = \eta(\lambda^{-\frac{1+\beta}{2(p+\beta)}} s)$, and

$$\tilde{v}(x) = v(x)\eta_\lambda\big(|x|\big), \quad x \in \Omega_{\lambda,\varepsilon}.$$

We claim that for every $t \in [0, t_0]$ (here t_0 is a large positive constant),

$$J_{\lambda,\varepsilon}(t\tilde{v}) \leqslant J_{\lambda,\varepsilon}(tv) + \exp\Big(-2\big(\big(\varepsilon\lambda^{\frac{1}{p+\beta}}\big)^{-1-\beta} - 1\big)^{1/2}\big((R_{\lambda,\varepsilon} + \delta')\lambda^{\frac{1+\beta}{2(p+\beta)}} - R_1\big) + C\log(1/\varepsilon) + C\log\lambda\Big). \tag{7.5}$$

Indeed,

$$\begin{aligned} J_{\lambda,\varepsilon}(t\tilde{v}) &= \int_{\Omega_{\lambda,\varepsilon}} \left(\frac{t^2}{2}|\nabla\tilde{v}|^2 + G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}(t\tilde{v}) - \frac{t^{p+1}}{p+1}\tilde{v}^{p+1}\right) \\ &\leqslant J_{\lambda,\varepsilon}(tv) + t^2\int_{\Omega_{\lambda,\varepsilon}} \eta_\lambda\big(|x|\big)|\eta_\lambda'\big(|x|\big)v|\nabla v| + \frac{t^2}{2}\int_{\Omega_{\lambda,\varepsilon}} v^2\big|\eta_\lambda'\big(|x|\big)\big|^2 \\ &\quad + \int_{\Omega_{\lambda,\varepsilon}} \big(G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}(tv) - G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}\big(tv\eta_\lambda\big(|x|\big)\big)\big) + \frac{t^{p+1}}{p+1}\int_{\Omega_{\lambda,\varepsilon}} v^{p+1}\big(1 - \eta_\lambda\big(|x|\big)^{p+1}\big). \end{aligned}$$

Using the same supersolutions as in Lemmas 6.2 and 6.3 we find R_1 and $\varepsilon_0 > 0$ such that

$$v(x) \leqslant \exp\Big(-\big(\big(\varepsilon\lambda^{\frac{1}{p+\beta}}\big)^{-1-\beta} - 1\big)^{1/2}\big(|x| - R_1\big)\Big) \tag{7.6}$$

for all $x \in \Omega_{\lambda,\varepsilon}$ with $|x| \geqslant R_1$, and all $0 < \varepsilon \leqslant \varepsilon_0$. This implies also a similar estimate for the gradient, namely

$$\big|\nabla v(x)\big| \leqslant \exp\Big(-\big(\big(\varepsilon\lambda^{\frac{1}{p+\beta}}\big)^{-1-\beta} - 1\big)^{1/2}\big(|x| - R_1\big) + C\log(1/\varepsilon) + C\log\lambda\Big),$$

for all $x \in \Omega_{\lambda,\varepsilon}$ with $|x| \geqslant R_1$, and all $0 < \varepsilon \leqslant \varepsilon_0$. We can write,

$$G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}(tv) = \frac{v^2}{2\varepsilon^{1+\beta}\lambda^{\frac{1+\beta}{p+\beta}}} + O\left(\frac{v^3}{\varepsilon^{2+\beta}\lambda^{\frac{2+\beta}{p+\beta}}}\right),$$

provided $\frac{v}{\varepsilon\lambda^{\frac{1}{p+\beta}}} \leqslant 1/2$. Using (7.6) we see that this holds in $\Omega_{\lambda,\varepsilon} \setminus B_{r(\lambda,\varepsilon)}(0)$, where $r(\lambda,\varepsilon) = (R_{\lambda,\varepsilon} + \delta')\lambda^{\frac{1+\beta}{2(p+\beta)}}$. So we may estimate

$$\begin{aligned} &\int_{\Omega_{\lambda,\varepsilon}} \big(G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}(tv) - G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}\big(tv\eta_\lambda(x)\big)\big) \\ &\quad \leqslant \frac{Ct^3}{\varepsilon^{2+\beta}\lambda^{\frac{2+\beta}{p+\beta}}} \int_{\Omega\lambda,\varepsilon \setminus B(0, R_{\lambda,\varepsilon}\lambda^{\frac{1+\beta}{2(p+\beta)}})} v^3 \\ &\quad \leqslant \exp\Big(-3\big(\big(\varepsilon\lambda^{\frac{1}{p+\beta}}\big)^{-1-\beta} - 1\big)^{1/2}\big(r(\lambda,\varepsilon) - R_1\big) + C\log(1/\varepsilon) + C\log\lambda\Big). \end{aligned}$$

Similarly we find the following estimates:

$$\begin{aligned} &\frac{t^{p+1}}{p+1}\int_{\Omega_{\lambda,\varepsilon}} v^{p+1}\big(1 - \eta_\lambda\big(|x|\big)^{p+1}\big) \\ &\quad \leqslant \exp\Big(-(p+1)\big(\big(\varepsilon\lambda^{\frac{1}{p+\beta}}\big)^{-1-\beta} - 1\big)^{1/2}\big(r(\lambda,\varepsilon) - R_1\big) + C\log(1/\varepsilon) + C\log\lambda\Big), \\ &t^2\int_{\Omega_{\lambda,\varepsilon}} \eta_\lambda\big(|x|\big)|\eta_\lambda'\big(|x|\big)v|\nabla v| \\ &\quad \leqslant \exp\Big(-2\big(\big(\varepsilon\lambda^{\frac{1}{p+\beta}}\big)^{-1-\beta} - 1\big)^{1/2}\big(r(\lambda,\varepsilon) - R_1\big) + C\log(1/\varepsilon) + C\log\lambda\Big), \end{aligned}$$

and

$$\frac{t^2}{2}\int\limits_{\Omega_{\lambda,\varepsilon}} v^2\left|\eta'_\lambda\left(|x|\right)\right|^2 \leqslant \exp\left(-2\left(\left(\varepsilon\lambda^{\frac{1}{p+\beta}}\right)^{-1-\beta}-1\right)^{1/2}\left(r(\lambda,\varepsilon)-R_1\right)+C\log(1/\varepsilon)+C\log\lambda\right),$$

which combined imply (7.5).

Let $R'_{\lambda,\varepsilon}>0$ be such that

$$vol(B_{R'_{\lambda,\varepsilon}}) = vol(\Omega\cap B_{R_{\lambda,\varepsilon}+\delta}).$$

Using the Schwarz symmetric rearrangement we find:

$$J_{\lambda,\varepsilon}(t\tilde{v}) \geqslant J_{\rho,\varepsilon\lambda^{\frac{1}{p+\beta}}}\left(t\tilde{v}^*\right)$$

where $c_{\rho,\varepsilon}$ is defined in (6.2), $\rho = R'_{\lambda,\varepsilon}\lambda^{\frac{1+\beta}{2(p+\beta)}}$ and $\tilde{v}^*$ is the radially decreasing rearrangement of $\tilde{v}$. We choose now $t\in[0,t_0]$ such that $J_{\rho,\varepsilon\lambda^{\frac{1}{p+\beta}}}(t\tilde{v}^*)$ is maximized, and find, using (7.5)

$$c_{\lambda,\varepsilon} \geqslant c_{\rho,\varepsilon\lambda^{\frac{1}{p+\beta}}} - \exp\left(-2\left(\left(\varepsilon\lambda^{\frac{1}{p+\beta}}\right)^{-1-\beta}-1\right)^{1/2}\left(\left(R_{\lambda,\varepsilon}+\delta'\right)\lambda^{\frac{1+\beta}{2(p+\beta)}}-R_1\right)+C\log(1/\varepsilon)+C\log\lambda\right).$$

Therefore, using (6.6),

$$\begin{aligned}
c_{\lambda,\varepsilon} &\geqslant \bar{c}_{\lambda,\varepsilon} + \exp\left(-2\left(\varepsilon^{-\frac{1+\beta}{2}}\lambda^{-\frac{1+\beta}{2(p+\beta)}}+C\right)\left(R'_{\lambda,\varepsilon}\lambda^{\frac{1+\beta}{2(p+\beta)}}-R_2\right)-C\log\lambda\right)\\
&\quad - \exp\left(-2\left(\left(\varepsilon\lambda^{\frac{1}{p+\beta}}\right)^{-1-\beta}-1\right)^{1/2}\left(\left(R_{\lambda,\varepsilon}+\delta'\right)\lambda^{\frac{1+\beta}{2(p+\beta)}}-R_1\right)+C\log(1/\varepsilon)+C\log\lambda\right)\\
&\geqslant \bar{c}_{\lambda,\varepsilon} + \exp\left(-2\left(\varepsilon^{-\frac{1+\beta}{2}}\lambda^{-\frac{1+\beta}{2(p+\beta)}}+C\right)(R_{\lambda,\varepsilon}+\delta)\lambda^{\frac{1+\beta}{2(p+\beta)}}-R_2\right)-C\log\lambda,
\end{aligned}$$

for $\varepsilon>0$ small. This establishes (7.3). □

8. Proof of Theorem 1.3

Let u_λ be the solution constructed in Theorem 1.1 and let

$$v_\lambda = \lambda^{\frac{1}{p-1}}u_\lambda.$$

Then v_λ satisfies:

$$\begin{cases} -\Delta v + \lambda^{\frac{1+\beta}{p-1}}v^{-\beta}\chi_{\{v>0\}} = v^p, & v\geqslant 0 \quad \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega. \end{cases} \tag{8.1}$$

Let us recall the classical equation:

$$\begin{cases} -\Delta v = v^p, & v\geqslant 0 \quad \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega, \end{cases} \tag{8.2}$$

which arises as the limit of (8.1) as $\lambda\to 0$. It is well known that (8.2) admits a nontrivial solution, for example by minimizing the $H_0^1(\Omega)$ norm on the unit sphere of $L^p(\Omega)$ or by using the mountain pass theorem [1]. If Ω is a ball, it is furthermore known that positive solutions of that (8.1) are radial [14] and that there is a unique positive solution, which is furthermore nondegenerate.

We will first prove:

Lemma 8.1. *Suppose Ω is the unit ball in $\mathbb{R}^N$. Then the solution v_λ to* (8.1) *converges in the $C^1(\overline{B}_1)$ sense to the unique positive solution of* (8.2).

Proof. Let v_0 denote the unique nontrivial solution of (8.2). We can show that for $\lambda > 0$ sufficiently close to 0, (8.1) in the ball has a solution close to v_0. One can construct this solution of the form $v = v_0 + w$, where w has now to satisfy, assuming w is small in $C^1(\overline{B}_1)$,

$$-\Delta w - p v_0^{p-1} w = (v_0 + w)^p - v_0^p - p v_0^{p-1} w - \lambda^{\frac{1+\beta}{p-1}} (v_0 + w)^{-\beta} \quad \text{in } \Omega,$$
$$w = 0 \quad \text{on } \partial\Omega.$$

Using the nondegeneracy of v_0 we may set up a fixed point argument, similar to the one in the proof of Theorem 1.5 and deduce that (8.1) has a solution v_λ for $\lambda > 0$. By construction $v_\lambda \to v_0$ as $\lambda \to 0$. But radial solutions to (8.1) are unique, see [7], and this proves the result. □

Proof of Theorem 1.3. An argument similar to that of the proof of Theorem 1.1 shows that $\|v_\lambda\|_{L^\infty(\Omega)}$ remains bounded as $\lambda \to 0$. Therefore, thanks to Proposition 2.1, v_λ is also bounded in $C^{1,\mu}(\overline{\Omega})$ where $\mu = \frac{1-\beta}{1+\beta}$. Up to subsequence v_λ then converges to a function $v \in C^1(\overline{\Omega})$. We have to discard the possibility that $v = 0$.

Let,

$$J_\lambda(v) = \int_\Omega \left(\frac{1}{2} |\nabla v|^2 + \frac{\lambda^{\frac{1+\beta}{p-1}}}{1-\beta} v_+^{1-\beta} - \frac{1}{p+1} |v|^{p+1} \right), \quad v \in H_0^1(\Omega),$$

so that its critical points give rise to solutions of (8.1). The solution u_λ constructed in Theorem 1.1 is a least energy solution and therefore

$$J_\lambda(v_\lambda) = \sup_{t \geqslant 0} J_\lambda(t\lambda v).$$

Let $R > 0$ be such that $vol(B_R(0)) = vol(\Omega)$ and let v_λ^* be the Schwarz symmetrization of v. Then for every $t \geqslant 0$,

$$J_\lambda(v_\lambda) \geqslant J_\lambda(t v_\lambda) \geqslant J_\lambda\big(t v_\lambda^*\big),$$

and therefore,

$$J_\lambda(v_\lambda) \geqslant \inf_{v \in H_0^1(B_R(0)),\, v \neq 0} \; \sup_{t \geqslant 0} J_\lambda\big(v, B_R(0)\big) = J_\lambda\big(v_{rad,\lambda}; B_R(0)\big),$$

where $v_{rad,\lambda}$ is the unique radial solution of (8.1) in the ball $B_R(0)$. By Lemma 8.1 $v_{rad,\lambda} \to v_{rad,0}$ as $\lambda \to 0$ in $C^1(\overline{B}_R(0))$, where $v_{rad,0}$ is the unique nontrivial solution of (8.2). Therefore there is $c > 0$ such that

$$J_\lambda(v_\lambda) \geqslant c$$

for $\lambda > 0$. It follows that $v = \lim_{\lambda \to 0} v_\lambda$ cannot be identically zero. Testing the equation with functions supported on the set $[v > 0]$ we see that

$$-\Delta v = v^p \quad \text{in } [v > 0].$$

But this and the Hopf lemma imply that $[v > 0] = \Omega$, that is, $v > 0$. Since $v_\lambda \to v$ in $C^1(\overline{\Omega})$, we deduce that in fact $v_\lambda > 0$ for $\lambda > 0$ small. □

9. Positive solutions for $p = 1$ and λ close to λ_1

We look for a solution of (1.10) of the form:

$$u = \varepsilon^{-\frac{1}{1+\beta}} \varphi \quad \text{for } \varepsilon = \lambda - \lambda_1 > 0 \text{ small.} \tag{9.1}$$

A calculation shows that u is a solution of (1.10) if and only if φ solves:

$$\begin{cases} -\Delta\varphi + \varepsilon\varphi^{-\beta} = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.2}$$

which we write as

$$\begin{cases} -\Delta\varphi - \lambda_1\varphi = -\varepsilon\varphi^{-\beta} + \varepsilon\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{9.3}$$

We are going to find a solution for (9.3) of the form $\varphi = c\varphi_1 + z$ with z orthogonal to φ_1 in $L^2(\Omega)$, and z small compared to φ_1 so that $(c\varphi_1 + z)^{-\beta} \leqslant c'\delta^{-\beta}$, for some constant $c' > 0$. Inserting the expression of φ into (9.2), one finds:

$$\begin{cases} -\Delta z - \lambda_1 z = \varepsilon\big[-(c\varphi_1 + z)^{-\beta} + c\varphi_1 + z\big] & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{9.4}$$

For a function $h : \Omega \to \mathbb{R}$ we introduce the norm,

$$\|h\|_\beta = \sup_{x\in\Omega} \big|h(x)\big|\delta(x)^\beta,$$

where

$$\delta(x) = \operatorname{dist}(x, \partial\Omega).$$

We recall, see [17], that if $\|h\|_\beta < \infty$ then the problem:

$$\begin{cases} \Delta u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution $u \in C(\overline{\Omega}) \cap C^{1,\nu}(\Omega)$ (for every $0 < \nu < 1$). Moreover by the results of Gui and Lin [18], see also del Pino [8], one has:

$$\|u\|_{C^{1,1-\beta}(\overline{\Omega})} \leqslant C\|h\|_\beta.$$

We need the following:

Lemma 9.1. *If $h : \Omega \to \mathbb{R}$ satisfies $\|h\|_\beta < \infty$ and $\int_\Omega h\varphi_1 = 0$ then there is a solution u to*

$$\begin{cases} -\Delta u - \lambda_1 u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.5}$$

with $\int_\Omega u\varphi_1 = 0$ and

$$\|u\|_{C^{1,1-\beta}(\overline{\Omega})} \leqslant C\|h\|_\beta. \tag{9.6}$$

Proof. If $h \in L^\infty(\Omega)$ and $\int_\Omega h\varphi_1 = 0$ then there is a solution $u \in C^{1,\nu}(\overline{\Omega})$ (for every $0 < \nu < 1$) to (9.5) with $\int_\Omega u\varphi_1 = 0$. Thus it is sufficient to prove (9.6) in this situation and then proceed by density.

First we prove that there is $C > 0$ depending only on Ω, β such that

$$\big|u(x)\big| \leqslant C\|h\|_\beta\delta(x)^{-N} \quad \text{for all } x \in \Omega. \tag{9.7}$$

By standard arguments, if u solves (9.5) and $\int_\Omega u\varphi_1 = 0$, then

$$\|u\|_{L^1(\Omega)} \leqslant C\|h\|_{L^1(\Omega)}$$

with C independent of u and h. Now let $x_0 \in \Omega$ and $r = \delta(x_0)/2$. Solve

$$-\Delta v - \lambda_1 v = h \quad \text{in } B_r(x_0),$$
$$v = 0 \quad \text{on } \partial B_r(x_0).$$

Then, by standard elliptic estimates,

$$\big|v(x_0)\big| \leqslant Cr^2 \sup_{x\in B_r(x_0)} \big|h(x)\big| \leqslant r^{2-\beta}\|h\|_\beta,$$

and

$$\|v\|_{L^1(B_r(x_0))} \leqslant Cr^2 \|h\|_{L^1(B_r(x_0))} \leqslant Cr^{N+2-\beta}\|h\|_\beta,$$

where C does not depend on h or r. Define $w = u - v$. Then

$$\Delta w + \lambda_1 w = 0 \quad \text{in } B_r(x_0).$$

Again by elliptic estimates

$$\begin{aligned} |w(x_0)| &\leqslant \frac{C}{r^N}\int_{B_r(x_0)} |w| \leqslant \frac{C}{r^N}\int_{B_r(x_0)} |v| + \frac{C}{r^N}\int_{B_r(x_0)} |u| \\ &\leqslant Cr^{2-\beta}\|h\|_\beta + Cr^{-N}\|u\|_{L^1(\Omega)} \leqslant Cr^{-N}\|h\|_\beta. \end{aligned}$$

This proves (9.7).

Fix a $\tau \in (0,1)$ and let $\eta > 0$ be small so that $\delta(x)$ is smooth in the region

$$N_\eta = \{x \in \Omega:\ \delta(x) < \eta\}.$$

Let $\bar u = k\|h\|_\beta \delta^\tau$. Then

$$-\Delta \bar u - \lambda_1 \bar u = k\|h\|_\beta\big(-\tau(\tau-1)|\nabla\delta|^2\delta^{\tau-2} - \tau\delta^{\tau-1}\Delta\delta - \lambda_1\delta^\tau\big) \geqslant |h| \quad \text{in } N_\eta,$$

if we take $\eta > 0$ small and k sufficiently large. Now we fix η. By increasing k if necessary we have $|u| \leqslant \bar u$ on ∂N_η. By the maximum principle, which is valid for the operator $-\Delta - \lambda_1$ in N_η, we deduce that $|u| \leqslant \bar u$ in N_η. Thus we have obtained

$$|\Delta u| \leqslant \lambda_1|u| + |h| \leqslant C\|h\|_\beta\delta^{-\beta}.$$

By the estimates of Gui and Lin [18] we deduce (9.6). □

Proof of Theorem 1.5. Let $0 < \underline{c} < \bar c$ be arbitrary constants, to be chosen later on. We are going to prove that for every $c \in [\underline{c}, \bar c]$ there is a solution z in an appropriate space and $\mu \in \mathbb{R}$ to the problem:

$$\begin{cases} -\Delta z - \lambda_1 z = \varepsilon\big[-(c\varphi_1 + z)^{-\beta} + c\varphi_1 + z\big] + \mu\varphi_1 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.8}$$

if $\varepsilon > 0$ is sufficiently small.

To do this consider the spaces $E = \{h : \Omega \to \mathbb{R}:\ \|h\|_\beta < \infty, h \perp \varphi_1\}$, $F = \{u \in C^{0,1}(\overline{\Omega}):\ u \perp \varphi_1\}$ where the orthogonality is with respect to the L^2 inner product. Define $S : E \to F$ by $Sh = u$ with u the solution of (9.5) such that $\int_\Omega u\varphi_1 = 0$. By Lemma 9.1, S is a linear bounded operator.

For $\rho > 0$ define $B_\rho := \{z \in C^{0,1}(\overline{\Omega}):\ \|z\|_{C^{0,1}} \leqslant \rho, z = 0 \text{ on } \partial\Omega\}$, where $\|z\|_{C^{0,1}(\overline{\Omega})}$ is the smallest Lipschitz constant of z.

Let $A > 0$, $B > 0$ be such that $A\delta \leqslant \varphi_1 \leqslant B\delta$ in Ω. If $\rho > 0$ is such that $\rho < \underline{c}A$ and $h \in B_\rho$ we have that $\|(c\varphi_1 + h)^{-\beta}\|_\beta < +\infty$. Thus, for such ρ we can define $\Phi : B_\rho \to C^{0,1}(\overline{\Omega})$ by:

$$\Phi(h) := S\big(\varepsilon\big[-(c\varphi_1 + h)^{-\beta} + c\varphi_1 + h\big] + \mu\varphi_1\big),$$

where $\mu \in \mathbb{R}$ is such that

$$\varepsilon\int_\Omega \big[-(c\varphi_1 + h)^{-\beta} + c\varphi_1 + h\big]\varphi_1 + \mu\int_\Omega \varphi_1^2 = 0.$$

Thus $z = \Phi(h)$ solves:

$$\begin{cases} -\Delta z - \lambda_1 z = \varepsilon\big[-(c\varphi_1 + h)^{-\beta} + c\varphi_1 + h\big] + \mu\varphi_1 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\int_\Omega z\varphi_1 = 0$.

For $\varepsilon > 0$ small enough Φ is a contraction in B_ρ. Indeed, if $h \in B_\rho$, then

$$\left\|\Phi(h)\right\|_{C^{0,1}} \leqslant \|S\|\left(\varepsilon\left\|-(c\varphi_1+h)^{-\beta}+c\varphi_1+h\right\|_\beta+\mu\|\varphi_1\|_\beta\right) \leqslant \rho,$$

for ε small enough, and

$$\left\|\Phi(z_1)-\Phi(z_2)\right\| \leqslant \varepsilon\|S\|\left\|(c\varphi_1+z_2)^{-\beta}-(c\varphi_1+z_1)^{-\beta}+z_1-z_2\right\| \leqslant \varepsilon C\|z_1-z_2\|.$$

Applying the Banach fixed point theorem, for $c \in [\underline{c}, \bar{c}]$ there is a solution $z(c) \in B_\rho$ of (9.8) with a corresponding $\mu(c) \in \mathbb{R}$, provided $\varepsilon > 0$ is small. Since Φ is continuous in c, by the fixed point characterization of $z(c)$ we deduce that it is continuous with respect to c, and hence $c \mapsto \mu(c)$ is also continuous.

We proceed to show that if we take $\underline{c} > 0$ small and $\bar{c} > 0$ is large, then for some $c \in [\underline{c}, \bar{c}]$ we have $\mu(c) = 0$. We have the expression:

$$\mu(c) = \frac{\int_\Omega -(c\varphi_1+z(c))^{-\beta}\varphi_1+(c\varphi_1+z(c))\varphi_1}{\int_\Omega \varphi_1^2}.$$

Let us work with $\underline{c} > 0$ small so that

$$-\underline{c}^{-\beta}(A/2+B)^{-\beta}A\int_\Omega \delta^{1-\beta}+\underline{c}(A/2+B)B\int_\Omega \delta^2 < 0,$$

and with $\rho < A\underline{c}/2$. Since $z \in B_\rho$ we have $|z| \leqslant \rho\delta$, and then $c\varphi_1 + z \leqslant (\frac{A}{2}+B)c\delta$. Hence

$$\left[-(c\varphi_1+z)^{-\beta}+c\varphi_1+z\right]\varphi_1 \leqslant -\left(\frac{A}{2}+B\right)^{-\beta}c^{-\beta}A\delta^{1-\beta}+\left(\frac{A}{2}+B\right)cB\delta^2.$$

This implies $\mu(\underline{c}) < 0$.

Similarly, using that $c\varphi + z \geqslant (cA-\rho)\delta \geqslant cA\delta/2$, we see that taking $\bar{c} > 0$ large enough $\mu(\bar{c}) > 0$. □

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