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Critical points of the regular part of the harmonic Green function with Robin boundary condition

Juan Dávila [a,*], Michał Kowalczyk [a], Marcelo Montenegro [b]

[a] *Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile*
[b] *Universidade Estadual de Campinas, IMECC, Departamento de Matematica, Caixa Postal 6065, CEP 13083-970, Campinas, SP, Brasil*

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Abstract

In this paper we consider the Green function for the Laplacian in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ with Robin boundary condition

$$\frac{\partial G_\lambda}{\partial \nu} + \lambda b(x) G_\lambda = 0, \quad \text{on } \partial\Omega,$$

and its regular part $S_\lambda(x, y)$, where $b > 0$ is smooth. We show that in general, as $\lambda \to \infty$, the Robin function $R_\lambda(x) = S_\lambda(x, x)$ has at least 3 critical points. Moreover, in the case $b \equiv const$ we prove that R_λ has critical points near non-degenerate critical points of the mean curvature of the boundary, and when $b \not\equiv const$ there are critical points of R_λ near non-degenerate critical points of b.

Keywords: Green's function; Regular part; Harmonic; Robin boundary condition; Critical points

* Corresponding author.
E-mail addresses: jdavila@dim.uchile.cl (J. Dávila), kowalczy@dim.uchile.cl (M. Kowalczyk), msm@ime.unicamp.br (M. Montenegro).

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $b(x) > 0$ a smooth function defined on $\partial\Omega$. We will consider the fundamental solution of the Laplacian in Ω with Robin boundary condition, that is,

$$\begin{cases} -\Delta G_\lambda = d_N \delta_y, & \text{in } \Omega, \\ \dfrac{\partial G_\lambda}{\partial \nu} + \lambda b(x) G_\lambda = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν denotes the exterior unit normal vector, $\lambda > 0$ is parameter and

$$d_N = \begin{cases} 2\pi, & N = 2, \\ N(N-2)\omega_N, & N \geqslant 3 \end{cases}$$

(ω_N denotes the volume of the unit ball in $\mathbb{R}^N$).

Let Γ be the fundamental solution to Δ in $\mathbb{R}^N$ i.e.

$$\Gamma(x-y) = \begin{cases} -\log|x-y|, & N = 2, \\ \frac{1}{|x-y|^{N-2}}, & N > 2. \end{cases}$$

The regular part of G_λ is then defined as

$$S_\lambda(x,y) = G_\lambda(x,y) - \Gamma(x-y). \tag{1.2}$$

In general we will be interested in the asymptotic behavior of S_λ as $\lambda \to +\infty$. More precisely, our goal is to understand the asymptotic behavior of critical points of the Robin function defined by

$$R_\lambda(x) = S_\lambda(x,x), \quad \text{as } \lambda \to \infty. \tag{1.3}$$

We notice that, formally, as $\lambda \to +\infty$ we have that G_λ approaches Green's function G_∞ for the Laplacian with zero Dirichlet boundary condition. The corresponding Robin function $R_\infty(x)$ turns out to play an important role in many applications. For instance in the context of singular perturbation problems the location of the critical points of $R_\infty(x)$ determines the location where concentration phenomena occur. To name a few examples, the locations of: a blow-up point in nonlinear elliptic problems near criticality [4,13,14], a single bubble in the Liouville problem [3, 7,9] and a single vortex in the Ginzburg–Landau equation [8] are all determined by the location of critical points of R_∞. An interesting relation between R_∞ and an isoperimetric inequality was established in [1]. Many other applications as well as the most important properties of the Robin function and its relation to the harmonic radius and harmonic center of a domain can be found in [2]. For other applications of the function R_∞ we refer the reader to [10]. When some of the problems mentioned above are considered with Robin instead of Dirichlet boundary condition it is expected that $R_\lambda(x)$ may play a similar role.

The first result we will establish says that in general R_λ possesses at least 3 critical points for λ sufficiently large.

Theorem 1.1. *There exists a $\lambda_0 > 0$ such that for any $\lambda \in [\lambda_0, \infty)$ there are at least 3 critical points of R_λ. Two of them are at distance $O(\lambda^{-1})$ from $\partial\Omega$.*

Note that $R_\infty(x) \to -\infty$ as $x \to \partial\Omega$ and therefore R_∞ has always a maximum. When $\Omega \subset \mathbb{R}^2$ is a bounded, smooth and *convex* domain the result of Caffarelli and Friedman [5] ($N = 2$) and Cardaliaguet and Tahraoui [6] ($N \geqslant 3$) implies that the level sets of R_∞ are convex. Hence generically for a convex domain R_∞ has a unique maximum point. A similar situation occurs when Ω is a symmetric domain [12], quite in contrast with the behavior of R_λ. In this sense Theorem 1.1 says that the set of critical points of R_λ is larger than the set of critical points of R_∞, with some of the critical points of R_λ approaching the boundary of the domain. This in turn implies that for $\lambda < \infty$ the set of points where concentration phenomena may occur is much richer than that corresponding to $\lambda = \infty$.

To explain our results we introduce the notation

$$d(x) = \operatorname{dist}(x, \partial\Omega), \quad x \in \Omega.$$

One of the major achievements in this paper is that we obtain a precise asymptotic formula for R_λ as $\lambda \to +\infty$ near $\partial\Omega$. As a consequence of this formula we will see that if $\lambda d(x) = o(1)$ then, formally,

$$R_\lambda(x) \sim \Gamma\big(2d(x)\big),$$

which means that for x very near $\partial\Omega$ the function R_λ blows up asymptotically (to leading order) as the Robin function with Neumann boundary condition, R_0. On the other hand it is not difficult to see that $R_\lambda \to R_\infty$ uniformly on compact subsets of Ω as $\lambda \to +\infty$. These facts imply that in the intermediate region $\lambda d(x) = O(1)$ the function R_λ attains a local minimum. Using a linking argument the existence of a second critical point can be obtained, and we show that in fact there is at least one more critical point of R_λ located away from the boundary and which corresponds to a local maximum, leading thus to the proof of Theorem 1.1.

Next we will consider two cases: (1) $b \equiv 1$; (2) b is not a constant function. In the first case we have the following, for any $N \geqslant 2$:

Theorem 1.2. *Assume $b \equiv 1$. Let $\kappa(x)$ denote the mean curvature of $\partial\Omega$ at x. If $x_0 \in \partial\Omega$ is a non-degenerate critical point of κ then for each $\beta \in (0, 1)$ there exists a $\lambda_0 > 0$ such that for any $\lambda \geqslant \lambda_0$ there exists a critical point $x_\lambda \in \Omega$ of R_λ such that $|x_\lambda - x_0| = O(\lambda^{-\beta})$.*

Theorem 1.2 is a consequence of a precise asymptotic formula for R_λ.

Theorem 1.3. *Assume $b \equiv 1$. For any $K > 1$ there exists λ_K such that for each $\lambda \geqslant \lambda_K$ and for each $x \in \Omega \subset \mathbb{R}^N$, such that $\lambda d(x) \in (K^{-1}, K)$ the following asymptotic expansion formula is true*

$$R_\lambda(x) = \lambda^{N-2}\mathrm{h}_\lambda\big(\lambda d(x)\big) + \lambda^{N-3}(N-1)\kappa(\hat{x})\mathrm{v}\big(\lambda d(x)\big) + O\big(\lambda^{N-3-\alpha}\big), \tag{1.4}$$

where $0 < \alpha < 1$ and $\kappa(\hat{x})$ is the mean curvature of $\partial\Omega$ at $\hat{x}$ and

$$\mathrm{h}_\lambda(\theta) = -\log\lambda - \log(2\theta) + 4\theta\int_0^\infty e^{-2\theta t}\log(1+t)\,dt,$$

when $N = 2$,

$$\mathrm{h}_\lambda(\theta) = (2\theta)^{2-N}\left[1 - 4\theta\int_0^\infty \frac{e^{-2\theta t}\,dt}{(1+t)^{N-2}}\right],$$

when $N > 2$. (1.5)

The function v *is given by*

$$\mathrm{v}(\theta) = -\frac{\theta}{2} - \theta\int_0^\infty e^{-2\theta s}\frac{1}{(1+s)^2}\,ds, \quad \text{when } N = 2,$$

$$\mathrm{v}(\theta) = (2\theta)^{2-N}(N-2)\left[N - 2 - \frac{3\theta}{2} + \left(2\theta^2 - (N-2)^2\right)I_{0,N-1}(2\theta)\right]$$

$$\text{when } N > 2, \quad \text{with } I_{0,N-1}(2\theta) = \int_0^\infty e^{-2\theta s}\frac{1}{(1+s)^{N-1}}\,ds. \tag{1.6}$$

For the proof of Theorem 1.1 it suffices to know the leading order term in (1.4). On the other hand Theorem 1.2 is more delicate and requires a very precise knowledge of the asymptotic behavior of R_λ, not only of its leading order, but also the next term in (1.4) which is of order $O(\lambda^{N-3})$ in the intermediate region $\lambda d(x) = O(1)$. A remarkable fact is that this term depends on the domain Ω only through the mean curvature of $\partial\Omega$. In particular $\mathrm{v}(\cdot)$ is a "universal" function depending only on the dimension, a property of R_λ which is of interest by itself.

Our approach to obtain Theorem 1.3 involves first the construction of an approximation of $S_\lambda(x, y)$ based on the corresponding Green function for a half space appropriately translated and rotated, and the use of a rescaling $\xi = \lambda x$ to analyze the behavior of $S_\lambda(x, y)$ for a point y such that $\lambda d(y) = O(1)$. To control the difference between $S_\lambda(x, y)$ and its approximation we use a suitable barrier in the new variables. This procedure leads to an expansion like (1.4) but not as explicit. To remedy this situation we compare this expansion with the corresponding one in a ball, where the Green function with Robin boundary condition can be explicitly written.

When b is *not* a constant we have:

Theorem 1.4. *Let* $x_0 \in \partial\Omega$ *be a non-degenerate critical point of* b. *Then there exists a* $\lambda_0 > 0$ *such that for any* $\lambda \geqslant \lambda_0$ *there exists an* $x_\lambda \in \Omega$ *which is a critical point of* R_λ *such that* $|x_\lambda - x_0| = O(\lambda^{-\beta})$, $0 < \beta < 1$, *as* $\lambda \to \infty$.

The proof of this last theorem is based on a formula similar to (1.4). We should point out here that when b is not a constant the relation between its critical points and those of R_λ is seen at the leading order of the expansion of R_λ as $\lambda \to \infty$.

The rest of this paper will be devoted to the proofs of the above theorems. In Section 2 we construct an approximation to S_λ and compute asymptotically the difference of the operator $\frac{\partial}{\partial\nu} + \lambda$ applied to S_λ and this approximation. This already gives the first term of R_λ and leads to a proof of Theorem 1.1 in Section 3. In Section 4, under the assumption $b \equiv 1$ we improve the expansion of R_λ to the next order, and in Section 5 we show that this expansion holds also for the derivatives of R_λ. Then in Section 6 we prove Theorems 1.2 and 1.3 and in Section 7 we present the proof of Theorem 1.4.

2. Asymptotic behavior of S_λ in Ω

In the sequel we will write $d(x) = \operatorname{dist}(x, \partial\Omega)$ and if $x \in \Omega$ is sufficiently close to $\partial\Omega$ we let $\hat{x} \in \partial\Omega$ be the unique point in $\partial\Omega$ for which $\mathrm{d}(x) = |x - \hat{x}|$.

The Green function for the Robin boundary condition in a half-space is well known [11] and will be important in our analysis. In order to define it we will denote $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and let $H = \{(x', x_N) \mid x_N > 0\}$ be the half-space. We recall (see [11, p. 121]) that if $y \in H$ and $a > 0$ the Green function for the Robin problem

$$\begin{cases} -\Delta G_a^H(x, y) = d_N \delta_y & \text{in } H, \\ -\dfrac{\partial G_a^H}{\partial x_N} + a G_a^H = 0 & \text{on } \partial H, \\ \lim\limits_{|x| \to +\infty} G_a^H(x, y) = 0 \end{cases}$$

is given by

$$G_a^H(x, y) = \Gamma(x - y) - \Gamma(x - y^*) - 2 \int_{-\infty}^{0} e^{as} \frac{\partial}{\partial x_N} \Gamma(x - y^* - e_N s)\, ds, \tag{2.1}$$

where y^* is the reflection of $y = (y', y_N)$ across ∂H, that is $y^* = (y', -y_N)$, and e_j, $j = 1, \dots, N$ denotes the canonical basis in $\mathbb{R}^N$.

To explain our definition of an approximation to G_λ let $y \in \Omega$ be close to $\partial\Omega$ and such that $\hat{y} = 0 \in \partial\Omega$, ∂H is tangent to $\partial\Omega$ at the origin and the outer normal unit vector to $\partial\Omega$ is $-e_N$. Thus we assume that $y = (0, y_N)$, where $y_N = d(y) > 0$. For such y we define the approximation as

$$\widehat{G}_\lambda(x, y) = \Gamma(x - y) - \Gamma(x - y^*) - 2 \int_{-\infty}^{0} e^{\lambda b(\hat{y}) s} \frac{\partial}{\partial x_N} \Gamma(x - y^* - e_N s)\, ds, \tag{2.2}$$

where $e_N = (0, 1)$ and $y^* = (0, -y_N)$.

We generalize this definition for an arbitrary $y \in \Omega$, sufficiently close to $\partial\Omega$ as follows. Locally, say near a point $\hat{y} \in \partial\Omega$ there exists a smooth rotation matrix $\mathcal{R}_{\hat{y}}$ such that

$$\mathcal{R}_{\hat{y}} \nu(\hat{y}) = -e_N,$$

where $\nu(\hat{y})$ denotes the outer unit normal to $\partial\Omega$ at $\hat{y}$. One such rotation can be written explicitly:

$$\mathcal{R}_{\hat{y}}(x) = \big(\tau_1(\hat{y}), \dots, \tau_{N-1}(\hat{y}), -\nu(\hat{y})\big)^{\mathrm{T}} \cdot x^{\mathrm{T}}, \tag{2.3}$$

where $\tau_1(\hat{y}), \dots, \tau_{N-1}(\hat{y})$ form an orthonormal basis of $T_{\hat{y}}\partial\Omega$ which can be assumed to be smooth. Then a precise way to define $\hat{G}_\lambda$ is as follows:

$$\widehat{G}_\lambda(x, y) = G^H_{\lambda b(\hat{y})}\big(\mathcal{R}_{\hat{y}}(x - \hat{y}), \big(0, d(y)\big)\big). \tag{2.4}$$

Observe that there is an ambiguity in the choice of the rotation $\mathcal{R}_{\hat{y}}$ since composing it with any other rotation that leaves $-e_N$ fixed may also be considered. But any choice of the rotation matrix with the above restriction leads to the same definition of $\widehat{G}_\lambda$, which allows us to define globally this function for any $y \in \Omega$ close to $\partial\Omega$, for instance $0 < d(x) < \delta$ and any $x \in \mathbb{R}^N$ except y and the line segment $\{y^* + \nu(\hat{y})s\colon\ s \geqslant 0\}$. Note that $\widehat{G}_\lambda(x, y)$ is also smooth for x in this region. In general the line segment $\{y^* + \nu(\hat{y})s\colon\ s \geqslant 0\}$ may have an intersection with Ω, but $\widehat{G}_\lambda(x, y)$ is smooth for $x \in (\Omega \cap B_\delta(\hat{y})) \setminus \{y\}$, if $\delta > 0$ is fixed suitably small.

We will write

$$S_\lambda(x, y) = u_\lambda + h_\lambda,$$

where

$$u_\lambda = G_\lambda(x, y) - \widehat{G}_\lambda(x, y)$$

and

$$h_\lambda = \widehat{G}_\lambda - \Gamma(x - y).$$

Lemma 2.1. *Let $x \in \Omega$ be such that there exists a unique $\hat{x} \in \partial\Omega$ for which $d(x) = |x - \hat{x}|$. Then the following formula holds*

$$h_\lambda(x, x) = \lambda^{N-2}\mathrm{h}_\lambda\big(\lambda d(x), b(\hat{x})\big), \tag{2.5}$$

where h_λ is defined by

$$\mathrm{h}_\lambda(\theta, b) = -\log\lambda - \log(2\theta) + 2\int_0^\infty e^{-t}\log(2\theta + t/b)\,dt, \quad \text{when } N = 2,$$

$$\mathrm{h}_\lambda(\theta, b) = (2\theta)^{2-N} - 2\int_0^\infty \frac{e^{-t}}{(2\theta + \frac{t}{b})^{N-2}}\,dt, \quad \text{when } N > 2. \tag{2.6}$$

Moreover the map $\theta \mapsto \mathrm{h}_\lambda(\theta, b)$ has the following properties:

1. *If $N = 2$, for fixed $\lambda > 0$*

$$\mathrm{h}_\lambda(\theta, b) \sim -\log(2\theta) \quad \text{as } \theta \to 0,$$
$$\mathrm{h}_\lambda(\theta, b) \sim \log(2\theta) \quad \text{as } \theta \to +\infty. \tag{2.7}$$

2. *If* $N \geqslant 3$

$$\mathrm{h}_\lambda(\theta, b) \sim (2\theta)^{2-N} \quad \text{as } \theta \to 0,$$
$$\mathrm{h}_\lambda(\theta, b) \sim -(2\theta)^{2-N} \quad \text{as } \theta \to +\infty. \tag{2.8}$$

3. *For any* $N \geqslant 2$ *and* $b > 0$ *the function* $\mathrm{h}_\lambda(\cdot, b)$ *has a unique minimum* θ_0 *in* $(0, \infty)$*. This minimum is non-degenerate.*
4. *If* $x_0 \in \partial\Omega$ *is a critical point of* b *then the function* $\mathrm{h}_\lambda(\lambda d(x), b(x))$ *has a critical point* $x_\lambda \in \Omega$ *such that* $\hat{x}_\lambda = x_0$ *and* $d(x_\lambda) = O(\lambda^{-1})$.

Remark 2.2. Note that in the case $b = 1$ formula (2.6) reduces to the one in (1.5).

Proof. With $y \in \Omega$ such that $y = (0, y_N)$ and $\nu = -(0, 1)$ we have by (2.2)

$$\begin{aligned} h_\lambda(x, y) &= -\Gamma(x - y^*) + 2e^{\lambda b(0)s}\Gamma(x - y^* - e_N s)\Big|_{s=-\infty}^{0} \\ &\quad - 2\lambda b(0)\int_{-\infty}^{0} e^{\lambda b(0)s}\Gamma(x - y^* - e_N s)\,ds \\ &= \Gamma(x - y^*) - 2\lambda b(0)\int_{-\infty}^{0} e^{\lambda b(0)s}\Gamma(x - y^* - e_N s)\,ds. \end{aligned}$$

Letting $y = x = (0, x_N)$ we get

$$\begin{aligned} h_\lambda(x, x) &= \Gamma(2x_N) - 2b(0)\lambda\int_{-\infty}^{0} e^{\lambda b(0)s}\Gamma(2x_N - e_N s)\,ds \\ &= \Gamma(2x_N) - 2\int_{-\infty}^{0} e^{t}\Gamma\left(2x_N - \frac{t}{\lambda b(0)}\right)dt. \end{aligned}$$

Identity (2.5) in the case of an arbitrary $y \in \Omega$ close to $\partial\Omega$ follows from the above formula after applying $\mathcal{R}_y^{-1}$ (cf. (2.3)) and translating.

Now we deal with the properties of $\mathrm{h}_\lambda(\theta, b)$. Assume first $N = 2$. Then

$$\mathrm{h}_\lambda(\theta, b) = -\log\lambda - \log(2\theta) + 2\int_0^\infty e^{-t}\log\left(2\theta + \frac{t}{b}\right)dt. \tag{2.9}$$

Integrating by parts we get

$$\mathrm{h}_\lambda(\theta, b) = -\log\lambda + \log(2\theta) + 2\int_0^\infty \frac{e^{-t}}{2\theta b + t}\,dt,$$

and these formulas imply (2.7).

When $N \geqslant 3$ the argument is similar. Indeed, in this case

$$\mathtt{h}_\lambda(\theta, b) = (2\theta)^{2-N} - 2\int_0^\infty \frac{e^{-t}}{(2\theta + \frac{t}{b})^{N-2}}\,dt. \tag{2.10}$$

Integrating by parts, we see from the formula above that

$$\mathtt{h}_\lambda(\theta, b) = (2\theta)^{2-N} + O\left(\log\frac{1}{\theta}\right), \quad \text{as } \theta \to 0, \text{ if } N = 3,$$
$$\mathtt{h}_\lambda(\theta, b) = (2\theta)^{2-N} + O\big(\theta^{3-N}\big), \quad \text{as } \theta \to 0, \text{ if } N > 3.$$

Integrating by parts (2.10), we also have

$$\mathtt{h}_\lambda(\theta, b) = -(2\theta)^{2-N} + O\big(\theta^{1-N}\big), \quad \text{as } \theta \to +\infty,$$

and these properties imply (2.7).

By the above considerations we deduce that $\mathtt{h}_\lambda(\cdot, b)$ has at least one minimum. To see that it is unique we may assume that $b = 1$ and consider

$$f_2(t) = \log t - 2\int_0^\infty e^{-s}\log(t+s)\,ds.$$
$$f_N(t) = t^{2-N} - 2\int_0^\infty \frac{e^{-s}}{(t+s)^{N-2}}\,ds.$$

Then

$$f_2' = f_3, \qquad f_N' = (2-N)f_{N+1} \quad \text{for all } N \geqslant 3.$$

We claim that for all $N \geqslant 3$ f_N has a unique zero t_N and that t_N is increasing. Indeed $f_N(t) = 0$ is equivalent to

$$k_N(t) = \frac{1}{2} \quad \text{where } k_N(t) = \int_0^\infty e^{-s}\left(1 - \frac{s}{t+s}\right)^{N-2} ds.$$

But $\frac{d}{dt}k_N(t) > 0$. To see that t_N is increasing note that $\frac{1}{2} = k_N(t_N) > k_{N+1}(t_N)$ so $t_{N+1} > t_N$. Finally t_N is a non-degenerate minimum of f_{N-1} because $f_{N-1}'' = (N-3)(N-2)f_{N+1}$ has its zero at t_{N+1} and is positive to the left of t_{N+1}.

The proof of the last property is direct from the previous considerations. $\square$

Remark 2.3. For x near $\partial\Omega$ consider $h_\lambda(x, x)$ as a function of the variables $(d(x), \hat{x})$ and let us still denote it by $h_\lambda(d(x), \hat{x})$. From the proof of the above lemma for $\hat{x} \in \partial\Omega$ held fixed the function $d \mapsto h_\lambda(d, \hat{x})$ has a local minimum at $d = d^*(\hat{x})$. Moreover there exist constants $0 < m < M$ such that for all $\hat{x} \in \partial\Omega$ we have $d^*(x) \in (m\lambda^{-1}, M\lambda^{-1})$ and as $\lambda \to +\infty$

$$h_\lambda\big(d^*(\hat{x}), \hat{x}\big) \sim -\log\lambda \quad \text{if } N = 2, \tag{2.11}$$

$$h_\lambda\big(d^*(\hat{x}), \hat{x}\big) \sim -\lambda^{N-2} \quad \text{if } N \geqslant 3. \tag{2.12}$$

In the sequel we investigate the asymptotic behavior of u_λ as $\lambda \to +\infty$. It is rather easy to see that u_λ satisfies

$$\begin{cases} \Delta u_\lambda = 0, & \text{in } \Omega \cap B_\delta(\hat{y}), \\ \dfrac{\partial u_\lambda}{\partial \nu} + \lambda b(x) u_\lambda = g_\lambda, & \text{on } \partial\Omega \cap B_\delta(\hat{y}), \end{cases} \tag{2.13}$$

where

$$g_\lambda(x, y) = -\left[\frac{\partial}{\partial \nu} + \lambda b(x)\right] \widehat{G}_\lambda(x, y).$$

A convenient way to describe the behaviors of u_λ and g_λ is using stretched variables. More precisely define

$$\begin{aligned} \tilde{u}_\lambda(\xi, \eta) &= u_\lambda(x, y), \\ \tilde{g}_\lambda(\xi, \eta) &= g_\lambda(x, y), \end{aligned} \tag{2.14}$$

where $\xi, \eta, \hat{y}$ are in 1–1 correspondence with x, y by relations

$$\xi = \lambda \mathcal{R}_{\hat{y}}(x - \hat{y}), \qquad \eta = \lambda d(y). \tag{2.15}$$

Notice that $\tilde{u}$, $\tilde{g}$ depend also on $\hat{y}$ and we may have to write $\tilde{u}_\lambda(\xi, \eta, \hat{y}) = u_\lambda(x, y)$, but we will avoid this notation.

With the purpose to keep the notation simple we write

$$\Omega_\lambda = \big\{\lambda \mathcal{R}_{\hat{y}}(x - \hat{y}) \mid x \in \Omega\big\}.$$

Note that in our definition Ω_λ depends also on $\hat{y}$ but we will not emphasize this dependence. For a fixed $\hat{y} \in \partial\Omega$, as $\lambda \to +\infty$ the set Ω_λ approaches the upper half-space. For $\tilde{g}$ it is similar except that the limit domain of definition is $\partial H = \{\xi \mid \xi = (\xi', 0)\}$.

Lemma 2.4. *Let $N \geqslant 2$. There exists $\lambda_0 > 0$ such that for each $\lambda \geqslant \lambda_0$, each constant $K > 0$ and each $y \in \Omega$ such that*

$$K^{-1} \leqslant \lambda d(y) \leqslant K \tag{2.16}$$

we have

$$\big\| g_\lambda(\cdot, y)\big\|_{L^\infty(\partial\Omega \cap B_\delta(\hat{y}))} \leqslant C(K)\lambda^{N-2}. \tag{2.17}$$

Proof. Let λ_0 be a large number such that for each $y \in \Omega$, with $d(y) \leqslant \lambda_0^{-1}$, its projection $\hat{y} \in \partial\Omega$ is uniquely determined. Let $y \in \Omega$ satisfying (2.16) be fixed. Since the linear isometry $\mathcal{R}_{\hat{y}}(x-\hat{y})$ takes $T_{\hat{y}}\partial\Omega$ onto ∂H and $\mathcal{R}_{\hat{y}}(y-\hat{y}) = (0, d(y))$, without loss of generality we can assume that $\hat{y} = 0$, $y = (0, y_N)$, $\nu(\hat{y}) = -e_N$ and $K^{-1} < \lambda y_N < K$. Let $\delta > 0$ be a small, fixed number such that in a δ-neighborhood of 0 $\partial\Omega$ is represented as a graph, i.e.

$$\partial\Omega \cap B_\delta(0) = \big\{x_N = \varphi(x')\big\},$$

where g is a smooth function. We have

$$\varphi(x') = \frac{1}{2}\langle Ax', x'\rangle + O\big(|x'|^3\big) \quad \text{as } |x'| \to 0, \tag{2.18}$$

where $A = D^2\varphi(0)$.

When $x \in \partial\Omega \cap B_\delta(0)$ we have

$$\frac{\partial}{\partial\nu} = -\frac{\partial}{\partial x_N} + a(x') \cdot \nabla, \tag{2.19}$$

where

$$a(x') = \big(D^2\varphi(0) \cdot x', 0\big) + O\big(|x'|^2\big). \tag{2.20}$$

Writing

$$\frac{\partial}{\partial\nu} + \lambda b(x) = -\frac{\partial}{\partial x_N} + \lambda b(0) + a(x') \cdot \nabla + \lambda\big(b(x) - b(0)\big),$$

we get at $x_N = \varphi(x')$,

$$\begin{aligned}
-g_\lambda(x,y) &= \left[\frac{\partial}{\partial\nu} + \lambda b(x)\right]\widehat{G}_\lambda(x,y) \\
&= \left[-\frac{\partial}{\partial x_N} + \lambda b(0)\right]\widehat{G}_\lambda + a(x') \cdot \nabla\widehat{G}_\lambda + \lambda\big[b(x) - b(0)\big]\widehat{G}_\lambda \\
&:= g_{1\lambda} + g_{2\lambda} + g_{3\lambda}.
\end{aligned}$$

Let us first consider $g_{1\lambda}$. Notice that after integration by parts we have

$$\widehat{G}_\lambda(x,y) = \Gamma(x-y) + \Gamma(x-y^*) - 2\lambda b(0)\int_{-\infty}^{0} e^{\lambda b(0)s}\,\Gamma(x-y^*-e_N s)\,ds.$$

Then

$$
\begin{aligned}
\frac{\partial \widehat{G}_\lambda(x,y)}{\partial x_N} &= \frac{\partial \Gamma}{\partial x_N}(x-y) + \frac{\partial \Gamma}{\partial x_N}(x-y^*) \\
&\quad - 2\lambda b(0) \int_{-\infty}^{0} e^{\lambda b(0)s} \frac{\partial \Gamma}{\partial x_N}(x - y^* - e_N s)\, ds \\
&= \frac{\partial \Gamma}{\partial x_N}(x-y) + \frac{\partial \Gamma}{\partial x_N}(x-y^*) + 2\lambda b(0)\Gamma(x-y^*) \\
&\quad - 2\lambda^2 b^2(0) \int_{-\infty}^{0} e^{\lambda b(0)s} \Gamma(x - y^* - e_N s)\, ds,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
g_{1\lambda} &= -\left[\frac{\partial \Gamma}{\partial x_N}(x-y) + \frac{\partial \Gamma}{\partial x_N}(x-y^*)\right] + \lambda b(0)\left[\Gamma(x-y) - \Gamma(x-y^*)\right] \\
&:= \tilde{g}_{1\lambda} + \hat{g}_{1\lambda}.
\end{aligned}
$$

In what follows we will consider stretched variables as defined in (2.14), (2.15). In terms of these new variables we have at $x_N = \varphi(x')$

$$
\begin{aligned}
\tilde{g}_{1\lambda} &= \gamma_N \left[\frac{x_N - y_N}{|(x', x_N - y_N)|^N} + \frac{x_N + y_N}{|(x', x_N + y_N)|^N}\right] \\
&= \lambda^{N-1}\gamma_N \left[\frac{\lambda\varphi(\xi'/\lambda) - \eta}{[|\xi'|^2 + (\lambda\varphi(\xi'/\lambda) - \eta)^2]^{N/2}} + \frac{\lambda\varphi(\xi'/\lambda) + \eta}{[|\xi'|^2 + (\lambda\varphi(\xi'/\lambda) + \eta)^2]^{N/2}}\right],
\end{aligned}
\tag{2.21}
$$

where

$$
\gamma_N = \begin{cases} 1 & \text{if } N = 2, \\ N-2 & \text{if } N \geqslant 3. \end{cases}
$$

We observe that

$$
\left|\lambda^2 \varphi\left(\frac{\xi'}{\lambda}\right)\right| \leqslant C|\xi'|^2,
$$

with some $C > 0$ independent on λ. Let us write

$$
\alpha(\xi') = \lambda^2 \varphi\left(\frac{\xi'}{\lambda}\right). \tag{2.22}
$$

Expanding then the term inside the brackets in (2.21) in powers of $\frac{1}{\lambda}$ we get:

$$
\tilde{g}_{1\lambda}(\xi', \eta) = \lambda^{N-2}\gamma_N \langle A\xi', \xi'\rangle \frac{|\xi'|^2 + \eta^2(1-N)}{(|\xi'|^2 + \eta^2)^{N/2+1}} \left(1 + O\left(\frac{|\xi'|}{\lambda}\right)\right). \tag{2.23}
$$

In order to estimate $\hat{g}_{1\lambda}$ we will separately consider the cases $N=2$ and $N \geqslant 3$. In the former case we claim that

$$\begin{aligned}\hat{g}_{1\lambda} &= \lambda b(0)\big(-\log|x-y| + \log|x-y^*|\big) \\ &= \frac{\lambda b(0)}{2}\log\left(\frac{|\xi'|^2 + (\alpha(\xi')/\lambda + \eta)^2}{|\xi'|^2 + (\alpha(\xi')/\lambda - \eta)^2}\right) \\ &= b(0)\langle A\xi', \xi'\rangle \frac{\eta}{|\xi'|^2+\eta^2} + O\big(\lambda^{-1}\big(1+|\xi'|\big)\big). \end{aligned} \tag{2.24}$$

Indeed, observe that

$$\log\left(\frac{|\xi'|^2 + (\alpha(\xi')/\lambda + \eta)^2}{|\xi'|^2 + (\alpha(\xi')/\lambda - \eta)^2}\right) = \log(1+\theta),$$

where

$$\theta = \frac{4\alpha(\xi')\eta}{\lambda(|\xi'|^2 + (\alpha(\xi')/\lambda - \eta)^2)} = O\big(\lambda^{-1}\big),$$

by (2.22). Hence

$$\log\left(\frac{|\xi'|^2 + (\alpha(\xi')/\lambda + \eta)^2}{|\xi'|^2 + (\alpha(\xi')/\lambda - \eta)^2}\right) = \theta + O\big(\lambda^{-2}\big).$$

Applying the Mean value theorem we get

$$\theta = \frac{4\alpha(\xi')\eta}{\lambda(|\xi'|^2+\eta^2)} + O\left(\lambda^{-2}\frac{|\xi'|^2(1+|\xi'|)}{|\xi'|^2+\eta^2}\right) = \frac{4\alpha(\xi')\eta}{\lambda(|\xi'|^2+\eta^2)} + O\big(\lambda^{-2}\big(1+|\xi'|\big)\big),$$

where $O(\cdot)$ is uniform for $|\xi'| \leqslant \delta\lambda$ and $K^{-1} \leqslant \eta \leqslant K$. From this we get our claim (2.24) if $N=2$.

When $N \geqslant 3$ we get by a similar argument

$$\hat{g}_{1\lambda} = (N-2)\lambda^{N-2} b(0)\langle A\xi', \xi'\rangle \frac{\eta}{(|\xi'|^2+\eta^2)^{N/2}} + O\left(\frac{\lambda^{N-3}}{(1+|\xi'|)^{N-4}}\right). \tag{2.25}$$

We will now compute $g_{2\lambda}$. From the definition of $g_{2\lambda}$ we have

$$\begin{aligned} g_{2\lambda} &= a(x')\cdot\nabla\widehat{G}_\lambda(x,y) \\ &= -\gamma_N a(x')\cdot\left[\frac{x-y}{|x-y|^N} + \frac{x-y^*}{|x-y^*|^N}\right. \\ &\qquad \left. - 2\lambda b(0)\int_{-\infty}^{0} e^{\lambda b(0)s}\frac{x-y^*-e_N s}{|x-y^*-e_N s|^N}\,ds\right]. \end{aligned} \tag{2.26}$$

We notice that, going from the original to stretched variables, we have

$$a(x') = \left(\frac{\nabla\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}}, \frac{-1}{\sqrt{1+|\nabla\varphi(x')|^2}} + 1 \right)$$
$$= \left(\frac{\frac{1}{\lambda}\nabla\alpha(\xi')}{\sqrt{1+\frac{1}{\lambda^2}|\nabla\alpha(\xi')|^2}}, \frac{-1}{\sqrt{1+\frac{1}{\lambda^2}|\nabla\alpha(\xi')|^2}} + 1 \right).$$

Noting that

$$\left| \frac{1}{\lambda}\nabla\alpha(\xi') \right| \leqslant C\frac{|\xi'|}{\lambda},$$

we get

$$a\left(\frac{\xi'}{\lambda}\right) = \left(\frac{1}{\lambda}\nabla\alpha(\xi'), \frac{1}{2\lambda^2}\left|\nabla\alpha(\xi')\right|^2 \right)\left(1 + O\left(\frac{|\xi'|^2}{\lambda^2}\right)\right).$$

Then we have, again changing to stretched variables,

$$a(x')\cdot\left[\frac{x-y}{|x-y|^N} + \frac{x-y^*}{|x-y^*|^N}\right] = 2\lambda^{N-2}\frac{\langle A\xi',\xi'\rangle}{(|\xi'|^2+\eta^2)^{N/2}} + O\left(\frac{\lambda^{N-3}}{(1+|\xi'|)^{N-3}}\right).$$

The second term in (2.26) can be written as follows:

$$a(x')\cdot\left[2\lambda b(0)\int_{-\infty}^{0} e^{\lambda b(0)s}\frac{x-y^*-e_N s}{|x-y^*-e_N s|^N}\,ds\right]$$
$$= 2\lambda^{N-2}b(0)\langle A\xi',\xi'\rangle\int_{-\infty}^{0}\frac{e^{b(0)t}}{(|\xi'|^2+(\eta-t)^2)^{N/2}}\,dt + O\left(\frac{\lambda^{N-3}}{(1+|\xi'|)^{N-3}}\right).$$

Summarizing we have for $g_{2\lambda}$

$$g_{2\lambda}(\xi',\eta) = \lambda^{N-2}\gamma_N\langle A\xi',\xi'\rangle\left[-\frac{2}{(|\xi'|^2+\eta^2)^{N/2}} + 2b(0)\int_{-\infty}^{0}\frac{e^{b(0)t}}{(|\xi'|^2+(\eta-t)^2)^{N/2}}\,dt\right]$$
$$+ O\left(\frac{\lambda^{N-3}}{(1+|\xi'|)^{N-3}}\right), \tag{2.27}$$

where the $O(\cdot)$ term is bounded uniformly in the region $\frac{|\xi'|}{\lambda} \leqslant \delta$ and $K^{-1} \leqslant \eta \leqslant K$.

To compute $g_{3\lambda}(\xi',\eta)$ we notice first that, denoting $\beta(\xi') = \lambda[b(\xi'/\lambda) - b(0)]$, we have

$$\left|\beta(\xi')\right| \leqslant C|\xi'|.$$

Using then the explicit formula for $\widehat{G}_\lambda(x,y)$ (2.2) expressed in stretched variables we get in case $N \geqslant 3$

$$\begin{aligned}
&\lambda\big[b(x)-b(0)\big]\big[\Gamma(x-y)-\Gamma(x-y^*)\big]\\
&\quad=\lambda^{N-2}\beta(\xi')\big[\Gamma\big(\xi',\eta-\alpha(\xi')/\lambda\big)-\Gamma\big(\xi',-\eta-\alpha(\xi')/\lambda\big)\big]\\
&\quad=2\gamma_N\beta(\xi')\lambda^{N-3}\frac{\alpha(\xi')\eta}{(|\xi'|^2+\eta^2)^{N/2}}+O\left(\frac{\lambda^{N-4}}{(1+|\xi'|)^{N-5}}\right),
\end{aligned}$$

while, when $N=2$, we get

$$\lambda\big[b(x)-b(0)\big]\big[\Gamma(x-y)-\Gamma(x-y^*)\big]=4\gamma_2\beta(\xi')\lambda^{-1}\frac{\alpha(\xi')\eta}{(|\xi'|^2+\eta^2)^{1/2}}+O\left(\frac{(1+|\xi'|)^2}{\lambda^2}\right).$$

Likewise, we get

$$\begin{aligned}
&-2\lambda\big[b(x)-b(0)\big]\int_{-\infty}^{0}e^{\lambda b(0)s}\frac{\partial}{\partial x_N}\Gamma(x-y^*-se_N)\,ds\\
&\quad=2\lambda^{N-2}\gamma_N\beta(\xi')\int_{-\infty}^{0}e^{b(0)t}\frac{\eta-t}{(|\xi'|^2+\eta^2)^{N/2}}\,dt\\
&\qquad+O\left(\frac{\lambda^{N-3}}{(1+|\xi'|)^{N-3}}\right).
\end{aligned}$$

Combining the last two estimates we get when $N\geqslant 3$

$$g_{3\lambda}(\xi',\eta)=2\lambda^{N-2}\gamma_N\beta(\xi')\int_{-\infty}^{0}e^{b(0)t}\frac{\eta-t}{(|\xi'|^2+\eta^2)^{N/2}}\,dt+O\left(\frac{\lambda^{N-3}}{(1+|\xi'|)^{N-4}}\right),\tag{2.28}$$

and when $N=2$ we get

$$g_{3\lambda}(\xi',\eta)=2\gamma_2\beta(\xi')\int_{-\infty}^{0}e^{b(0)t}\frac{\eta-t}{(|\xi'|^2+\eta^2)^{N/2}}\,dt+O\left(\frac{1+|\xi'|}{\lambda}\right).\tag{2.29}$$

The assertion of the lemma in the region

$$\frac{|\xi'|}{\lambda}\leqslant\delta,\qquad K^{-1}\leqslant\eta\leqslant K,$$

follows now from formulas (2.23)–(2.29). □

Observe that in the proof of Lemma 2.4 we have actually shown an asymptotic formula for $g_\lambda(\xi,\eta)$ which can be conveniently written in terms of powers of λ. The following corollary summarizes this observation.

Corollary 2.5. *Let ξ, η denote stretched variables defined in* (2.14), (2.15) *and let* $\tilde{g}_\lambda(\xi, \eta) = g_\lambda(x, y)$. *Let* $K > 0$. *In the region*

$$|\xi'| \leqslant C_1 \lambda, \qquad K^{-1} \leqslant \eta \leqslant K,$$

where $C_1 > 0$ *is fixed and small, we have*

$$\tilde{g}_\lambda(\xi, \eta) = g_0(\xi', \eta) + O\left(\frac{1 + |\xi'|}{\lambda}\right) \quad \text{if } N = 2, \tag{2.30}$$

$$\tilde{g}_\lambda(\xi, \eta) = \lambda^{N-2} g_0(\xi', \eta) + O\left(\frac{\lambda^{N-3}}{(1 + |\xi'|)^{N-4}}\right) \quad \text{if } N \geqslant 3, \tag{2.31}$$

where g_0 *is given by*

$$g_0(\xi', \eta) = \gamma_N \langle A\xi', \xi' \rangle \frac{|\xi'|^2 + (N+1)\eta^2}{(|\xi'|^2 + |\eta|^2)^{N/2+1}} + g_b(\xi', \eta), \tag{2.32}$$

where

$$\begin{aligned} g_b(\xi', \eta) = {} & -\gamma_N \langle A\xi', \xi' \rangle b(\hat{y}) \left[\frac{\eta}{(|\xi'|^2 + |\eta|^2)^{N/2}} + 2 \int_{-\infty}^{0} \frac{e^{b(\hat{y})t}}{(|\xi'|^2 + (\eta - t)^2)^{N/2}}\, dt \right] \\ & + 2\gamma_N \big\langle \nabla b(\hat{y}), \xi' \big\rangle \int_{-\infty}^{0} \frac{e^{b(\hat{y})t}(\eta - t)}{[|\xi'|^2 + (\eta - t)^2]^{N/2}}\, dt \end{aligned} \tag{2.33}$$

and $A = D^2\varphi(\hat{y})$.

Observe that in the above formulas $\tilde{g}_\lambda(\xi, \eta)$ is defined for ξ close to a part of $\partial\Omega_\lambda$ that asymptotically as $\lambda \to +\infty$ becomes ∂H. For such ξ we may write $\xi = (\xi', \xi_N)$ for unique ξ' and ξ_N, and the magnitudes of ξ and ξ' are comparable in the sense $\frac{1}{C}|\xi'| \leqslant |\xi| \leqslant C|\xi'|$.

We show now an a priori estimate which is essentially a version of the maximum principle with Robin boundary condition:

Lemma 2.6. *Let* $b : \partial\Omega \to \mathbb{R}$ *be a smooth such that* $b > 0$, $F : \partial\Omega \to \mathbb{R}$ *be a smooth function and* u *be the solution to*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \lambda b(x) u = F & \text{on } \partial\Omega, \end{cases} \tag{2.34}$$

where $\lambda > 0$. *Then*

$$\|u\|_{L^\infty(\Omega)} + \big\|d(x)\nabla u\big\|_{L^\infty} \leqslant \frac{C(N, b)}{\lambda} \|F\|_{L^\infty(\partial\Omega)}. \tag{2.35}$$

Proof. We may assume that $F \geqslant 0$ and then, by the maximum principle, $u \geqslant 0$. Let $j \geqslant 1$ and multiply (2.34) by u^j. Integrating and using Hölder's inequality we obtain

$$\lambda\Big(\min_{\partial\Omega} b\Big)\int_{\partial\Omega} u^{j+1} \leqslant \int_{\partial\Omega} Fu^j \leqslant \Big(\int_{\partial\Omega} u^{j+1}\Big)^{j/(j+1)}\Big(\int_{\partial\Omega} F^{j+1}\Big)^{1/(j+1)}$$

which implies

$$\lambda\Big(\min_{\partial\Omega} b\Big)\Big(\int_{\partial\Omega} u^{j+1}\Big)^{1/(j+1)} \leqslant \Big(\int_{\partial\Omega} F^{j+1}\Big)^{1/(j+1)}.$$

Letting $j \to +\infty$ we find

$$\lambda\Big(\min_{\partial\Omega} b\Big)\|u\|_{L^\infty(\partial\Omega)} \leqslant \|F\|_{L^\infty(\partial\Omega)}.$$

Using first the maximum principle and then the gradient estimate for the Poisson equation we deduce now estimate (2.35). □

As a consequence of estimate (2.17) and Lemma 2.6 we deduce that u_λ has the following uniform estimate:

Corollary 2.7. *We have*

$$\|u_\lambda\|_{L^\infty(\Omega\cap B_\delta(\hat{y}))} \leqslant C\lambda^{N-3}. \tag{2.36}$$

3. Existence of at least 3 critical points of R_λ

Proof of Theorem 1.1. The proof is based on the asymptotic formula for R_λ found in the previous section combined with a linking argument.

We have

$$R_\lambda(x) = h_\lambda(x,x) + u_\lambda(x,x).$$

When $x \in \Omega$ is sufficiently close to $\partial\Omega$, with some abuse of notation we can write

$$R_\lambda(x) = \lambda^{N-2}\mathrm{h}_\lambda\big(\lambda d(x), b(\hat{x})\big) + \tilde{u}\big(\lambda d(x), \hat{x}\big),$$

where h_λ is the function defined in (2.6) and $\tilde{u}(\lambda d(x), \hat{x}) = u_\lambda(x,x)$. Let m, M be the constants in Remark 2.3. Let us define

$$\mathcal{U}(m,M) \equiv \big\{x \in \Omega \;\big|\; \lambda d(x) \in (m,M)\big\} \subset \Omega. \tag{3.1}$$

Further, let $d^*(x)$ be the point at which h_λ achieves its minimum when we allow to vary $x \in \mathcal{U}(m,M)$ with $\hat{x}$ fixed. We define

$$S^* = \big\{x \in \mathcal{U}(m,M) \;\big|\; d(x) = d^*(x)\big\}. \tag{3.2}$$

Arguing as in Lemma 2.1 one can show that

$$\inf_{\partial\mathcal{U}(m,M)} \mathrm{h}_\lambda\big(\lambda d(x),\hat{x}\big) > \sup_{S^*} \mathrm{h}_\lambda\big(\lambda d(x),\hat{x}\big). \tag{3.3}$$

By (2.36) it follows that

$$\|\tilde{u}\|_{L^\infty(\Omega)} \leqslant C\lambda^{N-3},$$

and therefore from (3.3) and the formulas (2.9), (2.10) we get for λ large

$$\inf_{\partial\mathcal{U}(m,M)} R_\lambda(x) > \sup_{S^*} R_\lambda(x). \tag{3.4}$$

In particular we see that there exists $x_{\min} \in \mathcal{U}(m,M)$ such that

$$\inf_{\mathcal{U}(m,M)} R_\lambda(x) = R_\lambda(x_{\min}). \tag{3.5}$$

To find another critical point of R_λ in $\mathcal{U}(m,M)$ let us assume that there exists an $x_1 \in \mathcal{U}(m,M) \cap S^*$ such that

$$R_\lambda(x_1) > R_\lambda(x_{\min}). \tag{3.6}$$

If such a point does not exist then the theorem is proven. Let $\hat{x}_1$ be the projection of x_1 onto $\partial\Omega$ and let

$$Q \equiv \big\{x \in \mathcal{U}(m,M) \mid \hat{x} = \hat{x}_1\big\} \subset \mathcal{U}(m,M).$$

Then the sets S^* and ∂Q link in $\mathcal{U}(m,M)$. Moreover, by (3.4), we have

$$\inf_{\partial Q} R_\lambda > \sup_{S^*} R_\lambda. \tag{3.7}$$

Let

$$\mathcal{G} = \big\{f \in C^0\big(\overline{\mathcal{U}}(m,M),\overline{\mathcal{U}}(m,M)\big) \bigm| f|_{\partial Q} = \mathrm{id}\big\}.$$

Then

$$\beta \equiv \sup_{f\in\mathcal{G}} \inf_{Q} R_\lambda\big(f(x)\big)$$

is a critical value of R_λ which, by (3.6), is different than $R_\lambda(x_{\min})$.

The existence of a third critical point can be obtained by maximizing R_λ on the set $U_\lambda = \{x \in \Omega \mid d(x) > \delta\}$ where $\delta > 0$ is fixed suitably small. Indeed, we have by (2.11), (2.12) that $\sup_{\partial U_\lambda} R_\lambda \to -\infty$ as $\delta \to 0$ uniformly for all large $\lambda > 0$ while $R_\lambda \to R_\infty$ on compact sets of Ω. This shows that for sufficiently large λ the maximum of R_λ on $\overline{U}_\lambda$ is attained at some point $x_{\max} \in U_\lambda$, and hence is a critical point of R_λ. The proof of the theorem is complete. □

4. More on the asymptotic behavior of S_λ

To find the asymptotic behavior of u_λ as $\lambda \to +\infty$ we need a suitable candidate for an appropriately rescaled limit. According to Corollary 2.5 we need a function v which is harmonic in H and satisfies the boundary condition $-\frac{\partial v}{\partial \xi_N} + v = g_0$ on ∂H. For this purpose we have:

Lemma 4.1. *Let $K \geqslant 1$ be a fixed constant and let η be such that $K^{-1} < \eta < K$. There exists a smooth function v in $\overline{H}$ satisfying*

$$\Delta v = 0 \quad \text{in } H,$$
$$-\frac{\partial v}{\partial \xi_N} + v = g_0(\cdot, \eta) \quad \text{on } \partial H.$$

Moreover

$$\lim_{|\xi| \to +\infty} v(\xi) = -(1+\eta)\kappa(\hat{y}) \quad \text{if } N = 2 \tag{4.1}$$

and

$$\bigl|v(\xi, \eta)\bigr| \leqslant \frac{C(K)}{1 + |\xi|^{N-2}} \quad \forall \xi \in H, \text{ if } N \geqslant 3. \tag{4.2}$$

In (4.1) *by $\kappa(\hat{y})$ we denoted the curvature of the boundary at $\hat{y}$.*

Proof. If $N = 2$ we note that formula (2.32) for g_0 may be written in the form

$$g_0(\xi', \eta) = -(1+\eta)\kappa(\hat{y}) + g_1(\xi', \eta),$$

where g_1 has the property that

$$\bigl|g_1(\xi', \eta)\bigr| \leqslant \frac{C(K)}{1 + |\xi'|^2} \quad \forall \xi' \in \mathbb{R}, \text{ when } \nabla b(\hat{y}) = 0,$$
$$\bigl|g_1(\xi', \eta)\bigr| \leqslant \frac{C(K)}{1 + |\xi'|} \quad \forall \xi' \in \mathbb{R}, \text{ when } \nabla b(\hat{y}) \neq 0.$$

Thus, in dimension $N = 2$ we define

$$v = -(1+\eta)\kappa(\hat{y}) + v_1, \quad \text{where } v_1(\xi, \eta) = \frac{1}{d_2} \int_{\partial H} G(\zeta, \xi) g_1(\zeta, \eta)\, d\zeta$$

and $G(\zeta, \xi) = G_1^H(\zeta, \xi)$, where G_a^H is the function defined in (2.1) with $a = 1$.

In dimension $N \geqslant 3$ we define directly

$$v(\xi, \eta) = \frac{1}{d_N} \int_{\partial H} G(\zeta, \xi) g_0(\zeta, \eta)\, d\zeta. \tag{4.3}$$

Note that in all dimensions when $\zeta \in \partial H$ then $\Gamma(\zeta - \xi) = \Gamma(\zeta - \xi^*)$. Thus in all cases we are led to examine:

$$I_\mu(\xi) = \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{0} \frac{e^t(\xi_N - t)}{(|\zeta' - \xi'|^2 + (\xi_N - t)^2)^{N/2}} \frac{1}{1 + |\zeta'|^\mu} \, dt \, d\zeta',$$

where $\mu = 1, 2$ in dimension 2 and $\mu = N - 2$ if $N \geqslant 3$. Then, the assertion of the lemma follows directly from the following.

Lemma 4.2. *Let $\mu > 0$. Then if $\mu < N - 1$*

$$I_\mu(\xi) \leqslant \frac{C}{1 + |\xi|^\mu} \quad \forall \xi \in H, \tag{4.4}$$

if $\mu = N - 1$ then

$$I_\mu(\xi) \leqslant \frac{C \max(1, \log|\xi|)}{1 + |\xi|^{N-1}} \quad \forall \xi \in H, \tag{4.5}$$

and if $\mu > N - 1$ then

$$I_\mu(\xi) \leqslant \frac{C}{1 + |\xi|^{N-1}}. \tag{4.6}$$

We will prove Lemma 4.2 in Appendix A.

As a consequence of (4.6) we have in dimension $N = 2$

$$\left|v_1(\xi, \eta)\right| \leqslant \frac{C \max(1, \log|\xi|)}{1 + |\xi|} \quad \forall \xi \in H,$$

and this proves (4.1). Estimate (4.2) is a direct consequence of (4.4). The proof of Lemma 4.1 is complete. □

We will need a more explicit form of $v(\xi, \eta)$, when $\xi = (0, \eta)$, in particular in the way it depends on the geometry of $\partial\Omega$.

Corollary 4.3. *Under the assumptions of Lemma 4.1 we have*

$$v\big((0, \eta), \eta\big) = (N - 1)\kappa(\hat{y})\mathrm{v}(\eta), \tag{4.7}$$

where $\mathrm{v} : (0, +\infty) \to \mathbb{R}$ is a smooth function given by

$$\mathrm{v}(\eta) = \frac{1}{d_N} \int_{\mathbb{R}^{N-1}} G\big(\zeta, (0, \eta)\big) \zeta_1^2 \mathrm{g}\big(|\zeta'|, \eta\big) \, d\zeta', \tag{4.8}$$

$\mathrm{g}(|\zeta'|, \eta)$ is given by

$$\mathrm{g}\big(|\xi'|,\eta\big)=\gamma_N\frac{|\xi'|^2+(N+1)\eta^2}{(|\xi'|^2+|\eta|^2)^{N/2+1}}$$
$$-\gamma_N b(\hat y)\left[\frac{\eta}{(|\xi'|^2+|\eta|^2)^{N/2}}+2\int_{-\infty}^{0}\frac{e^{b(\hat y)t}}{(|\xi'|^2+(\eta-t)^2)^{N/2}}\,dt\right]$$
$$+2\gamma_N\big\langle\nabla b(\hat y),\xi'\big\rangle\int_{-\infty}^{0}\frac{e^{b(\hat y)t}(\eta-t)}{[|\xi'|^2+(\eta-t)^2]^{N/2}}\,dt$$

and $G(\zeta,\xi)=G_1^H(\zeta,\xi)$, *where* G_a^H *is the function defined in* (2.1) *with* $a=1$.

Observe that v is independent of Ω. Later on we shall give another formula for v.

Proof. The case $N=2$ is direct, so we focus only on the case $N>2$. Indeed, in this situation the function $g_0(\xi',\eta)$ can be written in the form

$$g_0(\xi,\eta)=\sum_{i,j=1}^{N-1}A_{ij}(\hat y)\xi_i\xi_j\,\mathrm{g}\big(|\xi'|,\eta\big),$$

where $A_{ij}(\hat y)$ are the coefficients of $A(\hat y)=D^2\varphi(\hat y)$. Thus, by the construction (4.3) of v in Lemma 4.1

$$v(\xi,\eta)=\frac{1}{d_N}\sum_{i,j=1}^{N-1}A_{ij}(\hat y)\int_{\mathbb{R}^{N-1}}G(\zeta,\xi)\zeta_i\zeta_j\mathrm{g}\big(|\zeta'|,\eta\big)\,d\zeta'$$
$$=\frac{1}{d_N}\sum_{i=1}^{N-1}A_{ii}(\hat y)\int_{\mathbb{R}^{N-1}}G(\zeta,\xi)\zeta_i^2\mathrm{g}\big(|\zeta'|,\eta\big)\,d\zeta'.$$

Observe that the value of the above integrals does not depend on i when evaluated at point ξ of the form $(0,\xi_N)$. In particular, with v defined as in (4.8) we see that

$$v\big((0,\eta),\eta\big)=\sum_{i=1}^{N-1}A_{ii}\,\mathrm{v}(\eta)=(N-1)\kappa(\hat y)\mathrm{v}(\eta).\qquad\square$$

Let us consider a fixed $y=(0,y_N)$ as in the proof of Lemma 2.4. Let $\delta>0$ be a small number. In order to relate $u_\lambda(x,y)$ for $x\in\Omega\cap B_\delta(0)$ with v we will pass to stretched variables and combine v with a change of variables so that v is defined in $\Omega_\lambda\cap B_{\delta\lambda}$:

$$\tilde v(\xi,\eta)=v\big(T_\lambda(\xi),\eta\big)\quad \xi\in\Omega_\lambda\cap B_{\delta\lambda},$$

where

$$T_\lambda(\xi,'\xi_N)=\big(\xi',\xi_N-\lambda g(\xi'/\lambda)\big).\tag{4.9}$$

We will also denote $\tilde u_\lambda(\xi,\eta)=u_\lambda(x,y)$ where (ξ,η) and (x,y) are related by relations (2.15).

Lemma 4.4. *Assume that $b \equiv 1$. For any $0 < \alpha < 1$ there exists a $C > 0$ independent of λ such that*

$$\left|\lambda^{3-N}\tilde{u}_\lambda(\xi,\eta) - \tilde{v}(\xi,\eta)\right| \leqslant C\frac{1+|\xi|^\alpha}{\lambda^\alpha} \quad \forall \xi \in \Omega_\lambda \cap B_{\delta\lambda}.$$

Proof. Note that estimate (2.17) and Lemma 2.6 imply

$$\lambda\|\tilde{u}\|_{L^\infty(\Omega_\lambda \cap B_{\delta\lambda})} \leqslant C. \tag{4.10}$$

It can be seen easily that the function $\tilde{u}$ satisfies

$$\begin{aligned} &\Delta\tilde{u} = 0, \quad \text{in } \Omega_\lambda \cap B_{\delta\lambda}, \\ &\lambda\left(\frac{\partial \tilde{u}}{\partial \nu} + \tilde{u}\right) = \begin{cases} g_0 + O(\frac{1+|\xi'|}{\lambda}), & N = 2, \\ g_0 + O(\frac{1}{\lambda(1+|\xi'|)^{N-4}}), & N \geqslant 3, \end{cases}, \quad \text{on } \partial\Omega_\lambda \cap B_{\delta\lambda}. \end{aligned} \tag{4.11}$$

We shall use a barrier to estimate the difference $\lambda\tilde{u} - \tilde{v}$. This barrier is given by

$$\bar{u} = \frac{(d_\lambda(\xi) + c_1)^\alpha}{\lambda^\alpha} + \frac{c_2(|\xi|^2+1)^{\alpha/2}}{\lambda^\alpha},$$

where $0 < \alpha < 1$ and $c_1, c_2 > 0$ are constants to be fixed later on and

$$d_\lambda(\xi) = \operatorname{dist}(\xi, \partial\Omega_\lambda).$$

We claim that there exists a $C > 0$ such that

$$|\lambda\tilde{u} - \tilde{v}| \leqslant C\bar{u} \quad \text{in } \Omega_\lambda \cap B_{\delta\lambda}, \tag{4.12}$$

provided that $\delta > 0$ is sufficiently small.

We compute

$$\begin{aligned} \Delta\bar{u} = {}& \frac{\alpha(d_\lambda + c_1)^{\alpha-1}\Delta d_\lambda}{\lambda^\alpha} + \frac{\alpha(\alpha-1)(d_\lambda+c_1)^{\alpha-2}}{\lambda^\alpha} \\ &+ \frac{c_2(N-1)\alpha(|\xi|^2+1)^{(\alpha/2-1)}}{\lambda^\alpha} + \frac{c_2\alpha(\alpha-2)(|\xi|^2+1)^{(\alpha/2-2)}|\xi|^2}{\lambda^\alpha}. \end{aligned}$$

Observing that $|\Delta d_\lambda| \leqslant C\lambda^{-1}$ in $\Omega_\lambda \cap B_{\lambda\delta}$ and fixing $\delta > 0$, $c_2 > 0$ small we see that

$$\Delta\bar{u} \leqslant -c\frac{(d_\lambda + c_1)^{\alpha-2}}{\lambda^\alpha} \quad \text{in } \Omega_\lambda \cap B_{\lambda\delta}, \tag{4.13}$$

for some $c > 0$. From the change of variables (4.9) we find

$$\Delta\tilde{v} = \frac{\partial^2 v}{\partial\xi_i\partial\xi_j}O\left(\frac{|\xi|}{\lambda}\right) + \frac{\partial v}{\partial\xi_i}O\left(\frac{1}{\lambda}\right) \quad \text{in } \Omega_\lambda \cap B_{\lambda\delta}.$$

Using the explicit formula for v in Lemma 4.1 and applying Lemma 4.2 we get

$$\frac{\partial^2 v}{\partial \xi_i \partial \xi_j} = O\big((1+|\xi|)^{-N}\big), \qquad \frac{\partial v}{\partial \xi_i} = O\big((1+|\xi|)^{1-N}\big),$$

and hence

$$\Delta \tilde{v} = O\left(\frac{1}{\lambda(1+|\xi|)^{N-1}}\right) \quad \text{in } \Omega_\lambda \cap B_{\lambda\delta}. \tag{4.14}$$

From (4.13), (4.14) we have, taking $\delta > 0$ sufficiently small,

$$\Delta\big(C\bar{u} - (\lambda\tilde{u} - \tilde{v})\big) \leqslant 0 \quad \text{in } \Omega_\lambda \cap B_{\lambda\delta}$$

for some fixed constant C.

Now let us compute the boundary condition on $\partial\Omega_\lambda \cap B_{\delta\lambda}$ where ν is the outer unit normal vector to $\partial\Omega_\lambda$. We note that from (2.19), (2.20), at $\xi_N = \lambda g(\xi'/\lambda)$,

$$\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial \xi_N} + O\left(\frac{|\xi'|}{\lambda}\right)\big|\nabla(\cdot)\big|.$$

Hence

$$\frac{\partial \bar{u}}{\partial \nu} \geqslant -\frac{\alpha c_1^{\alpha-1}}{\lambda^\alpha} - c_2 C \frac{(|\xi|^2+1)^{\alpha/2}}{\lambda^\alpha}\frac{|\xi|}{\lambda} \quad \text{on } \partial\Omega_\lambda \cap B_{\delta\lambda},$$

where C is some constant. We then find

$$\begin{aligned}\frac{\partial \bar{u}}{\partial \nu} + \bar{u} &\geqslant \frac{c_1^\alpha - \alpha c_1^{\alpha-1}}{\lambda^\alpha} - c_2 C \frac{(|\xi|^2+1)^{\alpha/2}}{\lambda^\alpha}\frac{|\xi|}{\lambda} + c_2 \frac{(|\xi|^2+1)^{\alpha/2}}{\lambda^\alpha} \\ &\geqslant c\frac{(|\xi|+1)^\alpha}{\lambda^\alpha},\end{aligned} \tag{4.15}$$

in the region of $\partial\Omega_\lambda$ such that $|\xi| \leqslant \delta\lambda$ where $c > 0$ provided $c_1 > 0$ is fixed large and $\delta > 0$ is taken small. On the other hand, (2.30), (2.31) and (4.11) imply

$$\lambda\left(\frac{\partial \tilde{u}}{\partial \nu} + \tilde{u}\right) = g_0 + O\left(\frac{(1+|\xi|)}{\lambda}\right) \quad \text{on } \partial\Omega_\lambda \cap B_{\delta\lambda}, \quad \text{if } N = 2, \tag{4.16}$$

$$\lambda\left(\frac{\partial \tilde{u}}{\partial \nu} + \tilde{u}\right) = g_0 + O\left(\frac{1}{\lambda(1+|\xi|)^{N-4}}\right) \quad \text{on } \partial\Omega_\lambda \cap B_{\delta\lambda}, \quad \text{if } N \geqslant 3. \tag{4.17}$$

For $\frac{\partial \tilde{v}}{\partial \nu}$ we have

$$\frac{\partial \tilde{v}}{\partial \nu} + \tilde{v} = g_0 + O\left(\frac{1}{\lambda(1+|\xi|)^{N-2}}\right) \quad \text{on } \partial\Omega_\lambda \cap B_{\delta\lambda}. \tag{4.18}$$

From (4.16)–(4.18) and (4.15) we obtain

$$\frac{\partial(C\bar{u} - (\lambda\tilde{u} - \tilde{v}))}{\partial\nu} + C\bar{u} - (\lambda\tilde{u} - \tilde{v}) \geqslant 0 \quad \text{on } \partial\Omega_\lambda \cap B_{\delta\lambda}.$$

Finally, by Lemma 4.1 and (4.10), we have

$$|\lambda\tilde{u} - \tilde{v}| \leqslant C\bar{u} \quad \text{on } \Omega_\lambda \cap \partial B_{\delta\lambda}.$$

The maximum principle now implies $\lambda\tilde{u} - \tilde{v} \leqslant C\bar{u}$ in $\Omega_\lambda \cap B_{\delta\lambda}$ and reversing the roles of $\lambda\tilde{u}$ and $\tilde{v}$ we obtain $\tilde{v} - \lambda\tilde{u} \leqslant C\bar{u}$ in $\Omega_\lambda \cap B_{\delta\lambda}$. This establishes (4.12) and the conclusion of the lemma follows from this inequality and the behavior of $\bar{u}$ on bounded sets. □

Using an elliptic estimate for the gradient we get from Lemma 4.4:

Lemma 4.5. *Assume that $b \equiv 1$. Then there is a fixed $\delta > 0$ such that for any $0 < \alpha < 1$ there exists a constant C such that the following estimate holds*:

$$\left|\lambda^{3-N}\nabla_\xi\tilde{u}_\lambda(\xi,\eta) - \nabla_\xi\tilde{v}(\xi,\eta)\right| \leqslant C\frac{1+|\xi|^\alpha}{\lambda^\alpha} \quad \forall\xi \in \Omega_\lambda \cap B_{\delta\lambda}. \tag{4.19}$$

In addition we will need an estimate for the derivatives of the function $\tilde{u}_\lambda(\xi,\eta)$ with respect to η.

Lemma 4.6. *Assume that $b \equiv 1$. For any $0 < \alpha < 1$ there is C independent of λ such that*

$$\left|\lambda^{3-N}\partial_\eta\tilde{u}_\lambda(\xi,\eta) - \partial_\eta\tilde{v}(\xi,\eta)\right| \leqslant C\frac{1+|\xi|^\alpha}{\lambda^\alpha} \quad \forall\xi \in \Omega_\lambda \cap B_{\delta\lambda}. \tag{4.20}$$

Proof. The proof of this lemma goes along the same lines as the proof of Lemma 4.4. Indeed, after rotation and translation as in the proof of Lemma 2.4 we get that the function $u_{\lambda,y_N}(x,y) \equiv \partial_{y_N}u_\lambda(x,y)$, where $y = (0, y_N)$, satisfies

$$\begin{cases} \Delta u_{\lambda,y_N} = 0, & \text{in } \Omega, \\ \dfrac{\partial u_{\lambda,y_N}}{\partial\nu} + \lambda u_{\lambda,y_N} = g_{\lambda,y_N}, & \text{on } \partial\Omega. \end{cases}$$

Calculations similar to those in the proof of Lemma 2.4 lead to the analogs of (2.30), (2.31) of Corollary 2.5 with g_0 replaced by $g_{0,\eta}$. Then Lemma 4.1 can be applied to find the function $\partial_\eta\tilde{v}(\xi,\eta)$. An application of a comparison argument as in Lemma 4.4 yields finally (4.20). □

5. Estimates for the derivatives of R_λ

Throughout this section $b \equiv 1$. Let us observe that combining Corollary 4.3, Lemma 4.4 and the change of variables (2.14), (2.15) we find

$$u_\lambda(x,x) = \lambda^{N-3}(N-1)\kappa(\hat{x})\,\mathrm{v}\big(\lambda d(x)\big) + O\big(\lambda^{N-3-\alpha}\big)$$

uniformly for $K^{-1} \leqslant \lambda d(x) \leqslant K$.

Let ∇_T denote the tangential which is defined in a neighborhood of $\partial\Omega$. The aim in this section is to show that the following estimates hold:

Proposition 5.1. *Let $K > 1$, $0 < \alpha < 1$. Then*

$$\nabla_T u_\lambda(x, x) = \lambda^{N-3}(N-1)\nabla\kappa(\hat{x})\mathrm{v}\big(\lambda d(x)\big) + O\big(\lambda^{N-3-\alpha}\big) \tag{5.1}$$

uniformly for $K^{-1} \leqslant \lambda d(x) \leqslant K$.

Proposition 5.2. *Let $K > 1$, $0 < \alpha < 1$. Then*

$$\big\langle \nabla u_\lambda(x, x), \nu(\hat{x})\big\rangle = -\lambda^{N-2}(N-1)\kappa(\hat{x})\,\mathrm{v}'\big(\lambda d(x)\big) + O\big(\lambda^{N-2-\alpha}\big) \tag{5.2}$$

uniformly for $K^{-1} \leqslant \lambda d(x) \leqslant K$, where $\nu(\hat{x})$ is the unit normal vector at $\hat{x}$.

For simplicity of the presentation we shall give the detailed calculations in dimension $N = 2$.

We rotate and translate Ω such that $0 \in \partial\Omega$ and the exterior unit normal vector at 0 points down, that is $\nu(0) = -e_2$.

Let us fix $\delta > 0$ small and let $\varphi : (-\delta, \delta) \to \mathbb{R}$ be a smooth function whose graph is $\partial\Omega$ near 0, or more precisely

$$\big\{(x_1, x_2) \in \partial\Omega \;\big|\; |x_1|, |x_2| < \delta\big\} = \big\{(x_1, x_2) \;\big|\; x_2 = \varphi(x_1),\ |x_1| < \delta\big\}$$

and note that

$$\varphi(0) = 0, \qquad \varphi'(0) = 0.$$

We shall write

$$a_0 = \varphi''(0), \qquad a_1 = \varphi'''(0)$$

so that

$$\varphi(y_1) = \frac{a_0}{2} y_1^2 + \frac{a_1}{6} y_1^3 + O\big(y_1^4\big) \quad \text{for } y_1 \text{ near } 0. \tag{5.3}$$

The exterior unit normal vector at a point $(y_1, \varphi(y_1))$ is then given by

$$\nu(y_1) = \frac{1}{\sqrt{1 + \varphi'(y_1)^2}}\big(\varphi'(y_1), -1\big)^{\mathrm{T}}.$$

Recall that the curvature at 0 is given by

$$\kappa(0) = \varphi''(0) = a_0$$

and

$$\kappa'(0) = \varphi'''(0) = a_1.$$

The smooth rotation matrix $\mathcal{R}$ introduced in (2.3) can be considered to depend on y_1:

$$\mathcal{R}(y_1) = \frac{1}{\sqrt{1+\varphi'(y_1)^2}} \begin{bmatrix} 1 & \varphi'(y_1) \\ -\varphi'(y_1) & 1 \end{bmatrix}$$

so that

$$\mathcal{R}(y_1)\nu(y_1) = -e_2.$$

As before, we introduce the change of variables

$$\xi = \lambda \mathcal{R}(y_1)\bigl(x - \bigl(y_1, \varphi(y_1)\bigr)\bigr), \qquad \eta = \lambda d(y),$$

and the functions

$$\tilde{u}_\lambda(\xi, \eta, y_1) = u_\lambda(x, y),$$
$$\tilde{g}_\lambda(\xi, \eta, y_1) = g_\lambda(x, y).$$

The difference with respect to the change of variables (2.14), (2.15) is that now $\tilde{u}_\lambda$ and $\tilde{g}_\lambda$ depend on y_1 rather than on $\hat{y}$.

To show (5.1) we will need:

Lemma 5.3. *Let $K > 1$, $0 < \alpha < 1$. Then for $K^{-1} \leqslant \eta \leqslant K$ we have*

$$\frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\bigl((0, \eta), \eta, 0\bigr) = O\left(\frac{1}{\lambda^{1+\alpha}}\right). \tag{5.4}$$

Lemma 5.4. *Let $K > 1$, $0 < \alpha < 1$. Then for $K^{-1} \leqslant \eta \leqslant K$ we have*

$$\partial_{y_1} \tilde{u}_\lambda\bigl((0, \eta), \eta, 0\bigr) = \frac{1}{\lambda}\kappa'(0)\mathrm{v}_0(\eta) + O\left(\frac{1}{\lambda^{1+\alpha}}\right). \tag{5.5}$$

Proof of Proposition 5.1. Assume for a moment that (5.4), (5.5) hold. At a point x close to $\partial\Omega$ of the form $x = (0, x_2)$ the tangential direction is given by e_1 and hence

$$\nabla_{\mathrm{T}} u_\lambda(x, x) = \frac{\partial u_\lambda(x, x)}{\partial x_1} + \frac{\partial u_\lambda(x, x)}{\partial y_1}. \tag{5.6}$$

By the chain rule

$$\frac{\partial u_\lambda}{\partial x_1} = \lambda \nabla_\xi \tilde{u}_\lambda \mathcal{R}(y_1) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\frac{\partial u_\lambda}{\partial y_1} = \lambda \nabla_\xi \tilde{u}_\lambda \left(\frac{d\mathcal{R}}{dy_1}\left(x - \begin{bmatrix} y_1 \\ \varphi(y_1) \end{bmatrix}\right) - \mathcal{R}\begin{bmatrix} 1 \\ \varphi'(y_1) \end{bmatrix}\right) + \lambda \partial_\eta \tilde{u}_\lambda \frac{\partial d(y)}{\partial y_1} + \partial_{y_1} \tilde{u}_\lambda.$$

We want to evaluate these expressions at $y_1 = 0$. For a point x close to $\partial\Omega$ of the form $x = (0, x_2)$ with $x_2 = \eta/\lambda$ we have

$$\frac{\partial u_\lambda}{\partial x_1}(x, x) = \lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\big((0, \eta), \eta, 0\big).$$

On the other hand, since $\frac{\partial d(y)}{\partial y_1} = 0$ at a point $y = (0, y_2)$, and

$$\frac{d}{dy_1}\mathcal{R}(0) = \begin{bmatrix} 0 & a_0 \\ -a_0 & 0 \end{bmatrix},$$

it follows that

$$\frac{\partial u_\lambda}{\partial y_1}(x, x) = a_0\lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\big((0, \eta), \eta, 0\big)x_2 - \lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\big((0, \eta), \eta, 0\big) + \partial_{y_1}\tilde{u}_\lambda\big((0, \eta), \eta, 0\big).$$

Therefore, for such x and since $x_2 = \eta/\lambda$

$$\frac{\partial u_\lambda}{\partial x_1}(x, x) + \frac{\partial u_\lambda}{\partial y_1}(x, x) = a_0 \frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\big((0, \eta), \eta, 0\big)\eta + \partial_{y_1}\tilde{u}_\lambda\big((0, \eta), \eta, 0\big). \tag{5.7}$$

Combining (5.4), (5.5) and (5.7) we find

$$\nabla_{\mathrm{T}} u_\lambda(x, x) = \frac{1}{\lambda}\kappa'(0)\mathrm{v}_0(\eta) + O\left(\frac{1}{\lambda^{1+\alpha}}\right)$$

for $K^{-1} \leqslant \eta \leqslant K$, which is the desired estimate (5.1). □

Proof of Lemma 5.3. We showed in Lemma 4.4 that for any fixed $R > 0$

$$\lambda\tilde{u}_\lambda - \tilde{v} = O\big(1/\lambda^\alpha\big) \quad \text{uniformly for } |\xi| \leqslant R$$

$(0 < \alpha < 1)$ and in Lemma 4.5

$$\nabla_\xi[\lambda\tilde{u}_\lambda - \tilde{v}] = O\big(1/\lambda^\alpha\big) \quad \text{uniformly for } |\xi| \leqslant R.$$

But observe that v is even with respect to ξ_1, which implies $\frac{\partial \tilde{v}}{\partial \xi_1}((0, \eta), \eta, y_1) = 0$ and therefore

$$\frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\big((0, \eta), \eta, 0\big) = O\left(\frac{1}{\lambda^{1+\alpha}}\right). \qquad \square$$

As before, let Ω_λ denote the set

$$\Omega_\lambda = \big\{\lambda\mathcal{R}(y_1)\big(x - \big(y_1, \varphi(y_1)\big)\big) \bigm| x \in \Omega\big\}.$$

Near 0 the boundary $\partial\Omega$ is represented as the graph of φ. Hence, near the origin $\partial\Omega_\lambda$ may be also represented by a graph of a function $\psi_\lambda(\xi_1, y_1)$, that is,

$$(\xi_1, \xi_2) \in \partial\Omega_\lambda \quad \Longleftrightarrow \quad \xi_2 = \psi_\lambda(\xi_1, y_1)$$

for $|\xi_1|, |\xi_1| < \lambda\delta$.

We shall need the following formula which can be obtained by a direct calculation:

Lemma 5.5.

$$\frac{\partial \psi_\lambda}{\partial y_1}(\xi_1, 0) = \frac{1}{\lambda}\big[\varphi'(\lambda\xi_1) - a_0\lambda\xi_1 - a_0^2\varphi'(\lambda\xi_1)\varphi(\lambda\xi_1)\big]$$

and hence, using (5.3),

$$\frac{\partial \psi_\lambda}{\partial y_1}(\xi_1, 0) = \frac{a_1}{2\lambda}\xi_1^2 + O\left(\frac{|\xi_1|^3}{\lambda^2}\right). \tag{5.8}$$

Before proving Lemma 5.4 we need the following expansion for $\frac{\partial \tilde{g}_\lambda}{\partial y_1}$ at $y_1 = 0$.

Lemma 5.6. *Assume $N = 2$ and let $K > 1$. Then for some suitably small $\delta > 0$ we have*

$$\frac{\partial \tilde{g}_\lambda}{\partial y_1}(\xi_1, \eta, 0) = \kappa'(0)\mathrm{g}(\xi_1, \eta) + O\left(\frac{1 + |\xi_1|}{\lambda}\right) \tag{5.9}$$

for $|\xi_1| \leqslant \delta\lambda$, $K^{-1} \leqslant \eta \leqslant K$, where

$$\mathrm{g}(\xi_1, \eta) = \xi_1^2 \frac{\xi_1^2 + 3\eta^2}{(\xi_1^2 + \eta^2)^2} - \frac{\eta\xi_1^2}{\xi_1^2 + \eta^2} - 2\xi_1^2 \int_0^\infty \frac{e^{-t}}{\xi_1^2 + (\eta + t)^2}\, dt.$$

Proof. The calculation is analogous to that in Lemma 2.4. In particular, recalling the notation in that lemma and using that $b \equiv 1$ we have

$$\begin{aligned} -\tilde{g}_\lambda &= \left[\frac{\partial}{\partial \nu} + \lambda b(x)\right]\widehat{G}_\lambda(x, y) = \left[-\frac{\partial}{\partial x_N} + \lambda\right]\widehat{G}_\lambda + a(x')\cdot\nabla\widehat{G}_\lambda \\ &:= g_{1\lambda} + g_{2\lambda}. \end{aligned}$$

For $g_{1\lambda}$ we had the formula

$$\begin{aligned} g_{1\lambda} &= -\left[\frac{\partial \Gamma}{\partial x_N}(x - y) + \frac{\partial \Gamma}{\partial x_N}(x - y^*)\right] + \lambda\big[\Gamma(x - y) - \Gamma(x - y^*)\big] \\ &:= \tilde{g}_{1\lambda} + \hat{g}_{1\lambda}. \end{aligned}$$

In terms of these new variables we have at $\xi_2 = \psi_\lambda(\xi_1, y_1)$

$$\tilde{g}_{1\lambda} = \lambda\left[\frac{\psi_\lambda(\xi_1, y_1) - \eta}{\xi_1^2 + (\psi_\lambda(\xi_1, y_1) - \eta)^2} + \frac{\psi_\lambda(\xi_1, y_1) + \eta}{\xi_1^2 + (\psi_\lambda(\xi_1, y_1) + \eta)^2}\right].$$

Differentiating with respect to y_1 and setting then $y_1 = 0$ yields

$$\begin{aligned} \frac{\partial \tilde{g}_{1\lambda}}{\partial y_1}(\xi_1, \eta, 0) = \lambda\frac{\partial \psi_\lambda(\xi_1, 0)}{\partial y_1}\bigg[&\frac{1}{\xi_1^2 + (\varphi_\lambda - \eta)^2} + \frac{1}{\xi_1^2 + (\varphi_\lambda + \eta)^2} \\ &- 2\frac{(\varphi_\lambda - \eta)^2}{(\xi_1^2 + (\varphi_\lambda - \eta)^2)^2} - 2\frac{(\varphi_\lambda + \eta)^2}{(\xi_1^2 + (\varphi_\lambda + \eta)^2)^2}\bigg], \end{aligned}$$

where for convenience we have written

$$\varphi_\lambda = \varphi_\lambda(\xi_1) = \lambda\varphi(\xi_1/\lambda) = \psi_\lambda(\xi_1, 0).$$

Expanding in powers of λ^{-1} yields:

$$\frac{1}{\xi_1^2 + (\varphi_\lambda - \eta)^2} + \frac{1}{\xi_1^2 + (\varphi_\lambda + \eta)^2} = \frac{2}{\xi_1^2 + \eta^2} + O\big(\lambda^{-2}\big)$$

and

$$\frac{(\varphi_\lambda - \eta)^2}{(\xi_1^2 + (\varphi_\lambda - \eta)^2)^2} + \frac{(\varphi_\lambda + \eta)^2}{(\xi_1^2 + (\varphi_\lambda + \eta)^2)^2} = \frac{2\eta^2}{(\xi_1^2 + \eta^2)^2} + O\big(\lambda^{-2}\big).$$

Therefore, using (5.8) we obtain

$$\frac{\partial \tilde{g}_{1\lambda}}{\partial y_1}(\xi_1, \eta, 0) = a_1 \xi^2 \frac{\xi_1^2 - \eta^2}{(\xi_1^2 + \eta^2)^2} + O\left(\frac{1 + |\xi_1|}{\lambda}\right).$$

The other terms are all similar:

$$\frac{\partial \hat{g}_{1\lambda}}{\partial y_1}(\xi_1, \eta, 0) = a_1 \frac{\eta \xi_1^2}{\xi_1^2 + \eta^2} + O\left(\frac{1 + |\xi_1|}{\lambda}\right)$$

and

$$g_{2\lambda} = -2a_1 \frac{\xi_1^2}{\xi_1^2 + \eta^2} + 2a_1 \xi_1^2 \int_0^\infty \frac{e^{-t}}{\xi_1^2 + (\eta + t)^2}\, dt + O\left(\frac{1 + |\xi_1|}{\lambda}\right). \qquad \square$$

Proof of Lemma 5.4. With the same argument as in Lemma 4.1 we can construct a smooth function v in $\overline{H}$ satisfying

$$\Delta v = 0 \quad \text{in } H,$$
$$-\frac{\partial v}{\partial \xi_2} + v = \kappa'(0) \mathrm{g}(\cdot, \eta) \quad \text{on } \partial H.$$

Indeed, since

$$\lim_{|\xi| \to +\infty} \mathrm{g}(\xi, \eta) = -1 - \eta$$

we define

$$v = \kappa'(0)(-1 - \eta + v_1) \quad \text{where } v_1(\xi, \eta) = \frac{1}{d_2} \int_{\partial H} G(\zeta, \xi) g_1(\zeta, \eta)\, d\zeta$$

with $g_1 = \kappa'(0)(\mathrm{g} + 1 + \eta)$. Then, using Lemma 4.2 we have

$$\lim_{|\xi| \to +\infty} v(\xi) = -\kappa'(0)(1+\eta).$$

Note that

$$v\big((0,\eta),\eta\big) = \kappa'(0)\mathrm{v}(\eta), \tag{5.10}$$

where v is the function defined in (4.8). Define

$$\tilde{v}(\xi,\eta) = v\big(T_\lambda(\xi),\eta\big), \quad \xi \in \Omega_\lambda \cap B_{\delta\lambda},$$

where

$$T_\lambda(\xi_1,\xi_2) = \big(\xi_1, \xi_2 - \varphi_\lambda(\xi_1)\big).$$

Let

$$w(\xi,\eta) = \lambda \frac{\partial \tilde{u}_\lambda}{\partial y_1}(\xi,\eta,0). \tag{5.11}$$

Then w is harmonic in $\Omega_\lambda \cap B_{\delta\lambda}$. Since $\lambda(\frac{\partial \tilde{u}_\lambda}{\partial \nu} + \tilde{u}) = \tilde{g}_\lambda$ we obtain the following boundary condition for w

$$\frac{\partial w}{\partial \nu} + w = \frac{\partial \tilde{g}_\lambda}{\partial y_1} - \lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\frac{\partial \nu_1}{\partial y_1} - \lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_2}\frac{\partial \nu_2}{\partial y_1} \quad \text{on } \partial\Omega_\lambda \cap B_{\delta\lambda},$$

where we have written $\nu = (\nu_1,\nu_2)$.

Observe that by (5.9), the definition of v and a calculation similar to (4.18) we have

$$\frac{\partial(\tilde{v} - w)}{\partial \nu} + \tilde{v} - w = -\lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\frac{\partial \nu_1}{\partial y_1} - \lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_2}\frac{\partial \nu_2}{\partial y_1} + O\left(\frac{1+|\xi_1|}{\lambda}\right) \quad \text{on } \partial\Omega_\lambda \cap B_{\delta\lambda}.$$

Now we just need to estimate $\lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\frac{\partial \nu_1}{\partial y_1}$ and $\lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_2}\frac{\partial \nu_2}{\partial y_1}$. By direct computation

$$\left.\frac{\partial \nu_1}{\partial y_1}\right|_{y_1=0} = \frac{\partial^2 \psi_\lambda}{\partial y_1 \xi_1}\left(1 + \left(\frac{\partial \psi_\lambda}{\partial \xi_1}\right)^2\right)^{-3/2} = O\left(\frac{1+|\xi_1|}{\lambda}\right) \tag{5.12}$$

by a formula similar to (5.8). Similarly

$$\left.\frac{\partial \nu_2}{\partial y_1}\right|_{y_1=0} = \frac{\partial^2 \psi_\lambda}{\partial y_1 \xi_1}\frac{\partial \psi_\lambda}{\partial \xi_1}\left(1 + \left(\frac{\partial \psi_\lambda}{\partial \xi_1}\right)^2\right)^{-3/2} = O\left(\frac{1+|\xi_1|^2}{\lambda^2}\right). \tag{5.13}$$

On the other hand, from (4.19) it follows that

$$\lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_1} = O\left(\frac{1+|\xi_1|^\alpha}{\lambda^\alpha}\right), \qquad \lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_2} = O\left(\frac{1+|\xi_1|^\alpha}{\lambda^\alpha}\right)$$

with $0 < \alpha < 1$. Hence

$$\lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_1} \frac{\partial \nu_1}{\partial y_1} + \lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_2} \frac{\partial \nu_2}{\partial y_1} = O\left(\frac{1 + |\xi_1|}{\lambda}\right).$$

Then using the barrier $\bar{u}$ constructed in Lemma 4.4 and the maximum principle we deduce

$$|w - \tilde{v}| \leqslant C\bar{u} \quad \text{in } \partial\Omega_\lambda \cap B_{\delta\lambda}$$

and this implies that for any $R > 0$

$$\left|w(\xi, \eta) - \tilde{v}(\xi, \eta)\right| \leqslant \frac{CR^\alpha}{\lambda^\alpha} \quad \text{for } \xi \in \Omega_\lambda,\ |\xi| \leqslant R.$$

Using now (5.10), (5.11) and the previous estimate we deduce (5.5). □

Now let us turn out attention to Proposition 5.2.

Proof of Proposition 5.2. Again we assume that $0 \in \partial\Omega$ and the exterior unit normal vector at 0 points down, that is $\nu(0) = -e_2$. At a point x close to $\partial\Omega$ of the form $x = (0, x_2)$ the normal direction is given by $-e_2$ and hence

$$\nabla u_\lambda(x, x) \cdot \nu(\hat{x}) = -\left[\frac{\partial u_\lambda(x, x)}{\partial x_2} + \frac{\partial u_\lambda(x, x)}{\partial y_2}\right].$$

By the chain rule, and evaluating at a point x close to $\partial\Omega$ of the form $x = (0, x_2)$ with $x_2 = \eta/\lambda$ we have

$$\frac{\partial u_\lambda}{\partial x_2}(x, x) = \lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_2}\big((0, \eta), \eta, 0\big)$$

and

$$\frac{\partial u_\lambda}{\partial y_1}(x, x) = \lambda \partial_\eta \tilde{u}_\lambda\big((0, \eta), \eta, 0\big).$$

By Lemmas 4.5 and 4.6

$$\frac{\partial u_\lambda}{\partial x_2}(x, x) = \frac{\partial \tilde{v}}{\partial \xi_2}\big((0, \eta), \eta, 0\big) + O\left(\frac{1}{\lambda^\alpha}\right),$$
$$\frac{\partial u_\lambda}{\partial y_1}(x, x) = \frac{\partial \tilde{v}}{\partial \eta}\big((0, \eta), \eta, 0\big) + O\left(\frac{1}{\lambda^\alpha}\right)$$

uniformly for $K^{-1} \leqslant \eta \leqslant K$. Hence

$$\nabla u_\lambda(x, x) \cdot \nu(\hat{x}) = -\kappa(\hat{x}) \mathrm{v}'(\eta) + O\left(\frac{1}{\lambda^\alpha}\right). \qquad \square$$

6. Locating critical points of R_λ when $b \equiv 1$

Proof of Theorem 1.3. Let $y_0 \in \partial\Omega$ be a fixed point. We work with x, y in a neighborhood $B_{R/\lambda}(y_0) \cap \Omega$ where $R > 0$ is fixed suitably large. For x, y in this neighborhood, by Lemma 4.4

$$u_\lambda(x, y) = \lambda^{N-3}(N-1)\kappa(\hat{x})\mathrm{v}\big(\lambda d(x)\big) + O\big(\lambda^{-\alpha}\big), \quad x, y \in B_{R/\lambda}(y_0). \tag{6.1}$$

We will show now asymptotic formulas (1.6). We begin by noticing that the right-hand side of (6.1) depends on Ω only through the mean curvature κ at $\hat{x}$, appearing as a multiplicative factor. Therefore replacing Ω with a ball B_R, such that ∂B_R is tangential to $\partial\Omega$ and $R = \frac{1}{\kappa(\hat{x})}$ will lead to the same formula for v_λ. To determine v_λ we will use the fact that the Green function $G_{\lambda,R}(x, y)$ for a ball with corresponding Robin boundary condition is known explicitly:

$$-\Delta G_{\lambda,R} = d_N \delta_y, \quad \text{in } B_R(0),$$
$$\frac{\partial G_{\lambda,R}}{\partial \nu} + \lambda G_{\lambda,R} = 0, \quad \text{on } \partial B_R.$$

Let us consider first the case $N = 2$. As it can be verified directly the following formula holds

$$G_{\lambda,R}(x, y) = -\log|x-y| + \log\left|(x-y^*)\frac{|y|}{R}\right| + \frac{1}{\lambda R} + 2\int_0^R \left(1-\frac{s}{R}\right)^{\lambda R} \frac{\partial}{\partial s}\log\left|x\left(1-\frac{s}{R}\right) - y^*\right| ds, \tag{6.2}$$

where $y^* = \frac{R^2 y}{|y|^2}$ (see Appendix B). We have

$$S_\lambda(x, y) = G_{\lambda,R} + \log|x-y|,$$

and $R_\lambda(y) = S_\lambda(y, y)$. We will find an asymptotic formula for R_λ in terms of powers of $1/\lambda$ assuming that y is a point such that $\lambda d(y) \in (K^{-1}, K)$, for some fixed $K > 0$. We can write

$$y = \hat{y} - \frac{\hat{y}}{R} d(y) = \hat{y}(1 - \varepsilon\delta),$$

where for convenience we have denoted $\varepsilon = \frac{1}{\lambda R}$ and $\delta = \lambda d(y)$. In terms of ε and δ we get the following formula

$$R_\lambda(y) = -\log 2\lambda + \log\delta + 2\int_0^\infty e^{-t}\frac{dt}{t+2\delta} + \varepsilon\left[1 - \frac{\delta}{2} - \int_0^\infty e^{-t}\frac{t^2\,dt}{t+2\delta} - 6\delta^2\int_0^\infty e^{-t}\frac{dt}{(t+2\delta)^2}\right] + O\big(\varepsilon^2\big).$$

Denoting the $O(\varepsilon)$ term above by $\tilde{\mathrm{v}}(\delta)$ we see, since $\varepsilon = 1/R\lambda$, that $\mathrm{v}_\lambda(d(y)) = \tilde{\mathrm{v}}(\lambda d(y))$ and the required formula follows. A straightforward calculation involving integration by parts shows that in fact

$$\tilde{\mathrm{v}}(\delta) = -\frac{\delta}{2} - 2\delta^2 \int_0^\infty e^{-t} \frac{dt}{(t+2\delta)^2}.$$

An important consequence of this last formula is that we have $\mathrm{v}_\lambda(d(y)) < 0$, for $d(y) \geqslant K^{-1}$.

Now let us assume that $N \geqslant 3$. The Green function $G_{\lambda,R}$ can be written explicitly (see Appendix B)

$$\begin{aligned} G_{\lambda,R}(x,y) &= |x-y|^{2-N} - \left(1 - \frac{N-2}{\lambda R}\right)\left(\frac{|y|}{R}\right)^{2-N} |x-y^*|^{2-N} \\ &\quad - \left(2 - \frac{N-2}{\lambda R}\right)\left(\frac{|y|}{R}\right)^{2-N} \int_0^R \left(1-\frac{s}{R}\right)^{\lambda R} \frac{\partial}{\partial s}\left|x\left(1-\frac{s}{R}\right) - y^*\right|^{2-N} ds. \end{aligned} \tag{6.3}$$

When $N \geqslant 3$ an argument similar to the previous one yields the formula

$$\mathrm{v}(\theta) = (2\theta)^{2-N}(N-2)\tilde{\mathrm{v}}(2\theta),$$

where

$$\begin{aligned} \tilde{\mathrm{v}}(t) &= 1 - \frac{\theta}{2} + \frac{1}{2}\int_0^\infty e^{-ts} \frac{t(N-1)(1+4s)}{(1+s)^N} \\ &\quad - \int_0^\infty e^{-ts} \frac{(N-2) + t(2+ts^2)}{(1+s)^{N-1}}\, ds. \end{aligned} \tag{6.4}$$

We write

$$\begin{aligned} \tilde{\mathrm{v}}(t) &= 1 - \frac{t}{4} + \frac{t(N-1)}{2} I_{0,N-1}(t) + \frac{3t(N-1)}{2} I_{1,N}(t) \\ &\quad - (N-2+2t) I_{0,N-1}(t) - t^2 I_{2,N-1}, \end{aligned} \tag{6.5}$$

where

$$I_{j,N}(t) = \int_0^\infty e^{-ts} \frac{s^j}{(1+s)^N}\, ds. \tag{6.6}$$

Using the relations

$$I_{j+1,N+1} = I_{j,N} - I_{j,N+1},$$
$$I_{0,N} = \frac{1}{N-1} - \frac{t}{N-1} I_{0,N-1},$$

and integration by parts we get

$$-t^2 I_{2,N-1}(t) = -N + 3 - t + \left[-2 - (N-4)(N-1) - 2t(N-2)t - t^2\right] I_{0,N-1}(t),$$

and

$$\tilde{\mathrm{v}}(t) = N - 2 - \frac{3t}{4} + \left[\frac{t^2}{2} - (N-2)^2\right] I_{0,N-1}(t). \tag{6.7}$$

The proof is completed. □

Before giving the proof of Theorem 1.2 we need a technical but crucial result.

Lemma 6.1. *Let $N \geqslant 3$ and consider the functions h_λ and v defined in* (1.5) *and* (1.6). *Then h'_λ and v have no zero in common in the positive real axis, that is $\mathrm{v}(\theta) \neq 0$, whenever $\mathrm{h}'_\lambda(\theta) = 0$ and $\theta > 0$.*

We give the proof of this fact in Appendix C.

Proof of Theorem 1.2. Let $y_0 \in \partial\Omega$ be a non-degenerate critical point of the mean curvature κ. In the proof of this theorem we take advantage of the asymptotic formula of Theorem 1.3 to relate the topological degree of the ∇R_λ in a suitable small set close to y_0 with that of the $\nabla\kappa$.

Note that as a consequence of (5.1) and (5.2) we have, writing ∇_{T} as the tangential gradient

$$\nabla_{\mathrm{T}} R_\lambda(x) = \lambda^{N-3}(N-1)\nabla\kappa(\hat{x})\mathrm{v}\big(\lambda d(x)\big) + O\big(\lambda^{N-3-\alpha}\big) \tag{6.8}$$

and

$$\big\langle \nabla R_\lambda(x), \nu\big\rangle = -\lambda^{N-1}\mathrm{h}'_\lambda\big(\lambda d(x)\big) - \lambda^{N-2}(N-1)\kappa(\hat{x})\mathrm{v}'\big(\lambda d(x)\big) + O\big(\lambda^{N-2-\alpha}\big) \tag{6.9}$$

uniformly for $K^{-1} \leqslant \lambda d(x) \leqslant K$.

Since $y_0 \in \partial\Omega$ is a non-degenerate critical point of κ, there exist $c > 0$, $\sigma > 0$ such that

$$\big|\nabla\kappa(\hat{x})\big| \geqslant c|\hat{x} - y_0| \quad \text{for all } \hat{x} \text{ such that } |\hat{x} - y_0| \leqslant \sigma.$$

On the other hand, we know that h_λ has a unique minimum $\theta_0 > 0$, which is non-degenerate, and hence by taking $c > 0$, $\sigma > 0$ smaller if necessary, we have

$$\big|\mathrm{h}'_\lambda(\theta)\big| \geqslant c|\theta - \theta_0| \quad \text{for all } |\theta - \theta_0| \leqslant \sigma.$$

Using that $\mathrm{v} < 0$ in $\mathbb{R}$ if $N = 2$ or Lemma 6.1 if $N \geqslant 3$, we see that selecting $\sigma > 0$ smaller we can achieve

$$\inf_{\theta \in [\theta_0 - \sigma, \theta_0 + \sigma]} \big|\mathrm{v}(\theta)\big| > 0. \tag{6.10}$$

We can also assume $\sigma < \theta_0$. Let $0 < \beta < \alpha$ and consider the compact set

$$\mathcal{K}_\lambda = \{x \in \Omega \mid |\lambda d(x) - \theta_0| \leqslant \sigma,\ |\hat{x} - y_0| \leqslant \lambda^{-\beta}\}.$$

Now define

$$R_\lambda^0(x) = \lambda^{N-2}\mathrm{h}_\lambda(\lambda d(x)) + \lambda^{N-3}(N-1)\kappa(\hat{x})\mathrm{v}(\lambda d(x))$$

and for $0 \leqslant t \leqslant 1$

$$R_\lambda^t = tR_\lambda + (1-t)R_\lambda^0.$$

We observe that there are $\lambda_0 > 0$ and $c' > 0$ such that if $\lambda \geqslant \lambda_0$ and $x \in \partial\mathcal{K}_\lambda$ then:

(1) if $|\lambda d(x) - \theta_0| = \sigma$ by (6.9) we have

$$|\nabla R_\lambda^t(x)| \geqslant \lambda^{N-1}c';$$

(2) if $|\hat{x} - y_0| = \lambda^{-\beta}$ from (6.8) and (6.10) we deduce

$$|\nabla R_\lambda^t(x)| \geqslant \lambda^{N-3-\beta}c'.$$

From (1) and (2), by degree theory $R_\lambda = R_\lambda^1$ has a critical point in the set $\mathcal{K}_\lambda$, and hence it lies at distance $\lambda^{-\beta}$ from y_0. This completes the proof of the theorem. □

7. Critical points of R_λ when b is not a constant

As a consequence of (2.36) we have the following expansion:

$$R_\lambda(x) = \lambda^{N-2}\mathrm{h}_\lambda(\lambda d(x), b(\hat{x})) + O(\lambda^{N-3})$$

uniformly for $K^{-1} \leqslant \lambda d(x) \leqslant K$, where $K > 1$ and $\mathrm{h}_\lambda(\theta, b)$ is defined in (2.6).

We need similar estimates for the gradient of R_λ.

Lemma 7.1. *Given $K > 1$, the following estimates*

$$\nabla_{\mathrm{T}} R_\lambda(x) = \lambda^{N-2}\frac{\partial \mathrm{h}_\lambda}{\partial b}(\lambda d(x), b(\hat{x}))\nabla b(\hat{x}) + O(\lambda^{N-3}), \tag{7.1}$$

$$\langle \nabla R_\lambda, \nu(\hat{x})\rangle = -\lambda^{N-1}\frac{\partial \mathrm{h}_\lambda}{\partial \theta}(\lambda d(x), b(\hat{x})) + O(\lambda^{N-2}) \tag{7.2}$$

hold uniformly for $K^{-1} \leqslant \lambda d(x) \leqslant K$.

Proof. To prove (7.1) it will be sufficient to show that

$$\nabla_{\mathrm{T}} u_\lambda(x, x) = O(\lambda^{N-3}) \tag{7.3}$$

uniformly for $K^{-1} \leqslant \lambda d(x) \leqslant K$.

As in the proof of Proposition 5.1 we will give the details only for dimension $N = 2$. We consider the geometric set up as in Section 5. We have by (5.6) and (5.7)

$$\nabla_T u_\lambda(x, x) = a_0 \frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\big((0,\eta),\eta,0\big)\,\eta + \partial_{y_1}\tilde{u}_\lambda\big((0,\eta),\eta,0\big)$$

for a point $x = (0, \eta/\lambda)$, with the estimate being uniform for $K^{-1} \leqslant \eta \leqslant K$. Observe that by standard elliptic estimates, and since $\tilde{u}_\lambda(\xi, \eta, y_1) = O(\lambda^{-1})$ for $|\xi| \leqslant \delta\lambda$, $K^{-1} \leqslant \eta \leqslant K$, we have

$$\frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\big((0,\eta),\eta,0\big) = O\big(\lambda^{-1}\big)$$

for η in this region. Now we need to estimate $\frac{\partial \tilde{u}_\lambda}{\partial y_1}$ in the case of non-constant b.

Define $\tilde{b}_\lambda$ by

$$\tilde{b}_\lambda(\xi, y_1) = b\bigg(\frac{1}{\lambda}\mathcal{R}(y_1)^{-1}\xi + \big(y_1, \varphi(y_1)\big)\bigg)$$

or equivalently

$$b(x) = \tilde{b}\big(\lambda\mathcal{R}(y_1)\big(x - \big(y_1, \varphi(y_1)\big)\big), y_1\big). \tag{7.4}$$

Differentiating with respect to y_1, setting $y_1 = 0$ yields

$$0 = a_0 \frac{\partial \tilde{b}_\lambda}{\partial \xi_2}\xi_1 + a_0 \frac{\partial \tilde{b}_\lambda}{\partial \xi_1}\xi_2 - \lambda a_0 \frac{\partial \tilde{b}_\lambda}{\partial \xi_1} + \frac{\partial \tilde{b}_\lambda}{\partial y_1}. \tag{7.5}$$

On the other hand, differentiating (7.4) with respect to x_j and setting $y_1 = 0$ gives

$$\frac{\partial b}{\partial x_j} = \lambda \frac{\partial \tilde{b}_\lambda}{\partial \xi_j}.$$

Since b is smooth

$$\frac{\partial \tilde{b}_\lambda}{\partial \xi_j} = O\big(\lambda^{-1}\big)$$

and this combined with (7.5) implies

$$\frac{\partial \tilde{b}_\lambda}{\partial y_1}(\xi, 0) = O(1), \quad \forall \xi \in \partial\Omega_\lambda,\ |\xi| \leqslant \delta\lambda. \tag{7.6}$$

Let $w = \frac{\partial \tilde{u}_\lambda}{\partial y_1}$ at $y_1 = 0$. Then w satisfies

$$\Delta w = 0 \quad \text{in } \Omega_\lambda \cap B_{\delta\lambda},$$
$$\lambda\bigg(\frac{\partial w}{\partial \nu} + \tilde{b}_\lambda w\bigg) = \frac{\partial \tilde{g}_\lambda}{\partial y_1} - \tilde{u}_\lambda \frac{\partial \tilde{b}_\lambda}{\partial y_1} - \lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\frac{\partial \nu_1}{\partial y_1} - \lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_2}\frac{\partial \nu_2}{\partial y_1} \quad \text{on } \partial\Omega_\lambda \cap B_{\delta\lambda}.$$

But observe that from (7.6) and since $\tilde{u}_\lambda(\xi,\eta,y_1)=O(\lambda^{-1})$

$$\tilde{u}_\lambda \frac{\partial \tilde{b}_\lambda}{\partial y_1} = O\big(\lambda^{-1}\big) \quad \text{on } \partial\Omega_\lambda \cap B_{\delta\lambda}.$$

Similarly, since $\nabla_\xi \tilde{u}_\lambda = O(\lambda^{-1})$ for $\xi \in \Omega_\lambda \cap B_{\delta\lambda}$ and using (5.12), (5.13) we see that

$$\lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_1}\frac{\partial \nu_1}{\partial y_1} + \lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_2}\frac{\partial \nu_2}{\partial y_1} = O(1) \quad \text{on } \partial\Omega_\lambda \cap B_{\delta\lambda}.$$

A calculation similar to the one in Lemma 5.6 gives, for the case of a non-constant b

$$\frac{\partial \tilde{g}_\lambda}{\partial y_1} = O(1) \quad \text{on } \partial\Omega_\lambda \cap B_{\delta\lambda}.$$

Then Lemma 2.6 implies that $w=O(1)$ in $\Omega_\lambda \cap B_{\delta\lambda}$, which proves that

$$\frac{\partial \tilde{u}_\lambda}{\partial y_1} = O\big(\lambda^{-1}\big) \quad \text{in } \Omega_\lambda \cap B_{\delta\lambda}.$$

Hence (7.3) follows.

The proof of (7.2) is analogous, using the formula

$$\nabla u_\lambda(x,x)\cdot \nu(\hat{x}) = -\left[\frac{\partial u_\lambda(x,x)}{\partial x_2} + \frac{\partial u_\lambda(x,x)}{\partial y_2}\right]$$

and that at a point x close to $\partial\Omega$ of the form $x=(0,x_2)$ with $x_2=\eta/\lambda$ we have

$$\frac{\partial u_\lambda}{\partial x_2}(x,x) = \lambda \frac{\partial \tilde{u}_\lambda}{\partial \xi_2}\big((0,\eta),\eta,0\big) \quad \text{and} \quad \frac{\partial u_\lambda}{\partial y_1}(x,x) = \lambda \partial_\eta \tilde{u}_\lambda\big((0,\eta),\eta,0\big).$$

This time one may verify

$$\frac{\partial u_\lambda}{\partial x_2}(x,x) = O(1) \quad \text{and} \quad \frac{\partial u_\lambda}{\partial y_1}(x,x) = O(1). \qquad \square$$

Proof of Theorem 1.4. The proof is similar to the one of Theorem 1.2. The main difference is that in this case the function $h_\lambda(x,x)$ is dominating over $u_\lambda(x,x)$. To set up the degree theory argument we define

$$R_\lambda^0(x) = \lambda^{N-2}\mathrm{h}_\lambda\big(\lambda d(x), b(\hat{x})\big)$$

and for $0 \leqslant t \leqslant 1$

$$R_\lambda^t = tR_\lambda + (1-t)R_\lambda^0.$$

Let $x_0 \in \partial\Omega$ be a non-degenerate critical point of b. Notice that by Lemma 2.1 the function $R_\lambda^0(x)$ has a critical point x_λ such that its projection $\hat{x}_\lambda$ is exactly x_0 and $d(x_\lambda)=O(\lambda^{-1})$. Then using degree theory in an appropriate set around x_λ (as in the proof of Theorem 1.2) and Lemma 7.1 the theorem follows. $\square$

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Appendix A

Proof of Lemma 4.2. We will denote

$$K(\zeta';x,t)=\frac{e^t(x-t)}{(|\zeta'|^2+(x-t)^2)^{N/2}},\quad \zeta'\in\mathbb{R}^{N-1},\ x>0.$$

To prove (4.4)–(4.6) we need first to estimate

$$J\big(|\xi'-\zeta'|,\xi_N\big)=\int_{-\infty}^{0}\frac{e^t(\xi_N-t)}{(|\zeta'-\xi'|^2+(\xi_N-t)^2)^{N/2}}\,dt=\int_{-\infty}^{0}K(\zeta'-\xi';\xi_N,t)\,dt.$$

We start with the case $\xi_N\geqslant 1$.

Claim 1. Assuming $\xi_N\geqslant 1$ we have

$$J\big(|\xi'-\zeta'|,\xi_N\big)\leqslant C\min\big(\xi_N/|\xi'-\zeta'|^N,\xi_N^{1-N}\big). \tag{A.1}$$

Proof of the claim. Assume $N\geqslant 3$ and write $J=J_1+J_2$ where

$$J_1=\int_{-\infty}^{-1}K(\zeta'-\xi';\xi_N,t)\,dt,\qquad J_2=\int_{-1}^{0}K(\zeta'-\xi';\xi_N,t)\,dt.$$

We estimate

$$J_1=\int_{-\infty}^{-1}\frac{e^t(\xi_N-t)}{(|\zeta'-\xi'|^2+(\xi_N-t)^2)^{N/2}}\,dt\leqslant\int_{-\infty}^{-1}\frac{e^t}{(\xi_N-t)^{N-1}}\,dt\leqslant\frac{C}{\xi^{N-1}},$$

and also

$$J_1=\int_{-\infty}^{-1}\frac{e^t(\xi_N-t)}{(|\zeta'-\xi'|^2+(\xi_N-t)^2)^{N/2}}\,dt\leqslant\int_{-\infty}^{-1}\frac{e^t(\xi_N-t)}{|\zeta'-\xi'|^N}\,dt\leqslant C\frac{\xi_N}{|\xi'-\zeta'|^N}.$$

These two inequalities show that

$$J_1\leqslant C\min\big(\xi_N/|\xi'-\zeta'|^N,\xi_N^{1-N}\big).$$

Now J_2 is bounded by

$$J_2 \leqslant \int_{-1}^{0} \frac{(\xi_N - t)}{(|\zeta' - \xi'|^2 + (\xi_N - t)^2)^{N/2}} \, dt.$$

Changing variables we get

$$J_2 \leqslant |\xi' - \zeta'|^{2-N} \int_{\xi_N/|\xi'-\zeta'|}^{(\xi_N+1)/|\xi'-\zeta'|} \frac{s}{(1+s^2)^{N/2}} \, ds.$$

Let us now assume that $N > 2$. If $\xi_N/|\xi' - \zeta'| \geqslant 1$ then

$$\begin{aligned}
J_2 &\leqslant \frac{|\xi' - \zeta'|^{N-2}}{2-N} \left[\left(1 + \frac{\xi_N^2}{|\zeta' - \xi'|^2}\right)^{-N/2+1} - \left(1 + \frac{(\xi_N + 1)^2}{|\zeta' - \xi'|^2}\right)^{-N/2+1} \right] \\
&\leqslant C|\xi' - \zeta'|^{2-N} \left(\frac{|\xi' - \zeta'|^N}{\xi_N^{N+1}} \right) \\
&\leqslant C\xi_N^{1-N}.
\end{aligned}$$

If $\xi_N/|\xi' - \zeta'| \leqslant 1$ then

$$J_2 \leqslant C|\xi' - \zeta'|^{2-N} \left(\frac{\xi_N}{|\xi' - \zeta'|^2} \right) = C \frac{\xi_N}{|\xi' - \zeta'|^N}.$$

Thus we deduce

$$J_2 \leqslant C \min\bigl(\xi_N/|\xi' - \zeta'|^N, \xi_N^{1-N}\bigr).$$

Case $N = 2$ is similar. This ends the proof of the claim. $\square$

Proof of (4.4)–(4.6) under the assumption $\xi_N \geqslant 1$. When $\xi_N \geqslant 1$ then of course

$$\int_{-\infty}^{0} K(\zeta' - \xi'; \xi_N, t) \, dt = J(|\xi' - \zeta'|, \xi_N) \leqslant \frac{1}{\xi_N^{N-1}},$$

and hence

$$\int_{|\zeta'-\xi'| \leqslant 1} \int_{-\infty}^{0} K(\zeta' - \xi'; \xi_N, t) \, dt \, d\zeta' \leqslant \frac{C}{(1 + |\xi'|^\mu)\xi_N^{N-1}}. \tag{A.2}$$

Thus, in the sequel we do not need to consider integrals over $|\zeta' - \xi'| \leqslant 1$.

Using (A.1) we see that we have to estimate

$$\int_{\mathbb{R}^{N-1}} \min\bigl(\xi_N/|\xi'-\zeta'|^N, \xi_N^{1-N}\bigr)\frac{1}{1+|\zeta'|^\mu}\,d\zeta' = A+B,$$

where

$$A = \xi_N^{1-N} \int_{1\leqslant|\xi'-\zeta'|\leqslant\xi_N} \frac{1}{1+|\zeta'|^\mu}\,d\zeta',$$

$$B = \xi_N \int_{|\xi'-\zeta'|\geqslant\xi_N} \frac{1}{|\xi'-\zeta'|^N(1+|\zeta'|^\mu)}d\zeta'.$$

Let us estimate first A changing variables $\zeta' = |\xi'|z$:

$$A = \xi_N^{1-N}|\xi'|^{N-1} \int_{1/|\xi'|\leqslant|z-e|\leqslant\xi_N/|\xi'|} \frac{1}{1+|\xi'|^\mu|z|^\mu}\,dz,$$

where $e = \xi'/|\xi'|$ is a unit vector.

Suppose first that $\xi_N/|\xi'|\leqslant 1/2$. Then in the region of integration $|z|\geqslant 1/2$ and estimating A by the volume of the ball times the maximum of the integrand we find

$$A \leqslant \frac{C}{1+|\xi'|^\mu} \leqslant \frac{C'}{1+|\xi|^\mu}.$$

Now suppose that $\xi_N/|\xi'|\geqslant 1/2$. Then

$$\begin{aligned} A &\leqslant \xi_N^{1-N}|\xi'|^{N-1} \int_{|z-e|\leqslant 3\xi_N/|\xi'|} \frac{1}{1+|\xi'|^\mu|z|^\mu}\,dz \\ &= \xi_N^{1-N}|\xi'|^{N-1} \int_0^{3\xi_N/|\xi'|} \frac{r^{N-2}}{1+|\xi'|^\mu r^\mu}\,dr \\ &= \xi_N^{1-N} \int_0^{3\xi_N} \frac{r^{N-2}}{1+r^\mu}\,dr. \end{aligned}$$

If $\mu < N-1$ then

$$A \leqslant C\xi_N^{-\mu} \leqslant \frac{C}{1+|\xi|^\mu}.$$

If $\mu = N-1$ then

$$A \leqslant C\xi_N^{-\mu}\max\bigl(1,\log|\xi_N|\bigr) \leqslant \frac{C\max(1,\log|\xi|)}{1+|\xi|^\mu}.$$

If $\mu > N-1$ then

$$A \leqslant C\xi_N^{1-N} \leqslant \frac{C}{1+|\xi|^{N-1}}.$$

With the same change of variables as before:

$$B = \frac{\xi_N}{|\xi'|} \int\limits_{|z-e|\geqslant \xi_N/|\xi'|} \frac{1}{|z-e|^N(1+|\xi'|^\mu|z|^\mu)}\,dz.$$

Suppose $\xi_N/|\xi'| \geqslant 2$. Then $|z-e| \geqslant |z|/2$ and since $1+\mu > 1$

$$B \leqslant C\frac{\xi_N}{|\xi'|^{1+\mu}} \int\limits_{|z|\geqslant \xi_N/(3|\xi'|)} \frac{1}{|z|^{N+\mu}}\,dz = C\xi_N^{-\mu} \leqslant \frac{C}{1+|\xi|^\mu}.$$

Next assume that $\xi_N/|\xi'| \leqslant 2$. We write

$$B = B_1 + B_2 + B_3,$$

where

$$B_1 = \frac{\xi_N}{|\xi'|} \int\limits_{\xi_N/|\xi'|\leqslant|z-e|\leqslant 1/2} \frac{1}{|z-e|^N(1+|\xi'|^\mu|z|^\mu)}\,dz,$$

$$B_2 = \frac{\xi_N}{|\xi'|} \int\limits_{1/2\leqslant|z-e|\leqslant 2} \frac{1}{|z-e|^N(1+|\xi'|^\mu|z|^\mu)}\,dz,$$

$$B_3 = \frac{\xi_N}{|\xi'|} \int\limits_{2\leqslant|z-e|} \frac{1}{|z-e|^N(1+|\xi'|^\mu|z|^\mu)}\,dz.$$

Arguing as in the previous case,

$$\begin{aligned} B_3 &= \frac{\xi_N}{|\xi'|} \int\limits_{2\leqslant|z-e|} \frac{1}{|z-e|^N(1+|\xi'|^\mu|z|^\mu)}\,dz \\ &\leqslant C\frac{\xi_N}{|\xi'|^{1+\mu}} \int\limits_{|z|\geqslant 1/3} \frac{1}{|z|^{N+\mu}}\,dz \\ &\leqslant C\frac{\xi_N}{|\xi'|^{1+\mu}} \\ &\leqslant C|\xi'|^{-\mu} \leqslant \frac{C}{1+|\xi|^\mu}. \end{aligned}$$

In the region of integration for B_1 we have, $|z| \geqslant 1/2$,

$$B_1 \leqslant C \frac{\xi_N}{|\xi'|^{1+\mu}} \int\limits_{\xi_N/|\xi'| \leqslant |z-e| \leqslant 1/2} \frac{1}{|z-e|^N} \, dz = \frac{C}{|\xi'|^\mu} \leqslant \frac{C}{1+|\xi|^\mu}.$$

Finally, for B_2:

$$B_2 \leqslant C \frac{\xi_N}{|\xi'|} \int\limits_{|z| \leqslant 3} \frac{1}{1+|\xi'|^\mu |z|^\mu} \, dz = C \frac{\xi_N}{|\xi'|^N} \int\limits_{3|\xi'|} \frac{r^{N-2}}{1+r^\mu} \, dr.$$

We see that if $\mu < N-1$ then

$$B_2 \leqslant C \frac{\xi_N}{|\xi'|^{1+\mu}} \leqslant \frac{C}{1+|\xi|^\mu}.$$

We see that if $\mu = N-1$ then

$$B_2 \leqslant C \frac{\max(1, \log|\xi|)}{1+|\xi|^{N-1}}$$

and if $\mu > N-1$ then

$$B_2 \leqslant \frac{C}{1+|\xi|^{N-1}}.$$

The proof in the case $\xi_N \leqslant 1$ is similar, using

Claim 2. Assume $\xi_N \leqslant 1$. Then, if $N = 2$

$$J\big(|\xi_1 - \zeta_1|, \xi_2\big) \leqslant C \begin{cases} 1 - \log|\zeta_1 - \xi_1| & \text{if } |\zeta_1 - \xi_1| \leqslant 1, \\ |\zeta_1 - \xi_1|^{-2} & \text{if } |\zeta_1 - \xi_1| \geqslant 1 \end{cases}$$

and if $N \geqslant 3$

$$J\big(|\xi' - \zeta'|, \xi_N\big) \leqslant C \begin{cases} |\zeta' - \xi'|^{2-N} & \text{if } |\zeta' - \xi'| \leqslant 1, \\ |\zeta' - \xi'|^{-N} & \text{if } |\zeta' - \xi'| \geqslant 1. \end{cases}$$

Proof of the claim. If $N = 2$ then

$$J\big(|\zeta_1 - \xi_1|, \xi_2\big) = \int_{-\infty}^{0} \frac{e^t(\xi_2 - t)}{(\zeta_1 - \xi_1)^2 + (\xi_2 - t)^2} \, dt = J_1 + J_2, \tag{A.3}$$

where

$$J_1 = \int_{-\infty}^{-1} \frac{e^t(\xi_2 - t)}{(\zeta_1 - \xi_1)^2 + (\xi_2 - t)^2} \, dt, \qquad J_2 = \int_{-1}^{0} \frac{e^t(\xi_2 - t)}{(\zeta_1 - \xi_1)^2 + (\xi_2 - t)^2} \, dt.$$

We have

$$J_1 \leqslant \frac{1}{(\zeta_1-\xi_1)^2+1}\int_{-\infty}^{-1} e^t\big(1+|t|\big)\,dt = \frac{C}{(\zeta_1-\xi_1)^2+1}$$

and

$$\begin{aligned} J_2 &\leqslant \int_0^1 \frac{(\xi_2+t)}{(\zeta_1-\xi_1)^2+(\xi_2+t)^2}\,dt = \int_{\xi_2}^{1+\xi_2} \frac{t}{(\zeta_1-\xi_1)^2+t^2}\,dt \\ &\leqslant \int_0^2 \frac{t}{(\zeta_1-\xi_1)^2+t^2}\,dt \\ &= \int_0^{2/|\zeta_1-\xi_1|} \frac{r}{1+r^2}\,dr \\ &\leqslant C\begin{cases} 1-\log|\zeta_1-\xi_1| & \text{if } |\zeta_1-\xi_1|\leqslant 1, \\ |\zeta_1-\xi_1|^{-2} & \text{if } |\zeta_1-\xi_1|\geqslant 1. \end{cases} \end{aligned}$$

The case $N \geqslant 3$ is similar. □

Appendix B

In this appendix we will verify formulas (6.2) and (6.3). Since the cases $N=2$ and $N>2$ are similar we will consider the case $N>2$. Integrating by parts (6.3) we have also the following formula for G_λ:

$$\begin{aligned} G_\lambda(x,y) &= |x-y|^{2-N} + \left(\frac{|y|}{R}\right)^{2-N}|x-y^*|^{2-N} \\ &\quad - \lambda\left(2-\frac{N-2}{\lambda R}\right)\left(\frac{|y|}{R}\right)^{2-N}\int_0^R \left(1-\frac{s}{R}\right)^{\lambda R-1}\left|x\left(1-\frac{s}{R}\right)-y^*\right|^{2-N} ds. \end{aligned} \tag{B.1}$$

To evaluate G_λ on ∂B_R we use formula (6.3), which yields, for $x \in \partial B_R$

$$\begin{aligned} G_\lambda(x,y) &= \frac{N-2}{\lambda R}|x-y|^{2-N} - (N-2)\left(2-\frac{N-2}{\lambda R}\right)\left(\frac{|y|}{R}\right)^{2-N} \\ &\quad \times \int_0^R \left[\left(1-\frac{s}{R}\right)^{\lambda R}\left|x\left(1-\frac{s}{R}\right)-y^*\right|^{-N}\left\langle x\left(1-\frac{s}{R}\right)-y^*, \frac{x}{R}\right\rangle\right] ds. \end{aligned}$$

To compute $\frac{\partial G_\lambda}{\partial \nu}$ on ∂B_R we use formula (B.1):

$$\frac{\partial G_\lambda}{\partial \nu}(x,y) = \frac{2-N}{R}|x-y|^{2-N} + \lambda(N-2)\left(2-\frac{N-2}{\lambda R}\right)\left(\frac{|y|}{R}\right)^{2-N}$$
$$\times \int_0^R \left[\left(1-\frac{s}{R}\right)^{\lambda R}\left|x\left(1-\frac{s}{R}\right)-y^*\right|^{-N}\left\langle x\left(1-\frac{s}{R}\right)-y^*, \frac{x}{R}\right\rangle\right] ds.$$

Hence

$$\frac{\partial G_\lambda}{\partial \nu} + \lambda G = 0 \quad \text{on } \partial B_R.$$

Appendix C

Proof of Lemma 6.1. Define $\tilde{\mathrm{v}}(t)$ as in (4.14) so that

$$\mathrm{v}(\theta) = (2\theta)^{2-N}(N-2)\tilde{\mathrm{v}}(2\theta)$$

and define also

$$\tilde{\mathrm{h}}_\lambda(t) = t^{2-N} - 2\int_0^\infty e^{-s}\frac{1}{(t+s)^{N-2}}\,ds$$

so that

$$\mathrm{h}_\lambda(\theta) = \lambda^{N-2}\tilde{\mathrm{h}}_\lambda(2\theta).$$

Then to show that h_λ' and v have no common positive zeros is equivalent to showing the same property for the functions $\tilde{\mathrm{h}}_\lambda'$ and $\tilde{\mathrm{v}}$.

Observe that

$$\begin{aligned}\tilde{\mathrm{h}}_\lambda'(t) &= (N-2)\left(-t^{1-N} + 2\int_0^\infty e^{-s}\frac{1}{(t+s)^{N-1}}\,ds\right)\\ &= (N-2)\big(-t^{1-N} + 2t^{2-N} I_{0,N-1}(t)\big), \qquad \text{(C.1)}\end{aligned}$$

where $I_{0,N-1}$ is defined in (6.6).

Now suppose that $\tilde{\mathrm{h}}_\lambda'(t_0) = 0$ for some $t_0 > 0$. Then thanks to (C.1) we have

$$I_{0,N-1}(t_0) = \frac{1}{2t_0}. \qquad \text{(C.2)}$$

Replacing this relation in (6.7) we then find that

$$N-2-\frac{3t_0}{4} + \left[\frac{t_0^2}{2} - (N-2)^2\right]\frac{1}{2t_0} = 0$$

which implies

$$t_0 = N - 2.$$

But we claim that

$$(N-2)I_{0,N-1}(N-2) < \frac{1}{2}. \tag{C.3}$$

Indeed, notice that

$$tI_{0,N-1}(t) = t^{1-N}\int_0^\infty e^{-s}\frac{1}{(t+s)^{N-1}}\,ds = \int_0^\infty e^{-s}\left(\frac{t}{t+s}\right)^{N-1} ds$$

and therefore

$$(N-2)I_{0,N-1}(N-2) = \int_0^\infty e^{-s}\left(\frac{1}{1+\frac{s}{N-2}}\right)^{N-1} ds.$$

But it is a standard inequality that

$$\left(1+\frac{s}{N-2}\right)^{N-2} < e^s \quad \forall N \geqslant 3,\ \forall s > 0.$$

This implies

$$(N-2)I_{0,N-1}(N-2) < \int_0^\infty \left(\frac{1}{1+\frac{s}{N-2}}\right)^{2N-3} ds = \frac{1}{2},$$

which proves our claim (C.3). But (C.3) contradicts (C.2). □

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