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Radial solutions of an elliptic equation with singular nonlinearity

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Abstract

For the equation

$$-\Delta u + u^{-\beta} = u^p, \qquad u > 0 \quad \text{in } B_R, \qquad u = 0 \quad \text{on } \partial B_R,$$

where $B_R \subseteq \mathbb{R}^N$, $0 < \beta < 1$ and $1 < p < \frac{N+2}{N-2}$ if $N \geqslant 3$, $1 < p < +\infty$ if $N = 2$, we show that there is $\bar{R} > 0$ such that a radial solution u_R exists if and only if $0 < R \leqslant \bar{R}$. It is unique in the class of radial solutions and $u'_R(R) < 0$ if $R < \bar{R}$, while $u'_{\bar{R}}(\bar{R}) = 0$. We also give a variational characterization of $u_{\bar{R}}$.

Keywords: Radial solutions; Uniqueness; Singular nonlinearity

1. Summary

In this work we are interested in radially symmetric solutions to the singular equation

$$\begin{cases} -\Delta u + u^{-\beta} = \lambda u^p & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1}$$

where B_1 is the unit ball in $\mathbb{R}^N$, $0 < \beta < 1$, $\lambda > 0$ is a parameter and $1 < p < \frac{N+2}{N-2}$ if $N \geqslant 3$, $1 < p < +\infty$ if $N = 2$. Solutions are understood as $u \in C^2(B_1) \cap C(\bar{B}_1)$.

Several authors [1,5,7,15,16] have studied existence and uniqueness of radial solutions for equations involving singular nonlinearities. Serrin and Tang [16] established uniqueness of radial solutions of $\Delta_m u + f(u) = 0$ in B_R with $u = u' = 0$ on ∂B_R provided $N > m > 1$ and f satisfies certain hypotheses, which allow $f(u) = -u^p + u^q$ with $p < q$ (no restriction on the sign of p, q).

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We prove

Theorem 1.1. *Let* $1 < p < \frac{N+2}{N-2}$ *if* $N \geqslant 3$, $1 < p < +\infty$ *if* $N = 2$. *There exists* $\bar{\lambda} > 0$ *such that* (1) *has a radial solution if and only if* $0 < \lambda \leqslant \bar{\lambda}$. *Moreover the radial solution* u_λ *is unique,* $u_\lambda \in C^1(\bar{B}_1)$ *and* $u'_\lambda(1) < 0$ *if* $0 < \lambda < \bar{\lambda}$ *and* $u'_\lambda(1) = 0$ *if* $\lambda = \bar{\lambda}$.

Eq. (1) is equivalent to

$$-\Delta u + u^{-\beta} = u^p, \qquad u > 0 \quad \text{in } B_R, \qquad u = 0 \quad \text{on } \partial B_R, \tag{2}$$

where $R > 0$ replaces the parameter λ. Thus we may restate Theorem 1.1 as follows.

Corollary 1.2. *There exists* $\bar{R} > 0$ *such that* (2) *has a radial solution* u *if and only if* $0 < R \leqslant \bar{R}$, *and it is unique in the class of radial solutions. Moreover the solution to* (2) *with* $R = \bar{R}$ *has vanishing gradient on the boundary and hence satisfies the equation*

$$-\Delta u + \chi_{\{u>0\}} u^{-\beta} = u^p, \qquad u \geqslant 0 \quad \text{in } \mathbb{R}^N. \tag{3}$$

For $0 < R < \bar{R}$ *the solution* u *to* (2) *satisfies* $u'(R) < 0$.

If $N \geqslant 3$ part of Corollary 1.2 is contained in [16]. More precisely, the result of [16] implies the existence of a unique $\bar{R}$ such that (2) admits a radial solution with zero gradient on the boundary, and that this radial solution is unique. Our contribution is that we prove the uniqueness for (2) for any R, with an alternative proof which is valid in dimension 2.

The case $p = 1$ has been considered in [1,7]. Chen [1] showed that there exist λ^*, $\bar{\lambda}$ with $\lambda_1 \leqslant \lambda^* < \bar{\lambda}$, λ_1 being the first eigenvalue of $-\Delta$ under Dirichlet boundary conditions, such that there exists a positive radial solution of (1) if and only if $\lambda^* < \lambda \leqslant \bar{\lambda}$. Moreover, whenever a solution exists, it is unique. In [1] it was also proved that the solution corresponding to $\bar{\lambda}$ has vanishing gradient on the boundary of the ball. Hirano and Shioji [7] obtained existence results for variants of this problem in a ball or annulus. In particular they clarified that $\lambda^* = \lambda_1$ in the result of [1]. Ouyang, Shi and Yao [14] studied the case $0 < p < 1$ finding zero, 1 or 2 positive solutions for λ in different intervals.

From the previous discussion (3) possesses a unique radially symmetric solution whose support is a ball. We are interested in the uniqueness question in a broader class. Define

$$\mathcal{N} = \{ u \in H^1(\mathbb{R}^N) \cap L^{1-\beta}(\mathbb{R}^N) \colon u \geqslant 0,\ u \not\equiv 0,\ G(u) = 0 \},$$

where

$$G(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + u^{1-\beta} - u^{p+1}.$$

For $u \in \mathcal{N}$ let

$$J(u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 + \frac{u^{1-\beta}}{1-\beta} - \frac{u^{p+1}}{p+1}.$$

Theorem 1.3. *Let* $\bar{u}$ *be the radial solution to* (2) *with* $R = \bar{R}$ *extended by zero to* $\mathbb{R}^N$. *Then* $\bar{u} \in \mathcal{N}$ *and satisfies*

$$J(\bar{u}) \leqslant J(\varphi) \quad \forall \varphi \in \mathcal{N}. \tag{4}$$

Moreover if $u \in \mathcal{N}$ *is any other function satisfying* (4) *then up to translation* $u = \bar{u}$.

Cortázar, Elgueta and Felmer [4] studied a similar problem:

$$\Delta u - u^q + u^p = 0, \qquad u \geqslant 0 \quad \text{in } \mathbb{R}^N,$$

where $0 < q < 1 < p < \frac{N+2}{N-2}$ and $N \geqslant 3$. They showed that if $u \in H^1(\mathbb{R}^N)$ is a solution such that $\{x\colon u(x) > 0\}$ is connected then u is radial about some point. The proof relies on the moving plane method, which works well even with the non-Lipschitz nonlinearity $f(u) = -u^q + u^p$ because it is nonincreasing on some interval $[0, \delta]$, $\delta > 0$. This result raises the question whether a solution $u \in H^1(\mathbb{R}^N) \cap L^{1-\beta}(\mathbb{R}^N)$ to (3) such that $\{x\colon u(x) > 0\}$ is connected is radial about some point. Since our nonlinearity $f(u) = -u^{-\beta} + u^p$ is increasing and singular the moving plane method is difficult to apply.

2. Some properties of radial solutions

We study the initial value problem

$$u'' + \frac{N-1}{r}u' + f(u) = 0, \qquad u > 0, \quad r \in (0, T), \tag{5}$$

$$u(0) = q, \qquad u'(0) = 0 \tag{6}$$

for a given $q > 0$ where

$$f(u) = u^p - u^{-\beta}.$$

The solution $u(r, q)$ to (5)–(6) is defined on a maximal interval $[0, T(q))$ where $T(q) > 0$ or $T(q) = +\infty$. We shall write just $u(r)$ when the initial condition q is clear from the context.

In this section we deal with some properties of solutions to (5)–(6). We give some basic properties in Lemma 2.1. In Lemma 2.2 and Remark 2.3 we find the behavior of the solution near $T(q)$ and in Lemma 2.4 we prove a uniqueness result. Lemma 2.5 is the differentiability of $T(q)$ when $u'(T(q)) < 0$. Properties similar to those mentioned here have appeared in Chen [1], Kwong [10], Ouyang, Shi and Yao [14], Serrin and Tang [16].

Given a solution u to (5) it will be useful to define

$$E_u(r) = \frac{1}{2}u'(r)^2 + F\big(u(r)\big),$$

where

$$F(u) = \frac{u^{p+1}}{p+1} - \frac{u^{1-\beta}}{1-\beta}.$$

Then if u is a solution of (5) we have

$$\frac{d}{dr}E_u(r) = -\frac{N-1}{r}u'(r)^2.$$

In particular E_u is nonincreasing. Actually, $\frac{d}{dr}E_u(\bar r) = 0$ for some $\bar r > 0$ if and only if $u'(\bar r) = 0$. If this happens and $u(\bar r) = 1$ then $u \equiv 1$ which is the case only for $q = 1$. If $u(\bar r) \neq 1$ then $u''(\bar r) \neq 0$ and hence $\frac{d}{dr}E_u(r) < 0$ for r close to $\bar r$, $r \neq \bar r$. This shows that E_u is strictly decreasing if $u \not\equiv 1$.

Lemma 2.1. *Let $q_1 = (\frac{p+1}{1-\beta})^{\frac{1}{p+\beta}} > 1$ so that $F(q_1) = 0$, and let u be a solution to* (5)–(6). *Then*

(a) $0 \leqslant u(r) \leqslant \max(q, q_1)$ *for all $r \in [0, T(q))$.*
(b) *If $E_u(r_0) < 0$ for some $r_0 \in [0, T(q))$ then $T(q) = \infty$.*
(c) *If $0 < q \leqslant q_1$ then $T(q) = \infty$.*
(d) *If $T(q) = \infty$ then $\lim_{r\to\infty} u(r) = 1$.*
(e) *If $T(q) < \infty$ then u is decreasing, u is C^1 up to $T(q)$ with $u(T(q)) = 0$ and $u'(T(q)) \leqslant 0$.*

Proof. (a) Suppose this fails and define $r_1 = \inf\{r > 0\colon u(r) = \max(u(0), q_1)\}$. If $u(0) \leqslant 1$ then $r_1 > 0$. If $u(0) > 1$ then from the equation $u(r)$ is decreasing for small r and hence $r_1 > 0$ also in this case. But then $E_u(r_1) \geqslant F(\max(u(0), q_1)) \geqslant F(u(0))$ which is impossible because E_u is strictly decreasing (unless $u \equiv 1$, in which case the proof of this part is trivial).

(b) By contradiction assume $T = T(q) < \infty$. By hypothesis $E_u(r) \leqslant E_u(r_0) < 0$ for $r \in [r_0, T(q))$. Suppose that for some sequence $r_n \to T$, $u(r_n) \to 0$. Then $\liminf E_u(r_n) \geqslant 0$ which is impossible. By standard results on ODE $\lim_{r\to T} u(r)$ exists and is positive. Then the solution can be continued beyond T and hence $T = \infty$.

(c) We have $E_u(0) = F(u(0)) \leqslant 0$ and therefore $E_u(r) \leqslant 0$ for all $0 \leqslant r < T$. If $u \not\equiv 1$ then E_u is strictly decreasing and therefore there exists $\delta > 0$ such that $E_u(r) \leqslant -\delta$ for all $r \in [\delta, T)$. Thus we may apply (b).

(d) We first show that if $E_u(r_0) < 0$ for some $r_0 \in [0, T(q))$ then $\lim_{r\to\infty} u(r) = 1$. Suppose that for some sequence $r_n \to \infty$ we have $u(r_n) \to 0$. Then $\liminf E_u(r_n) \geqslant 0$ which is impossible. Let u_0 be any accumulation point of $u(r)$ as $r \to \infty$. Then $u_0 > 0$ and by the ODE $u''(r)$ remains also bounded. We must have $u'(r) \to 0$, because otherwise, integrating $E'_u(r) = -(N-1)u'(r)^2/r$ and using u'' bounded we would get that $E_u(r) \to -\infty$ as $r \to \infty$. Hence the accumulation point must be a positive zero of f and then $u_0 = 1$. Actually one may check that $u(r)$ oscillates around 1 as $r \to \infty$.

Now let us deal with the general case. If $u(r) > 1$ for all r then the same argument as before implies $\lim_{r\to\infty} u(r) = 1$.

Now suppose that $u(r) \leqslant 1$ for some $r > 0$ and let $\bar r > 0$ be the first one. Then $u'(\bar r) < 0$.

If $u'(r) = 0$ for some $r \in (\bar r, T)$, let r_0 be the first one. Then $u(r_0) < 1$ and we deduce $E_u(r_0) < 0$. In this case we have already proved that $\lim_{r\to\infty} u(r) = 1$.

Let us analyze the case $u'(r) < 0$ for $r \in (\bar r, T)$. From

$$\left(r^{N-1}u'\right)' = r^{N-1}\left(u^{-\beta} - u^p\right) > 0 \tag{7}$$

we deduce that $r^{N-1}u'$ is increasing in $(\bar r, \infty)$ and since $r^{N-1}u' \leqslant 0$ this quantity remains bounded as $r \to \infty$. It follows that $\lim_{r\to\infty} u'(r) = 0$. Since u is decreasing $\lim_{r\to\infty} u(r)$ exists. From the ODE we deduce that $u''(r) \leqslant -C$ for all r large where $C > 0$ is some constant, which is impossible.

(e) If $T(q) < \infty$ then necessarily $\lim_{r\to T(q)} u(r) = 0$, because otherwise u can be continued beyond T. Applying the symmetry result of Gidas, Ni, Nirenberg [6] in the ball B_{R_ε} where given $\varepsilon > 0$ we define $R_\varepsilon = \inf\{r \in [0, T(q)):\ u(r) \leqslant \varepsilon\}$ and deduce that u is decreasing. Then by (7) $r^{N-1}u'$ is increasing near $T(q)$ which shows that $\lim_{r\to T(q)} u'(r)$ exists. □

Lemma 2.2. *Suppose u is solution to* (5)–(6) *with $R = T(q) < \infty$ and such that $u'(R) = 0$. Then for some $\delta > 0$ we have as $r \to R$,*

$$u(r) = c(R-r)^\alpha + O\left((R-r)^{\alpha+\delta}\right), \tag{8}$$

$$u'(r) = -c\alpha(R-r)^{\alpha-1} + O\left((R-r)^{\alpha-1+\delta}\right), \tag{9}$$

$$u''(r) = c\alpha(\alpha-1)(R-r)^{\alpha-2} + O\left((R-r)^{\alpha-2+\delta}\right), \tag{10}$$

where $\alpha = \frac{2}{1+\beta}$ and $c > 0$ is given by the relation $c^{-1-\beta} = \alpha(\alpha-1)$.

Proof. Since $u' \leqslant 0$ for some $\delta > 0$,

$$u'' = -\frac{N-1}{r}u' - f(u) \geqslant 0 \quad \text{in } (R-\delta, R)$$

which implies that u is convex near R. Let us change $R - r = t$ and write $u' = \frac{du}{dt}$. Then

$$u'' - \frac{N-1}{R-t}u' + f(u) = 0 \quad \text{in } (0, R) \tag{11}$$

and u is increasing and convex near 0. Multiplying by u' and integrating on $(0, t)$ we obtain

$$\frac{1}{2}u'(t)^2 - (N-1)\int_0^t \frac{u'(s)^2}{R-s}\,ds + F\left(u(t)\right) = 0 \quad \text{in } (0, R).$$

By convexity

$$(N-1)\int_0^t \frac{u'(s)^2}{R-s}\,ds \leqslant Ctu'(t)^2 \quad \forall t \in (0, R/2),$$

and hence

$$u'(t)^2\big(1+O(t)\big)+2F\big(u(t)\big)=0 \quad \text{as } t\to 0.$$

After the change of variables $R-r=t$ we have $u'>0$, and therefore we can rewrite this as

$$\frac{u'(t)}{(-2F(u(t)))^{1/2}}=1+O(t) \quad \text{as } t\to 0,$$

and integrate

$$\int_0^t \frac{u'(s)}{(-2F(u(s)))^{1/2}}\,ds=t+O\big(t^2\big).$$

But

$$\int_0^u \frac{1}{(-2F(s))^{1/2}}\,ds=\left(\frac{2(1-\beta)}{(1+\beta)^2}\right)^{1/2}u^{\frac{1+\beta}{2}}+O\big(u^{\frac{1+\beta}{2}+p+\beta}\big).$$

This gives

$$u(t)=\left(\frac{(1+\beta)^2}{2(1-\beta)}\right)^{\frac{1}{1+\beta}}\big[\big(t+O\big(t^2\big)\big)^{\frac{2}{1+\beta}}\big(1+O\big(t^{\frac{2(p+\beta)}{1+\beta}}\big)\big)\big]=\left(\frac{(1+\beta)^2}{2(1-\beta)}\right)^{\frac{1}{1+\beta}}t^{\frac{2}{1+\beta}}\big(1+O\big(t^{\frac{2\beta}{1+\beta}}\big)\big).$$

This proves (8). By standard elliptic estimates we find $u'(t)=O(t^{\alpha-1})$ and $u''(t)=O(t^{\alpha-2})$ as $t\to 0$. Going back to (11) we obtain (10) and then by integration (9). □

Remark 2.3. Suppose u is solution to (5)–(6) with $R=T(q)<\infty$ and such that $u'(R)<0$. Then as $r\to R$,

$$u(r)=O(R-r), \tag{12}$$

$$u'(r)=O(1), \tag{13}$$

$$u''(r)=O\big((R-r)^{-\beta}\big). \tag{14}$$

The first 2 assertions are direct, since $\lim_{r\to R}u'(r)$ exists and is negative. The third statement follows from the equation.

Lemma 2.4. *Suppose $u_1(r,q_1)$, $u_2(r,q_2)$ are solutions to (5)–(6) such that $T(q_1)=T(q_2)=T<\infty$ and $u_1(T)=u_1'(T)=0$ and $u_2(T)=u_2'(T)=0$. Then $u_1\equiv u_2$ in $(0,T)$.*

Proof. First we transform the problem. Assume that u is a solution to (5) in $(0,T)$ such that $u(T)=u'(T)=0$. Changing variables $t=T-r$ and writing $u'=\frac{du}{dt}$ we have

$$u''-\frac{N-1}{T-t}u'+f(u)=0, \qquad u>0 \quad \text{in } (0,T),$$
$$u(0)=u'(0)=0.$$

Let $v(t)=t^{-\alpha}u(t)$. Then $v>0$ and satisfies in $(0,T)$,

$$v''+2\alpha t^{-1}v'+\alpha(\alpha-1)t^{-2}v-\frac{N-1}{T-t}\big(v'+\alpha t^{-1}v\big)+t^{-\alpha}f\big(t^\alpha v\big)=0.$$

Moreover, by (8), $v(t)=c+O(t^\delta)$ as $t\to 0$. Set

$$w(t)=v(t)-c.$$

Then the equation for w becomes

$$Lw=E(w,w',t)+t^{-2}Q(w),$$

where L is the linear differential operator

$$Lw = w'' + 2\alpha t^{-1} w' + 2(\alpha - 1)w$$

and

$$E(w, w', t) = \frac{N-1}{T-t}\big(w' + \alpha t^{-1} w + \alpha t^{-1} c\big),$$
$$Q(w) = (w+c)^{-\beta} - c^{-\beta} + \beta c^{-1-\beta} w - t^{\alpha(p-1)+2}(w+c)^p.$$

The operator L has 2 linearly independent elements in its kernel given by: if $\alpha \neq 3/2$,

$$\varphi_1(t) = t^{\gamma_1} \quad \text{and} \quad \varphi_2(t) = t^{\gamma_2},$$
$$\gamma_1 = -1 \quad \text{and} \quad \gamma_2 = 2 - 2\alpha,$$

and if $\alpha = 3/2$ then

$$\varphi_1(t) = t^{-1} \quad \text{and} \quad \varphi_2(t) = t^{-1}\log(t).$$

By the variation of parameters formula a solution to

$$Lw = h \quad \text{in } (0, T),$$

can be written as

$$w(t) = c_1\varphi_1(t) + c_2\varphi_2(t) - \varphi_1(t)\int_{t_0}^{t} \frac{\varphi_2 h}{W}\, ds + \varphi_2(t)\int_{t_0}^{t} \frac{\varphi_1 h}{W}\, ds,$$

where $W = \varphi_1\varphi_2' - \varphi_1'\varphi_2$, $t_0 \in (0, T)$ is arbitrary, and c_1, c_2 are given by

$$c_1 = \frac{w(t_0)\varphi_2'(t_0) - w'(t_0)\varphi_2(t_0)}{W(t_0)},$$
$$c_2 = -\frac{w(t_0)\varphi_1'(t_0) - w'(t_0)\varphi_1(t_0)}{W(t_0)}.$$

From now on we will assume that $\alpha \neq 3/2$. The case $\alpha = 3/2$ can be treated analogously. We know by Lemma 2.2 that $w(t) = O(t^\delta)$ and $w'(t) = O(t^{-1+\delta})$ for some $\delta > 0$, which implies that $c_1(t_0), c_2(t_0) \to 0$ as $t_0 \to 0$. Thus letting $t_0 \to 0$ we find that in the case $\alpha \neq 3/2$,

$$w(t) = \frac{t^2}{\gamma_2 - \gamma_1}\int_0^1 \big(\tau^{1-\gamma_2} - \tau^{1-\gamma_1}\big)\big(E(w, w', t\tau) + (t\tau)^{-2} Q(w)\big)\, d\tau.$$

Thus, to show uniqueness for solutions to (5) which together with the first derivative vanish at T it suffices to prove that the above fixed point equation has at most one solution. We do this in the space X of C^1 functions on $(0, T_1)$ for which the following norm is finite

$$\|w\|_X = \sup_{t\in[0,T_1]} t^{-\delta}\big|w(t)\big|,$$

where $T_1 > 0$ is a small constant to be fixed later on. Define the linear operator

$$Sh(t) = \frac{t^2}{\gamma_2 - \gamma_1}\int_0^1 \big(\tau^{1-\gamma_2} - \tau^{1-\gamma_1}\big) h(t\tau)\, d\tau,$$

and the mapping

$$\mathcal{A}(w) = S\big[E(w, w', t) + t^{-2} Q(w)\big].$$

The space X is not complete but verifying that $\mathcal{A}$ is a contraction on an appropriate ball is sufficient to prove uniqueness. Since $\gamma_1, \gamma_2 < 0$ we have

$$\left|Sh(t)\right| \leqslant C\|h\|_X t^{2+\delta}, \qquad \left|Sh(t)\right| \leqslant C\left\|s^{-1}h(s)\right\|_X t^{1+\delta},$$
$$\left|Sh(t)\right| \leqslant C\left\|s^{-2}h(s)\right\|_X t^{\delta}.$$

Hence

$$\left\|\mathcal{A}(w_1) - \mathcal{A}(w_2)\right\|_X \leqslant C\left\|Q(w_1) - Q(w_2)\right\|_X.$$

But

$$\begin{aligned}\left|Q(w_1)(t) - Q(w_2)(t)\right| &\leqslant C\left|w_1(t) - w(t)\right|^2 + t^{\alpha(p-1)+2}\left|\left(w_1(t)+c\right)^p - \left(w_2(t)+c\right)^p\right| \\ &\leqslant C\left|w_1(t) - w_2(t)\right|^2 + Ct^{\alpha(p-1)+2}\left|w_1(t) - w_2(t)\right| \\ &\leqslant Ct^{2\delta}\|w_1 - w_2\|_X^2 + Ct^{\alpha(p-1)+2+\delta}\|w_1 - w_2\|_X\end{aligned}$$

so that

$$\begin{aligned}\left\|\mathcal{A}(w_1) - \mathcal{A}(w_2)\right\|_X &\leqslant CT_1^{\delta}\|w_1 - w_2\|_X^2 + CT_1^{\alpha(p-1)+2}\|w_1 - w_2\|_X \\ &\leqslant CT_1^{\delta}\left(\|w_1\|_X + \|w_2\|_X\right)\|w_1 - w_2\|_X.\end{aligned}$$

Given w_1, w_1 solutions of the fixed point equation $\mathcal{A}(w) = w$ with $\|w_1\|_X + \|w_2\|_X < \infty$, by decreasing T_1 we see that $w_1 \equiv w_2$ in $(0, T_1)$. This shows that if u_1 and u_2 are solutions to (5) and satisfy $u_1(T) = u_1'(T) = 0$ and $u_2(T) = u_2'(T) = 0$ then $u_1 \equiv u_2$ is a neighborhood to the left of T. Then by the standard uniqueness result for ODE's we deduce that $u_1 \equiv u_2$ in $(0, T)$, which proves the lemma. □

Lemma 2.5. *Suppose $u(r, \bar{q})$ is a solution to* (5)–(6) *such that $T(\bar{q}) < \infty$ and $u'(T) < 0$. Then the map $q \to T(q)$ is finite and differentiable for q near $\bar{q}$.*

Proof. Write $T = T(\bar{q})$. By standard results on ODE, given $\varepsilon > 0$ there is $\delta > 0$ such that if $|q - \bar{q}| \leqslant \delta$ then $u(r, q)$ is defined in $[0, T - \varepsilon]$ and the map $q \to u(\cdot, q)$ is differentiable into the space $C([0, T - \varepsilon])$.

Changing variables $t = T - r$ and writing $u' = \frac{du}{dt}$ we study the initial value problem

$$u'' - \frac{N-1}{T-t}u' + f(u) = 0, \qquad u > 0 \quad \text{in } (0, T_1), \tag{15}$$
$$u(0) = 0, \qquad u'(0) = c, \tag{16}$$

where $T, c > 0$ are parameters and $T_1 > 0$ is fixed suitably small. We will establish:

Claim. *Given $\bar{T}, \bar{c} > 0$, problem* (15)–(16) *has a solution $u(t; \bar{T}, \bar{c})$ defined for $t \in [0, T_1]$, $T_1 > 0$. Moreover for T, c close to $\bar{T}, \bar{c}$ this solution is well defined up to same fixed T_1 and $T, c \to u(\cdot; T, c)$ is differentiable into the space $C^1([0, T_1])$. The conclusion of the lemma then follows from the implicit function theorem.*

To prove the claim fix $0 < \delta < 1 - \beta$ and define the initial approximation for the solution as

$$u_0(t) = ct + c't^{2-\beta},$$

where $c' > 0$ is such that

$$c'(2-\beta)(1-\beta) = c^{-\beta}.$$

We seek a solution to (15)–(16) of the form $u = u_0 + \phi$ where $\phi \in X$:

$$X = \left\{\phi \in C^1\left([0, T_1]\right): \|\phi\|_X < \infty\right\},$$

where

$$\|\phi\|_X = \sup_{t\in[0,T_1]} t^{\beta-2-\delta}\big|\phi(t)\big| + \sup_{t\in[0,T_1]} t^{\beta-1-\delta}\big|\phi'(t)\big|.$$

Given $h:[0,T_1]\to\mathbb{R}$ integrable define $T(h)=\phi$ by $\phi(t)=\int_0^t (t-s)h(s)\,ds$. This means just that $\phi''=h$ and $\phi(0)=\phi'(0)=0$.

Then (15)–(16) is equivalent to the following fixed point equation:

$$\phi = T\big[A(\phi)+E\big],$$

where

$$A(\phi) = \frac{N-1}{T-t}\phi' - (u_0+\phi)^p + u_0^p + (u_0+\phi)^{-\beta} - u_0^{-\beta}$$

and

$$E = u_0^{-\beta} - u_0'' + \frac{N-1}{T-t}u_0' - u_0^p.$$

Then we have $E = O(t^{1-2\beta})$. Define $\|h\|_Y = \sup_{t\in[0,T_1]} t^{\beta-\delta}|h(t)|$. Then $\|E\|_Y \leqslant CT_1^{1-\beta-\delta}$. The other terms can be estimated as follows

$$\left\|\frac{N-1}{T-t}\phi'\right\|_Y \leqslant CT_1\|\phi\|_X,$$
$$\big\|-(u_0+\phi)^p + u_0^p\big\|_Y \leqslant CT_1^2\|\phi\|_X,$$
$$\big\|(u_0+\phi)^{-\beta} - u_0^{-\beta}\big\|_Y \leqslant CT_1^{1-\beta}\|\phi\|_X.$$

Then for small $T_1>0$ the operator $T(A(\phi)+E)$ is a contraction in the closed unit ball of X, and therefore a unique fixed point exists in this ball. The fixed point characterization of ϕ and the differentiability of this operator with respect to T, c imply the desired differentiability of ϕ. □

3. Uniqueness of radial solutions

The proof here is similar to the work of Cortázar, Elgueta and Felmer [4] with ideas that go back to Kolodner [9], Coffman [3], Ni and Nussbaum [13], McLeod and Serrin [12], Kwong [10], Kwong and Zhang [11], Chen and Lin [2], and Yanagida [17].

The uniqueness proof of [16] is carried out by studying the function $t(u)=\rho_1(u)-\rho_2(u)$, where $\rho_i=\rho_i(u)$ are the inverses of two existing solutions $u_1(\rho)$ and $u_2(\rho)$ defined on $(0,\alpha_i)$ with $\rho_i(\alpha_i)=0$. This analysis, as well as their Separation Lemma stating that $t(u)t'(u)<0$, require $N>2$. Here we are able to obtain the same result of the paper [16] for $N\geqslant 2$, that is, we can handle the case $N=2$ not treated before, for a more restricted nonlinearity. Our approach relies on the estimate and regularity with respect to the initial data q of the maximal time $T(q)$ of existence of a solution. We prove that $T'(q)<0$ and $\lim_{q\to\infty} T(q)=0$.

The main result in this section is

Proposition 3.1. *There exists $\bar q>0$ such that*

- *if $0<q<\bar q$ then $T(q)=+\infty$;*
- *if $q=\bar q$ then $T(q)<\infty$ and the corresponding solution satisfies*

$$u'\big(T(\bar q)\big)=0;$$

- *if $q>\bar q$ then $T(q)<+\infty$ and the corresponding solution satisfies*

$$u'\big(T(q)\big)<0.$$

By Lemma 2.1 we know that $T(q)=\infty$ if $q\leqslant 1$. So in the rest of the section we will work only with $q>1$.

Let

$$\varphi(r,q)=\frac{\partial u}{\partial q}(r,q)\quad \text{for all } r\in\big[0,T(q)\big).$$

Again, when it is clear from the context we will write just $\varphi(r)$.

Lemma 3.2. *If $q>1$ is such that $T(q)<\infty$ then φ has at least one zero in $(0,T(q))$.*

For the proof we need the next computation.

Lemma 3.3. *If $a>0$ is small then*

$$f'(u)(u-a)-f(u)>0\quad \forall u\geqslant a. \tag{17}$$

Proof. Let $a>0$ and compute

$$f'(u)(u-a)-f(u)=u^{p-1}\big[(p-1)u-ap+au^{-p-\beta}\big].$$

Note that

$$\min_{u\geqslant 0}\big[(p-1)u-ap+au^{-p-\beta}\big]$$

is attained at a unique point u_* by convexity. This point is given by

$$u_*=\left(\frac{a(p+\beta)}{p-1}\right)^{\frac{1}{p+\beta+1}}$$

and replacing this value we find

$$\min_{u\geqslant 0}\big[(p-1)u-ap+au^{-p-\beta}\big]=-ap+\frac{(p-1)(p+\beta+1)}{p+\beta}\left(\frac{a(p+\beta)}{p-1}\right)^{\frac{1}{p+\beta+1}}.$$

This number is positive provided we take $a>0$ suitably small. □

Proof of Lemma 3.2. Let $a>0$ be such that (17) holds. Then choose $r_0\in(0,T(q))$ such that $u(r_0)=a$. Then $u(r)>a$ for all $r\in[0,r_0)$. Using Green's identity we find

$$\int_0^{r_0}\varphi\big(f'(u)(u-a)-f(u)\big)t^{N-1}\,dt=r_0^{N-1}\varphi(r_0)u'(r_0).$$

If $\varphi>0$ in $(0,r_0)$ then the integral above is positive, which is not possible because $\varphi(r_0)\geqslant 0$ and $u'(r_0)<0$. □

Lemma 3.4. *Suppose $q>1$ is such that $u(T(q))<\infty$. Then $\frac{ru'(r)}{u(r)}$ is strictly decreasing on $(0,T(q))$.*

Proof. The proof is essentially the same as in [4]. Let $R=T(q)$ and $v(r)=ru'(r)$. Then

$$r^{N-1}\left(-\frac{ru'(r)}{u(r)}\right)'u(r)^2=r^{N-1}\big(v(r)u'(r)-u(r)v'(r)\big)$$

for all $r\in[0,R)$. Integrating in $(0,r)$ we obtain

$$\begin{aligned}r^{N-1}\big(v(r)u'(r)-u(r)v'(r)\big)&=\big(f\big(u(r)\big)u(r)-2F\big(u(r)\big)\big)r^N\\&\quad+\int_0^r\big[2NF\big(u(t)\big)-(N-2)f\big(u(t)\big)u(t)\big]t^{N-1}\,dt.\end{aligned}\tag{18}$$

We have

$$f(u)u - 2F(u) = u^{p+1}\left(1 - \frac{2}{p-1}\right) + u^{1-\beta}\left(\frac{2}{1-\beta} - 1\right) > 0 \quad \forall u > 0,$$

so we obtain

$$r^{N-1}\big(v(r)u'(r) - u(r)v'(r)\big) > \int_0^r \big[2NF\big(u(t)\big) - (N-2)f\big(u(t)\big)u(t)\big]t^{N-1}\,dt. \tag{19}$$

We claim that if $r \in (0, R)$ then

$$\int_0^r \big[2NF\big(u(t)\big) - (N-2)f\big(u(t)\big)u(t)\big]t^{N-1}\,dt > 0. \tag{20}$$

To prove this we observe that $2NF(u) - (N-2)f(u)u$ has a unique positive zero which we write as d and satisfies $2NF(u) - (N-2)f(u)u > 0$ for all $u > d$ and $2NF(u) - (N-2)f(u)u < 0$ for $0 < u < d$. If $u(r) \geqslant d$ then (20) holds. If $u(r) < d$ then

$$\int_0^r \big[2NF\big(u(t)\big) - (N-2)f\big(u(t)\big)u(t)\big]t^{N-1}\,dt > \int_0^R \big[2NF\big(u(t)\big) - (N-2)f\big(u(t)\big)u(t)\big]t^{N-1}\,dt.$$

To compute the above quantity we let $r \to R$ in (18). Note that

$$\lim_{r\to R}\big(f\big(u(r)\big)u(r) - 2F\big(u(r)\big)\big)r^N = 0.$$

If $u'(T(q)) = 0$ then by (8)–(10)

$$v(r)u'(r) = O\big((R-r)^{2\alpha-2}\big) \quad \text{as } r \to R,$$

and

$$v'(r)u(r) = O\big((R-r)^{2\alpha-2}\big) \quad \text{as } r \to R.$$

Hence

$$\lim_{r\to R} v(r)u'(r) = \lim_{r\to R} v'(r)u(r) = 0. \tag{21}$$

If $u'(T(q)) < 0$ then by (12)–(14)

$$v(r)u'(r) = O\big((R-r)^{1-\beta}\big), \qquad v'(r)u(r) = O\big((R-r)^{1-\beta}\big) \quad \text{as } r \to R,$$

and hence (21) also holds in this case. Thus

$$\int_0^R \big[2NF\big(u(t)\big) - (N-2)f\big(u(t)\big)u(t)\big]t^{N-1}\,dt = 0$$

and (20) follows. From (19) and (20) we deduce

$$r^{N-1}\big(v(r)u'(r) - u(r)v'(r)\big) > 0 \quad \forall r \in (0, R),$$

and this proves the lemma. □

Proposition 3.5. *Suppose $q > 1$ is such that $T(q) < \infty$.*

(a) *Then $\varphi = \frac{\partial u}{\partial q}$ has exactly one zero in $(0, T(q))$ which we call r_0. Moreover $\varphi > 0$ in $[0, r_0)$, $\varphi < 0$ in $(r_0, T(q))$.*
(b) *If $u'(T(q)) < 0$ then $\varphi(T(q)) < 0$.*

(c) *If* $u'(T(q)) = 0$ *then*

$$\lim_{r \to T(q)} \varphi(r) = 0$$

and there exists a unique $r_1 \in (r_0, T(q))$ *such that* $\varphi'(r_1) = 0$ *and we have* $\varphi' < 0$ *in* (r_0, r_1) *and* $\varphi' > 0$ *in* $(r_1, T(q))$.

Proof. Write $R = T(q)$.

(a) Let

$$v(r) = ru' + cu,$$

where $c \in \mathbb{R}$ is to be determined. Then

$$v'' + \frac{N-1}{r} v' + f'(u)v = -(2+c)f(u) + cf'(u)u.$$

Let $r_0 \in (0, R)$ denote the smallest zero of φ. We know by Lemma 3.2 that it exists. Choose $c \in \mathbb{R}$ such that

$$-(2+c)f\big(u(r_0)\big) + cf'\big(u(r_0)\big)u(r_0) = 0.$$

The value of c is given explicitly by

$$c = \frac{2(u(r_0)^{p+\beta} - 1)}{(p-1)u(r_0)^{p+\beta} + \beta + 1}.$$

Note that $c > 0$ if and only if $u(r_0) > 1$, which we cannot assert in our situation as opposed to the work [4]. Having fixed c as above define

$$\phi(u) = -(2+c)f(u) + cf'(u)u.$$

We claim that

$$\begin{cases} \text{if } u(r) > u(r_0) \text{ then } \phi(u(r)) < 0, \\ \text{if } u(r) < u(r_0) \text{ then } \phi(u(r)) > 0. \end{cases} \tag{22}$$

Indeed, $\phi(u)$ is given by

$$\phi(u) = u^\beta\big(u^{p+\beta}\big(c(p-1) - 2\big) + c(\beta+1) + 2\big)$$

and hence (22) is valid if $c(p-1) - 2 < 0$, which can be easily checked.

Now suppose that φ has another zero in $(0, R)$ and let r_1 denote the next one, that is, the smallest zero bigger than r_0. Then, integrating by parts and using that $\Delta\varphi + f'(u)\varphi = 0$ for $r \in (0, r_1)$ we have

$$\int_0^r \varphi(t)\phi\big(u(t)\big)t^{N-1}\,dt = r^{N-1}\big(\varphi(r)v'(r) - \varphi'(r)v(r)\big). \tag{23}$$

By (22) we have

$$\int_0^r \varphi(t)\phi\big(u(t)\big)t^{N-1}\,dt < 0 \quad \forall r \in (0, r_1).$$

Hence, evaluating (23) at r_0 we deduce that $-\varphi'(r_0)v(r_0) < 0$. But $\varphi'(r_0) \leqslant 0$ and therefore $v(r_0) < 0$. By Lemma 3.4 we deduce $v(r_1) < 0$ and therefore, using (23), we obtain $\varphi'(r_1) < 0$, which is not possible. This shows that φ has only one zero in $(0, R)$.

(b) Assume $u'(R) < 0$ and $\varphi(R) = 0$. Then using (12)–(14) we see that $\varphi(r) = O(R - r)$ as $r \to R$. Then $\lim_{r \to R} \varphi(r)v'(r) = 0$. On the other hand $\lim_{r \to R} \varphi'(r) \geqslant 0$. But letting $r \to R$ in (23) we find as in the previous case $\lim_{r \to R} \varphi'(r) < 0$ which is a contradiction.

(c) Assume $u'(R) = 0$. We first verify that $\varphi' > 0$ on some point in (r_0, R). Suppose on the contrary that $\varphi' \leqslant 0$ in $[r_0, R)$. Then $L = \lim_{r\to R} \varphi(r)$ exists and $L < 0$. But

$$\varphi'' + \frac{N-1}{r}\varphi' + f'(u)\varphi = 0$$

and by (8)

$$f'(u) = \beta\alpha(\alpha-1)(R-r)^{-2}\bigl(1 + O\bigl((R-r)^\delta\bigr)\bigr) \quad \text{as } r \to R,$$

for some fixed $\delta > 0$. This shows that $\varphi'' \geqslant b(R-r)^2$ for some $b > 0$ and r close to R, which implies that $\varphi(r) \to +\infty$ as $r \to R$, which is impossible.

Since $f'(u) < 0$ we see that φ cannot have a local maximum at points where $\varphi < 0$ and cannot have a local minimum at point where $\varphi > 0$. Thus in $(0, r_0)$ we must have $\varphi' \leqslant 0$. In (r_0, R) we have seen that $\varphi < 0$ and φ' changes sign, because $\varphi'(r_0) < 0$. Let r_1 denote the smallest zero of φ' in (r_0, R). Then φ' cannot have another zero in (r_1, R). Hence $\varphi' > 0$ near R and hence $\lim_{r\to R} \varphi(r) = L$ exists. Suppose $L < 0$. Then the argument in the previous paragraph gives that $\varphi(r) \to +\infty$ as $r \to R$, which is impossible. This shows that $\lim_{r\to R} \varphi(r) = 0$. □

Lemma 3.6. *Suppose $q > 1$ is such that $T(q) < \infty$ and $u'(T(q)) = 0$. Let $r_0 \in (0, T(q))$ be the zero of φ and $r_1 \in (r_0, T(q))$ such that $\varphi'(r_1) = 0$. Then there exists $r^* \in (r_0, r_1)$ such that $u(r^*) < 1$.*

Proof. Suppose that $u \geqslant 1$ on $[r_0, r_1]$. Let $a = u(r_0) \geqslant 1$ so that $1 \leqslant u \leqslant a$ on $[r_0, r_1]$. Then using Green's identity we find

$$\int_{r_0}^{r_1} \varphi\bigl(f'(u)(u-a) - f(u)\bigr)t^{N-1}\,dt = r_1^{N-1}\varphi(r_1)u'(r_1) < 0.$$

But for $1 \leqslant u \leqslant a$ we have

$$\begin{aligned} f'(u)(u-a) - f(u) &= u^{p-1}\bigl[(p-1)u - ap + au^{-p-\beta}\bigr] \\ &\leqslant u^{p-1}\bigl[(p-1)u + a(1-p)\bigr] = (p-1)u^{p-1}(u-a) \leqslant 0 \end{aligned}$$

and since $\varphi < 0$ on (r_0, r_1) we obtain that $\int_{r_0}^{r_1} \varphi(f'(u)(u-a) - f(u))t^{N-1}\,dt \geqslant 0$, a contradiction. □

Lemma 3.7. *Suppose $q > 1$ is such that $T(q) < \infty$ and $u'(T(q)) = 0$. Then for $q_1 \in (1, q)$ with $q - q_1$ sufficiently small we have $T(q_1) = \infty$.*

Proof. Let $R = T(q)$, $u(r) = u(r, q)$ and $u_1(r) = u(r, q_1)$. As before let $\varphi(r) = \frac{\partial u}{\partial q}(r, q)$ and let r_0 be the unique zero of φ in $(0, R)$. Fix $r_1 \in (r_0, R)$ such that $u(r_1) < 1$, $\varphi(r_1) < 0$ and $\varphi'(r_1) < 0$. Then for $q_1 < q$, $q - q_1$ small we have

$$1 > u_1(r_1) > u(r_1) \quad \text{and} \quad u_1'(r_1) > u'(r_1).$$

These inequalities imply that

$$E_{u_1}(r_1) < E_u(r_1). \tag{24}$$

Step 1.

$$u_1 > u \quad \forall r \in (r_1, R).$$

Suppose that this claim is false and define

$$r_2 = \inf\bigl\{r \in (r_1, R):\ u(r) = u_1(r)\bigr\}.$$

Then $u_1'(r_2) \leqslant u'(r_2)$ and equality cannot hold for otherwise by standard uniqueness results for ODE's we would have $u \equiv u_1$ in $[r_1, r_2]$. Since $u_1(r_2) = u(r_2)$ we find

$$E_{u_1}(r_2) > E_u(r_2).$$

On the other hand we have (24) and hence we may define

$$r_3 = \inf\{r \in (r_1, r_2):\ E_{u_1}(r) = E_u(r)\}.$$

In this way we have

$$E'_u(r_3) \leqslant E'_{u_1}(r_3),$$

which implies

$$u'(r_3)^2 \geqslant u'_1(r_3)^2. \tag{25}$$

By definition of r_2 we have $u_1 > u$ in (r_1, r_2) and in particular $u_1(r_3) > u(r_3)$. This yields $F(u_1(r_3)) < F(u(r_3))$ and together with (25) we find

$$E_u(r_3) > E_{u_1}(r_3),$$

contradicting the definition of r_3.

Step 2. We have

$$E_{u_1} < E_u \quad \text{in } (r_1, R).$$

The argument is almost the same as in the previous claim. Suppose by contradiction that this claim is false and define

$$r_2 = \inf\{r \in (r_1, R):\ E_{u_1}(r) = E_u(r)\}.$$

Then $E'_u(r_2) \leqslant E'_{u_1}(r_2)$ which implies

$$u'(r_2)^2 \geqslant u'_1(r_2)^2.$$

Since $u_1(r_2) > u(r_2)$ we have $F(u_1(r_2)) < F(u(r_2))$ and we deduce

$$E_u(r_2) > E_{u_1}(r_2),$$

contradicting the definition of r_2.

Step 3.

$$u_1(R) > 0.$$

Suppose that $u_1(R) = 0$. Then since $E_{u_1}(R) \leqslant E_u(R) = 0$ we also deduce $u'_1(R) = 0$. By Lemma 2.4 $u_1 \equiv u$ in $(0, R)$, which leads to a contradiction, since $u_1(0) = q_1 \neq q = u(0)$, and proves the claim.

We deduce that $u_1(R) > 0$ with $E_{u_1}(R) < 0$. This shows that u_1 is defined for all t, that is $T(q_1) = +\infty$. □

Lemma 3.8. *Suppose $q > 1$ is such that $T(q) < \infty$ and $u'(T(q)) = 0$. Then for $q_1 > q$ with $q_1 - q$ sufficiently small we have $T(q_1) < \infty$ and $u'(R(q_1), q_1) < 0$.*

Proof. The proof is analogous to that of Lemma 3.7. Let $R = T(q)$, $u(r) = u(r, q)$, $u_1(r) = u(r, q_1)$ and $\varphi(r) = \frac{\partial u}{\partial q}(r, q)$ and let r_0 be the unique zero of φ in $(0, R)$. Fix $r_1 \in (r_0, R)$ such that $u(r_1) < 1$, $\varphi(r_1) < 0$ and $\varphi'(r_1) < 0$. Then for $q_1 > q$, $q_1 - q$ small we have

$$1 > u(r_1) > u_1(r_1) > 0 \quad \text{and} \quad 0 > u'(r_1) > u'_1(r_1).$$

These inequalities imply that

$$E_{u_1}(r_1) > E_u(r_1). \tag{26}$$

Step 1. Let $[0, T_1)$ be the interval of existence of u_1. Then $T_1 \leqslant R$ and

$$u_1 < u \quad \text{for all } r \in [r_1, T_1).$$

It is enough to establish that $u_1 < u$ in $[r_1, \min(R, T_1))$ since this property forces u_1 to vanish before (or at the same time) as u. Suppose this is not true and define

$$r_2 = \inf\{r \in [r_1, \min(T_1, R)):\ u_1(r) > u(r)\}.$$

Then $u(r_2) = u_1(r_2)$ and $u'(r_2) < u_1'(r_2) < 0$. But then $F(u(r_2)) > F(u_1(r_2))$ and $u'(r_2)^2 > u_1'(r_2)^2$ which implies that $E_u(r_2) > E_{u_1}(r_2)$. Since (26) holds, we may define

$$r_3 = \inf\{r \in (r_1, r_2):\ E_u(r) = E_{u_1}(r)\}.$$

Then $E_u(r_3) = E_{u_1}(r_3)$ and $\frac{d}{dr} E_{u_1}(r_3) \leqslant \frac{d}{dr} E_u(r_3)$. This implies $u_1'(r_3)^2 \geqslant u'(r_3)^2$. On the other hand, since $u_1(r_3) < u(r_3) < 1$ we have $E_{u_1}(r_3) > E_u(r_3)$, which contradicts the definition of r_3.

Step 2.

$$E_{u_1} > E_u \quad \text{in } [r_1, T_1).$$

Suppose the contrary and define

$$r_2 = \inf\{r \in [r_1, T_1):\ E_{u_1}(r) > E_u(r)\}.$$

Then $E_{u_1}(r_2) = E_u(r_2)$ and $\frac{d}{dr} E_{u_1}(r_2) \leqslant \frac{d}{dr} E_u(r_2)$. This gives $u_1'(r_2)^2 \geqslant u'(r_2)^2$. By the previous step $u_1(r_2) < u(r_2) < 1$ and therefore $F(u_1(r_2)) > F(u(r_2))$. We deduce then that $E_{u_1}(r_2) > E_u(r_2)$, a contradiction.

Step 3. We have $R(q_1) = T_1 < R$ and $u_1'(R(q_1)) < 0$.

Let us write $R_1 = R(q_1)$. Observe that $R_1 \leqslant T_1 \leqslant R$ and also $R_1 > r_1$. If $R_1 < T_1$ then $E_{u_1}(R_1) > E_u(R_1) \geqslant 0$. Since $1 > u_1(r_1) > u_1(R_1)$ we must have $F(u_1(R_1)) \leqslant 0$ and we conclude that $u_1'(R_1) \neq 0$. But then $u_1(R_1) = 0$ and $T_1 = R_1$.

If $T_1 = R$ then $u_1(R) = u_1'(R) = 0$ and then by the uniqueness result Lemma 2.4 we would have $u_1 = u$ in $[0, R]$ which is not possible. Thus $T_1 < R$. Then the same argument as in the previous paragraph leads to $u_1'(R_1) \neq 0$. □

Proof of Proposition 3.1. Define

$$\mathcal{P} = \{q > 1:\ T(q) = \infty\},$$
$$\mathcal{C} = \{q > 1:\ T(q) < \infty,\ u'(T(q)) < 0\},$$
$$\mathcal{Q}_0 = \{q > 1:\ T(q) < \infty,\ u'(T(q)) = 0\}$$

so that $(1, \infty) = \mathcal{Q}_0 \cup \mathcal{P} \cup \mathcal{C}$ and these sets are disjoint. The set $\mathcal{C}$ is open by Lemma 2.5. An argument using E_u similar to the proof of Lemma 2.1 implies that $\mathcal{P}$ is open. By Lemma 3.7 if $q \in \mathcal{Q}_0$ then for some $\delta > 0$ we have $(q - \delta, q) \subset \mathcal{P}$. Similarly, by Lemma 3.8 if $q \in \mathcal{Q}_0$ then for some $\delta > 0$ we have $(q, q + \delta) \subset \mathcal{C}$. Then the same argument as in [4] implies that $\mathcal{Q}_0$ consists of only one point $\mathcal{Q}_0 = \{\bar{q}\}$, $\mathcal{P} = (1, \bar{q})$ and $\mathcal{C} = (\bar{q}, \infty)$. □

Proof of Theorem 1.1. For $q \in \mathcal{C}$ the map $T(q)$ is differentiable (Lemma 2.5) and differentiating $u(T(q), q)$ yields

$$u'(T(q), q)T'(q) + \varphi(T(q), q) = 0,$$

which shows that $T'(q) < 0$, because by Proposition 3.5(b) $\varphi(T(q)) < 0$. To finish, we claim that

$$\lim_{q \to \infty} T(q) = 0.$$

One way to prove this is to assume that $\lim_{q \to \infty} T(q) > 0$. Let

$$v_q(x) = \frac{1}{q} u\big(q^{\frac{1-p}{2}}\big),$$

which is then defined for $|x| \leqslant T(q) q^{\frac{p-1}{2}} \to \infty$ as $q \to \infty$. Then

$$\Delta v_q - q^{-p-\beta} v_q^{-\beta} + v_q^p = 0.$$

Using the same arguments as in Lemmas 4.2 and 4.3 we can show that v_q converges locally uniformly as $q \to \infty$ in $\mathbb{R}^N$ to $v > 0$ satisfying $\Delta v + v^p = 0$ in $\mathbb{R}^N$, which is impossible. □

4. Proof of Theorem 1.3

Given $\varepsilon > 0$ and a fixed large $R > 0$, we study the problem

$$\begin{cases} -\Delta u + g_\varepsilon(u) = u^p & \text{in } B_R, \\ u > 0 & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R, \end{cases} \tag{27}$$

where for $\varepsilon > 0$,

$$g_\varepsilon(u) = \begin{cases} \frac{u}{(u+\varepsilon)^{1+\beta}}, & u \geqslant 0, \\ 0, & u < 0. \end{cases} \tag{28}$$

Then we prove that (27) has a solution u_ε, which is radial and bounded in $L^\infty(B_R)$. Then we show that $u = \lim_{\varepsilon\to 0} u_\varepsilon$ is a minimizer of J on $\mathcal{N}$.

Lemma 4.1. *Problem* (27) *admits a solution u_ε. Moreover u_ε is radial and radially nonincreasing.*

Proof. Define the following functional in $H_0^1(B_R)$:

$$J_\varepsilon(u) = \int_{B_R} \frac{1}{2}|\nabla u|^2 + G_\varepsilon(u) - \frac{|u|^{p+1}}{p+1}, \tag{29}$$

where

$$G_\varepsilon(u) = \int_0^u g_\varepsilon(t)\,dt = \frac{\beta u + \varepsilon}{\beta(1-\beta)(u+\varepsilon)^\beta} - \frac{\varepsilon^{1-\beta}}{\beta(1-\beta)} \quad \text{for all } u \geqslant 0.$$

Let $\varphi_1 > 0$ denote the first eigenfunction of $-\Delta$ in B_R with Dirichlet boundary conditions, normalized such that $\|\varphi_1\|_L^2 = 1$. Let $A > 0$ be fixed sufficiently large and fixed to ensure

$$\frac{1}{2}\int_\Omega \left|\nabla(A\varphi_1)\right|^2 - \frac{1}{p+1}\int_\Omega (A\varphi_1)^{p+1} < 0.$$

Then

$$J_\varepsilon(A\varphi_1) < 0 \tag{30}$$

for all $\varepsilon > 0$.

We solve (27) using the mountain pass theorem for the functional J_ε. Since $G_\varepsilon \geqslant 0$ this functional satisfies:

$$\text{there exist } \rho > 0,\ c > 0 \text{ such that } J_{\lambda,\varepsilon}(u) \geqslant c\ \forall \|u\|_{H_0^1(\Omega)} = \rho.$$

This and (30) give the geometric condition for the mountain pass theorem, and the Ambrosetti–Rabinowitz condition

$$\exists\theta > 2 \text{ such that } \theta\left(\frac{\lambda}{p+1}u^{p+1} - G_\varepsilon(u)\right) \leqslant \lambda u^p - g_\varepsilon(u) \text{ for sufficiently large } |u|$$

is satisfied since the term that dominates in the nonlinearity for large u is u^p. Therefore there exists a critical point u_ε of J_ε in $H_0^1(B_R)$. By standard regularity theory u_ε is $C^2(\bar{B}_R)$. We claim that $u > 0$ in B_R. To prove this it suffices to verify that $u_\varepsilon \geqslant 0$ in B_R. Suppose to the contrary that $\omega = \{x \in B_R\colon u_\varepsilon(x) < 0\}$ is nonempty. Then

$$-\Delta u_\varepsilon = |u_\varepsilon|^p > 0 \quad \text{in } \omega, \qquad u_\varepsilon = 0 \quad \text{on } \partial\omega,$$

and we deduce $u_\varepsilon > 0$ in ω, a contradiction. Thus we have produced a positive solution u of (27). By the result of Gidas, Ni and Nirenberg [6] u_ε is radially symmetric and radially nonincreasing. □

Lemma 4.2. *Let u_ε denote any radial solution of* (27). *Then there is some constant $C > 0$ such that*

$$\|u_\varepsilon\|_{L^\infty(B_R)} \leqslant C \quad \text{as } \varepsilon \to 0. \tag{31}$$

Proof. Define

$$m_\varepsilon = \sup u_\varepsilon$$

and assume by contradiction that $m_\varepsilon \to \infty$ as $\varepsilon \to 0$. Let

$$v_\varepsilon(x) = \frac{u_\varepsilon(\rho_\varepsilon x)}{m_\varepsilon},$$

where $\rho_\varepsilon = m_\varepsilon^{\frac{1-p}{2}}$. Then v_ε is radially symmetric, radially nonincreasing and uniformly bounded by 1. We also have $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$ and v_ε satisfies

$$-\Delta v_\varepsilon + \frac{\rho_\varepsilon^2}{m_\varepsilon^{1+\beta}} v_\varepsilon^{-\beta} = v_\varepsilon \quad \text{in } B_{R/\rho_\varepsilon}, \qquad v_\varepsilon = 0 \quad \text{on } \partial B_{R/\rho_\varepsilon}. \tag{32}$$

Multiplying this equation by v_ε and integrating we obtain $\|\nabla v_\varepsilon\|_{L^2(B_{R/\rho_\varepsilon})} \leqslant C$. Since v_ε is radial, it has a subsequence such that v_ε convergence locally uniformly in $\mathbb{R}^N - \{0\}$ to a radially symmetric, radially nonincreasing function $v \in H_0^1(\mathbb{R}^N)$, $v \geqslant 0$. We claim that $v_\varepsilon(r) \geqslant \frac{1}{2}$ for $0 \leqslant r \leqslant 1$. Indeed, let us rewrite (32) as

$$\left(r^{N-1} v_\varepsilon'\right)' = \frac{\rho_\varepsilon^2}{m_\varepsilon^{1+\beta}} r^{N-1} v_\varepsilon^{-\beta} - r^{N-1} v_\varepsilon^p.$$

Hence

$$r^{N-1} v_\varepsilon(r)' \geqslant -\int_0^r s^{N-1} v_\varepsilon(s)^p \, ds \tag{33}$$

from which we deduce that $r^{N-1} v_\varepsilon(r)' \geqslant -r^N$ and therefore $v_\varepsilon(r) \geqslant 1 - \frac{1}{2} r^2$ for $r \geqslant 0$. This proves our claim and, using elliptic regularity, shows that $v_\varepsilon \to v$ locally uniformly in $\mathbb{R}^N$. If $v(r) > 0$ for all $r \geqslant 0$ then v satisfies

$$-\Delta v = v^p, \qquad v > 0 \quad \text{in } \mathbb{R}^N,$$

which is not possible. Define

$$R_0 = \sup\{r > 0\colon\ v(r) > 0\}.$$

Then $R_0 > 0$ is well defined and finite and v satisfies

$$-\Delta v = v^p, \qquad v > 0 \quad \text{in } B_{R_0}, \qquad v = 0 \quad \text{on } \partial B_{R_0}.$$

By the Hopf lemma $v'(R_0) < 0$. We will find a contradiction with this fact as follows. Let $R_0 - 1 < r < R_0$ and $\eta(x) = r + 1 - x$ if $x \leqslant r + 1$ and $\eta(x) = 0$ for $x \geqslant r + 1$. Multiplying Eq. (32) by $v_\varepsilon' \eta$ and integrating we find

$$\begin{aligned} &-\frac{1}{2} v_\varepsilon'(r)^2 - \frac{1}{2} \int_r^{r+1} \left(v_\varepsilon'\right)^2 \eta' + (N-1) \int_r^{r+1} \left(v_\varepsilon'\right)^2 \eta \frac{ds}{s} \\ &\quad = -\frac{\rho_\varepsilon^2}{m_\varepsilon^{1+\beta}} \frac{v_\varepsilon(r)^{1-\beta}}{1-\beta} - \frac{\rho_\varepsilon^2}{m_\varepsilon^{1+\beta}} \int_r^{r+1} \frac{v_\varepsilon(r)^{1-\beta}}{1-\beta} \eta' + \frac{v_\varepsilon(r)^{p+1}}{p+1} + \int_r^{r+1} \frac{v_\varepsilon^{p+1}}{p+1} \eta'. \end{aligned} \tag{34}$$

Letting $\varepsilon \to 0$,

$$-\frac{1}{2} v'(r)^2 - \frac{1}{2} \int_r^{r+1} (v')^2 \eta' + (N-1) \int_r^{r+1} (v')^2 \eta \frac{ds}{s} = \frac{v(r)^{p+1}}{p+1} + \int_r^{r+1} \frac{v^{p+1}}{p+1} \eta'.$$

Since $v = v' = 0$ to the right of R_0 the previous formula shows that

$$v'(r) \to 0 \quad \text{as } r \to R_0,\ r < R_0.$$

This contradicts Hopf's lemma, and proves the claim (31). □

Lemma 4.3. *Let $\bar{R}$ be as in Corollary 1.2 and $\bar{u}$ the solution to (2) extended by 0 to $\mathbb{R}^N$. If $R > \bar{R}$ then $u_\varepsilon \to \bar{u}$ uniformly in B_R and in $H_0^1(B_R)$.*

Proof. Multiplying (27) by u_ε and integrating by parts we find that ∇u_ε is bounded in $L^2(\mathbb{R}^N)$. Also, inequality (33) is also valid for u_ε and since u_ε is uniformly bounded we deduce that $u_\varepsilon(r) \geqslant \frac{1}{2}$ in a neighborhood of 0. Thus up to subsequence $u = \lim_{\varepsilon\to 0} u_\varepsilon$ exists and the convergence is locally uniformly in B_R. Moreover $u \geqslant 0$, $u > 0$ near the origin, u is radially symmetric and radially nonincreasing. Define

$$R_0 = \sup\{r > 0:\ u(r) > 0\}.$$

Then u satisfies

$$-\Delta u + u^{-\beta} = u^p, \qquad u > 0 \quad \text{in } B_{R_0}, \qquad u = 0 \quad \text{on } \partial B_{R_0}.$$

By Theorem 1.1 $R_0 \leqslant \bar{R}$ (actually we should argue that R_0 is finite, which follows from the results of the previous section). We will verify now that $u'(R_0) = 0$. Let $R_0 - 1 < r < R_0$ and $\eta(x) = r + 1 - x$ if $x \leqslant r + 1$ and $\eta(x) = 0$ for $x \geqslant r + 1$. Multiplying Eq. (27) by $u'_\varepsilon \eta$ and following a calculation similar to (34) we find

$$-\frac{1}{2}u'(r)^2 - \frac{1}{2}\int_r^{r+1} (u')^2\eta' + (N-1)\int_r^{r+1} (u')^2\eta\frac{ds}{s} = -\frac{v(r)^{1-\beta}}{1-\beta} - \int_r^{r+1} \frac{v^{1-\beta}}{1-\beta}\eta' + \frac{v(r)^{p+1}}{p+1} + \int_r^{r+1} \frac{v^{p+1}}{p+1}\eta'.$$

Since $u = u' = 0$ to the right of R_0 the previous formula shows that

$$u'(r) \to 0 \quad \text{as } r \to R_0,\ r < R_0.$$

By Corollary 1.2 $R_0 = \bar{R}$, $u = \bar{u}$ and by uniqueness it is the complete sequence that converges. We have seen that the convergence is locally uniformly in B_R and by the previous estimate it is actually uniformly in B_R. The convergence in $H^1(B_R)$ follows from the weak convergence in this space and the equality:

$$\int_{B_R} |\nabla u_\varepsilon|^2 = \int_{B_R} G_\varepsilon(u_\varepsilon) - u_\varepsilon^{p+1}. \qquad \square$$

Define

$$(mp)_\varepsilon = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J_\varepsilon\big(\gamma(t)\big),$$

where

$$\Gamma = \big\{\gamma : [0,1] \to H_0^1(B_R):\ \gamma \text{ is continuous},\ \gamma(0) = 0,\ \gamma(1) = A\varphi_1\big\}.$$

In the above definition the constant A is fixed such that $J_\varepsilon(A\varphi_1) < 0$ for all $\varepsilon > 0$. Then by construction of u_ε,

$$J_\varepsilon(u_\varepsilon) = (mp)_\varepsilon.$$

We also define

$$(mp)_0 = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J\big(\gamma(t)\big)$$

and

$$\mathcal{N}_R = \big\{u \in H_0^1(B_R):\ u \geqslant 0,\ u \not\equiv 0,\ G(u) = 0\big\}.$$

Lemma 4.4. *Let $\bar u$ be the solution to* (2) *extended by 0 to $\mathbb{R}^N$. Then*

$$J(\bar u) = \lim_{\varepsilon\to 0}(mp)_\varepsilon = (mp) = \inf_{\mathcal{N}_R} J.$$

Proof. Let u_ε denote the solution of (27) constructed in Lemma 4.1 with the mountain pass theorem. Multiplying Eq. (27) by u_ε and integrating we have

$$\int_{B_R} |\nabla u_\varepsilon|^2 + g_\varepsilon(u_\varepsilon)u_\varepsilon - u_\varepsilon^{p+1} = 0.$$

Since $g_\varepsilon(u)u \to u_+^{1-\beta}$ uniformly for u on compact sets of $\mathbb{R}$ we have

$$\int_{B_R} |\nabla u_\varepsilon|^2 + u_\varepsilon^{1+\beta} - u_\varepsilon^{p+1} = o(\varepsilon),$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus there exists $t(\varepsilon) = 1 + o(\varepsilon)$ with the property $t(\varepsilon)u_\varepsilon \in \mathcal{N}_R$. Hence

$$\inf_{\mathcal{N}_R} J \leqslant J\big(t(\varepsilon)u_\varepsilon\big).$$

But

$$J\big(t(\varepsilon)u_\varepsilon\big) = J(u_\varepsilon) + o(\varepsilon) = J_\varepsilon(u_\varepsilon) + o(\varepsilon) = (mp)_\varepsilon + o(\varepsilon).$$

Thus

$$\inf_{\mathcal{N}_R} J \leqslant (mp)_\varepsilon + o(\varepsilon). \tag{35}$$

For any fixed $\gamma \in \Gamma$,

$$(mp)_\varepsilon \leqslant \sup_{t\in[0,1]} J_\varepsilon\big(\gamma(t)\big)$$

and letting $\varepsilon \to 0$

$$\limsup_{\varepsilon\to 0}(mp)_\varepsilon \leqslant \sup_{t\in[0,1]} J\big(\gamma(t)\big).$$

Therefore

$$\limsup_{\varepsilon\to 0}(mp)_\varepsilon \leqslant (mp)_0. \tag{36}$$

To prove the converse let $u \in \mathcal{N}_R$. Given $c_1 > 0$, $c_2 \geqslant 0$, $c_3 > 0$, we consider the function

$$f(t) = c_1\frac{t^2}{2} + c_2\frac{t^{1-\beta}}{1-\beta} - c_3\frac{t^{p+1}}{p+1} \quad \text{for } t > 0.$$

Note that

$$\frac{f'(t)}{t} = c_1 + c_2 t^{-\beta-1} - c_3 t^{p-1}$$

is a decreasing function with limit $+\infty$ as $t \to 0$ and $-\infty$ as $t \to +\infty$. Thus f has a unique critical point, which corresponds to a maximum and is nondegenerate. Thus there is a unique $t^*(u) > 0$ which is critical point of

$$t \mapsto J(tu)$$

and hence $t^*(u) = 1$. Therefore $J(tu) \leqslant J(t^*(u)u)$ for all $t \geqslant 0$. Let $t_1 > t^*(u)$ be large such that $J(t_1 u) < 0$. We take as γ the path that connects 0 with $t_1 u$ through a straight line and then $t_1 u$ with $A\varphi_1$ on the affine space $\{s_1(t_1 u) + s_2 A\varphi_1\colon s_1, s_2 \in \mathbb{R}\}$ along which J is negative. Then $\max_{t\in[0,1]} J(\gamma(t)) = J(u)$ and hence

$$(mp)_0 \leqslant \inf_{\mathcal{N}_R} J. \tag{37}$$

Collecting (35), (36) and (37) we find

$$\lim_{\varepsilon\to 0}(mp)_\varepsilon = (mp) = \inf_{\mathcal{N}_R} J.$$

On the other hand $(mp)_\varepsilon = J_\varepsilon(u_\varepsilon) = J(\bar u) + o(\varepsilon)$ and the result follows. □

Proof of Theorem 1.3. By density it is sufficient to show that $J(\bar u) \leqslant J(\varphi)$ for any $\varphi \in \mathcal{N}$ with compact support. But then $\varphi \in \mathcal{N}_R$ with $R > 0$ large and the conclusion follows from Lemma 4.4.

For the uniqueness part, we assume that $u \in \mathcal{N}$ minimizes J on $\mathcal{N}$. Let u^* denote the Schwarz symmetrization of u. Then u^* is radially symmetric and radially nonincreasing and it is well known [8] that $\int_{\mathbb{R}^N}(u^*)^{p+1} = \int_{\mathbb{R}^N} u^{p+1}$, $\int_{\mathbb{R}^N}(u^*)^{1-\beta} = \int_{\mathbb{R}^N} u^{1-\beta}$ and

$$\int_{\mathbb{R}^N} \left|\nabla u^*\right|^2 \leqslant \int_{\mathbb{R}^N} |\nabla u|^2$$

with equality if and only if $u = u^*$ (after translating). As a consequence $G(u^*) \leqslant 0$ and we can select $t^* > 0$ such that $G(t^* u^*) = 0$. This number t^* is the one that maximizes $t \mapsto J(tu^*)$, that is,

$$J\left(t^* u^*\right) = \sup_{t\geqslant 0} J(tu).$$

Similarly

$$J(u) = \sup_{t\geqslant 0} J(tu).$$

Given $b, c > 0$ the function $a \in (0,\infty) \mapsto \sup_{t\geqslant 0}(at^2 + bt^{1-\beta} - ct^{p+1})$ is increasing and therefore

$$J\left(t^* u^*\right) \leqslant J(u),$$

with strict inequality unless $\int_{\mathbb{R}^N}|\nabla u^*|^2 = \int_{\mathbb{R}^N}|\nabla u|^2$, that is, $u = u^*$. Since u minimizes J in $\mathcal{N}$ we deduce that also t^*u^* minimizes J in $\mathcal{N}$ and $J(t^*u^*) = J(u)$ and therefore $u = u^*$ after translating. By Corollary 1.2 $u = \bar u$. □

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