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Self-similar solutions of the porous medium equation in a half-space with a nonlinear boundary condition: existence and symmetry

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Abstract

We find existence of a nonnegative compactly supported solution of the problem $\Delta u = u^\alpha$ in $\mathbb{R}^N_+$, $\partial u/\partial \nu = u$ on $\partial\mathbb{R}^N_+$. Moreover, we prove that every nonnegative solution with finite energy is compactly supported and radially symmetric in the tangential variables.

Keywords: Elliptic equations; Nonlinear boundary conditions; Symmetry

1. Introduction

We study existence of nonnegative solutions of the following problem:

$$\begin{cases} \Delta u = u^\alpha & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial \nu} = u & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{1.1}$$

where $\partial/\partial\nu$ is the outer unit normal derivative and $0 < \alpha < 1$.

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This elliptic problem appears naturally when one considers self-similar blowing up solutions of the porous medium equation ($m > 1$)

$$\begin{cases} v_t = \Delta v^m & \text{in } \mathbb{R}^N_+ \times (0,T), \\ \frac{\partial v^m}{\partial \nu} = v^m & \text{on } \partial\mathbb{R}^N_+ \times (0,T). \end{cases} \tag{1.2}$$

The blow-up problem for the porous medium equation has deserved a great deal of attention; see, for example, [3,10–12,19].

In the study of blow-up problems, self-similar profiles are used to study the fine asymptotic behavior of a solution of the parabolic equation near its blow-up time; see, for example, [14,15]. It often happens that the spatial shape of the solution near blow-up is close to a self-similar profile [5,6,12,15].

In our case, assume that $v(x,t)$ is a solution of (1.2) with blow-up time T. Then the rescaled function $z(x,t) = (T-t)^{1/(m-1)} v(x,t)$ should converge as $t \nearrow T$ to a stationary profile $z(x)$ satisfying

$$\begin{cases} \Delta z^m = \frac{1}{m-1} z & \text{in } \mathbb{R}^N_+, \\ \frac{\partial z^m}{\partial \nu} = z^m & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

as is often the case when dealing with parabolic problems; see [5–7,10]. Then $u(x) = cz(x)^m$ is a solution of (1.1) with $\alpha = 1/m$ for a suitable choice of the constant c.

On the other hand, given a nonnegative solution $u(x)$ of (1.1), $z(x) = (u(x)/c)^{1/m}$ gives rise to a special solution to (1.2) (in self-similar form) blowing up at time T, of the form

$$v(x,t) = (T-t)^{-1/(m-1)} z(x). \tag{1.3}$$

Remark that in our case the self-similar scaling does not change the spatial variable, and hence the blow-up set of (1.3) is given by the support of $z(x)$.

Therefore there is an interest in studying self-similar profiles, in our case solutions of (1.1).

In order to motivate our study, let us recall what is known for the problem

$$v_t = \Delta v^m + v^m \quad \text{in } \mathbb{R}^N \times (0,T). \tag{1.4}$$

Problem (1.4) admits self-similar solutions of the form (1.3). In this case the profile $z(x)$ is a solution of

$$0 = \Delta z^m + z^m - \frac{1}{m-1} z \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

One way to look for solutions of (1.5) is to search for radial ones. The existence of a radial compactly supported nontrivial solution reduces to the study of an ODE and was done in [7,17]. Moreover, a symmetry analysis using moving planes implies that every solution with finite energy has compact support and is composed by a finite number of radial "bumps" located such that their supports do not intersect; see [7,18].

Concerning the existence of solutions of (1.1), let us observe that in one space dimension we are facing an ODE that can be solved explicitly and it turns out that there exists only one compactly supported solution in $\mathbb{R}_+$,

$$u(x) = c_1\big((c_2 - x)_+\big)^{2/(1-\alpha)}. \tag{1.6}$$

Unfortunately, for $N \geqslant 2$, an easy inspection of problem (1.1) shows that there is no hope to look for radial solutions since they can not verify the boundary condition. Therefore, in the case under study, the elliptic problem remains a PDE that can not be solved by ODE methods.

However, the problem has still some natural symmetry in the tangential variables. In fact, if we call a point $x \in \mathbb{R}^N_+$, $x = (x', x_N)$ $(x' \in \mathbb{R}^{N-1})$, we can search for solutions that are radial in the tangential variables, i.e.,

$$u(x) = u\big(|x'|, x_N\big). \tag{1.7}$$

It has to be noted that this symmetry assumption does not reduce the problem to an ODE.

Our first result reads as follows.

Theorem 1.1. *There exists a nontrivial, nonnegative compactly supported solution of* (1.1) *of the form* (1.7).

Next, we use the moving planes device (with a moving plane parallel to the x_N direction) to prove the following result.

Theorem 1.2. *Let $u \in H^1(\mathbb{R}^N_+)$ be a nonnegative solution of* (1.1) *with connected support. Then u is compactly supported and radial in the tangential variables, that is it has the form* (1.7).

Remark that this theorem justifies our symmetry assumption in Theorem 1.1.

When this analysis is performed we can obtain some easy corollaries concerning problem (1.2).

Corollary 1.1. *Every nonnegative nontrivial solution of* (1.2) *blows up in finite time.*

The proof of this fact follows by contradiction. Assume that v is a global nontrivial solution. As v is a supersolution of the porous medium equation its support expands [20], and eventually covers the support of a self-similar profile z. The proof ends just with the use of a comparison argument using a solution of the form (1.3) with T large enough as subsolution.

Corollary 1.2. *There exists a solution of* (1.2) *with a blow-up set composed by an arbitrary number of connected components.*

In fact, we may consider a solution of the form (1.3) with a profile $z(x)$ composed by n disjoint copies of the compactly supported solution provided by Theorem 1.1.

Moreover, we conjecture that the self-similar solutions that we have constructed give the asymptotic behavior of any solution of (1.2) as it happens in one space dimension; see [9].

The problem of uniqueness of solutions to (1.1) with compact support remains open. In the case of Eq. (1.5) it is known that solutions with compact support are unique except for translations, see [8], but the argument relies strongly on ODE techniques.

The rest of the paper is organized as follows. In Section 2 we prove our existence result, Theorem 1.1, and in Section 3, we prove our symmetry result, Theorem 1.2.

Throughout the paper, by C we mean a constant that may vary from line to line but remains independent of the relevant quantities.

2. Existence of a symmetric solution

In this section we obtain the existence of a nontrivial nonnegative compactly supported solution of (1.1).

The main idea of the proof is to consider the problem in a large half ball $B(0,R)_+ = \{x, |x| < R,\ x_N > 0\}$ with mixed boundary conditions, namely,

$$\begin{cases} \Delta u_R = (u_R)^\alpha & \text{in } B(0,R)_+, \\ \frac{\partial u_R}{\partial \nu} = u_R & \text{on } \partial B(0,R)_+ \cap \{x_N = 0\}, \\ u_R = 0 & \text{on } \partial B(0,R)_+ \cap \{x_N > 0\}. \end{cases} \tag{2.1}$$

And then obtain the desired solution proving that the support of u_R verifies

$$\max_{x \in \operatorname{supp}(u_R)} |x| < R.$$

Therefore u_R is a solution of (1.1).

This approach has already been employed by other authors. For instance, in [4] they prove existence of positive solutions to a nonlinear problem in a half-space by first solving a related problem in a half ball B_R^+ and then letting $R \to \infty$. Our problem is different in that we deal with a non Lipschitz nonlinearity and the solutions we find have compact support.

For $R > 0$ let us introduce the notation:

$$B_R^+ = B(0,R)_+, \quad \partial_1 B_R^+ = \partial B_R^+ \cap \{x_N = 0\}, \quad \partial_2 B_R^+ = \partial B_R^+ \cap \{x_N > 0\}.$$

To prove existence of a solution to (2.1) we consider the functional

$$I_R(u) = \frac{\int_{B_R^+} |\nabla u|^2 - \int_{\partial_1 B_R^+} u^2}{(\int_{B_R^+} |u|^{\alpha+1})^{2/(\alpha+1)}}$$

on the space

$$H = \{u \in H^1(B_R^+) \mid \text{such that } u = 0 \text{ on } \partial_2 B_R^+\}$$

equipped with the norm

$$\|u\|_H^2 = \int_{B_R^+} |\nabla u|^2.$$

This is indeed a norm on H by Poincaré's inequality, which is valid for functions in H since they vanish on a nontrivial part of the boundary of B_R^+.

Lemma 2.1. *For every R large enough I_R attains a minimum and there is a minimizer $u \geqslant 0$, $u \not\equiv 0$ which is a solution of* (2.1)*.*

Proof. First, let us verify that

$$\inf_{u\in H,\, u\neq 0} I_R(u) > -\infty.$$

This statement is equivalent to establish the following Sobolev inequality:

$$\int_{B_R^+} |\nabla u|^2 + K\left(\int_{B_R^+} |u|^{\alpha+1}\right)^{2/(\alpha+1)} \geqslant \int_{\partial_1 B_R^+} u^2 \quad \forall u \in H, \tag{2.2}$$

where K is a constant (it may depend on R). If (2.2) fails there exists a sequence $u_n \in H$ with $\int_{\partial_1 B_R^+} u_n^2 = 1$ such that

$$\int_{B_R^+} |\nabla u_n|^2 + n\left(\int_{B_R^+} |u_n|^{\alpha+1}\right)^{2/(\alpha+1)} \leqslant 1 \quad \forall n \geqslant 1. \tag{2.3}$$

But then, up to a subsequence, we have $u_n \rightharpoonup u$ weakly in H, $u_n \to u$ strongly in $L^{\alpha+1}(B_R^+)$ and $u_n|_{\partial_1 B_R^+} \to u|_{\partial_1 B_R^+}$ strongly in $L^2(\partial_1 B_R^+)$. Since $\int_{\partial_1 B_R^+} u_n^2 = 1$ we must have $\int_{\partial_1 B_R^+} u^2 = 1$ on one hand, but (2.3) implies that $u = 0$, a contradiction.

Let $\lambda_1(R)$ denote the first eigenvalue for the problem

$$\begin{cases} \Delta u = 0 & \text{in } B_R^+, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial_1 B_R^+, \\ u = 0 & \text{on } \partial_2 B_R^+, \end{cases} \tag{2.4}$$

and let $\varphi_{1,R} > 0$ be the eigenfunction associated to $\lambda_1(R)$. Then $\lambda_1(R) > 0$ and

$$I_R(\varphi_{1,R}) = \big(\lambda_1(R) - 1\big)\frac{\int_{\partial_1 B_R^+} \varphi_{1,R}^2}{(\int_{B_R^+} \varphi_{1,R}^{\alpha+1})^{2/(\alpha+1)}}.$$

We claim that if R is sufficiently large then the expression above is negative. In fact, observe that $\lambda_1(R)$ is given by

$$\lambda_1(R) = \min_{\varphi\in H\setminus\{0\}} \frac{\int_{B_R^+} |\nabla\varphi|^2}{\int_{\partial_1 B_R^+} \varphi^2}$$

and a change of variables shows that

$$\frac{\int_{B_R^+} |\nabla\varphi|^2}{\int_{\partial_1 B_R^+} \varphi^2} = \frac{1}{R}\frac{\int_{B_1^+} |\nabla\tilde\varphi|^2}{\int_{\partial_1 B_1^+} \tilde\varphi^2},$$

where $\tilde\varphi(x) = \varphi(Rx)$. Therefore

$$\lambda_1(R) = \frac{\lambda_1(1)}{R} \tag{2.5}$$

and this establishes that

$$\inf I_R(u) < 0 \tag{2.6}$$

for R sufficiently large.

Let (u_n) be a minimizing sequence for I_R. We can assume that $\int_{\partial_1 B_R^+} u_n^2 = 1$. Since $\inf I_R < 0$ we can also assume that $I_R(u_n) \leqslant 0$. Therefore $\int_{B_R^+} |\nabla u_n|^2 \leqslant 1$ and hence up to subsequence $u_n \rightharpoonup u$ weakly in H, $u_n \to u$ strongly in $L^{\alpha+1}(B_R^+)$ and $u_n|_{\partial_1 B_R^+} \to u|_{\partial_1 B_R^+}$ strongly in $L^2(\partial_1 B_R^+)$. Since $\int_{\partial_1 B_R^+} u_n^2 = 1$ we conclude that $u \not\equiv 0$ and by the lower semi-continuity of $\|\cdot\|_H$ under weak convergence in H we see that

$$\inf I_R \leqslant I_R(u) \leqslant \liminf_{n\to\infty} I_R(u_n) = \inf I_R.$$

Thus I_R has minimizer $u \not\equiv 0$ and we can assume that $u \geqslant 0$. There is a Lagrange multiplier λ such that

$$\int_{B_R^+} \nabla u \nabla \varphi - \int_{\partial_1 B_R^+} u\varphi = \lambda \int_{B_R^+} u^\alpha \varphi \quad \forall \varphi \in H.$$

Using this with $\varphi = u$ we see that λ has the same sign as $I_R(u)$, and thus $\lambda < 0$. Choosing $\theta = (-\lambda)^{\alpha-1} > 0$ it is easy to verify that θu solves (2.1). Finally note that θu is also a minimizer of I_R. □

Remark 2.1. From the previous proof we may observe that $\inf I(u) < 0$ if and only if there exists a nontrivial nonnegative solution of (2.1). Moreover this occurs if and only if $\lambda_1(R) < 1$.

Lemma 2.2. *Let u_R be a nonnegative minimizer of I_R. Then for R large enough there exists C independent of R such that $\|u_R\|_{L^{\alpha+1}(B_R^+)} \leqslant C$, $\|u_R\|_{L^\infty(B_R^+)} \leqslant C$, and $\|\nabla u_R\|_{L^\infty(B_{R/2}^+)} \leqslant C$.*

Proof. The first step is to show that

$$\int_{B_R^+} u_R^{\alpha+1} \leqslant C \tag{2.7}$$

with C independent of R.

Indeed, multiplying (2.1) by u_R and integrating by parts we obtain

$$\int_{B_R^+} |\nabla u_R|^2 + u_R^{\alpha+1} = \int_{\partial_1 B_R^+} u_R^2. \tag{2.8}$$

On the other hand we have shown in (2.6) that $I_R < 0$ for R large enough, but in fact we have more. Indeed fix R_0 so that $\lambda_1(R_0) - 1 < 0$, where $\lambda_1(R_0)$ is the first eigenvalue for (2.4). Let φ_{1,R_0} be the first eigenfunction associated to $\lambda_1(R_0)$ and extend it by zero to B_R^+. Then for $R > R_0$,

$$\inf I_R \leqslant I_R(\varphi_{1,R_0}) = \big(\lambda_1(R_0) - 1\big) \frac{\int_{\partial_1 B_{R_0}^+} \varphi_{1,R_0}^2}{(\int_{B_{R_0}^+} \varphi_{1,R_0}^{\alpha+1})^{2/(\alpha+1)}} = -C_0.$$

Thus

$$\int_{B_R^+} |\nabla u_R|^2 + C_0 \left(\int_{B_R^+} u_R^{\alpha+1} \right)^{2/(\alpha+1)} \leqslant \int_{\partial_1 B_R^+} u_R^2. \tag{2.9}$$

From (2.8) and (2.9) we see that (2.7) follows.

The proof of the uniform estimates $\|u_R\|_{L^\infty(B_R^+)} \leqslant C$ and $\|\nabla u_{R/2}\|_{L^\infty(B_R^+)} \leqslant C$ is standard. For simplicity let us assume first that $B_1(x_0) \subset B_R^+$. Since $u_R^\alpha \in L^{(\alpha+1)/\alpha}(B_1(x_0))$ by L^p regularity theory $u_R \in W^{2,(\alpha+1)/\alpha}(B_{1/2}(x_0))$ and then by the Sobolev embedding $u_R \in L^q(B_{1/2}(x_0))$ with $1/q = \alpha/(\alpha+1) - 2/n$. Repeating this argument a finite number of times we deduce the bound in L^∞. The bound for ∇u_R in L^∞ is similar, using Schauder estimates. Finally the same proof works if $x_0 \in \partial\mathbb{R}_+^N \cap B_R$. The only point that deserves an explanation is the L^p regularity theory for the Laplace equation with the boundary condition $\partial u/\partial\nu = u$ on $\partial\mathbb{R}_+^N$. This is well known, but for completeness we present a short proof in Appendix A. □

Remark 2.2. The mountain pass theorem of Ambrosetti and Rabinowitz [2] can also be used to prove the existence of a solution to (2.1). Indeed, the functional

$$F(u) = \frac{1}{2} \int_{B_R^+} |\nabla u|^2 + \frac{1}{1+\alpha} \int_{B_R^+} |u|^{1+\alpha} - \frac{1}{2} \int_{\partial_1 B_R^+} u^2$$

satisfies the hypotheses of the theorem. An estimate similar to (2.2),

$$\frac{1}{2} \int_{B_R^+} |\nabla u|^2 + K \left(\int_{B_R^+} |u|^{1+\alpha} \right)^{2/(1+\alpha)} \geqslant \int_{\partial_1 B_R^+} u^2 \quad \forall u \in H,$$

shows that if r is small enough and $\|u\|_H = r$ then

$$F_R(u) \geqslant \frac{1}{4} \int_{B_R^+} |\nabla u|^2 = \frac{1}{4} r^2.$$

On the other hand $F_R(u_1) < r^2/4$ and $\|u_1\|_H \geqslant r$, where $u_1 = t\varphi_{1,R}$ with $\varphi_{1,R}$ the first eigenfunction for (2.4) and t_1 is large.

Finally, the estimates of Lemma 2.2 can also be obtained for the mountain pass solution u_{mp}. It suffices to verify that the critical value of the mountain pass solution $F(u_{\mathrm{mp}})$ is bounded independently of R.

Remark 2.3. Let us write $x \in \mathbb{R}^N$ as $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. Minimizing I_R in the subspace of H consisting of functions u such that

$$u(x', x_N) = u\big(|x'|, x_N\big), \tag{2.10}$$

that is, functions that are radial with respect to x', we can find a solution to (2.1) with this property.

Definition 2.1. From now on we let $u_R \geqslant 0$ denote a nontrivial solution of (2.1) that satisfies (2.10), obtained by minimizing I_R on the space of functions in H satisfying (2.10).

We need now a result which will be proved in the next section.

Lemma 2.3. *Let u_R be the solution of Definition* 2.1. *Then $u(|x'|, x_N)$ is decreasing in $|x'|$ and x_N.*

The next result will establish Theorem 1.1.

Lemma 2.4. *Let u_R denote the solution of Definition* 2.1. *Then for R large enough u_R has compact support.*

Proof. As before we will write $x \in \mathbb{R}^N_+$ as $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and $x_N > 0$. First let us show that u_R satisfies

$$u_R(x', x_N) \leqslant \frac{C}{|x'|^{(N-1)/(\alpha+1)}|x_N|^{1/(\alpha+1)}}. \tag{2.11}$$

In fact, by Lemma 2.3,

$$u_R(x', x_N) \leqslant u_R(y', y_N) \quad \forall |y'| \leqslant |x'|,\ 0 < y_N \leqslant x_N.$$

Raising to the power $\alpha + 1$ on both sides, integrating in the region $\{(y', y_N)\colon\ |y'| \leqslant |x'|, 0 < y_N \leqslant x_N\}$ and using the estimate of Lemma 2.2 we deduce (2.11).

Let L denote a constant such that $\|\nabla u_R\|_{L^\infty(B^+_{R/2})} \leqslant L$ for all R large. By (2.11) there is R_1 such that for $R \geqslant R_1$,

$$u_R(x', x_N) \leqslant 1/2 \quad \forall |x'| \geqslant \frac{1}{2L},\ x_N \geqslant R_1.$$

This together with the Lipschitz bound implies that

$$u_R(x', R_1) \leqslant 1 \quad \forall x'.$$

Consider

$$w_1 = a\big((b - x_N)^+\big)^{2/(1-\alpha)}, \tag{2.12}$$

where f^+ denotes the positive part of f, that is $f^+ = \max(f, 0)$ and a, b are determined by

$$a^{\alpha-1} = \frac{(1-\alpha)^2}{2(1+\alpha)}, \qquad a(b - R_1)^{2/(1-\alpha)} = 1. \tag{2.13}$$

Then $\Delta w_1 = w_1^\alpha$ and $w_1(R_1) = 1$. We claim that from the maximum principle it follows that $u_R \leqslant w_1$ in $\{x_N > R_1\} \cap B^+_R$. In fact first note that $w_1 \geqslant u$ on $\partial(\{x_N > R_1\} \cap B^+_R)$. Then observe that

$$0 = -\Delta(w_1 - u) + w_1^\alpha - u^\alpha = -\Delta(w_1 - u) + c(x)(w_1 - u),$$

where

$$c = \begin{cases} \frac{w_1^\alpha - u^\alpha}{w_1 - u} \geqslant 0 & \text{if } w_1 \neq u, \\ 0 & \text{if } w_1 = u. \end{cases}$$

Thus the maximum principle can be applied and from $u_R \leqslant w_1$ in $\{x_N > R_1\} \cap B_R^+$ we deduce that $u(x', x_N) = 0$ if $x_N \geqslant R_2$, $R \geqslant R_2$.

Finally, to prove that the support of u_R is bounded in the direction of x' we need to apply the maximum principle in a region which has part of its boundary on $\{x_N = 0\}$. For an arbitrary region as before the maximum principle may not hold, because of the boundary condition $\partial u/\partial \nu = u$ on $\{x_N = 0\}$. However if the part of the boundary on $\{x_N = 0\}$ is small enough the maximum principle is valid.

Lemma 2.5. *Let $U \subset \mathbb{R}^N_+$ be open, bounded with a Lipschitz boundary. Suppose that $w \in H^1(U)$ satisfies*

$$\begin{cases} \Delta w \leqslant a(x) w & \text{in } U, \\ w \geqslant 0 & \text{on } \partial U \cap \{x_N > 0\}, \\ \frac{\partial w}{\partial \nu} \geqslant w & \text{on } \partial U \cap \{x_N = 0\}, \end{cases} \tag{2.14}$$

where $a(x) \geqslant 0$. Then there exists δ such that, if the $N-1$ dimensional measure $|\partial U \cap \{x_N = 0\}| < \delta$ then we have $w \geqslant 0$ in U.

Proof of Lemma 2.4 continued. Let $x_0 \in \partial \mathbb{R}^N_+$. We shall show that if $|x_0|$ and R are large enough then $u_R = 0$ in a neighborhood of x_0. We utilize Lemma 2.5 with $U = \{x_N > 0\} \cap B(x_0, r_0)$, with $0 < r_0 < 1$ small enough. We are going to construct a suitable comparison function w_2 which satisfies the following properties:

$$\Delta w_2 \leqslant w_2^\alpha \quad \text{in } D, \tag{2.15}$$

$$\frac{\partial w_2}{\partial \nu} \geqslant w_2 \quad \text{on } \partial D \cap x_N = 0, \tag{2.16}$$

$$w_2 \equiv 0 \text{ in a neighborhood of } x_0, \tag{2.17}$$

and

$$\inf_{\partial D \cap x_N > 0} w_2 > 0. \tag{2.18}$$

Write $x_0 = (x_0', 0)$ and define the coordinate $r = |x' - x_0'|$. Set

$$w_2 = a\big((r^2 + (x_N - d)^2 - b)^+\big)^{2/(1-\alpha)}, \tag{2.19}$$

where $a, b, d > 0$ are going to be fixed below depending only on r_0, N and α (w_2 is just a radial function about the point (x_0', d)).

First we deal with (2.16). On $\partial D \cap \{x_N = 0\}$ we have

$$\frac{\partial w_2}{\partial \nu} = -\frac{\partial w_2}{\partial x_N} = \frac{4ad}{1-\alpha}\big((r^2 + d^2 - b)^+\big)^{(1+\alpha)/(1-\alpha)}$$

so that $\partial w_2/\partial \nu \geqslant w_2$ is equivalent to

$$\frac{4d}{1-\alpha} \geqslant (r^2 + d^2 - b)^+. \tag{2.20}$$

We choose d such that

$$\frac{r_0}{2} > d > \frac{r_0^2}{4} \tag{2.21}$$

and thus

$$\frac{4d}{1-\alpha} > r_0^2.$$

Then pick b so that

$$d^2 - b < 0. \tag{2.22}$$

Therefore for $0 \leqslant r \leqslant r_0$, (2.20) holds. Note that condition (2.22) also implies that $r^2 + (x_N - d)^2 - b < 0$ in a neighborhood of $r = 0$ and $x_N = 0$, so that (2.17) holds.

To verify (2.18) observe that if $r^2 + x_N^2 = r_0^2$ then

$$r^2 + (x_N - d)^2 - b = r_0^2 - 2x_N d + d^2 - b \geqslant r_0(r_0 - 2d) + d^2 - b.$$

Because of (2.21), $r_0(r_0 - 2d) > 0$ so we restrict b to have $r_0(r_0 - 2d) + d^2 - b > 0$ in addition to (2.22).

To achieve (2.15) let us compute

$$\begin{aligned}
\Delta w_2 &= (w_2)_{x_N x_N} + (w_2)_{rr} + \frac{N-2}{r}(w_2)_r \\
&= a\big(\big(r^2 + (x_N - d)^2 - b\big)^+\big)^{2\alpha/(1-\alpha)} \\
&\quad \times \left[\frac{4N}{1-\alpha}\big(r^2 + (x_N - d)^2 - b\big)^+ + \frac{8(1+\alpha)}{(1-\alpha)^2}\big(r^2 + (x_N - d)^2\big)\right] \\
&= a^{1-\alpha} w_2^\alpha \left[\frac{4N}{1-\alpha}\big(r^2 + (x_N - d)^2 - b\big)^+ + \frac{8(1+\alpha)}{(1-\alpha)^2}\big(r^2 + (x_N - d)^2\big)\right].
\end{aligned}$$

If we choose $a > 0$ small enough then $\Delta w_2 \leqslant w_2^\alpha$ in D.

Let

$$\varepsilon := \inf_{\partial D \cap \{x_N > 0\}} w_2 > 0.$$

By (2.11) we can find R_3 such that for all $R > R_3$,

$$u_R(x', x_N) \leqslant \varepsilon/2 \quad \forall |x'| \geqslant R_3,\ x_N \geqslant \frac{\varepsilon}{2L},$$

where L is a uniform Lipschitz constant for u_R. As argued before, we deduce that

$$u_R(x', x_N) \leqslant \varepsilon \quad \forall |x'| > R_3,\ x_N > 0.$$

Now let $x_0 \in \partial\mathbb{R}^N_+$ be such that $|x_0| = R_3 + r_0$ and let $R \geqslant R_3 + 2r_0$. Then we have the hypotheses of Lemma 2.5 and since $-\Delta(w_2 - u_R) = c(x)(w_2 - u_R)$ with $c(x) \geqslant 0$, we conclude that $u_R \leqslant w_2$ in $U = B(x_0, r_0) \cap \mathbb{R}^N_+$. Since w_2 vanishes in a neighborhood of x_0 and x_0 was chosen arbitrarily in $\partial B_{R_3+r_0} \cap \partial\mathbb{R}^N_+$, we conclude that u_R vanishes in a neighborhood of that set. By monotonicity of u_R with respect to $|x'|$ and x_N we reach the desired conclusion. □

Finally we provide a short argument for Lemma 2.5.

Proof of Lemma 2.5. Let us multiply (2.14) by $w^- = -\min(w, 0)$ and integrate in U,

$$\int_U |\nabla w^-|^2 + a(x)(w^-)^2 - \int_{\partial U \cap \{x_N=0\}} (w^-)^2 \leqslant 0.$$

Then

$$\int_U |\nabla w^-|^2 + a(x)(w^-)^2 \leqslant \int_{\partial U \cap \{x_N=0\}} (w^-)^2$$
$$\leqslant \left(\int_{\partial U \cap \{x_N=0\}} (w^-)^{\tilde{2}} \right)^{2/\tilde{2}} \left|\partial U \cap \{x_N = 0\}\right|^{1/(N-1)},$$

where $\tilde{2} = 2(N-1)/(N-2)$ is the critical exponent for the Sobolev trace embedding. Using the Sobolev trace embedding we can bound

$$\left(\int_{\partial U \cap \{x_N=0\}} (w^-)^{\tilde{2}} \right)^{2/\tilde{2}} \leqslant C \int_U |\nabla w^-|^2,$$

where the constant C can be chosen independent of U. Hence

$$\int_U |\nabla w^-|^2 + a(x)(w^-)^2 \leqslant C\left|\partial U \cap \{x_N = 0\}\right|^{1/(N-1)} \int_U |\nabla w^-|^2$$

and if $C|\partial U \cap \{x_N = 0\}|^{1/(N-1)} < 1$ then $w^- \equiv 0$ in U. □

3. Symmetry properties

In this section we study symmetry properties of solutions of (1.1). In particular we will show that every solution with finite energy is compactly supported and radial in the tangential variables.

Lemma 3.1. *Let $u \in H^1(\mathbb{R}^N_+)$ be a solution of* (1.1). *Then the following norms are finite*: $\|u\|_{L^{\alpha+1}(\mathbb{R}^N_+)}$, $\|u\|_{L^\infty(\mathbb{R}^N_+)}$, *and* $\|\nabla u\|_{L^\infty(\mathbb{R}^N_+)}$.

Proof. By the equation

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 + u^{\alpha+1} = \int_{\partial \mathbb{R}^N_+} u^2,$$

and since $u \in H^1(\mathbb{R}^N_+)$, we deduce that $\int_{\mathbb{R}^n_+} u^{\alpha+1} < \infty$. From here the estimates for u and ∇u in L^∞ are obtained as in Lemma 2.2. □

Lemma 3.2. *Let $u \in H^1(\mathbb{R}^N_+)$ be a solution of* (1.1). *Then u is compactly supported.*

Proof. First we remark that

$$\lim_{R\to\infty} \sup_{\mathbb{R}^N_+\setminus B_R} u = 0. \tag{3.1}$$

We prove this by contradiction, that is we suppose that (3.1) fails. Then there exists $\varepsilon > 0$ and a sequence of points $x_n \in \overline{\mathbb{R}^N_+}$ such that $|x_n| \to \infty$ and $u(x_n) \geqslant \varepsilon$ for all n. But, by Lemma 3.1, $\sup_{\mathbb{R}^N_+} |\nabla u| < \infty$ and therefore there exists $r > 0$ independent of n such that $u \geqslant \varepsilon/2$ on $B_r(x_n) \cap \mathbb{R}^N_+$ for all n. By taking a subsequence we can assume that the balls $B_r(x_n)$ are disjoint. But this implies that $\int_{\mathbb{R}^N_+} u^{\alpha+1} \geqslant \sum_n \int_{B_r(x_n)\cap\mathbb{R}^N_+} u^{\alpha+1} = \infty$, contradicting Lemma 3.1.

We proceed now with an argument similar to the one of Lemma 2.4. First, by (3.1) we can find $R_1 > 0$ such that

$$u(x', R_1) \leqslant 1 \quad \text{for all } x' \in \mathbb{R}^{N-1}.$$

Consider now the function w_1 defined in (2.12). Since $w_1 \geqslant u$ in $\{x_N = R_1\}$ and $\liminf_{|x|\to\infty} w_1 - u \geqslant 0$, by the maximum principle we deduce that $u \leqslant w_1$ in $\{x_N > R_1\}$ and thus there exists $R_2 > 0$ such that $u(x', x_N) = 0$ for all x' and $x_N > R_2$. (A direct way of verifying that the maximum principle holds in this situation is as follows: suppose that $\sup_{\{x_N>R_1\}} u - w_1 > 0$. Then this supremum is attained at a point $x_0 = (x_0', x_{0N})$ with $x_{0N} > R_1$. Hence $\Delta(u - w_1)(x_0) \leqslant 0$ but on the other hand $\Delta(u - w_1)(x_0) = u(x_0)^\alpha - w_1(x_0)^\alpha > 0$, a contradiction.)

Let us show now that if $x' \in \mathbb{R}^{N-1}$ with $|x'|$ large enough then $u(x', 0) = 0$. Indeed, first choose $r_0 > 0$ small so that the comparison principle of Lemma 2.5 holds in $B_{r_0}(x) \cap \mathbb{R}^N_+$ for all balls $B_{r_0}(x)$ with $x \in \overline{\mathbb{R}^N_+}$. Given $x_0 \in \partial\mathbb{R}^N_+$ we constructed a function w_2 in (2.19). It satisfies $\inf_{\partial D\cap\{x_N>0\}} w_2 = \varepsilon > 0$ (see (2.18)). Using (3.1) we can find $R_3 > 0$ large so that if $x_0 \in \partial\mathbb{R}^N_+$ and $|x_0| > R_3$ then $u \leqslant \varepsilon$ on $B_{r_0}(x_0) \cap \mathbb{R}^N_+$. Using the comparison principle Lemma 2.5 in $B_{r_0}(x_0) \cap \mathbb{R}^N_+$ we conclude that $u \leqslant w_2$ in this domain and hence $u = 0$ in a neighborhood of x_0.

Finally, to see that u has compact support we take the same expression of (2.12) but we consider it as a function of x_k for a direction $k = 1, \dots, N-1$,

$$w_3 = a\big((b - x_k)^+\big)^{2/(1-\alpha)},$$

where the constants a, b are as in (2.13) and R_1 is large enough so that $u(x) \leqslant 1$ if $x_k \geqslant R_1$, $x_N > 0$. We argue as before, using the maximum principle in the region $\{x_k > R_1\} \cap \mathbb{R}^N_+$ and conclude that $u \leqslant w_3$ in $\{x_k > R_1\} \cap \mathbb{R}^N_+$. Therefore $u(x) = 0$ for x_k large and $x_N > 0$. Applying the same procedure in the other directions we reach the conclusion of the lemma. □

To prove radial symmetry in the tangential variables, we will use the moving planes technique introduced in [13]; see also [7]. To this end first we need to introduce some notation. We will call $\Sigma_\lambda = \{x \in \mathbb{R} \mid x_1 > \lambda\}$, T_λ is the hyperplane $\partial\Sigma_\lambda$, x^λ is the reflection of x across the plane T_λ, that is $x^\lambda = 2(\lambda - x_1)e_1 + x$, $u_\lambda(x) = u(x^\lambda)$ and finally $w_\lambda = u_\lambda - u$. Also we assume that $D = \operatorname{supp}(u)$ is connected.

Lemma 3.3. *If there exists a point $x_0 \in \Sigma_\lambda \cap D$ such that $w_\lambda(x_0) = 0$, then $w_\lambda(x) = 0$ for all $x \in \Sigma_\lambda \cap D$.*

Proof. The proof is almost identical to the one of Lemma 2.1 in [7], with the only remark that if $x_0 \in \partial\mathbb{R}^N_+$ is such that $w_\lambda(x_0) = 0$, then we can use Hopf boundary lemma to deduce that $w_\lambda \equiv 0$. □

Proof of Theorem 1.2. Let us define λ_0 as follows:

$$\lambda_0 = \inf\big\{\lambda \mid w_\lambda(x) \geqslant 0 \text{ for all } x \in \Sigma_\lambda\big\}.$$

This value λ_0 is well defined and finite due to the compactness of the support of u (Lemma 3.2).

Step 1. First, we observe that $-\infty < \lambda_0 < \infty$ and $\Sigma_{\lambda_0} \cap D \neq \emptyset$.

The first assertion follows from the fact that u is compactly supported. The second one is a direct consequence of the maximum principle in small domains, Lemma 2.5. In fact for λ large we have that $\Sigma_\lambda \cap D = \emptyset$ therefore $w_\lambda \geqslant 0$. While for $-\lambda$ large $(\mathbb{R}^N_+ \setminus \Sigma_\lambda) \cap D = \emptyset$ therefore $w_\lambda \ngeqslant 0$. Moreover, there exists $\tilde\lambda$ such that $\Sigma_{\tilde\lambda} \cap D \cap \partial\mathbb{R}^N_+$ has small measure, therefore we can apply Lemma 2.5 in $\Sigma_{\tilde\lambda} \cap D \cap \mathbb{R}^N_+$ getting $w_{\tilde\lambda} \geqslant 0$.

Step 2. $w_{\lambda_0} \equiv 0$ in $\Sigma_{\lambda_0} \cap \mathbb{R}^N_+$.

We prove this by contradiction. If $w_{\lambda_0} \not\equiv 0$ then, by Lemma 3.3, $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0} \cap D$. The objective is to show that if $\lambda < \lambda_0$ but very close, then $w_\lambda \geqslant 0$ in $\Sigma_\lambda \cap D$, which is a contradiction with the definition of λ_0. If $\Sigma_\lambda \cap D \cap \partial\mathbb{R}^N_+ \neq \emptyset$ let us fix a compact set $K \subset \Sigma_\lambda \cap D \cap \partial\mathbb{R}^N_+$ such that $\Sigma_\lambda \cap D \cap \partial\mathbb{R}^N_+ \setminus K$ has measure less than $\delta/2$. Since $w_{\lambda_0} > 0$ in K then $w_\lambda > 0$ in K for λ sufficiently close to λ_0. By the definition of λ_0 for $\lambda < \lambda_0$,

$$D^- = \big\{x \in \Sigma_\lambda,\ w_\lambda(x) < 0\big\} \neq \emptyset,$$

and, by our previous considerations, we have that the measure of $D^- \cap \partial\mathbb{R}^N_+$ is small. Therefore we may apply Lemma 2.5 in D^-, obtaining that $w_\lambda \geqslant 0$ in D^-, a contradiction.

Step 3. To end the proof of the theorem we just observe that, by Step 2, for any given direction perpendicular to $\partial\mathbb{R}^N_+$ there exists a plane T_{λ_0} such that u is symmetric with respect to T_{λ_0}. Since this holds for any direction perpendicular to $\partial\mathbb{R}^N_+$ we conclude that u must be radial in the tangential variables, $u = u(|x'|, x_N)$. □

Proof of Lemma 2.3. The same argument as in the proof of Theorem 1.2, using the moving plane method with planes parallel to the x_N direction, shows that if $u_R \in H^1(B_R^+)$ is a solution to (2.1) then u_R is symmetric with respect to the tangential variables x' and that it is decreasing with respect to $|x'|$.

Next we prove that u_R is decreasing with respect to x_N. For this we consider the half space $\Sigma_\lambda = \{x \in \mathbb{R} \mid x_N > \lambda\}$ and the hyperplane $T_\lambda = \partial\Sigma_\lambda$. The reflection across T_λ is given by $x \mapsto x^\lambda = 2(\lambda - x_N)e_N + x$ and we define $u_\lambda(x) = u(x^\lambda)$ and $w_\lambda = u_\lambda - u$.

For $\lambda \in (R/2, R)$ w_λ satisfies $\Delta w_\lambda = c(x) w_\lambda$ with $c(x) \geqslant 0$ in the region $\Sigma_\lambda \cap B_R$, and $w_\lambda = 0$ on $T_\lambda \cap B_R$, $w_\lambda \geqslant 0$ on $\Sigma_\lambda \cap \partial B_R$. Hence $w_\lambda \geqslant 0$ in $\Sigma_\lambda \cap B_R$ and we deduce that u_R is decreasing with respect to x_N in the region $\{x_N > R/2\} \cap B_R$.

If $\lambda \in (0, R/2)$ w_λ is defined in $\{\lambda < x_N < 2\lambda\} \cap B_R$ and satisfies $w_\lambda \geqslant 0$ on $\{\lambda < x_N < 2\lambda\} \cap \partial B_R$, $w_\lambda = 0$ on $\{x_N = \lambda\} \cap B_R$. Suppose now that $\lambda \in (R/4, R/2)$. Then using that u_R is decreasing with respect to x_N for $x_N > R/2$ we see that $\partial w_\lambda/\partial\nu \geqslant 0$ on $\{x_N = 2\lambda\} \cap B_R$. By the maximum principle we deduce that $w_\lambda \geqslant 0$ in $\{\lambda < x_N < 2\lambda\} \cap B_R$ and therefore u_R is decreasing in this region. Repeating this process we obtain the conclusion. □

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Appendix A. An L^p estimate

Let $B_1 := B_1(x_0)$ be a ball with $x_0 \in \partial\mathbb{R}^N_+$ and consider the linear elliptic equation

$$\begin{cases} \Delta u = f & \text{in } B_1 \cap \mathbb{R}^N_+, \\ \frac{\partial u}{\partial \nu} = u & \text{on } B_1 \cap \partial\mathbb{R}^N_+. \end{cases} \tag{A.1}$$

What is needed in the proof of Lemma 2.2 is the following result from L^p regularity theory.

Lemma A.1. *Let $1 < p < \infty$ and assume that $f \in L^p(B_1 \cap \mathbb{R}^N_+)$ and $u \in W^{2,p}(B_1 \cap \mathbb{R}^N_+)$ satisfies* (A.1). *Then*

$$\|u\|_{W^{2,p}(B_{1/2}\cap\mathbb{R}^N_+)} \leqslant C(n,p)\big(\|f\|_{L^p(B_1\cap\mathbb{R}^N_+)} + \|u\|_{L^p(B_1\cap\mathbb{R}^N_+)}\big). \tag{A.2}$$

We present a proof using the following L^p estimate which can be found in [1, Theorem 14.1, p. 701].

Theorem A.1. *Let $1 < p < \infty$, suppose that $g \in W^{1-1/p,p}(\partial\mathbb{R}^N_+)$ and let $v \in W^{2,p}(\mathbb{R}^N_+)$ with support in B_1 satisfy*

$$\begin{cases} \Delta v = f & \text{in } \mathbb{R}^N_+, \\ \frac{\partial v}{\partial \nu} = g & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

Then

$$\|v\|_{W^{2,p}(\mathbb{R}^N_+)} \leqslant C(n,p)\big(\|f\|_{L^p(\mathbb{R}^N_+)} + \|g\|_{W^{1-1/p,p}(\partial\mathbb{R}^N_+)} + \|v\|_{L^p(\mathbb{R}^N_+)}\big). \tag{A.3}$$

Proof of (A.2). (This is an argument adapted from [16, Theorem 9.11].) Let $1/2 < \rho < 1$ and $\eta \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ in B_ρ, $\eta \equiv 0$ on $\mathbb{R}^N \setminus B_{(1+\rho)/2}$, $|\nabla\eta| \leqslant C/(1-\rho)$ and $|D^2\eta| \leqslant C/(1-\rho)^2$ with C independent of ρ. Let $v = \eta u$. Then

$$\Delta v = f\eta + 2\nabla u\nabla\eta + u\Delta\eta$$

and

$$\frac{\partial v}{\partial \nu} = \left(\frac{\partial \eta}{\partial \nu} + \eta\right)u.$$

Applying (A.3)

$$\|u\|_{W^{2,p}(B_\rho^+)} \leqslant C\biggl(\|f\|_{L^p(B_1^+)} + \frac{1}{1-\rho}\|u\|_{W^{1,p}(B^+_{(1+\rho)/2})} + \frac{1}{(1-\rho)^2}\|u\|_{L^p(B^+_{(1+\rho)/2})} + \bigl\|(\partial\eta/\partial\nu + \eta)u\bigr\|_{W^{1-1/p,p}(\partial\mathbb{R}^N_+)} + \|u\|_{L^p(B^+_{(1+\rho)/2})}\biggr)$$

and by the trace inequality

$$\|u\|_{W^{2,p}(B_\rho^+)} \leqslant C\biggl(\|f\|_{L^p(B_1^+)} + \frac{1}{1-\rho}\|u\|_{W^{1,p}(B^+_{(1+\rho)/2})} + \frac{1}{(1-\rho)^2}\|u\|_{L^p(B^+_{(1+\rho)/2})}\biggr). \tag{A.4}$$

Define the weighted norm

$$|[u]|_{k,p} = \sup_{1/2<\rho<1} (1-\rho)^k \|u\|_{W^{k,p}(B_\rho^+)}.$$

Then from (A.4) we get

$$|[u]|_{2,p} \leqslant C\bigl(\|f\|_{L^p(B_1^+)} + |[u]|_{1,p} + |[u]|_{0,p}\bigr).$$

Using the following interpolation inequality (see [16]):

$$|[u]|_{1,p} \leqslant \varepsilon|[u]|_{2,p} + \frac{C}{\varepsilon}|[u]|_{0,p},$$

we get (A.2). □

References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, Comm. Pure Appl. Math. 12 (1959) 623–727.

[2] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.

[3] C. Bandle, H. Brunner, Blow-up in diffusion equations: a survey, J. Comput. Appl. Math. 97 (1998) 3–22.

[4] M. Chipot, M. Chlebík, M. Fila, I. Shafrir, Existence of positive solutions of a semilinear elliptic equation in $\mathbf{R}^n_+$ with a nonlinear boundary condition, J. Math. Anal. Appl. 223 (1998) 429–471.

[5] C. Cortázar, M. Del Pino, M. Elgueta, On the blow-up set for $u_t = \Delta u^m + u^m$, $m > 1$, Indiana Univ. Math. J. 47 (1998) 541–561.

[6] C. Cortázar, M. Del Pino, M. Elgueta, Uniqueness and stability of regional blow-up in a porous-medium equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002) 927–960.

[7] C. Cortázar, M. Elgueta, P. Felmer, Symmetry in an elliptic problem and the blow-up set of a quasilinear heat equation, Comm. Partial Differential Equations 21 (1996) 507–520.

[8] C. Cortázar, M. Elgueta, P. Felmer, On a semilinear elliptic problem in $\mathbb{R}^N$ with a non-Lipschitzian nonlinearity, Adv. Differential Equations 1 (1996) 199–218.

[9] C. Cortázar, M. Elgueta, O. Venegas, On the blow-up set for $u_t = (u^m)_{xx}$, $m > 1$, with nonlinear boundary conditions, preprint.

[10] V. Galaktionov, On the blow-up set for the quasilinear heat equation $u_t = (u^\sigma u_x)_x + u^{\sigma+1}$, J. Differential Equations 101 (1993) 66–79.

[11] V. Galaktionov, J.L. Vazquez, Continuation of blow-up solutions of nonlinear heat equations in several space dimensions, Comm. Pure Appl. Math. 50 (1997) 1–67.

[12] V. Galaktionov, J.L. Vázquez, The problem of blow-up in nonlinear parabolic equations, Discrete Contin. Dynam. Systems A 8 (2002) 399–433.

[13] B. Gidas, W.M. Ni, L. Niremberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979) 209–243.

[14] Y. Giga, R.V. Kohn, Nondegeneracy of blow up for semilinear heat equations, Comm. Pure Appl. Math. 42 (1989) 845–884.

[15] Y. Giga, R.V. Kohn, Characterizing blow-up using similarity variables, Indiana Univ. Math. J. 42 (1987) 1–40.

[16] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., in: Grundlehren der Mathematischen Wissenschaften, vol. 224, Springer-Verlag, Berlin, 1998.

[17] C. Gui, Symmetry of the blow-up set of a porous medium type equation, Comm. Pure Appl. Math. 48 (1995) 471–500.

[18] H.G. Kaper, M.K. Kwong, Y. Li, Symmetry results for reaction diffusion equations, Differential Integral Equations 6 (1993) 1045–1056.

[19] A. Samarski, V.A. Galaktionov, S.P. Kurdyunov, A.P. Mikailov, Blow-Up in Quasilinear Parabolic Equations, de Gruyter, Berlin, 1995.

[20] J.L. Vazquez, An introduction to the mathematical theory of the porous medium equation, in: Shape Optimization and Free Boundaries (Montreal, PQ, 1990), in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 380, Kluwer Academic, Dordrecht, 1992, pp. 347–389.