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Concentrating solutions of the Liouville equation with Robin boundary condition

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ABSTRACT

We construct solutions of the Liouville equation

$$\Delta u + \epsilon^2 e^u = 0 \quad \text{in } \Omega$$

with Ω a smooth bounded domain in $\mathbb{R}^2$, with Robin boundary condition

$$\frac{\partial u}{\partial \nu} + \lambda u = 0 \quad \text{on } \partial\Omega.$$

The solutions constructed exhibit concentration as $\epsilon \to 0$ and simultaneously as $\lambda \to +\infty$, at points that get close to the boundary, and shows that in general the set of solutions of this problems exhibits a richer structure than the problem with Dirichlet boundary conditions.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. In this paper we construct solutions to the Liouville equation with Robin boundary condition:

$$\begin{cases} \Delta u + \epsilon^2 e^u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \lambda u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\epsilon > 0$ is small and $\lambda > 0$ is large.

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The Robin boundary condition has been considered in nonlinear equations in biological models, see [11]. Concentration phenomena for the least energy solution of equations of Ni–Takagi type with Robin boundary condition has been studied in [2]. Later on we shall compare our results to [2].

Intuitively, as $\lambda \to \infty$ the boundary condition in (1.1) tends to the homogeneous Dirichlet boundary condition $u|_{\partial\Omega} = 0$ and (1.1) becomes

$$\begin{cases} \Delta u + \epsilon^2 e^u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It is known, after the works [3,15,16,21], that if (u_ϵ) is an unbounded family of solutions of (1.2) and $\epsilon^2 \int_\Omega e^{u_\epsilon}$ remains bounded as $\epsilon \to 0$ then after passing to a subsequence there exists an integer $m \geqslant 1$ such that u_ϵ blows up at m points in Ω. More precisely, there exist points $\xi_1^\epsilon, \dots, \xi_m^\epsilon$ in Ω that stay uniformly separated from each other and from the boundary, such that for any $\delta > 0$, u_ϵ stays bounded on $\Omega \setminus \bigcup_{j=1}^m B(\xi_j^\epsilon, \delta)$, and

$$\sup_{B(\xi_j^\epsilon, \delta)} u_\epsilon \to \infty \quad \text{as } \epsilon \to 0.$$

Moreover,

$$\epsilon^2 e^{u_\epsilon} \rightharpoonup 8\pi \sum_{i=1}^m \delta_{\xi_i} \quad \text{as } \epsilon \to 0$$

in the weak sense of measures and $u_\epsilon \to \sum_{i=1}^m G_\infty(x, \xi_i)$ where G_∞ is the Green function with Dirichlet boundary condition

$$\begin{cases} -\Delta_x G_\infty(x, y) = 8\pi \delta_y, & \text{in } \Omega, \\ G_\infty(\cdot, y) = 0, & \text{on } \partial\Omega \end{cases}$$

(the subscript ∞ means it is associated to $\lambda = \infty$). Additionally, the vector $(\xi_1, \dots, \xi_m)$ of concentration points must be a critical point of the function

$$\varphi_{m,\infty}(\xi_1, \dots, \xi_m) = -\sum_{j=1}^m H_\infty(\xi_j, \xi_j) - \sum_{i \neq j} G_\infty(\xi_i, \xi_j)$$

where H_∞ is the regular part of G_∞:

$$H_\infty(x, y) = G_\infty(x, y) - 4 \log \frac{1}{|x - y|}.$$

The construction of solutions to (1.2) has been addressed in [22,1,9,12]. In [1] the authors showed that if $(\xi_1, \dots, \xi_m)$ is a non-degenerate critical point of $\varphi_{m,\infty}$ then for $\epsilon > 0$ small enough there is a solution concentrating at $\xi_1, \dots, \xi_m$. Then, in [12] and [9] the authors proved that if the domain is not simply connected, then for any integer $k \geqslant 1$ there are solutions concentrating at k points. In the case of a single point of concentration, it must be a critical point of $R_\infty(x) = H_\infty(x, x)$. In a convex domain R_∞ has a single critical point, see [4,5]. In particular, if solutions develop a single point of concentration, that point is uniquely determined in a convex domain. Under some assumptions on the domain, solutions to (1.2) can develop only a single point of concentration. This is the case for a domain which is convex and symmetric in each variable, and also small perturbations of them, see [14,20]. In [23] the authors studied an inhomogeneous Liouville equation.

In contrast, we will see that for any bounded smooth domain, when $\lambda < \infty$ is large, the set of solutions of (1.1) is much richer.

For problem (1.1) the Green function also plays a fundamental role. Given $\lambda > 0$, let G_λ denote the Green function

$$\begin{cases} -\Delta_x G_\lambda(\cdot, y) = 8\pi \delta_y, & \text{in } \Omega, \\ \dfrac{\partial G_\lambda}{\partial \nu}(\cdot, y) + \lambda G_\lambda(\cdot, y) = 0, & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

and H_λ its regular part:

$$H_\lambda(x, y) = G_\lambda(x, y) - 4 \log \frac{1}{|x - y|}. \tag{1.4}$$

As for the case of Dirichlet boundary condition, to understand the critical points of the Robin function $R_\lambda(x) = H_\lambda(x, x)$ is crucial to analyze solutions with a single blow up. In [8] the authors found that in any smooth domain $\Omega \subseteq \mathbb{R}^2$, for $x \in \Omega$ satisfying $a/\lambda \leqslant dist(x, \partial\Omega) \leqslant b/\lambda$ for some constants $0 < a < b$, for large $\lambda > 0$ one has the expansion

$$R_\lambda(x) = h_\lambda\big(\lambda d(x)\big) + \lambda^{-1} \kappa(\hat{x}) \mathrm{v}\big(\lambda d(x)\big) + O\big(\lambda^{-1-\alpha}\big) \tag{1.5}$$

where $0 < \alpha < 1$, $\kappa(\hat{x})$ is the mean curvature of $\partial\Omega$ at $\hat{x}$, which is the point in $\partial\Omega$ closest to x, and h_λ, v are explicitly given by

$$h_\lambda(\theta) = -\log\lambda - \log(2\theta) + 4\theta \int_0^\infty e^{-2\theta t} \log(1 + t)\, dt, \tag{1.6}$$

$$\mathrm{v}(\theta) = -\frac{\theta}{2} - \theta \int_0^\infty e^{-2\theta s} \frac{1}{(1 + s)^2}\, ds. \tag{1.7}$$

The function $h_\lambda : (0, +\infty) \to \mathbb{R}$ has a unique minimum $\theta_0 \in (0, +\infty)$, which is non-degenerate (see [8]). Therefore, formula (1.5) suggests that there exist solutions of (1.1) with a concentration point located at distance $O(1/\lambda)$ from $\partial\Omega$. For a fixed large λ this can be proved using the same approach as in [1,9,12]. Our interest here is to analyze whether this solution persists as $\epsilon \to 0$ and $\lambda \to +\infty$.

Let

$$S^* = \left\{ x \in \Omega \colon dist(x, \partial\Omega) = \frac{\theta_0}{\lambda} \right\}, \tag{1.8}$$

where θ_0 is the minimum of h_λ.

Theorem 1.1. *There exist $\lambda_0 > 0$ and $\epsilon_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $\epsilon > 0$ satisfying $0 < \epsilon\sqrt{\lambda} \leqslant \epsilon_0$, problem* (1.1) *has at least* 2 *different solutions, u_i, $i = 1, 2$, concentrating at a point $\xi_{i,\lambda,\epsilon} \in \Omega$ such that*

$$dist\big(\xi_{i,\lambda,\epsilon}, S^*\big) = O\big(\lambda^{-3/2}\big), \quad i = 1, 2 \text{ as } \lambda \to \infty.$$

Actually there is a third solution u_3 concentrating a point $\xi_{3,\lambda,\epsilon}$ with distance to the boundary not approaching zero, and with no restriction on the growth of λ. We will not address the construction of this solution, as it is very similar to previous work, [1,9,12].

We can generalize Theorem 1.1 and find solutions with multiple points of concentration near the boundary, at the expense of requiring a smaller growth of λ.

Theorem 1.2. *Let $m \geqslant 1$ be an integer. There exist $\lambda_0 > 0$ and $\epsilon_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $\epsilon > 0$ satisfying $0 < \epsilon^2\lambda^2 \log(\lambda) \leqslant \epsilon_0$, problem* (1.1) *has 2 solutions u_i, $i = 1, 2$. The solution u_i concentrates at points $\xi_{i,j,\lambda,\epsilon}$ for $j = 1, \dots, m$ in Ω such that*

$$dist\big(\xi_{i,j,,\lambda,\epsilon}, S^*\big) = O\big(\lambda^{-3/2}\big) \quad as\ \lambda \to \infty.$$

Let κ denote the curvature of $\partial\Omega$.

Theorem 1.3. *Suppose $x_0 \in \partial\Omega$ is a non-degenerate critical point of κ. Set $\alpha \in (0, \frac{1}{2})$. There exist $\lambda_0 > 0$ and $\epsilon_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $\epsilon > 0$ satisfying*

$$\epsilon^\alpha \lambda \leqslant \epsilon_0$$

problem (1.1) *has a solution u that concentrates at a point x_ϵ located at distance $O(1/\lambda)$ from x_0.*

Let us explain the restrictions on the growth of λ as $\epsilon \to 0$. The results are proved using a Lyapunov–Schmidt reduction, based on the family of solutions

$$w_\mu(r) = \log \frac{8\mu^2}{(\mu^2 + r^2)^2}, \quad \text{with } r = |x|,\ x \in \mathbb{R}^2, \tag{1.9}$$

where $\mu > 0$, of the Liouville equation:

$$\Delta u + e^u = 0 \quad \text{in } \mathbb{R}^2. \tag{1.10}$$

To construct a solution with concentration at $\xi \in \Omega$, it is natural to consider a first approximation of the form $w_\mu(x - \xi) - 2\log\epsilon$ with $\mu \to 0$. For x far from ξ, evaluation of this function at x suggests that μ should be taken of order ϵ, and therefore it is more convenient to write this approximation as $w_{\mu\epsilon}(x - \xi) - 2\log\epsilon$ for a new parameter $\mu > 0$. Nevertheless, this function still requires a large correction and it is convenient to take as initial approximation $u(x) = w_{\mu\epsilon}(x - \xi) - 2\log\epsilon + H(x)$, where H is harmonic in Ω and such that the appropriate boundary condition is satisfied. A computation will then show that at main order $H(x) \sim -\log(8\mu^2) - H_\lambda(x, \xi)$. Then u becomes a good approximation of a solution if $H(\xi) = 0$ which yields $8\mu^2 = e^{H_\lambda(\xi,\xi)}$. In the case of Robin boundary condition, from (1.5) and (1.6), this gives $\mu = O(\lambda^{-1/2})$, and we are led to consider $w_{\mu\epsilon\lambda^{-1/2}}(x - \xi) - 2\log\epsilon + H(x)$ for a new parameter $\mu = O(1)$. We observe that $w_{\mu\epsilon\lambda^{-1/2}}(r) = \log(8\mu^2\epsilon^2\lambda^{-1}) - 2\log(\mu^2\epsilon^2\lambda^{-1} + r^2)$. If ξ is at distance $1/\lambda$ from the boundary and x is on the boundary, to be able to expand this quantity we need $\epsilon^2\lambda \ll 1$. This indicates that the reduction in Theorem 1.1 can be carried out if $\epsilon\lambda^{1/2}$ is sufficiently small, and this gives the growth restriction for λ in this result.

In Theorems 1.2 and 1.3 more precise estimates of the energy of the ansatz are required and this leads to a stronger growth assumption on λ. One consideration that helps us to improve the estimates, is to work with concentration points close to the set S^*. A first calculation using (1.5) implies that if $x \in \Omega$ is such that $|\lambda\, dist(x, \partial\Omega) - \theta_0| = O(\lambda^{-1/2})$, then we have

$$\big|\nabla_x R_\lambda(x)\big| = O(\sqrt{\lambda}). \tag{1.11}$$

This estimate plays a key role, as it can be seen in the following section.

Let us compare Theorem 1.1 with the results of [2], where the following equation was studied

$$\begin{cases} \epsilon^2\Delta u + u^p - u = 0, \quad u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \lambda u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

where $\epsilon, \lambda > 0$, Ω is a domain in $\mathbb{R}^N$, $N \geqslant 2$ and $1 < p < \frac{N+2}{N-2}$. This equation with boundary condition $\partial u/\partial \nu = 0$ on Ω was analyzed in [17,18] and in [19] with Dirichlet boundary condition, proving that for Neumann condition least energy solution concentrates at a point in the boundary, while for Dirichlet concentration takes place at a point that maximizes distance to the boundary, see also [10]. The results of [2] roughly speaking assert that the minimal energy solution of (1.12) will behave like in the case of Neumann boundary condition if $\lambda < \bar{\lambda}/\epsilon$ and like in the Dirichlet boundary condition if $\lambda > \bar{\lambda}/\epsilon$, where $\bar{\lambda} > 0$ is a parameter associated to an auxiliary problem. Therefore $\lambda \sim 1/\epsilon$ represents a drastic change in behavior. Our results suggest that for least energy solutions of (1.1) the critical range for λ is $\lambda \sim 1/\epsilon^2$.

In Section 2 we provide the first approximation, and in Section 3 we analyze the linearization around this initial approximation. Then in Section 4 we solve a projected version of the nonlinear equation. We show in Section 5 that the projected problem reduces to the original one if $(\xi_1, \ldots, \xi_m)$ is a critical point of a functional close to the energy ansatz. Then Section 6 contains the expansion of the energy of the ansatz. With the aid of these expansion we prove Theorems 1.1, 1.2 and 1.3 in Section 7. Finally, in Appendix A we prove some estimates that were necessary in the expansion of the energy.

2. Initial approximation

In this section we describe the initial approximation used in the Lyapunov–Schmidt reduction.

Given $m \in \mathbb{N}$, $\{\xi_j\}_{j=1}^m \subset \Omega$ and $\mu_j > 0$ for $j = 1, \ldots, m$, we define:

$$u_j(x) = w_{\mu_j}\left(\frac{\sqrt{\lambda}}{\epsilon}|x - \xi_j|\right) - 4\log\epsilon + \log\lambda, \tag{2.1}$$

where w_μ is defined in (1.9), which satisfies

$$\Delta u_j + \epsilon^2 e^{u_j} = 0 \quad \text{in } \mathbb{R}^2.$$

Let $\delta_0 > 0$ be fixed suitably small. We will assume for the rest of the article the following separation conditions:

$$|\xi_i - \xi_j| \geqslant \delta_0 \quad \text{for all } i \neq j, \tag{2.2}$$

$$d_j := dist(\xi_j, \partial\Omega) \geqslant \frac{\delta_0}{\lambda} \quad \text{for all } j = 1, \ldots, m, \tag{2.3}$$

$$dist\big(\xi_i, S^*\big) \leqslant \lambda^{-3/2} \quad \text{for all } i = 1, \ldots, m, \tag{2.4}$$

where S^* is defined in (1.8).

For each $j = 1, \ldots, m$ let

$$\begin{cases} \Delta H_j = 0, & \text{in } \Omega, \\ \dfrac{\partial H_j}{\partial \nu} + \lambda H_j = -\left(\dfrac{\partial u_j}{\partial \nu} + \lambda u_j\right), & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

We will take as a first approximation to a solution of (1.1) the function

$$U(x) = \sum_{j=1}^m \big(u_j(x) + H_j(x)\big). \tag{2.6}$$

We will define $\rho := \epsilon/\sqrt{\lambda}$. For many of the calculations it is convenient to work in expanded variables in terms of ρ. Given $x \in \Omega$, consider $y = \frac{1}{\rho}x$, and denote $\Omega_\rho = \frac{1}{\rho}\Omega$. Let u be a function defined in Ω and let

$$v(y) = u(\rho y) + 4\log\epsilon - \log\lambda \quad \text{for } y \in \Omega_\rho.$$

Then u solves (1.1) if and only if v is a solution of

$$\begin{cases} \Delta v + e^v = 0, & \text{in } \Omega_\rho, \\ \dfrac{\partial v}{\partial \nu} + \rho\lambda v = \rho\lambda(4\log\epsilon - \log\lambda), & \text{on } \partial\Omega_\rho. \end{cases} \tag{2.7}$$

We also define $\xi_j' = \frac{1}{\rho}\xi_j$ and write the initial approximation of the solution in expanded variables as $V(y) = U(\rho y) + 4\log\epsilon - \log\lambda$. We look for a solution v of the problem (2.7) with the form

$$v = V + \phi,$$

with ϕ small in an adequate norm. Problem (2.7) can be viewed in terms of ϕ as the nonlinear problem

$$\begin{cases} L(\phi) = -\big(R + N(\phi)\big), & \text{in } \Omega_\rho, \\ \dfrac{\partial \phi}{\partial \nu} + \rho\lambda\phi = 0, & \text{on } \partial\Omega_\rho, \end{cases} \tag{2.8}$$

where

$$L(\phi) = \Delta\phi + W\phi, \quad \text{with} \quad W = e^V, \tag{2.9}$$

$$N(\phi) = W\big[e^\phi - 1 - \phi\big],$$

and

$$R = \Delta V + e^V.$$

Next we estimate the size of R.

Lemma 2.1. *If μ_j are given by*

$$\log\big(8{\mu_j}^2\big) = H_\lambda(\xi_j, \xi_j) + \sum_{i \neq j} G_\lambda(\xi_i, \xi_j) + \log\lambda, \tag{2.10}$$

we have:

$$\big|R(y)\big| \leqslant C\epsilon \sum_{j=1}^{m} \frac{1}{1 + |y - \xi_j'|^3} \quad \textit{for all } y \in \Omega_\rho. \tag{2.11}$$

In the proof of Lemma 2.1 we need an a priori estimate which is essentially a version of the maximum principle with Robin boundary condition. For a proof see [8].

Lemma 2.2. *Let $b : \partial\Omega \to \mathbb{R}$ be a smooth such that $b > 0$, $F : \partial\Omega \to \mathbb{R}$ be a smooth function and u be the solution to*

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \lambda b(x) u = F, & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$. Then

$$\|u\|_{L^\infty(\Omega)} + \big\| dist(\cdot, \partial\Omega)\nabla u \big\|_{L^\infty(\Omega)} \leqslant \frac{C(N,b)}{\lambda} \|F\|_{L^\infty(\partial\Omega)}.$$

Remark 2.3. We note that by (1.5) and (1.6), $H_\lambda(\xi_j, \xi_j) + \log\lambda$ remains bounded as $\lambda \to +\infty$. It follows that for some constant $C > 1$

$$\frac{1}{C} \leqslant \mu_j \leqslant C, \quad \forall j = 1, \dots, m. \tag{2.12}$$

The reason to introduce the initial approximation with the form (2.1) is so that μ_j satisfies (2.12).

Proof of Lemma 2.1. Let us analyze the behavior of the function $H_j(x)$. Note that since $H_j(x)$ satisfies Eq. (2.5), if we define $\tilde{H}_j = H_j + \log(8\mu_j^2) - \log\lambda$, then $\tilde{H}_j$ satisfies

$$\begin{cases} -\Delta \tilde{H}_j(x) = 0, & \text{in } \Omega, \\ \dfrac{\partial \tilde{H}_j}{\partial \nu} + \lambda \tilde{H}_j = 4\dfrac{(x-\xi_j)\nu}{{\mu_j}^2\rho^2 + |x-\xi_j|^2} - \lambda \log\left(\dfrac{1}{({\mu_j}^2\rho^2 + |x-\xi_j|^2)^2}\right), & \text{on } \partial\Omega. \end{cases}$$

The regular part of the Green function for homogeneous Robin boundary condition $H(x, \xi_j)$ satisfies the equation

$$\begin{cases} -\Delta H_\lambda(x, \xi_j) = 0, & \text{in } \Omega, \\ \dfrac{\partial H_\lambda(x,\xi_j)}{\partial \nu} + \lambda H_\lambda(x, \xi_j) = 4\dfrac{(x-\xi_j)\nu}{|x-\xi_j|^2} - \lambda \log\left(\dfrac{1}{|x-\xi_j|^4}\right), & \text{on } \partial\Omega. \end{cases}$$

Using the maximum principle applied to $H_\lambda(x, \xi_j) - \tilde{H}_j(x)$ for the problem with Robin boundary condition (Lemma 2.2), we conclude that

$$H_j(x) = H_\lambda(x, \xi_j) - \log\big(8\mu_j^2\big) + \log\lambda + O\left(\frac{\mu_j^2\rho^2}{\lambda d_j^3}\right) + O\left(\frac{\mu_j^2\rho^2}{d_j^2}\right) \tag{2.13}$$

where the term O is uniform in $\overline{\Omega}$ and also in the C^2 sense for compact subsets of Ω.

Observe that, away from the points ξ_j we can expand the expression given in (2.1) and obtain

$$u_j(x) = \log\big(8\mu_j^2\big) + 4\log\frac{1}{|x-\xi_j|} - \log\lambda + O\left(\frac{\mu_j^2\rho^2}{|x-\xi_j|^2}\right).$$

Using this and the expression given in (2.13) we get the following estimate

$$u_j(x) + H_j(x) = G_\lambda(x, \xi_j) + \mu_j^2 \rho^2 O\left(\frac{1}{\lambda {d_j}^3} + \frac{1}{|x - \xi_j|^2}\right), \tag{2.14}$$

where the term O is in the C^2 sense on compact sets of $\overline{\Omega} \setminus \{\xi_j\}$.

Let $\delta > 0$ be fixed, small compared with δ_0. Note that $e^{V(y)} = \rho^2 \epsilon^2 e^{U(x)}$, where $x = \rho y$. Then, we have

$$e^{V(y)} = O\left(\rho^2 \epsilon^2\right), \quad \text{if } \left|y - {\xi_j}'\right| > \frac{\delta}{\rho}, \ \forall j = 1, \ldots, m. \tag{2.15}$$

Also, thanks to $\Delta V(y) = \epsilon^2 \Delta U(x)$ and (2.14) we get

$$\Delta V(y) = O\left(\epsilon^4\right), \quad \text{if } \left|y - {\xi_j}'\right| > \frac{\delta}{\rho}, \ \forall j = 1, \ldots, m.$$

Now we consider $|y - \xi'_j| < \frac{\delta}{\rho}$ for some j. We will center our system of coordinates at ξ'_j writing $y = \xi'_j + z$. Then

$$e^{V(y)} = \frac{8{\mu_j}^2}{({\mu_j}^2 + |z|^2)^2} \times \exp\left\{H_j(\xi_j + \rho z) + \sum_{l \neq j} \left(u_l(\xi_j + \rho z) + H_l(\xi_j + \rho z)\right)\right\}.$$

Using the asymptotic relations (2.13), (2.14), (1.11) and the definition of the numbers μ_j given in (2.10), we obtain

$$e^{V(y)} = \frac{8{\mu_j}^2}{({\mu_j}^2 + |y - {\xi_j}'|^2)^2} \left[1 + O(\epsilon z) + O\left(\frac{\mu_j^2 \rho^2}{\lambda d_j^3}\right)\right]$$

for $|y - {\xi_j}'| < \frac{\delta}{\rho}$.

In the same region, we have

$$\Delta_y V(y) = \rho^2 \sum_{l=1}^m \Delta_x u_l(\rho y) = -\frac{8\mu_j^2}{(\mu_j^2 + |y - \xi'_j|^2)^2} + O\left(\rho^4\right). \tag{2.16}$$

Then, using (2.15)–(2.16) we deduce (2.11). □

3. The linearized operator around *V*

As before, we are considering here $\rho = \epsilon / \sqrt{\lambda}$.

We assume that the function $W : \Omega_\rho \to \mathbb{R}$ has the form

$$W(y) = \sum_{j=1}^m \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi'_j|^2)^2} \left(1 + \theta_\epsilon(y)\right) \tag{3.1}$$

where $\xi'_j = \xi_j / \rho \in \Omega_\rho = \Omega / \rho$ and $\xi_1, \ldots, \xi_m \in \Omega$ are different points. We assume that

$$\left|\theta_\epsilon(y)\right| \leqslant C\epsilon \sum_{j=1}^m \left(\left|y - \xi'_j\right| + 1\right)$$

and

$$\frac{1}{C} \leqslant \mu_j \leqslant C, \quad \forall j = 1, \ldots, m,$$

where C is independent of ϵ and λ.

Note that for each $j = 1, \ldots, m$, if we center the coordinate system around ξ'_j by setting $z = y - \xi'_j$, then formally the operator $L(\phi)$ has the form as $\rho \to 0$,

$$\Delta\phi + \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2}\phi,$$

which is the linearization of Eq. (1.10) around the function $w_{\mu_j}(|z|)$ given by (1.9). The kernel of this operator is given by the family of functions

$$z_{ij}(z) = \frac{\partial}{\partial \zeta_i}\big(w_{\mu_j}\big(|z + \zeta|\big)\big)\Big|_{\zeta=0}, \quad i = 1, 2.$$

$$z_{0j}(z) = \frac{\partial}{\partial s}\big(w_{\mu_j}\big(|sz|\big) + 2\log(s)\big)\Big|_{s=0}.$$

In this section we study the invertibility of the operator L defined in (2.9). For this, given $h \in C^{0,\alpha}(\Omega_\rho)$ we consider the linear problem of finding $\phi : \Omega_\rho \to \mathbb{R}$ and $c_{ij} \in \mathbb{R}$, $i = 1, 2$, $j = 1, \ldots, m$, such that:

$$\begin{cases} \Delta\phi + W(y)\phi = h + \displaystyle\sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\chi_j Z_{ij}, & \text{in } \Omega_\rho, \\ \dfrac{\partial \phi}{\partial \nu} + \rho\lambda\phi = 0, & \text{on } \partial\Omega_\rho, \\ \displaystyle\int_{\Omega_\lambda} \chi_j Z_{ij}\phi = 0, & \forall i = 1, 2,\ j = 1, \ldots, m, \end{cases} \tag{3.2}$$

where are defined as $Z_{ij}(y) = z_{ij}(|y - \xi'_j|)$ for $j = 1, \ldots, m$ and $i = 1, 2$. The functions χ_j appearing in (3.2) are defined by $\chi_j(y) = \chi(|y - \xi'_j|)$ with χ a nonnegative smooth function on $\mathbb{R}$ such that

$$\chi(r) = 1 \quad \text{if } r \leqslant R_0 \quad \text{and} \quad \chi(r) = 0 \quad \text{if } r \geqslant R_0 + 1 \tag{3.3}$$

where R_0 is a positive constant.

We will prove that (3.2) is solvable and find an estimate for the solution in $L^\infty(\Omega_\rho)$ in terms of the following weighted norm for h:

$$\|h\|_* = \sup_{y \in \Omega_\rho}\left(\sum_{j=1}^{m}\big(1 + \big|y - \xi'_j\big|\big)^{-2-\sigma} + \rho^2\right)^{-1}\big|h(y)\big|,$$

where $\sigma > 0$ is fixed and small.

Proposition 3.1. *There exist $\epsilon_0 > 0$ and $C > 0$ such that for any $\epsilon > 0$, $\lambda \geqslant 1$ such that*

$$\lambda\rho \leqslant \epsilon_0$$

any set of points that verify (2.2) *and* (2.3) *and $h \in L^\infty(\Omega_\rho)$ there is a unique solution $\phi \in L^\infty(\Omega_\rho)$, $c_{ij} \in \mathbb{R}$, $i = 1, 2$, $j = 1, \dots, m$, to* (3.2)*. Moreover, one has*

$$\|\phi\|_\infty \leqslant C\left|\log(\lambda\rho)\right|\|h\|_*. \tag{3.4}$$

Remark that the hypothesis $\lambda\rho$ small means that $\epsilon\sqrt{\lambda}$ has to be small, which is the same assumption of Theorem 1.1.

The first step is to find a priori bounds for the solution of the following problem:

$$\Delta\phi + W(y)\phi = h, \quad \text{in } \Omega_\rho, \tag{3.5}$$

$$\frac{\partial\phi}{\partial\nu} + \rho\lambda\phi = g, \quad \text{on } \partial\Omega_\rho, \tag{3.6}$$

$$\int_{\Omega_\rho} \chi_j Z_{ij}\phi = 0, \quad \forall i = 0, 1, 2, \ j = 1, \dots, m, \tag{3.7}$$

which includes orthogonality conditions with respect to all functions $\chi_j Z_{ij}$ and a right-hand side for the boundary condition (3.6).

Lemma 3.2. *There exist $\epsilon_0 > 0$ and $C > 0$ such that for any $0 < \epsilon < \epsilon_0$, $\lambda \geqslant 1$ such that*

$$\lambda\rho \leqslant \epsilon_0$$

any set of points which verify (2.2) *and* (2.3) *and any solution ϕ of* (3.5)–(3.7) *one has*

$$\|\phi\|_\infty \leqslant C\left(\|h\|_* + \frac{1}{\lambda\rho}\|g\|_{L^\infty(\partial\Omega_\rho)}\right).$$

Proof. We first prove that there exists a fixed number $R > 0$ so that

$$\|\phi\|_{L^\infty(\Omega_\rho)} \leqslant C\left(\max_{j=1,\dots,m}\sup_{B(\xi_j',R)}|\phi| + \|h\|_* + \frac{1}{\lambda\rho}\|g\|_{L^\infty(\partial\Omega_\rho)}\right) \tag{3.8}$$

where C does not depend on ϵ and λ.

To prove (3.8) we first show the $\Delta + W$ satisfies the following maximum principle in the region $\tilde{\Omega}_\rho = \Omega_\rho \setminus \bigcup_{j=1}^m B(\xi_j', R)$: if v satisfies

$$\Delta v + Wv \geqslant 0 \quad \text{in } \tilde{\Omega}_\rho,$$

$$v \leqslant 0 \quad \text{on } \bigcup_{j=1}^m \partial B\left(\xi_j', R\right) \quad \text{and} \quad \frac{\partial v}{\partial\nu} + \lambda\rho v \leqslant 0 \quad \text{on } \partial\Omega_\rho,$$

then $v \leqslant 0$ in $\tilde{\Omega}_\rho$. To prove this, it is sufficient to exhibit a positive C^2 function Z on $\tilde{\Omega}_\rho$ such that

$$\Delta Z + WZ < 0 \quad \text{in } \tilde{\Omega}_\rho, \tag{3.9}$$

$$Z > 0 \quad \text{on} \bigcup_{j=1}^{m} \partial B\left(\xi_j', R\right) \quad \text{and} \quad \frac{\partial Z}{\partial \nu} + \lambda\rho Z > 0 \quad \text{on } \partial\Omega_\rho. \tag{3.10}$$

Let $z_0 = \frac{r-1}{r+1}$, $r = |x|$, $x \in \mathbb{R}^2 \setminus \{(0,0)\}$, which satisfies

$$\Delta z_0 + \frac{2}{r(r+1)^2} z_0 = 0 \quad \text{in } \mathbb{R}^2 \setminus \{(0,0)\}.$$

Define

$$Z(y) = \sum_{j=1}^{m} z_0\left(a\left|y - \xi_j'\right|\right), \quad y \in \Omega_\rho,$$

where $a > 0$. Then

$$-\Delta Z = \sum_{j=1}^{m} \frac{2a(a|y - \xi_j'| - 1)}{|y - \xi_j'|(1 + a|y - \xi_j'|)^3}$$

If $a|y - \xi_j'| \geqslant 3$ then $\frac{a|y-\xi_j'|-1}{a|y-\xi_j'|+1} \geqslant 1/2$ and then

$$-\Delta Z \geqslant \sum_{j=1}^{m} \frac{a^{-1}}{|y - \xi_j'|^3}.$$

In the same region

$$WZ \leqslant C \sum_{j=1}^{m} \frac{1}{|y - \xi_j'|^4}\left(1 + \epsilon\left|y - \xi_j'\right|\right)$$

for some fixed constant C. Hence, tanking $a > 0$ small but fixed, we conclude that (3.9) holds. Besides, we have

$$Z \geqslant \frac{1}{2} \quad \text{on } \partial B\left(\xi_j', R\right),\ \forall j = 1, \ldots, m \quad \text{and on } \partial\Omega_\rho$$

taking R larger if it is necessary. With fixed a we have

$$|\nabla Z| \leqslant C \sum_{j=1}^{m} \frac{1}{|y - \xi_j'|^2}.$$

Using this and (2.3) we have on $\partial\Omega_\rho$,

$$\frac{\partial Z}{\partial \nu} + \lambda\rho Z \geqslant O\left(\sum_{j=1}^{m} dist\left(\xi_j', \partial\Omega_\rho\right)^{-2}\right) + \frac{\lambda\rho}{2} = O\left(\lambda^2\rho^2\right) + \frac{\lambda\rho}{2} \geqslant 0$$

if we choose $\epsilon_0 > 0$ small. Therefore Z satisfies (3.10) too.

Let $M > 0$ be large so that $\Omega_\rho \subset B(\xi_j', \frac{M}{2\rho})$ for all $j = 1, \dots, m$. Let ψ_j be the solution to the following problem:

$$-\Delta \psi_j = \frac{2}{|y-\xi_j'|^3} + 2\rho^2, \quad R < \left|y-\xi_j'\right| < \frac{M}{\rho},$$

$$\psi_j(y) = 0, \quad \text{for } \left|y-\xi_j'\right| = R, \ \left|y-\xi_j'\right| = \frac{M}{\rho},$$

which can be explicitly written:

$$\psi_j(r) = 2\left(\frac{1}{R} - \frac{1}{r}\right) - \frac{\rho^2}{2}\left(r^2 - R^2\right) - \left[\frac{M^2}{2} - \frac{2}{R} - \rho^2\left(\frac{R^2}{2} - \frac{2}{\rho M}\right)\right] \frac{\log(\frac{r}{R})}{\log(\frac{\rho R}{M})}.$$

Then $\max_{R \leqslant |y-\xi_j'| \leqslant M/\rho} \psi_j$ remains uniformly bounded as $\rho \to 0$, always assuming $1 \leqslant R \leqslant \frac{M}{2\rho}$. Moreover

$$\psi_j > 0, \quad \text{in } R < \left|y-\xi_j'\right| < \frac{M}{\rho}.$$

Since

$$|\nabla \psi_j| = O\left(\left|y-\xi_j'\right|^{-2} + \rho^2\left|y-\xi_j'\right| + \frac{1}{|y-\xi_j'||\log(\rho)|}\right)$$

we also have

$$|\nabla \psi_j| = O\left(\lambda^2\rho^2 + \frac{\rho}{\lambda} + \frac{\lambda\rho}{|\log(\rho)|}\right) \quad \text{on } \partial\Omega_\rho.$$

Furthermore

$$\Delta\psi_j + W\psi_j = -\frac{2}{|y-\xi_j'|^3} - 2\rho^2 + O\left(\left|y-\xi_j'\right|^{-4}\left(1+\epsilon\left|y-\xi_j'\right|\right)\right) \leqslant -\frac{1}{|y-\xi_j'|^3} - \rho^2$$

on $R < |y-\xi_j'| < \frac{M}{\rho}$, by fixing R larger if necessary. Let

$$\psi = C_0 Z + \sum_{j=1}^m \psi_j.$$

Then

$$\Delta\psi + W\psi \leqslant -\sum_{j=1}^m \frac{1}{|y-\xi_j'|^3} - \rho^2 \quad \text{in } \tilde{\Omega}_\rho,$$

$$\psi \geqslant \frac{1}{2} \quad \text{on } \partial B\left(\xi_j', R\right), \ \forall j = 1, \dots, m \quad \text{and on } \partial\Omega_\rho$$

choosing C_0 large enough, and then

$$\frac{\partial\psi}{\partial\nu}+\lambda\rho\psi \geqslant O\left(\lambda^2\rho^2+\frac{\rho}{\lambda}+\frac{\lambda\rho}{|\log(\rho)|}\right)+\frac{C_0\lambda\rho}{2}\geqslant\frac{C_0\lambda\rho}{4}\quad \text{on } \partial\Omega_\rho$$

if we choose ϵ_0 small. Set

$$\bar\phi = C\psi\left(\max_{j=1,\dots,m}\sup_{B(\xi_j',R)}|\phi|+\|h\|_*+\frac{1}{\lambda\rho}\|g\|_{L^\infty(\partial\Omega)}\right),$$

where $C=\max(2,4/C_0)$. Then $\Phi=\bar\phi-\phi$ satisfies

$$\Delta\Phi+W\Phi\leqslant 0\quad \text{in } \tilde\Omega_\rho,$$
$$\frac{\partial\Phi}{\partial\nu}+\lambda\rho\Phi\geqslant 0\quad \text{on } \partial\Omega_\rho,$$
$$\Phi\geqslant 0\quad \text{on } \partial B\big(\xi_j',R\big),\ \forall j=1,\dots,m.$$

Since the maximum principle is valid in $\tilde\Omega_\rho$ for this problem we conclude that $\Phi\geqslant 0$ in $\tilde\Omega_\rho$ and therefore $\phi\leqslant\bar\phi$ in $\tilde\Omega_\rho$. In a similar way, $-\phi\leqslant\bar\phi$ in $\tilde\Omega_\rho$. This proves (3.8).

Now we prove the lemma, arguing by contradiction. Assume that there exist sequences (ρ_n), (λ_n), $(\xi_j^{(n)})$, (h_n), (g_n), (ϕ_n), which solve (3.5)–(3.7), such that the conditions (2.2), (2.3) hold,

$$\lambda_n\epsilon_n\to 0 \tag{3.11}$$

and such that

$$\|\phi_n\|_\infty=1,\qquad \|h_n\|_*\to 0,\qquad \frac{1}{\lambda_n\rho_n}\|g\|_{L^\infty(\partial\Omega_{\rho_n})}\to 0\quad \text{as } n\to\infty. \tag{3.12}$$

Thanks to (3.8), (3.12) we can find $c>0$ and a fixed index $j\in\{1,\dots,m\}$ such that by passing to a subsequence

$$\sup_{B(\xi_j',R)}|\phi_n|\geqslant c\quad \text{for all } n. \tag{3.13}$$

Define $\hat\phi_n(z)=\phi_n(\xi_j^n+z)$. By (3.11) and (2.3) we see that

$$\frac{1}{\rho_n}\min_{j=1,\dots,m} dist\big(\xi_j^n,\partial\Omega\big)\to+\infty$$

and this implies that the domain of definition of $\hat\phi_n$ approaches $\mathbb{R}^2$ as $n\to\infty$. Since $\hat\phi_n$ is uniformly bounded, by standard elliptic regularity theory, by passing to another subsequence $\hat\phi_n\to\hat\phi$ uniformly on compact sets of $\mathbb{R}^2$ where $\hat\phi$ is a bounded solution of

$$\Delta\hat\phi+\frac{8\mu_j^2}{(\mu_j^2+|z|^2)^2}\hat\phi=0. \tag{3.14}$$

The orthogonality conditions (3.7) become

$$\int_{\mathbb{R}^2}\chi_j Z_{ij}\hat\phi=0,\quad \forall i=0,1,2. \tag{3.15}$$

We know that the only bounded solutions of (3.14) are linear combinations of z_{ij}, $i=0,1,2$. This together with (3.15) implies that $\hat{\phi} \equiv 0$. But this is not possible by (3.13). □

We now obtain an a priori estimate for the solution assuming that it satisfies orthogonality conditions only with respect to Z_{ij} with $i=1,2$ and $j=1,\dots,m$, that is, solutions to

$$\Delta\phi + W(y)\phi = h, \quad \text{in } \Omega_\rho, \tag{3.16}$$

$$\frac{\partial\phi}{\partial\nu} + \lambda\rho\phi = 0, \quad \text{on } \partial\Omega_\rho, \tag{3.17}$$

$$\int_{\Omega_\rho} \chi_j Z_{ij}\phi = 0, \quad \forall i=1,2,\ j=1,\dots,m. \tag{3.18}$$

Lemma 3.3. *There exist* $\epsilon_0 > 0$ *and* $C > 0$ *such that for any* $\epsilon > 0$, $\lambda \geqslant 1$ *such that*

$$\lambda\rho \leqslant \epsilon_0$$

any set of points which verify (2.2) *and* (2.3) *and any solution* ϕ *of* (3.16)–(3.18) *one has*

$$\|\phi\|_\infty \leqslant C\left|\log(\lambda\rho)\right|\|h\|_*.$$

Proof. Recall that $\xi_j \in \Omega$ and $d_j = dist(\xi_j, \partial\Omega)$ satisfies (2.3).

Given a solution ϕ to (3.2) we modify it so that it satisfies the orthogonality condition with respect to Z_{0j} by letting

$$\tilde{\phi} = \phi + \sum_{j=1}^m b_j \tilde{z}_{0j}$$

where $\tilde{z}_{0j}$ are suitable functions that we will construct next and we choose b_j such that

$$b_j \int_{\Omega_\rho} \chi_j |Z_{0j}|^2 + \int_{\Omega_\rho} \chi_j Z_{0j}\phi = 0. \tag{3.19}$$

Let us construct $\tilde{z}_{0j}$ in the case $d_j \leqslant \delta/10$. Later on we give the construction when $d_j \geqslant \delta/10$. We write $\hat{\xi}_j$ the point on $\partial\Omega$ closest to ξ_j. By taking $\delta > 0$ small, $\hat{\xi}_j$ is uniquely determined and depends smoothly on ξ_j.

We need the Green function for the Robin boundary condition in a half space. Let

$$\Gamma(x) = -\log|x|$$

so that $-\Delta\Gamma = 2\pi\delta_0$ in $\mathbb{R}^2$. Let $H = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$ be the half-space. We recall (see [13, p. 121]) that if $y \in H$ and $a > 0$ the Green function for the Robin problem

$$\begin{cases} -\Delta G_a(x,y) = 2\pi\delta_y, & \text{in } H, \\ -\dfrac{\partial G_a}{\partial x_N} + aG_a = 0, & \text{on } \partial H, \\ \lim\limits_{|x|\to+\infty} G_a(x,y) = 0 \end{cases}$$

is given by

$$G_a(x,y)=\Gamma(x-y)-\Gamma\big(x-y^*\big)-2\int_0^\infty e^{-as}\frac{\partial}{\partial x_N}\Gamma\big(x-y^*+e_2 s\big)\,ds, \tag{3.20}$$

where y^* is the reflection of $y=(y_1,y_2)$ across ∂H, that is $y^*=(y_1,-y_2)$, and $e_2=(0,1)$.

We take a smooth conformal change of variables $F_j:\overline{\Omega}\cap B(\hat{\xi}_j,\delta)\to H$ whose image is a neighborhood of 0 in $\overline{H}$ such that $F(\hat{\xi}_j)=0$, $F'(\hat{\xi}_j)$ is a rotation. We also let

$$F_{j,\rho}(x)=F_j(\rho x)/\rho,\quad x\in\Omega_\rho\cap B(\hat{\xi}_j/\rho,\delta/\rho). \tag{3.21}$$

We define

$$\hat{z}_{0j}(x)=\frac{1}{\log(d_j/\rho)}z_{0j}(x)G_{\lambda\rho}\big(F_{j,\rho}(x),F_{j,\rho}\big(\xi_j'\big)\big).$$

Now we take $R>R_0+1$ (cf. (3.3)). Let $\eta_1:\mathbb{R}\to\mathbb{R}$ be a smooth function such that

$$\eta_1(r)=1\quad\text{for } r\leqslant R,\qquad \eta_1(r)=0\quad\text{for } r\geqslant R+1,\qquad \big|\eta_1'(r)\big|\leqslant 2,\qquad \big|\eta_1''(r)\big|\leqslant C$$

and define

$$\eta_{1j}(y)=\eta_1\big(\big|y-\xi_j'\big|\big).$$

We need also smooth functions $\eta_{2j}:\mathbb{R}^2\to\mathbb{R}$ such that

$$\eta_{2j}(y)=1\quad\text{for }\big|y-\xi_j'\big|\leqslant\frac{\delta}{4\rho},\qquad \eta_{2j}(y)=0\quad\text{for } r\geqslant\big|y-\xi_j'\big|\leqslant\frac{\delta}{3\rho},$$
$$|\nabla\eta_{2j}|\leqslant C\rho,\qquad |\Delta\eta_{2j}|\leqslant C\rho^2,$$
$$\frac{\partial\eta_{2j}}{\partial\nu}=0\quad\text{on }\partial\Omega_\rho,$$

which can be constructed as composition of a cut-off function and a change of variables in Ω that flattens its boundary.

In the case $d_j\leqslant\delta/10$, set

$$\tilde{z}_{0j}=\eta_{1j}Z_{0j}+(1-\eta_{1j})\eta_{2j}\hat{z}_{0j}. \tag{3.22}$$

If $d_j\geqslant\delta/10$ the construction of $\tilde{z}_{0j}$ is the same as in [9]. Namely, we take the same formula as in (3.22) with new functions $\hat{z}_{0j}$ and η_{2j}. The new function $\hat{z}_{0j}$ is given by the solution to the problem

$$\Delta\hat{z}_{0j}+\frac{8\mu_j^2}{(\mu_j^2+|x-\xi_j'|^2)^2}\hat{z}_{0j}=0\quad\text{in } R<\big|x-\xi_j'\big|<\frac{\delta}{30\rho},$$
$$\hat{z}_{0j}(x)=0\quad\text{for }\big|x-\xi_j'\big|=R,\qquad \hat{z}_{0j}=0\quad\text{for }\big|x-\xi_j'\big|=\frac{\delta}{30\rho}.$$

The new function $\eta_{2j}:\mathbb{R}^2\to\mathbb{R}$ is such that

$$\eta_{2j}(y)=1 \quad \text{for } \left|y-\xi_j'\right| \leqslant \frac{\delta}{40\rho}, \qquad \eta_{2j}(y)=0 \quad \text{for } r \geqslant \left|y-\xi_j'\right| \leqslant \frac{\delta}{30\rho},$$
$$|\nabla\eta_{2j}| \leqslant C\rho, \qquad |\Delta\eta_{2j}| \leqslant C\rho^2.$$

Now suppose that ϕ is a solution to (3.2). Define

$$\tilde{\phi}=\phi+\sum_{j=1}^m b_j\tilde{z}_{0j}$$

where we choose b_j as in (3.19). We observe that $\tilde{\phi}$ satisfies

$$(\Delta+W)\tilde{\phi}=h+\sum_{j=1}^m b_j(\Delta+W)\tilde{z}_{0j} \quad \text{in } \Omega_\rho,$$
$$\left(\frac{\partial}{\partial\nu}+\lambda\rho\right)\tilde{\phi}=\sum_{j=1}^m b_j\left(\frac{\partial}{\partial\nu}+\lambda\rho\right)\tilde{z}_{0j} \quad \text{on } \partial\Omega_\rho \tag{3.23}$$

and the orthogonality conditions

$$\int_{\Omega_\rho}\chi_j Z_{ij}\tilde{\phi}=0, \quad \forall i=0,1,2,\ j=1,\dots,m.$$

By Lemma 3.2 we deduce the estimate

$$\|\tilde{\phi}\|_\infty \leqslant C\left(\|h\|_*+\sum_{j=1}^m|b_j|\left\|(\Delta+W)\tilde{z}_{0j}\right\|_*+\frac{1}{\lambda\rho}\sum_{j=1}^m|b_j|\left\|\left(\frac{\partial}{\partial\nu}+\lambda\rho\right)\tilde{z}_{0j}\right\|_{L^\infty(\partial\Omega_\rho)}\right). \tag{3.24}$$

We claim that the following inequalities hold:

$$\left\|(\Delta+W)\tilde{z}_{0j}\right\|_* \leqslant \frac{C}{|\log(\lambda\rho)|} \quad \text{for all } j=1,\dots,m, \tag{3.25}$$
$$\left\|\left(\frac{\partial}{\partial\nu}+\lambda\rho\right)\tilde{z}_{0j}\right\|_{L^\infty(\partial\Omega_\rho)} \leqslant \frac{C\lambda\rho}{|\log(\lambda\rho)|} \quad \text{for all } j=1,\dots,m, \tag{3.26}$$
$$|b_j| \leqslant C\left|\log(\lambda\rho)\right|\|h\|_* \quad \text{for all } j=1,\dots,m. \tag{3.27}$$

Using that $\tilde{\phi}=\phi+\sum_{j=1}^m b_j\tilde{z}_{0j}$ and the estimates (3.24), (3.25), (3.26) and (3.27) we obtain the conclusion of the lemma.

In the sequel we will give the proof of estimates (3.25)–(3.27) in the case $d_j \leqslant \delta/10$. For points such that $d_j \geqslant \delta/10$ the proofs of (3.25) and (3.27) are contained in the proof of Lemma 3.2 in [9], while (3.26) is trivial.

Proof of (3.25). We will need a more accurate estimate than (3.25), namely, we will prove that

$$\left\|(\Delta+W)\tilde{z}_{0j}\right\|_* \leqslant \frac{C}{\log(d_j/\rho)} \tag{3.28}$$

where $d_j = dist(\xi_j, \partial\Omega)$. Then this inequality and $d_j \geqslant \frac{1}{C\lambda}$ for all $j = 1, \dots, m$ (i.e. condition (2.3)) yield (3.25).

By (3.1),

$$\Delta\tilde{z}_{0j} + W\tilde{z}_{0j} = \Delta\tilde{z}_{0j} + \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2}\tilde{z}_{0j} + O\left(\frac{\epsilon}{1 + |y - \xi_j'|^3}\right).$$

We compute

$$\begin{aligned}
&\Delta\tilde{z}_{0j} + \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2}\tilde{z}_{0j} \\
&\quad = \Delta\eta_1(z_{0j} - \hat{z}_{0j}) + 2\nabla\eta_1\nabla(z_{0j} - \hat{z}_{0j}) \\
&\qquad + \Delta\eta_2\hat{z}_{0j} + 2\nabla\eta_2\nabla\hat{z}_{0j} + (1 - \eta_1)\eta_2\left(\Delta\hat{z}_{0j} + \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2}\hat{z}_{0j}\right).
\end{aligned}$$

For $x \in \Omega_\rho$ with $R \leqslant |x - \xi_j'| \leqslant R + 1$ and $y = F_{j,\rho}(x)$, $\eta_j' = F_{j,\rho}(\xi_j')$ we have

$$z_{0j}(x) - \hat{z}_{0j}(x) = z_{0j}(x)\left(1 - \frac{1}{\log(d_j/\rho)}G_{\lambda\rho}(y, \eta_j')\right) = O\left(\frac{1}{\log(d_j/\rho)}\right).$$

Indeed, for such points

$$\begin{aligned}
G_{\lambda\rho}(y, \eta_j') &= -\log|y - \eta_j'| + \log|y - \eta_j'^{*}| + 2\int_0^\infty e^{-\lambda\rho t}\frac{x_2 + \eta_{j,2}' + t}{(y_2 + \eta_{j,2}' + t)^2 + y_1^2}\,ds \\
&= \log(d_j/\rho) + O(1)
\end{aligned}$$

where $O(1)$ contains the first term $-\log(R)$, the integral, and part of the second term, and $y = (y_1, y_2)$, $\eta_j' = (\eta_{j,1}', \eta_{j,2}')$.

A similar estimate for its derivative implies

$$\left\|\Delta\eta_1(z_{0j} - \hat{z}_{0j}) + 2\nabla\eta_1\nabla(z_{0j} - \hat{z}_{0j})\right\|_* \leqslant \frac{C}{\log(d_j/\rho)}.$$

Similarly

$$\|\Delta\eta_2\hat{z}_{0j} + 2\nabla\eta_2\nabla\hat{z}_{0j}\|_* \leqslant \frac{C}{\log(d_j/\rho)}.$$

The last term is

$$\Delta\hat{z}_{0j} + \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2}\hat{z}_{0j} = \frac{2}{\log(d_j/\rho)}\nabla z_{0j}\nabla\left(G_{\lambda\rho}\left(F_{j,\rho}(\cdot), F_{j,\rho}(\xi_j')\right)\right)$$

away from ξ_j', and this implies

$$\left\| \Delta \hat{z}_{0j} + \frac{8\mu_j^2}{(\mu_j^2 + |y-\xi_j'|^2)^2} \hat{z}_{0j} \right\|_* \leqslant \frac{C}{\log(d_j/\rho)}. \qquad \square$$

Proof of (3.26). We will derive the estimate

$$\left\| \left(\frac{\partial}{\partial \nu} + \lambda\rho \right) \tilde{z}_{0j} \right\|_{L^\infty(\partial\Omega_\rho)} \leqslant \frac{C\lambda\rho}{\log(d_j/\rho)} \quad \text{for all } j = 1, \dots, m$$

from which (3.26) follows. On $\partial\Omega_\rho$ we have $\eta_1 = 0$ and hence $\tilde{z}_{0j} = \eta_{2j}\hat{z}_{0j}$. Therefore,

$$\left(\frac{\partial}{\partial \nu} + \lambda\rho \right) \tilde{z}_{0j} = \eta_{2j} \left(\frac{\partial \hat{z}_{0j}}{\partial \nu} + \lambda\rho \hat{z}_{0j} \right) + \lambda\rho \frac{\partial \eta_{2j}}{\partial \nu} \hat{z}_{0j}. \tag{3.29}$$

We compute

$$\begin{aligned} \frac{\partial \hat{z}_{0j}}{\partial \nu} + \lambda\rho \hat{z}_{0j} = {} & \frac{1}{\log(d_j/\rho)} \frac{\partial z_{0j}}{\partial \nu} G_{\lambda\rho}\big(F_{j,\rho}(\cdot), F_{j,\rho}(\xi_j')\big) \\ & + \frac{1}{\log(d_j/\rho)} z_{0j} \left(\frac{\partial}{\partial \nu} + \lambda\rho \right) G_{\lambda\rho}\big(F_{j,\rho}(\cdot), F_{j,\rho}(\xi_j')\big). \end{aligned}$$

Since $\nabla z_{0j}(x) = O(|x-\xi_j'|^{-3})$ and $G_{\lambda\rho}(F_{j,\rho}(x), F_{j,\rho}(\xi_j'))$ is bounded for $|x-\xi_j'| \geqslant d_j/\rho$ we have

$$\left\| \eta_{2j} \frac{1}{\log(d_j/\rho)} \frac{\partial z_{0j}}{\partial \nu} G_{\lambda\rho}\big(F_{j,\rho}(\cdot), F_{j,\rho}(\xi_j')\big) \right\|_{L^\infty(\partial\Omega_\rho)} \leqslant \frac{C\rho^3}{d_j^3 \log(d_j/\rho)} \leqslant \frac{C\lambda\rho}{\log(d_j/\rho)}.$$

Since F_j is conformal and smooth in the original domain $\overline{\Omega} \cap B(\hat{\xi}_j, \delta)$, we can write

$$\frac{\partial}{\partial \nu} G_{\lambda\rho}\big(F_{j,\rho}(x), F_{j,\rho}(\xi_j')\big) = -\frac{\partial}{\partial y_2} G_{\lambda\rho}\big(y, \eta_j'\big) \theta_{j,\rho}(y)$$

where $y = F_{j,\rho}(x)$, $\eta_j' = F_{j,\rho}(\xi_j')$ and $\theta_{j,\rho}(y)$ is the conformal factor of $F_{j,\rho}$, which has an expansion of the form $\theta_{j,\rho}(y) = 1 + O(\rho|y|)$. Then

$$\left(\frac{\partial}{\partial \nu} + \lambda\rho \right) G_{\lambda\rho}\big(F_{j,\rho}(\cdot), F_{j,\rho}(\xi_j')\big) = \big(1 - \theta_{j,\rho}(y)\big) \lambda\rho G_{\lambda\rho}\big(y, \eta_j'\big).$$

Since $G_{\lambda\rho}$ is bounded in the considered region we obtain

$$\left\| \frac{1}{\log(d_j/\rho)} \eta_{2j} z_{0j} \left(\frac{\partial}{\partial \nu} + \lambda\rho \right) G_{\lambda\rho}\big(F_{j,\rho}(\cdot), F_{j,\rho}(\xi_j')\big) \right\|_{L^\infty(\partial\Omega_\rho)} \leqslant \frac{C\lambda\rho}{\log(d_j/\rho)}.$$

Finally we also have $|\hat{z}_{0j}| \leqslant C/\log(d_j/\rho)$ for points in $\partial\Omega_\rho$ and hence

$$\left\| \lambda\rho \frac{\partial \eta_{2j}}{\partial \nu} \hat{z}_{0j} \right\|_{L^\infty(\partial\Omega_\rho)} \leqslant \frac{C\lambda\rho^2}{\log(d_j/\rho)} \leqslant \frac{C\lambda\rho}{\log(d_j/\rho)}. \qquad \square \tag{3.30}$$

Proof of (3.27). We multiply (3.23) by $\tilde z_{0k}$ and integrate in Ω_ρ:

$$\int_{\Omega_\rho} \tilde\phi(\Delta\tilde z_{0k} + W\tilde z_{0k}) - \int_{\partial\Omega_\rho} \tilde\phi\left(\frac{\partial\tilde z_{0k}}{\partial\nu} + \lambda\rho\tilde z_{0k}\right) + b_k \int_{\partial\Omega_\rho}\left(\frac{\partial\tilde z_{0k}}{\partial\nu} + \lambda\rho\tilde z_{0k}\right)\tilde z_{0k}$$
$$= \int_{\Omega_\rho} h\tilde z_{0k} + b_k\int_{\Omega_\rho}(\Delta\tilde z_{0k} + W\tilde z_{0k})\tilde z_{0k}. \tag{3.31}$$

Using (3.28) we find

$$\left|\int_{\Omega_\rho}\tilde\phi(\Delta\tilde z_{0k} + W\tilde z_{0k})\right| \leqslant \|\tilde\phi\|_{L^\infty(\Omega_\rho)}\|\Delta\tilde z_{0k} + W\tilde z_{0k}\|_* \leqslant \frac{C}{\log(d_k/\rho)}\|\tilde\phi\|_{L^\infty(\Omega_\rho)}. \tag{3.32}$$

We estimate

$$\left|\int_{\partial\Omega_\rho}\tilde\phi\left(\frac{\partial\tilde z_{0k}}{\partial\nu} + \lambda\rho\tilde z_{0k}\right)\right| \leqslant \|\tilde\phi\|_{L^\infty(\Omega_\rho)}\int_{\partial\Omega_\rho}\left|\frac{\partial\tilde z_{0k}}{\partial\nu} + \lambda\rho\tilde z_{0k}\right|.$$

By estimates as in (3.29)–(3.30) we have

$$\int_{\partial\Omega_\rho}\left|\frac{\partial\tilde z_{0k}}{\partial\nu} + \lambda\epsilon\tilde z_{0k}\right| \leqslant \frac{C}{\log(d_k/\rho)}. \tag{3.33}$$

Analogously, we have

$$\int_{\partial\Omega_\rho}\left|\left(\frac{\partial\tilde z_{0k}}{\partial\nu} + \lambda\epsilon\tilde z_{0k}\right)\tilde z_{0k}\right| \leqslant \frac{C}{\log^2(d_k/\rho)}. \tag{3.34}$$

From (3.31)–(3.34)

$$b_k\int_{\Omega_\epsilon}(\Delta\tilde z_{0k} + W\tilde z_{0k})\tilde z_{0k} \leqslant C\|h\|_* + \frac{Cb_k}{\log^2(d_k/\rho)} + \frac{C}{\log(d_k/\rho)}\|\tilde\phi\|_{L^\infty(\Omega_\rho)}.$$

Using (3.24), (3.25) and (3.26) we see that

$$\|\tilde\phi\|_{L^\infty(\Omega_\rho)} \leqslant C\,\|h\|_* + C\sum_{j=1}^m \frac{|b_j|}{\log(d_j/\rho)}.$$

Therefore

$$b_k\int_{\Omega_\rho}(\Delta\tilde z_{0k} + W\tilde z_{0k})\tilde z_{0k} \leqslant \|h\|_* + \frac{Cb_k}{\log^2(d_k/\rho)} + \frac{C}{\log(d_k/\rho)}\sum_{j=1}^m\frac{|b_j|}{\log(d_j/\rho)}. \tag{3.35}$$

We claim that

$$\left| \int_{\Omega_\rho} (\Delta \tilde{z}_{0k} + W \tilde{z}_{0k}) \tilde{z}_{0k} \right| \geqslant \frac{c}{\log(d_k/\rho)} \tag{3.36}$$

for some $c > 0$ independent of λ and ϵ.

Indeed, first we note that

$$\int_{|x-\xi_j'| \leqslant R} (\Delta \tilde{z}_{0j} + W \tilde{z}_{0j}) \tilde{z}_{0j} = O\left(\frac{\epsilon}{R}\right).$$

Next we compute in the region $R \leqslant |x - \xi_j'| \leqslant R + 1$. Here we have

$$\tilde{z}_{0j} = \eta_{1j} z_{0j} + (1 - \eta_1) \hat{z}_{0j} \tag{3.37}$$

and therefore

$$\begin{aligned} \Delta \tilde{z}_{0j} + W \tilde{z}_{0j} = {} & \Delta \eta_{1j} (z_{0j} - \hat{z}_{0j}) + 2 \nabla \eta_{1j} \nabla (z_{0j} - \hat{z}_{0j}) + \eta_{1j} (\Delta z_{0j} + W z_{0j}) \\ & + (1 - \eta_{1j})(\Delta \hat{z}_{0j} + W \hat{z}_{0j}). \end{aligned}$$

We obtain

$$\int_{R \leqslant |x-\xi_j'| \leqslant R+1} (\Delta \tilde{z}_{0j} + W \tilde{z}_{0j}) \tilde{z}_{0j} = I_1 + I_2 + I_3$$

where

$$\begin{aligned} I_1 &= \int_{R \leqslant |x-\xi_j'| \leqslant R+1} \Delta \eta_{1j} (z_{0j} - \hat{z}_{0j}) \tilde{z}_{0j} + 2 \nabla \eta_{1j} \nabla (z_{0j} - \hat{z}_{0j}) \tilde{z}_{0j}, \\ I_2 &= \int_{R \leqslant |x-\xi_j'| \leqslant R+1} \eta_{1j} (\Delta z_{0j} + W z_{0j}) \tilde{z}_{0j}, \\ I_3 &= \int_{R \leqslant |x-\xi_j'| \leqslant R+1} (1 - \eta_{1j})(\Delta \hat{z}_{0j} + W \hat{z}_{0j}) \tilde{z}_{0j}. \end{aligned}$$

Integrating by parts

$$\begin{aligned} I_1 &= \int_{R \leqslant |x-\xi_j'| \leqslant R+1} \nabla \eta_{1j} \nabla (z_{0j} - \hat{z}_{0j}) \tilde{z}_{0j} - \int_{R \leqslant |x-\xi_j'| \leqslant R+1} \nabla \eta_{1j} \nabla \tilde{z}_{0j} (z_{0j} - \hat{z}_{0j}) \\ &= - \int_{|x-\xi_j'| = R} \eta_{1j} \tilde{z}_{0j} \nabla (z_{0j} - \hat{z}_{0j}) \cdot \nu \end{aligned}$$

$$
\begin{aligned}
&- \int\limits_{R \leqslant |x-\xi_j'| \leqslant R+1} \eta_{1j}\big(\Delta(z_{0j}-\hat{z}_{0j})\tilde{z}_{0j} + \nabla(z_{0j}-\hat{z}_{0j})\nabla\tilde{z}_{0j}\big) \\
&- \int\limits_{R \leqslant |x-\xi_j'| \leqslant R+1} \nabla\eta_{1j}\nabla\tilde{z}_{0j}(z_{0j}-\hat{z}_{0j}) \\
&= A + B + C.
\end{aligned}
$$

We compute

$$
\begin{aligned}
A &= -\int\limits_{|x-\xi_j'|=R} \tilde{z}_{0j}\nabla(z_{0j}-\hat{z}_{0j})\cdot\nu \\
&= -\int\limits_{|x-\xi_j'|=R} z_{0j}\left(1-\frac{G_{\lambda\rho}(F_{j,\rho})}{\log(d_j/\rho)}\right)\nabla z_{0j}\cdot\nu + \int\limits_{|x-\xi_j'|=R} z_{0j}^2\frac{\nabla(G_{\lambda\rho}(F_{j,\rho}))}{\log(d_j/\rho)}\cdot\nu \\
&= A_1 + A_2,
\end{aligned}
$$

where we have omitted the second argument in $G_{\lambda,\rho}$, which is $F_{j,\rho}(\xi_j')$. For A_1 note that $|\nabla z_0| = O(1/R^3)$ and $(1-\frac{G_{\lambda\rho}(F_{j,\rho})}{\log(d_j/\rho)}) = O(\frac{1}{\log(d_j/\rho)})$ in the considered region. Therefore

$$
A_1 = O\left(\frac{1}{R^2\log(d_j/\rho)}\right).
$$

For points $x \in \Omega_\rho$ such that $|x-\xi_j'| = R$, thanks to (3.21), we may expand $F_{j,\rho}(x) = F_{j,\rho}(\xi_j') + x + O(\rho d_j R) + O(\rho^2R^2)$ and $DF_{j,\rho}(x) = I + O(d_j)$.

Using this information and the definition of $G_{\lambda\rho}$, (3.20), we find

$$
A_2 = \frac{1}{\log(d_j/\rho)}\left[2\pi + O\left(\frac{1}{R^2}\right) + O(d_j) + O(\rho R) + O\left(\frac{\rho^2R^2}{d^2}\right)\right].
$$

Using similar arguments we obtain

$$
B = \frac{1}{\log(d_j/\rho)}\left(O\left(\frac{1}{\log(d_j/\rho)}\right) + O\left(\frac{1}{R^3}\right)\right)
$$

and

$$
C = \frac{1}{\log(d_j/\rho)}\left(O\left(\frac{R}{\log(d_j/\rho)}\right) + O\left(\frac{1}{R^2}\right)\right).
$$

Hence

$$
I_1 = \frac{1}{\log(d_j/\rho)}\left[2\pi + O\left(\frac{1}{R^2}\right) + O(d_j) + O(\epsilon R) + O\left(\frac{\epsilon^2R^2}{d^2}\right) + O\left(\frac{R}{\log(d_j/\rho)}\right)\right].
$$

Similar estimates show that

$$I_2 = O\left(\frac{\epsilon}{R^2}\right)$$

and

$$I_3 = O\left(\frac{\epsilon}{R^2}\right) + O\left(\frac{1}{R^3 \log(d_j/\rho)}\right)$$

so that

$$\int\limits_{R \leqslant |x-\xi_j'| \leqslant R+1} (\Delta \tilde{z}_{0j} + W\tilde{z}_{0j})\tilde{z}_{0j}$$
$$= \frac{1}{\log(d_j/\rho)}\left[2\pi + O\left(\frac{1}{R^2}\right) + O(d_j) + O(\rho R) + O\left(\frac{\rho^2 R^2}{d^2}\right) + O\left(\frac{R}{\log(d_j/\rho)}\right) + O\left(\frac{\epsilon \log(d_j/\rho)}{R^2}\right)\right]. \tag{3.38}$$

We can also estimate

$$\int\limits_{R+1 \leqslant |x-\xi_j'| \leqslant \delta/(4\rho)} (\Delta \tilde{z}_{0j} + W\tilde{z}_{0j})\tilde{z}_{0j} = O\left(\frac{\epsilon}{R^2}\right) + O\left(\frac{1}{R^3 \log(d_j/\rho)}\right) \tag{3.39}$$

and

$$\int\limits_{\delta/(4\rho) \leqslant |x-\xi_j'| \leqslant \delta/(3\rho)} (\Delta \tilde{z}_{0j} + W\tilde{z}_{0j})\tilde{z}_{0j} = O\left(\frac{1}{\log(d_j/\rho)^2}\right). \tag{3.40}$$

In view of the estimates (3.37)–(3.40) we can select $R > 0$ large, $\delta > 0$ small, so that for $\lambda\rho$ sufficiently small (3.36) holds. Using then (3.35) and (3.36) we deduce the validity of (3.27). □

Proof of Proposition 3.1. First we prove that if $\phi \in L^\infty(\Omega_\rho)$, $c_{ij} \in \mathbb{R}$, $i = 1, 2,\ j = 1, \ldots, m$, solve (3.2), then the estimate (3.4) holds. Indeed, by Lemma 3.3 we have

$$\|\phi\|_{L^\infty(\Omega_\rho)} \leqslant C\left|\log(\lambda\rho)\right|\left[\|h\|_* + \sum_{i=1}^m \sum_{j=1}^m |c_{ij}|\right]. \tag{3.41}$$

Let $\eta_{3j} : \mathbb{R}^2 \to \mathbb{R}$ be smooth cut-off functions with the properties

$$\eta_{3j}(y) = 1 \quad \text{for } \left|y - \xi_j'\right| \leqslant \frac{1}{2C\lambda\rho}, \qquad \eta_{3j}(y) = 0 \quad \text{for } \left|y - \xi_j'\right| \geqslant \frac{1}{C\lambda\rho},$$
$$|\nabla \eta_{3j}| \leqslant C\lambda\rho, \qquad |\Delta \eta_{3j}| \leqslant C(\lambda\rho)^2,$$

where C is the constant that appears in the separation condition (2.3). Multiplying the equation in (3.2) by $Z_{ij}\eta_{3j}$ we find

$$\int_{\Omega_\rho} \phi\big[\Delta(\eta_{3j}Z_{ij}) + W\eta_{3j}Z_{ij}\big]\,dx = \int_{\Omega_\rho} h\eta_{3j}Z_{ij} + c_{ij}\int_{\Omega_\rho} \chi_j Z_{ij}^2.$$

Since $Z_{ij} = O(1/(1+r))$, $\nabla Z_{ij} = O(1/(1+r^2))$ where $r = |y - \xi_j'|$ we get

$$\Delta(\eta_{3j}Z_{ij}) + W\eta_{3j}Z_{ij} = O\big((\lambda\rho)^3\big) + O\left(\frac{\epsilon}{(1+r)^3}\right).$$

Therefore

$$|c_{ij}| \leqslant C\big(\|h\|_* + \epsilon\|\phi\|_{L^\infty(\Omega_\rho)}\big).$$

Using this and (3.41) we deduce that if $\lambda\epsilon$ is small enough, then

$$|\phi|_{L^\infty(\Omega_\rho)} \leqslant C\big|\log(\lambda\rho)\big|\|h\|_*,$$

and therefore (3.4) holds.

To prove the existence of solutions, consider the Hilbert space H of functions $u \in H^1(\Omega_\rho)$ such that $\int_{\Omega_\rho} \chi_j Z_{ij} u = 0$, for all $i = 1, 2$, $j = 1, \dots, m$, with the inner product

$$\langle u, v\rangle = \int_{\Omega_\rho} \nabla u \nabla v + \lambda\rho \int_{\partial\Omega_\epsilon} uv.$$

Then we weak formulation of (3.2) is to find $\phi \in H$ such that

$$\langle \phi, \psi\rangle = \int_{\Omega_\rho} (W\phi - h)\psi, \quad \forall\psi \in H.$$

Using the Riesz representation theorem, we can write this problem as follows: find $\phi \in H$ such that $\phi = K\phi + \tilde{h}$ where K is a compact operator in H and $\tilde{h} \in H$. By the Fredholm alternative, we obtain existence of a solution if the corresponding homogeneous problem $\phi = K\phi$ has no non-trivial solution. This is guaranteed by the estimate (3.4). The solution constructed in this way belongs to $H^1(\Omega_\rho)$, but by standard elliptic regularity it is also bounded. Therefore it satisfies the estimate (3.4). □

Let L_* denote the space of bounded functions $h : \Omega_\rho \to \mathbb{R}$ with norm $\|\ \|_*$. Let $T : L_* \to L^\infty(\Omega_\rho)$ be the operator constructed in Proposition 3.1, that to a function $h \in L_*$ assigns the solution $\phi \in L^\infty(\Omega_\rho)$ to (3.2). This operator depends on the points $\xi_1, \dots, \xi_m \in \Omega$ satisfying (2.2), (2.3), or the corresponding dilated variables $\xi_j' = \xi_j/\rho$. We claim that $(\xi_1', \dots, \xi_m') \mapsto T$ is C^1 in the region defined by (2.2), (2.3) and that

$$\big\|\partial_{\xi_j'} T(h)\big\|_{L^\infty(\Omega_\rho)} \leqslant C\big|\log(\lambda\rho)\big|^2\|h\|_* \tag{3.42}$$

provided $\lambda \geqslant 1$ and $\lambda\rho$ is sufficiently small. The proof of this statement is analogous to the corresponding one in [9].

4. The nonlinear problem

We return to the nonlinear problem (2.8), but through the associated problem

$$\begin{cases} L(\phi) = -\big[R + N(\phi)\big] + \sum_{i=1}^{2} \sum_{j=1}^{m} c_{ij} \chi_j Z_{ij}, & \text{in } \Omega_\rho, \\ \frac{\partial \phi}{\partial \nu} + \lambda \rho \phi = 0, & \text{on } \partial \Omega_\rho, \\ \int_{\Omega_\rho} \chi_j Z_{ij} \phi = 0, & \forall i = 1, 2;\ j = 1, \ldots, m. \end{cases} \tag{4.1}$$

This intermediate formulation gives us a framework to use the previous results. We have:

Lemma 4.1. *Under the separation conditions* (2.2) *and* (2.3) *on the points* ξ_j*, there exist constants* C, ϵ_0, $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$, $\epsilon > 0$ *with* $\lambda \rho \leqslant \epsilon_0$*, problem* (4.1) *has a unique solution* ϕ *satisfying*

$$\|\phi\|_\infty \leqslant C \lambda \rho \big|\log(\lambda \rho)\big|. \tag{4.2}$$

Moreover the map $\xi'_1, \ldots, \xi'_m \in \Omega_\rho \mapsto \phi \in L^\infty(\Omega_\rho)$ *is* C^1 *and we have the estimate*

$$\|\partial_{\xi'_{kj}} \phi\|_\infty \leqslant C \lambda \rho \big|\log(\lambda \rho)\big|^2. \tag{4.3}$$

Proof. Let

$$A(\phi) := T\big(-\big(N(\phi) + R\big)\big),$$

where T is the continuous linear map such defined on the set of all $h \in L^\infty(\Omega_\rho)$ satisfying $\|h\|_* < +\infty$, so that $\phi = T(h)$ corresponds to the unique solution of the problem (3.2). With this, problem (4.1) can be regarded as a fixed point problem

$$\phi = A(\phi).$$

For $\gamma > 0$, define the set

$$\mathcal{F}_\gamma = \big\{\phi \in C(\overline{\Omega}):\ \|\phi\|_\infty \leqslant \gamma \lambda \rho \big|\log(\lambda \rho)\big|\big\}.$$

Using the definition of the operator A and Proposition (3.2), we have

$$\big\|A(\phi)\big\|_\infty \leqslant C \big|\log(\lambda \rho)\big| \big(\big\|N(\phi)\big\|_* + \|R\|_*\big).$$

It can be proved that $\|N(\phi)\|_* \leqslant C\|\phi\|_\infty^2$ and $\|R\|_* \leqslant C\epsilon$, so we can conclude that $A(\mathcal{F}_\gamma) \subset \mathcal{F}_\gamma$ and A is a contraction, provided γ small. The fixed point theorem assures the existence of a unique fixed point of A in $\mathcal{F}_\gamma$.

Using the Implicit Function Theorem, one can justify the differentiability of the solution ϕ of the problem (4.1) as a function of the points $\xi'_j \in \Omega_\rho$. Formally, differentiating we have

$$\partial_{\xi'_{kj}} \phi = (\partial_{\xi'_{kj}} T)\big(-\big(N(\phi) + R\big)\big) - T\big(\partial_{\xi'_{kj}}\big(N(\phi) + R\big)\big).$$

So, by (3.42), the estimates for $\|N(\phi)\|_*$, $\|R\|_*$ given above and

$$\left\|\partial_{\xi'_{kj}} N(\phi)\right\|_* \leqslant C\big(\lambda\rho\big|\log(\lambda\rho)\big| + \|\partial_{\xi'_{kj}}\phi\|_\infty\big)\lambda\rho\big|\log(\lambda\rho)\big|,$$

we conclude the estimate (4.3). □

5. The reduced problem

In the past section, we proved existence of a solution of the nonlinear projected problem (4.1). The idea is to find a condition on the points $\xi_1, \dots, \xi_m$ that implies $c_{ij}(\xi') = 0$, for all i, j.

Eq. (1.1) is the Euler–Lagrange equation of the functional $J_{\epsilon,\lambda} : H^1(\Omega) \to \mathbb{R}$ defined by

$$J_{\epsilon,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \epsilon^2\int_\Omega e^u\,dx + \frac{\lambda}{2}\int_{\partial\Omega} u^2\,d\sigma(x). \tag{5.1}$$

Let

$$F(\xi) = J_{\epsilon,\lambda}(U + \tilde{\phi}) \tag{5.2}$$

where U is the ansatz defined in (2.6) and $\tilde{\phi} = \tilde{\phi}(x,\xi) = \phi(\frac{x}{\rho}, \xi')$, with ϕ the solution of the nonlinear problem (4.1) given in the last section. The following lemma characterizes the condition $c_{ij}(\xi') = 0$, for all i, j in (4.1).

Lemma 5.1. *The functional $F(\xi)$ is of class C^1 in the region determined by* (2.2)–(2.4)*. Moreover, for $\lambda\rho$ sufficiently small, $D_\xi F(\xi) = 0$ implies that ξ satisfies*

$$c_{ij}\big(\xi'\big) = 0, \quad \forall i, j.$$

Proof. Recall $\xi' = \xi/\rho$. We will work in the expanded variables and write the energy associated functional as

$$I_{\epsilon,\lambda}(v) = \frac{1}{2}\int_{\Omega_\rho} |\nabla v|^2\,dy - \int_{\Omega_\rho} e^v\,dy + \frac{\lambda\rho}{2}\int_{\partial\Omega_\rho} \big(v - \log\big(\epsilon^4/\lambda\big)\big)^2\,d\sigma(y).$$

Note that $F(\xi) = J_\epsilon(U + \tilde{\phi}) = I_\epsilon(V + \phi)$. The smoothness in terms of ξ of the function F is inherited by the solution ϕ of the nonlinear problem and the definition of the approximation V. Hence

$$\begin{aligned}
\partial_{\xi_{kl}} F(\xi) &= \rho^{-1} D I_{\epsilon,\lambda}(V + \phi)\big[\partial_{\xi'_{kl}}(V + \phi)\big] \\
&= \rho^{-1}\Bigg(\int_{\Omega_\rho} \big\langle\nabla(V + \phi), \nabla\partial_{\xi'_{kl}}(V + \phi)\big\rangle\,dy - \int_{\Omega_\rho} e^{V+\phi}\partial_{\xi'_{kl}}(V + \phi)\,dy \\
&\quad + \lambda\rho\int_{\partial\Omega_\rho} \big(V + \phi - \log\big(\epsilon^4/\lambda\big)\big)\partial_{\xi'_{kl}}(V + \phi)\,d\sigma(y)\Bigg)
\end{aligned}$$

using the equation satisfied by $V + \phi$, we can conclude that

$$\partial_{\xi_{kl}} F(\xi) = -\rho^{-1} \sum_{i=1}^{2} \sum_{j=1}^{m} \int_{\Omega_\rho} c_{ij} \chi_j Z_{ij} [\partial_{\xi'_{kl}} V + \partial_{\xi'_{kl}} \phi].$$

Let us assume that $D_\xi F(\xi) = 0$. Then

$$\sum_{i=1}^{2} \sum_{j=1}^{m} \int_{\Omega_\rho} c_{ij} \chi_j Z_{ij} [\partial_{\xi'_{kl}} V + \partial_{\xi'_{kl}} \phi] = 0, \quad k = 1, 2;\ l = 1, \dots, m. \tag{5.3}$$

As we saw at the end of the last section, we have $\|D_{\xi'_{kl}} \phi\|_\infty \leqslant C\lambda\rho|\log(\lambda\rho)|^2$.

On the other hand,

$$\partial_{\xi'_{kl}} V = -Z_{kl}(y) + \partial_{\xi'_{kl}} H_j(y) = -Z_{kl}(y) + O(\lambda\rho)$$

where the term $O(\lambda\rho)$ is uniformly in Ω. Indeed, to estimate $\partial_{\xi'_{kl}} H_j = \frac{\epsilon}{\sqrt{\lambda}} \partial_{\xi_{kl}} H_j$ note that $g = \partial_{\xi_{kl}} H_j$ satisfies

$$\Delta g = 0 \quad \text{in } \Omega, \qquad \frac{\partial g}{\partial \nu} + \lambda g = O\big(\lambda^2\big) \quad \text{on } \partial\Omega,$$

since $dist(\xi_j, \partial\Omega) \geqslant \delta/\lambda$ and we are assume $\rho > 0$ small, i.e., $\epsilon^2\lambda$ small. By Lemma 2.2 we obtain $\|g\|_{L^\infty(\Omega)} \leqslant C\lambda$. Hence $|\partial_{\xi'_{kl}} H_j| \leqslant C\epsilon\sqrt{\lambda} = C\rho\lambda$ in Ω.

Then, we can rewrite the system (5.3) as

$$\sum_{i=1}^{2} \sum_{j=1}^{m} \int_{\Omega_\rho} c_{ij} \chi_j Z_{ij} \big[Z_{kl} + O(1)\big] = 0, \quad k = 1, 2;\ l = 1, \dots, m.$$

For $\lambda\rho$ sufficiently small, this $2m \times 2m$ system is diagonal dominant. Hence, its unique solution is $c_{ij}(\xi') = 0$, for all i, j. □

We finish this section with an expansion of the function F as a perturbation of the energy of the ansatz.

Lemma 5.2. *Under the assumptions on the points ξ_j given by* (2.2)–(2.4)*, the following expansion holds*:

$$F(\xi) = J_{\epsilon,\lambda}(U) + \theta_{\epsilon,\lambda}(\xi),$$

where the term $|\theta_{\epsilon,\lambda}(\xi)| + |\nabla\theta_{\epsilon,\lambda}(\xi)| \to 0$ uniformly as $\lambda\rho \to 0$ in the region described by (2.2)–(2.4).

Proof. Working in expanded variables, by definitions (5.1) and (5.2) we have $F(\xi) = I_{\epsilon,\lambda}(V + \phi)$. Since $V + \phi$ is a solution of Eq. (2.7), the weak formulation of the problem give us $DI_{\epsilon,\lambda}(V + \phi)[\phi] = 0$. Then

$$\begin{aligned} \theta_{\epsilon,\lambda}(\xi) &= I_{\epsilon,\lambda}(V + \phi) - I_{\epsilon,\lambda}(V) \\ &= \int_0^1 t D^2 I_{\epsilon,\lambda}(V + t\phi)\phi^2 \, dt \end{aligned}$$

$$= \int_0^1 \left(\int_{\Omega_\rho} \left(|\nabla\phi|^2 - e^{V+t\phi}\phi^2 \right) dy + \lambda\rho \int_{\partial\Omega_\rho} \phi^2 \, d\sigma(y) \right) t \, dt$$

$$= \int_0^1 \left(\int_{\Omega_\rho} - \left[N(\phi) + R \right] \phi dy + \int_{\Omega_\rho} e^V \left(e^{t\phi} - 1 \right) \phi^2 \, dy \right) t \, dt, \tag{5.4}$$

after an integration by parts and the use of the equation satisfied by ϕ. Using the estimate $\|\phi\|_\infty \leqslant C\lambda\rho|\log(\lambda\rho)|$ found in the previous section, we get

$$I_{\epsilon,\lambda}(V+\epsilon) - I_{\epsilon,\lambda}(V) = C\left(\left(\lambda\rho\left|\log(\lambda\rho)\right|\right)^3 + \epsilon\lambda\rho\left|\log(\lambda\rho)\right| \right).$$

The continuity in ξ' of the all these expressions is inherited from that of ϕ in the L^∞ norm.

Note that $\nabla_\xi \theta_{\epsilon,\lambda}(\xi) = \rho^{-1}\nabla_{\xi'}\theta_{\epsilon,\lambda}(\rho\xi')$. Differentiating with respect to ξ'_{kl} under the integral sign in (5.4), we obtain

$$\partial_{\xi'_{kl}}\left[I_{\epsilon,\lambda}(V+\phi) - I_{\epsilon,\lambda}(V) \right]$$
$$= \int_0^1 \left(\int_{\Omega_\rho} - \partial_{\xi'_{kl}}\left[\left(N(\phi) + R \right)\phi \right] dy + \int_{\Omega_\rho} \partial_{\xi'_{kl}}\left[e^V \left(e^{t\phi} - 1 \right)\phi^2 \right] dy \right) t \, dt,$$

and using the estimates for $N(\phi)$, R and W and its derivatives with respect to ξ'_{kl} given in the previous section, we get

$$\partial_{\xi_{kl}}\theta_{\epsilon,\lambda}(\xi) = \rho^{-1}\partial_{\xi'_{kl}}\left[I_{\epsilon,\lambda}(V+\phi) - I_{\epsilon,\lambda}(V) \right]$$
$$= \epsilon\left(\left|\log(\lambda\rho)\right| + (\lambda\rho)^2\left|\log(\lambda\rho)\right|^4 \right) \to 0 \tag{5.5}$$

as $\lambda\rho \to 0$. □

6. An expression for the energy of the ansatz

Given the asymptotic expansion of the functional F in terms of the energy of the ansatz $J_\epsilon(U)$, we are interested in the form of this energy in order to find the critical points of F. The following result gives us an expression which will be useful for this purpose.

Define

$$d = \min\left\{ dist(\xi_j, \partial\Omega): \ j = 1, \ldots, m \right\}.$$

Proposition 6.1. *Let U be the function defined in* (2.6)*. There exists $\epsilon_0 > 0$, such that for all $0 < \epsilon < \epsilon_0$ we have*

$$J_\epsilon(U) = -16m\pi - 16m\pi\log(\epsilon) + 8m\pi\log(8) - 4\pi\varphi_m(\xi) + \Theta(\epsilon, \lambda, d)$$

where the function φ_m is defined as

$$\varphi_m(\xi_1, \ldots, \xi_m) = \sum_{j=1}^m H_\lambda(\xi_j, \xi_j) + \sum_{i \neq j} G_\lambda(\xi_i, \xi_j) \tag{6.1}$$

with G_λ and H_λ the Green function for the Laplacian in Ω with Robin boundary condition and its regular part (cf. (1.3), (1.4)). The term Θ has an order $O(\epsilon^2\lambda\log(\lambda))$ and $O(\epsilon^2\lambda^3)$ for its derivative, when the points $\xi_1,\dots,\xi_m$ are such that $|\xi_i-\xi_j|>\delta$ for each $i\neq j$ and $dist(\xi_j,S^*)\leqslant c\lambda^{-3/2}$ for some constant $c>0$.

Proof. We will divide the analysis looking each term appearing in the development of $J_\epsilon(U)$ individually.

Gradient squared. This term is given by

$$\frac{1}{2}\int_\Omega|\nabla U|^2\,dx=\frac{1}{2}\left\{\sum_{j=1}^m\int_\Omega|\nabla U_j|^2\,dx+\sum_{i\neq j}\int_\Omega\nabla U_i\nabla U_j\,dx\right\} \tag{6.2}$$

where $U_j=u_j+H_j$.

We have

$$\frac{1}{2}\int_\Omega|\nabla U_j|^2\,dx=\frac{1}{2}\int_\Omega|\nabla u_j|^2\,dx+\int_\Omega\langle\nabla u_j,\nabla H_j\rangle\,dx+\frac{1}{2}\int_\Omega|\nabla H_j|^2\,dx. \tag{6.3}$$

Taking the last two terms in this expansion, using integration by parts and the definition of U_j we obtain

$$\int_\Omega\langle\nabla u_j,\nabla H_j\rangle\,dx+\frac{1}{2}\int_\Omega|\nabla H_j|^2\,dx=\int_{\partial\Omega}U_j\frac{\partial H_j}{\partial\nu}\,d\sigma-\frac{1}{2}\int_{\partial\Omega}H_j\frac{\partial H_j}{\partial\nu}\,d\sigma \tag{6.4}$$

where ν represents the unit normal exterior of $\partial\Omega$.

Recall that d_j denotes the distance of the point ξ_j to $\partial\Omega$. For the first term on the right-hand side of (6.3), we will use the explicit expression of u_j given in (2.1):

$$\begin{aligned}\int_\Omega|\nabla u_j|^2\,dx={}&\int_{B(\xi_j,\frac{d_j}{2})}\left|\nabla w_{\mu_j}\left(\frac{\sqrt{\lambda}|x-\xi_j|}{\epsilon}\right)\right|^2dx\\&+\int_{\Omega\setminus B(\xi_j,\frac{d_j}{2})}\left|\nabla w_{\mu_j}\left(\frac{\sqrt{\lambda}|x-\xi_j|}{\epsilon}\right)\right|^2dx.\end{aligned} \tag{6.5}$$

For the first term in (6.5) we have by explicit calculation

$$\begin{aligned}&\int_{B(\xi_j,\frac{d_j}{2})}\left|\nabla w_{\mu_j}\left(\frac{\sqrt{\lambda}|x-\xi_j|}{\epsilon}\right)\right|^2dx\\&=16\pi\left[\log\left(\frac{\epsilon^2\mu_j^2}{\lambda}+\left(\frac{d_j}{2}\right)^2\right)-2\log\left(\frac{\epsilon\mu_j}{\sqrt{\lambda}}\right)+\frac{(\epsilon^2\mu_j^2)/\lambda}{(\epsilon^2\mu_j^2)/\lambda+(\frac{d_j}{2})^2}-1\right].\end{aligned} \tag{6.6}$$

Using the definition of w_{μ_j}

$$\int_{\Omega\setminus B(\xi_j,\frac{d_j}{2})} \left|\nabla w_{\mu_j}\left(\frac{\sqrt{\lambda}|x-\xi_j|}{\epsilon}\right)\right|^2 dx$$

$$= 16 \int_{\Omega\setminus B(\xi_j,\frac{d_j}{2})} \frac{1}{|x-\xi_j|^2}\,dx - 32\frac{\mu_j^2\epsilon^2}{\lambda} \int_{\Omega\setminus B(\xi_j,\frac{d_j}{2})} \frac{|x-\xi_j|^2}{(\tau+|x-\xi_j|^2)^3}\,dx \tag{6.7}$$

with $\tau \in [0, (\mu_j^2\epsilon^2)/\lambda]$. Denote θ_{11} the second term in the RHS of the last equality. We estimate θ_{11} in the following way

$$|\theta_{11}| \leqslant 32\frac{\mu_j^2\epsilon^2}{\lambda} \int_{\Omega\setminus B(\xi_j,\frac{d_j}{2})} \frac{1}{|x-\xi_j|^4}\,dx$$

$$= 16\frac{\mu_j^2\epsilon^2}{\lambda}\left(\int_{\partial\Omega} \frac{\partial|x-\xi_j|}{\partial\nu}|x-\xi_j|^{-3} + \int_{\partial B(\xi_j,d_j/2)} \frac{1}{|x-\xi_j|^3}\right)$$

and conclude that θ_{11} has order $O(\frac{\mu_j^2\epsilon^2}{\lambda d_j^2})$. For $\partial_\xi\theta_{11}$ we have

$$\partial_\xi\theta_{11} = O\big((\epsilon\lambda)^2\big) + \frac{\mu_j^2\epsilon^2}{\lambda}\left(\int_{\partial B(0,d_j/2)} \frac{2|z|(\tau-2|z|^2)}{(\tau+|z|^2)^4}\nu_k(z)\,dz\right.$$

$$+ \frac{\nu_k(\hat{\xi}_j)}{2}\int_{\partial B(\xi_j,d_j/2)} \frac{|x-\xi_j|^2}{(\tau+|x-\xi_j|^2)^3}\,dx$$

$$\left. - \int_{\Omega\setminus B(\xi_j,d_j/2)} \frac{2|x-\xi_j|(\tau-2|x-\xi_j|^2-3|x-\xi_j|\partial\tau)}{(\tau+|x-\xi_j|^2)^4}\right)$$

$$= O\big((\epsilon\lambda)^2\big) + O\big((\epsilon\lambda)^4\big) + O\big(\epsilon^2\lambda^3\big) + O\big((\epsilon\lambda)^2\big).$$

On the other hand, note that $|\nabla\Gamma(x,\xi_j)|^2 = \frac{16}{|x-\xi_j|^2}$, where $\Gamma(x,y) = 4\log(\frac{1}{|x-y|})$ is the fundamental solution of the Laplacian in $\mathbb{R}^2$. Hence

$$16\int_{\Omega\setminus B(\xi_j,\frac{d_j}{2})} \frac{1}{|x-\xi_j|^2}\,dx = \int_{\partial(\Omega\setminus B(\xi_j,\frac{d_j}{2}))} \Gamma(x,\xi_j)\frac{\partial\Gamma(x,\xi_j)}{\partial\nu}\,d\sigma$$

$$= \int_{\partial\Omega} \Gamma(x,\xi_j)\frac{\partial\Gamma(x,\xi_j)}{\partial\nu}\,d\sigma + \int_{\partial B(\xi_j,\frac{d_j}{2})} \Gamma(x,\xi_j)\frac{\partial\Gamma(x,\xi_j)}{\partial\nu}\,d\sigma$$

$$= \int_{\partial\Omega} G(x,\xi_j)\frac{\partial\Gamma}{\partial\nu} - \int_{\partial\Omega} H(x,\xi_j)\frac{\partial\Gamma}{\partial\nu}\,d\sigma + 32\pi\log\frac{1}{\frac{d_j}{2}} \tag{6.8}$$

where we have used that $H_\lambda(x,\xi_j) = G_\lambda(x,\xi_j) - \Gamma(x,\xi_j)$. Then, combining (6.7) and (6.8) we have

$$\begin{aligned}
\int\limits_{\Omega\setminus B(\xi_j,\frac{d_j}{2})} |\nabla u_j|^2\,dx &= \int\limits_{\partial\Omega} G_\lambda(x,\xi_j)\frac{\partial\Gamma}{\partial\nu} - \int\limits_{\partial\Omega} H_\lambda(x,\xi_j)\frac{\partial\Gamma}{\partial\nu}\,d\sigma + 32\pi\log\frac{1}{\frac{d_j}{2}} + \theta_{11} \\
&= -\int\limits_{\partial\Omega} \Gamma(x,\xi_j)\frac{\partial H}{\partial\nu}\,d\sigma - \int\limits_{\Omega} \Delta\Gamma(x,\xi_j)H_\lambda(x,\xi_j)\,dx \\
&\quad + \int\limits_{\partial\Omega} G(x,\xi_j)\frac{\partial\Gamma}{\partial\nu}\,d\sigma + 32\pi\log\frac{1}{\frac{d_j}{2}} + \theta_{11} \\
&= 8\pi H(\xi_j,\xi_j) - \int\limits_{\partial\Omega} \Gamma\frac{\partial H}{\partial\nu}\,d\sigma + \int\limits_{\partial\Omega} G_\lambda(x,\xi_j)\frac{\partial\Gamma}{\partial\nu}\,d\sigma \\
&\quad + 32\pi\log\frac{1}{\frac{d_j}{2}} + \theta_{11}.
\end{aligned} \tag{6.9}$$

Finally, using (6.9) and (6.6) we have

$$\begin{aligned}
\frac{1}{2}\int\limits_{\Omega} |\nabla u_j|^2\,dx &= -8\pi - 16\pi\log\left(\frac{\epsilon\mu_j}{\sqrt{\lambda}}\right) - \frac{1}{2}\int\limits_{\partial\Omega} \Gamma(x,\xi_j)\frac{\partial H}{\partial\nu}\,d\sigma \\
&\quad + \frac{1}{2}\int\limits_{\partial\Omega} G(x,\xi_j)\frac{\partial\Gamma}{\partial\nu}\,d\sigma + 4\pi H_\lambda(\xi_j,\xi_j) + \tilde\theta_1
\end{aligned} \tag{6.10}$$

where $\tilde\theta_1 = \theta_{11} + \theta_{12}$ and θ_{12} is the error term associated to (6.6). We can estimate θ_{12} noting that

$$\begin{aligned}
\theta_{12} &= -32\pi\log(d_j/2) + 16\pi\left[\log\left(\frac{\epsilon^2\mu_j^2}{\lambda} + \left(\frac{d_j}{2}\right)^2\right) + \frac{(\epsilon^2\mu_j^2)/\lambda}{(\epsilon^2\mu_j^2)/\lambda + (\frac{d_j}{2})^2}\right] \\
&= 5\times 16\pi\frac{1}{d_j}\frac{\mu_j^2\epsilon^2}{\lambda} - 16\pi\frac{\mu_j^4\epsilon^4}{\lambda^2}\frac{16}{d_j^4} \\
&= O\big(\epsilon^2\lambda\big) + O\big(\epsilon^4\lambda^2\big).
\end{aligned}$$

Meanwhile, if we denote $\rho^2 = \epsilon^2/\lambda$, we can estimate $\partial_\xi\theta_{12}$ using that

$$\begin{aligned}
\partial_{\xi_j}\theta_{12} &= 16\pi\left[-\frac{2}{d_j}\partial d_j + \frac{\rho^2\partial\mu^2 + d_j/2\partial d_j}{\rho^2\mu_j^2 + (d_j/2)^2} + \frac{\rho^2\partial\mu^2}{\rho^2\mu_j^2 + (d_j/2)^2} - \frac{\rho^2\mu_j^2(\rho^2\partial\mu^2 + d_j/2\partial d_j)}{(\rho^2\mu_j^2 + (d_j/2)^2)^2}\right] \\
&= O\big((\epsilon\lambda)^2\big).
\end{aligned}$$

Then, we conclude that $\tilde\theta_1$ has order $O(\epsilon^2\lambda)$ and $O(\epsilon^2\lambda^3)$ for its derivative.

We will need the following lemma to complete the estimate of (6.3). The proof of this estimate is given in Appendix A.

Lemma 6.2. *In virtue of the relation between $H_j(x)$ and $H_\lambda(x,\xi_j)$ we have*

$$\int_{\partial\Omega} H_j \frac{\partial H_j}{\partial \nu}\, d\sigma = \int_{\partial\Omega} H_\lambda(x,\xi_j)\frac{\partial H_\lambda(x,\xi_j)}{\partial \nu}\, d\sigma + O\big(\lambda\epsilon^2\big). \tag{6.11}$$

And the derivative of the error term has an order $O((\epsilon\lambda)^2\log(\lambda))$.

Continuing with the proof of Proposition 6.1, we see that thanks to (6.4), (6.10) and (6.11), we have

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla U_j|^2\, dx = {}& -8\pi - 16\pi\log(\mu_j\epsilon) + 4\pi H_\lambda(\xi_j,\xi_j)\\ &+\int_{\partial\Omega} U_j\frac{\partial H_j}{\partial\nu}\, d\sigma - \frac{1}{2}\int_{\partial\Omega} H_\lambda(x,\xi_j)\frac{\partial H(x,\xi_j)}{\partial\nu}\, d\sigma\\ &+\frac{1}{2}\int_{\partial\Omega} G_\lambda(x,\xi_j)\frac{\partial\Gamma}{\partial\nu}\, d\sigma - \frac{1}{2}\int_{\partial\Omega}\Gamma\frac{\partial H_\lambda(x,\xi_j)}{\partial\nu}\, d\sigma + \theta_1(\epsilon,\lambda,d_j)\end{aligned} \tag{6.12}$$

where $\theta_1(\epsilon,\lambda,d_j)$ includes all the error terms seen so far and has an order $O(\epsilon^2\lambda)$, and derivative of order $O(\epsilon^2\lambda^3)$.

For the crossed terms of (6.2), using the Robin boundary condition we have

$$\begin{aligned}\int_\Omega \nabla U_i\nabla U_j\, dx &= \int_{\partial\Omega} U_j\frac{\partial U_i}{\partial\nu} - \int_\Omega U_j\Delta U_i\\ &= -\lambda\int_{\partial\Omega} U_jU_i - \int_\Omega U_j\Delta U_i.\end{aligned} \tag{6.13}$$

Using the definition of the functions U_j and centering the coordinate system on ξ_i', the second integral of the last expression can be separated as follows

$$\begin{aligned}-\int_\Omega U_j\Delta U_i\, dx &= \epsilon^{-2}\lambda\int_\Omega \frac{8\mu_i^2}{(\mu_i^2+\frac{\lambda|x-\xi_i|^2}{\epsilon^2})^2}\left\{w_j\left(\frac{\sqrt{\lambda}|x-\xi_j|}{\epsilon}\right)+\log\frac{1}{\epsilon^4}+\log(\lambda)+H_j(x)\right\}dx\\ &= \int_{\frac{\sqrt{\lambda}}{\mu_i\epsilon}(\Omega-\xi_i)} \frac{8}{(1+|y|^2)^2}\left\{\log\frac{1}{(\mu_j^2\rho^2+|\xi_i-\xi_j+\frac{\epsilon\mu_i}{\sqrt{\lambda}}y|^2)^2}+\log\big(8\mu_j^2\big)-\log(\lambda)\right\}dy\\ &\quad+\int_{\frac{\sqrt{\lambda}}{\mu_i\epsilon}(\Omega-\xi_i)} \frac{8}{(1+|y|^2)^2}H_j\left(\xi_i+\frac{\epsilon\mu_i}{\sqrt{\lambda}}y\right)dy\\ &= I_1+I_2+I_3+I_4,\end{aligned} \tag{6.14}$$

where

$$I_1 = \int\limits_{\frac{\sqrt{\lambda}}{\mu_i \epsilon}(\Omega - \xi_i)} \frac{8}{(1+|y|^2)^2}\left(\log \frac{1}{(\mu_j^2\rho^2 + |\xi_i - \xi_j + \frac{\epsilon\mu_i}{\sqrt{\lambda}}y|^2)^2} - 4\log\frac{1}{|\xi_i - \xi_j|}\right) dy,$$

$$I_2 = \int\limits_{\frac{\sqrt{\lambda}}{\mu_i \epsilon}(\Omega - \xi_i)} \frac{8}{(1+|y|^2)^2}\left(H_j\left(\xi_i + \frac{\epsilon\mu_i}{\sqrt{\lambda}}y\right) - H_j(\xi_i)\right) dy,$$

$$I_3 = \int\limits_{\frac{\sqrt{\lambda}}{\mu_i \epsilon}(\Omega - \xi_i)} \frac{8}{(1+|y|^2)^2}\left(H_j(\xi_i) - H(\xi_i, \xi_j) + \log\left(8\mu_j^2\right) - \log(\lambda)\right),$$

$$I_4 = \int\limits_{\frac{\sqrt{\lambda}}{\mu_i \epsilon}(\Omega - \xi_i)} \frac{8}{(1+|y|^2)^2}\left(H(\xi_i, \xi_j) + 4\log\frac{1}{|\xi_i - \xi_j|}\right) dy.$$

We need to estimate each of the last four integrals. Since the points ξ_i, ξ_j are uniformly separated each other, we have I_1 and I_2 of order $O(\epsilon/\sqrt{\lambda})$ with the same order for its derivatives with respect to ξ_j. The asymptotic estimate (2.13) implies $I_3 = O(\frac{\mu_j^2\rho^2}{\lambda d_j^3})$ and $O(\frac{\mu_j^2\rho^2}{\lambda d_i d_j^3})$ for its derivative. Finally, for I_4 we have

$$\begin{aligned} I_4 &= \int\limits_{\frac{\sqrt{\lambda}}{\mu_i \epsilon}(\Omega - \xi_i)} \frac{8}{(1+|y|^2)^2}\left(H(\xi_i, \xi_j) + 4\log\frac{1}{|\xi_i - \xi_j|}\right) dy \\ &= \int\limits_{\frac{\sqrt{\lambda}}{\mu_i \epsilon}(\Omega - \xi_i)} \frac{8}{(1+|y|^2)^2} G(\xi_i, \xi_j)\, dy \\ &= 8G(\xi_i, \xi_j) \int\limits_{\frac{\sqrt{\lambda}}{\mu_i \epsilon}(\Omega - \xi_i)} \frac{1}{(1+|y|^2)^2}\, dy \\ &= 8\pi G(\xi_i, \xi_j) + O\left(\lambda\epsilon^2\right), \end{aligned}$$

and derivative with respect to ξ_j for the last error term of the same order.

Hence, the second term on the right-hand side of (6.13) can be estimated as

$$-\int\limits_{\Omega} U_j \Delta U_i = 8\pi G(\xi_i, \xi_j) + \theta_2(\epsilon, \lambda, d) \tag{6.15}$$

where θ_2 is $O(\lambda\epsilon^2)$ and order $O((\lambda\epsilon)^2)$ for its derivative.

For the first term in the right-hand side of (6.13), using the asymptotic relation (2.14) we have

$$-\lambda \int\limits_{\partial\Omega} U_j U_i = -\lambda \int\limits_{\partial\Omega} G_\lambda(x, \xi_i) G_\lambda(x, \xi_j) + O\left(\frac{\mu_i^2\rho^2}{\lambda d_i^3}\right) + O\left(\frac{\mu_j^2\rho^2}{\lambda d_j^3}\right) \tag{6.16}$$

where the derivative of the error term has an order $O(\frac{\mu_i^2\rho^2}{\lambda d^4})$.

Finally, with the estimates (6.12), (6.15) and (6.16), the expression for the term with the gradient squared in (6.2) can be written as follows

$$\begin{aligned}\frac{1}{2}\int_{\Omega}|\nabla U|^2\,dx = {} & 4\pi\left(\sum_{j=1}^m H_\lambda(\xi_j,\xi_j)+\sum_{i\neq j}G_\lambda(\xi_i,\xi_j)\right)-8m\pi-16\pi\sum_{j=1}^m\log(\mu_j\epsilon)\\ & +\sum_{j=1}^m\int_{\partial\Omega}U_j\frac{\partial H_j}{\partial\nu}-\frac{1}{2}\int_{\partial\Omega}H_\lambda(x,\xi_j)\frac{\partial H_\lambda(x,\xi_j)}{\partial\nu}\\ & +\frac{1}{2}\int_{\partial\Omega}G_\lambda(x,\xi_j)\frac{\partial\Gamma(x,\xi_j)}{\partial\nu}-\frac{1}{2}\int_{\partial\Omega}\Gamma\frac{\partial H_\lambda(x,\xi_j)}{\partial\nu}\\ & -\frac{\lambda}{2}\sum_{i\neq j}\int_{\partial\Omega}G_\lambda(x,\xi_i)G_\lambda(x,\xi_j)+\Theta_1(\epsilon,\lambda,d)\end{aligned}\tag{6.17}$$

where $\Theta_1(\epsilon,\lambda,d)$ includes all the error terms considered in the previous analysis and is $O(\lambda\epsilon^2)$ with derivative of order $O(\epsilon^2\lambda^3)$.

Exponential term. Now we will consider the exponential part of the energy. We can divide it in the following way

$$\epsilon^2\int_{\Omega}e^U\,dx=\epsilon^2\sum_{j=1}^m\int_{B(\xi_j,\frac{d_j}{2})}e^U\,dx+\epsilon^2\int_{\Omega\setminus\bigcup_{j=1}^m B(\xi_j,\frac{d_j}{2})}e^U\,dx.\tag{6.18}$$

For the first term on the right-hand side of (6.18) for each j we have

$$\begin{aligned}\epsilon^2\int_{B(\xi_j,\frac{d_j}{2})}e^U\,dx & =\epsilon^2\int_{B(\xi_j,\frac{d_j}{2})}e^{U_j}e^{\sum_{i\neq j}U_i}\,dx\\ & =\epsilon^2\int_{B(\xi_j,\frac{d_j}{2})}\frac{1}{((\mu_j\epsilon/\sqrt{\lambda})^2+|x-\xi_j|^2)^2}\exp\bigl(\log\bigl(8\mu_j^2\bigr)-\log(\lambda)+H_j(x)\bigr)\\ & \quad\times\exp\left(\sum_{i\neq j}\left(\log\frac{8\mu_i^2}{(\mu_i^2\epsilon^2+\lambda|x-\xi_i|^2)^2}+\log(\lambda)+H_i(x)\right)\right)dx\\ & =\frac{\lambda}{\mu_j^2}\int_{B(0,\frac{\sqrt{\lambda}d_j}{2\epsilon\mu_j})}\frac{1}{(1+|y|^2)^2}\exp\left[H_\lambda(\xi_j,\xi_j+\mu_j\epsilon/\sqrt{\lambda}y)+O\left(\frac{\mu_j^2\rho^2}{\lambda d_j^3}\right)\right]\\ & \quad\times\exp\left[\sum_{i\neq j}\left(G_\lambda(\xi_i,\xi_j+\mu_j\epsilon/\sqrt{\lambda}y)-4\frac{\mu_j\epsilon y}{\sqrt{\lambda}|\xi_i-\xi_j|}+O\left(\frac{\mu_i^2\rho^2}{\lambda d_i^3}\right)\right)\right]\\ & \quad\times\exp\left(-2\frac{\mu_i^2\epsilon^2}{\lambda|\xi_j-\xi_i+\mu_j\epsilon/\sqrt{\lambda}y|}\right)dy.\end{aligned}$$

Using the definition of the numbers μ_i, we can conclude

$$\epsilon^2 \sum_{j=1}^{m} \int\limits_{B(\xi_j, \frac{d_j}{2})} e^U \, dx = 8m\pi + O(\epsilon), \tag{6.19}$$

where we have used (1.11). The derivative of the error has an order $O(\lambda\epsilon)$.

Using the estimates given above, it is easy to see that the second part in the right-hand side of (6.18) becomes

$$\epsilon^2 \int\limits_{\Omega \setminus \bigcup_{j=1}^{m} B(\xi_j, \frac{d_j}{2})} e^U \, dx = O\left(\frac{\epsilon^2}{d}\right) \tag{6.20}$$

with $O((\lambda\epsilon)^2)$ for the derivative of the error.

Finally, with (6.19) and (6.20) we can write

$$\epsilon^2 \int\limits_{\Omega} e^U \, dx = 8m\pi + \Theta_2(\epsilon, \lambda, d) \tag{6.21}$$

where $\Theta_2(\epsilon, d)$ has an order $O(\epsilon)$ and $O(\lambda\epsilon)$ for its derivative.

Boundary term. For the boundary term of the energy, we use the asymptotic expansion (2.14) and the Robin boundary condition of the Green function to obtain

$$\begin{aligned} \frac{\lambda}{2} \int\limits_{\partial\Omega} U^2 \, d\sigma &= \frac{\lambda}{2} \int\limits_{\partial\Omega} \left(\sum_{j=1}^{m} \left(G_\lambda(x, \xi_j) + O\left(\frac{\rho^2 \mu_j^2}{\lambda d_j^3} \right) \right) \right)^2 d\sigma \\ &= \frac{\lambda}{2} \sum_{j=1}^{m} \int\limits_{\partial\Omega} G_\lambda^2(x, \xi_j) + \frac{\lambda}{2} \sum_{i \neq j} \int\limits_{\partial\Omega} G_\lambda(x, \xi_i) G_\lambda(x, \xi_j) \, d\sigma + \Theta_3(\epsilon, \lambda, d) \end{aligned} \tag{6.22}$$

where $\Theta_3(\epsilon, \lambda, d)$ has an order $O(\lambda\epsilon^2)$ and order $O((\epsilon\lambda)^2 \log(\lambda))$ for its derivative.

Taking into account the final expressions (6.17), (6.21) and (6.22) for each part of the energy, we can conclude that

$$\begin{aligned} J_\epsilon(U) = {} & 4\pi \left(\sum_{j=1}^{m} H_\lambda(\xi_j, \xi_j) + \sum_{i \neq j} G_\lambda(\xi_i, \xi_j) \right) - 16m\pi - 16\pi \sum_{j=1}^{m} \log(\mu_j \epsilon) \\ & + \sum_{j=1}^{m} \int\limits_{\partial\Omega} U_j \frac{\partial H_j}{\partial \nu} - \frac{1}{2} \int\limits_{\partial\Omega} H_\lambda(x, \xi_j) \frac{\partial H_\lambda(x, \xi_j)}{\partial \nu} + \frac{1}{2} \int\limits_{\partial\Omega} G_\lambda(x, \xi_j) \frac{\partial \Gamma(x, \xi_j)}{\partial \nu} \\ & - \frac{1}{2} \int\limits_{\partial\Omega} \Gamma \frac{\partial H_\lambda(x, \xi_j)}{\partial \nu} + \frac{\lambda}{2} \sum_{j=1}^{m} \int\limits_{\partial\Omega} G_\lambda^2(x, \xi_j) + \tilde{\Theta}(\epsilon, \lambda, d) \end{aligned} \tag{6.23}$$

where the error term $\tilde{\Theta}$ is $O(\epsilon)$ and $O(\epsilon^2\lambda^3)$ for its derivative. This term includes all the error terms Θ_i, $i = 1, 2, 3$. Using the definition of the regular part of the Green function and the Robin boundary condition, we can write

$$J_\epsilon(U) = 4\pi\left(\sum_{j=1}^m H_\lambda(\xi_j,\xi_j) + \sum_{i\neq j} G_\lambda(\xi_i,\xi_j)\right) - 16m\pi - 16\pi\sum_{j=1}^m \log(\mu_j\epsilon)$$
$$+\sum_{j=1}^m \int_{\partial\Omega} U_j\frac{\partial H_j}{\partial\nu} - \int_{\partial\Omega} G_\lambda(x,\xi_j)\frac{\partial H_\lambda(x,\xi_j)}{\partial\nu} + \tilde{\Theta}(\epsilon,\lambda,d).$$

To give the correct bound for the error term, we will need the following:

Lemma 6.3. *Under the assumptions* (2.2) *and* (2.3)*, for each* $j = 1,\dots,m$ *we have*

$$\int_{\partial\Omega} U_j\frac{\partial H_j}{\partial\nu} - \int_{\partial\Omega} G_\lambda(x,\xi_j)\frac{\partial H_\lambda(x,\xi_j)}{\partial\nu} = O\big(\epsilon^2\lambda\log(\lambda)\big) \tag{6.24}$$

and order $O((\epsilon\lambda)^2\log(\lambda))$ *for its derivative.*

The proof of this lemma is postponed to Appendix A. Using (6.24), we have

$$J_\epsilon(U) = 4\pi\left(\sum_{j=1}^m H_\lambda(\xi_j,\xi_j) + \sum_{i\neq j} G_\lambda(\xi_i,\xi_j)\right) - 16m\pi - 16\pi\sum_{j=1}^m \log(\mu_j\epsilon) + \Theta(\epsilon,\lambda,d)$$

with $\Theta(\epsilon,\lambda,d) = O(\epsilon^2\lambda\log(\lambda))$ and $O(\epsilon^2\lambda^3)$ for its derivative.

The definition of the numbers μ_j given in (2.10) allows us to conclude the following expression for the energy of the ansatz:

$$J_\epsilon(U) = -16m\pi - 16m\pi\log(\epsilon) + 8m\pi\log(8) - 4\pi\varphi_m(\xi_1,\dots,\xi_m) + \Theta(\epsilon,\lambda,d)$$

where $\varphi_m(\xi_1,\dots,\xi_m)$ is the function given by (6.1). □

7. Proof of the theorems

To prove the main theorems in this paper it is useful to recall here a few properties of the Green function G_λ, and its regular part H_λ (cf. (1.3), (1.4)). The proof of these estimates can be found in [8].

We have the following expression for $H_\lambda(\xi,\xi)$,

$$H_\lambda(\xi,\xi) = h_\lambda\big(\lambda d(\xi)\big) + O\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda\to+\infty \tag{7.1}$$

where $\xi\in\Omega$ has to satisfy $\lambda d(\xi)\in(M_1,M_2)$, and the function $h_\lambda(\theta)$ has the explicit representation

$$h_\lambda(\theta) = -\log(\lambda) - \log(2\theta) + 2\int_0^\infty e^{-t}\log(2\theta+t)\,dt.$$

This implies that the function $h_\lambda(\theta)$ has the following properties:

$$h_\lambda(\theta) = -\log(\theta) + O(1) \quad \text{as } \theta\to 0,$$
$$h_\lambda(\theta) = \log(\theta) + O(1) \quad \text{as } \theta\to+\infty.$$

Moreover, it is known that $h_\lambda(\theta)$ has a unique non-degenerate minimum $\theta_0 \in (0, +\infty)$ and we have $h_\lambda(\theta_0) = -\log(\lambda) + O(1)$. It can be seen from the formula for $h_\lambda(\theta)$ that the location of the minimum does not depend on λ.

Proof of Theorem 1.1. For the case $m = 1$, we look for critical points $\xi \in \Omega$ of the function

$$F(\xi) = -4\pi H_\lambda(\xi, \xi) + 8\pi \log(8) - 16\pi - 16\pi \log(\epsilon) + \Theta(\epsilon, \lambda, d), \tag{7.2}$$

with $\Theta(\epsilon, \lambda, d) = O(\frac{\epsilon^2}{\lambda d^3})$ and $d = dist(\xi, \partial\Omega)$. Finding critical points of F is equivalent to finding critical points of

$$\tilde{F}(\xi) = H_\lambda(\xi, \xi) + \tilde{\Theta}(\epsilon, \lambda, d),$$

where $\tilde{\Theta} = -\frac{1}{4\pi}\Theta$. Under the assumption $\lambda\epsilon \leqslant \epsilon_0$ we see that the *error* $\tilde{\Theta}$ can be made arbitrarily small by taking $\epsilon_0 > 0$ small, since $\Theta = O(\epsilon^2 \lambda \log(\lambda))$ uniformly in Ω.

Let

$$S^* = \left\{\xi^* \in \mathcal{U}\colon d(\xi^*) = \frac{\theta_0}{\lambda}\right\}$$

and for $0 < M$ to be fixed, consider the set

$$\mathcal{U} = \left\{\xi \in \Omega\colon -M\lambda^{-3/2} < d(\xi) - \frac{\theta_0}{\lambda} < M\lambda^{-3/2}\right\}.$$

Recall that for each $\xi \in \Omega$ sufficiently close to $\partial\Omega$, we define $\hat{\xi}$ the unique point in $\partial\Omega$ such that $|\xi - \hat{\xi}| = d(\xi)$. We can take M so that for each $x \in \partial\Omega$ there exists $\xi_x^* \in \mathcal{U}$ such that $\widehat{\xi_x^*} = x$ and $\lambda d(\xi_x^*) = \theta_0$.

Using that θ_0 is a non-degenerate critical point of h_λ, it is possible to take $0 < M$ large such that

$$\inf_{\partial\mathcal{U}} h_\lambda\big(\lambda d(\xi)\big) > \sup_{S^*} h_\lambda\big(\lambda d(\xi)\big) = h_\lambda(\theta_0).$$

Using the separation condition (2.3) and (7.1), taking λ_0 large enough and ϵ_0 sufficiently small we have

$$\inf_{\partial\mathcal{U}} \tilde{F} > \sup_{S^*} \tilde{F}, \tag{7.3}$$

for $\lambda \geqslant \lambda_0$, $\epsilon > 0$ satisfying $\lambda\epsilon \leqslant \epsilon_0$. This implies that the function $\tilde{F}$ has a minimum $\xi_1 \in \mathcal{U}$ which corresponds to a first critical point to F.

We now argue that $\tilde{F}$ has a second critical point in $\mathcal{U}$. For each $x \in \partial\Omega$ consider the set

$$Q_x = \{\xi \in \mathcal{U}\colon \hat{\xi} = x\}.$$

If for all $x \in \partial\Omega$,

$$\inf_{\xi \in Q_x} \tilde{F}(\xi) = \min_{\mathcal{U}} \tilde{F}$$

then actually $\tilde{F}$ has infinitely many critical points in $\mathcal{U}$, and we are done. So assume that there is $x \in \partial\Omega$ such that

$$\inf_{\xi \in Q_x} \tilde{F}(\xi) > \min_{\mathcal{U}} \tilde{F}. \tag{7.4}$$

Let ∂Q_x denote the relative boundary of Q_x. By (7.3) we have

$$\inf_{\partial Q_x} \tilde{F} > \sup_{S^*} \tilde{F}.$$

But S^* and ∂Q_x link in $\mathcal{U}$, so, if we define the set

$$\mathcal{P} = \left\{ p \in C^0(\overline{Q}_x; \overline{\mathcal{U}}):\ p|_{\partial Q_x} = Id_{\partial Q_x} \right\},$$

then, the real number

$$\beta = \sup_{p \in \mathcal{P}} \inf_{\xi \in Q_x} \tilde{F}(\xi)$$

is a critical value of $\tilde{F}$ which is different from $\tilde{F}(\xi_1)$ in virtue of (7.4). This implies the existence of a second critical point ξ_2 in $\mathcal{U}$ of F which is different from ξ_1. $\square$

To prove Theorem 1.2 will need the following definitions and computations.
Given $M > 0$ and $\delta > 0$ define

$$\Omega_0 = \left\{ (\xi_1, \ldots, \xi_m) \in \Omega^m:\ \lambda d(\xi_i) \in \left(\theta_0 - M\lambda^{-1/2}, \theta_0 + M\lambda^{-1/2}\right),\ i = 1, \ldots, m; \\ |\xi_i - \xi_j| > \delta_0,\ i \neq j \right\}.$$

We will sometimes write $\Omega_0(M, \delta)$ to make the dependence of this definition on M, δ explicit. Then Ω_0 is a smooth manifold with boundary $\partial\Omega_0$.

Lemma 7.1. *There is $c_0 > 0$, $\delta_0 > 0$, $M_0 > 0$, λ_0 such that for $0 < \delta \leqslant \delta_0$, $M \geqslant M_0$ one has*

$$\inf_{\partial\Omega_0} \varphi_m(\xi) \geqslant m h_\lambda(\theta_0) + \frac{c_0}{\lambda} \min\left(M^2, 1/\delta\right)$$

for all $\lambda \geqslant \lambda_0$.

Proof. If $\xi = (\xi_1, \ldots, \xi_m) \in \partial\Omega_0$ then either $\lambda d(\xi_i) = \theta_0 - M\lambda^{-1/2}$, or $\lambda d(\xi_i) = \theta_0 + M\lambda^{-1/2}$ for some i, or $|\xi_i - \xi_j| = \delta$ for some $i \neq j$. If $\lambda d(\xi_i) = \theta_0 - M\lambda^{-1/2}$, then by (7.1)

$$\begin{aligned} \varphi_m(\xi) &= \sum_{l=1}^{m} H_\lambda(\xi_l, \xi_l) + \sum_{l \neq j} G_\lambda(\xi_l, \xi_j) \\ &\geqslant h_\lambda\left(\theta_0 - M\lambda^{-1/2}\right) + (m-1) h_\lambda(\theta_0) \\ &\geqslant m h_\lambda(\theta_0) + cM^2\lambda^{-1} \end{aligned}$$

where we have used the positivity of the Green function. This implies, choosing $M > 0$ large

$$\varphi_m(\xi) \geqslant mh_\lambda(\theta_0) + \frac{c_0 M^2}{\lambda}$$

(for some fixed value of $c_0 > 0$). We get a similar conclusion if $\lambda d(\xi_i) = \theta_0 + M\lambda^{-1/2}$.

So let us consider the case $|\xi_i - \xi_j| = \delta$ for some $i \neq j$. Using expansion (7.1) we obtain in this case

$$\begin{aligned}\varphi_m(\xi) &= \sum_{i=1}^{m} H_\lambda(\xi_i, \xi_i) + \sum_{i \neq j} G_\lambda(\xi_i, \xi_j) \\ &\geqslant mh_\lambda(\theta_0) + O\left(\frac{1}{\lambda}\right) + \sum_{i \neq j} G(\xi_i, \xi_j).\end{aligned}$$

In this case, we use the following claim: For points ξ_i, ξ_j satisfy $|\xi_i - \xi_j| = \delta$ and the separation condition (2.3), then there exists $c_0 > 0$ such that

$$G(\xi_i, \xi_j) \geqslant \frac{c_0}{\delta\lambda}$$

for some δ fixed small and all λ sufficiently large. This claim concludes the proof.

To prove the claim, we consider after a rotation and translation $\xi_j = (0, d(\xi_j))$, the projection of ξ_j to $\partial\Omega$ is the origin and the outer normal vector to the boundary at the origin is $(0, -1)$.

Denote by $\hat{G}_\lambda$ the Green function in the half-space $\{(x, y)\colon\ y > 0\}$ associated to the Robin boundary condition. Fix $\bar{\delta} > 0$ small. It is proven in [8] that

$$\|G_\lambda - \hat{G}_\lambda\|_{L^\infty(B(\xi_j, \bar{\delta}) \cap \Omega)} \leqslant C_{\bar{\delta}} \frac{1}{\lambda}.$$

We recall that

$$\hat{G}_\lambda(\xi_i, \xi_j) = \Gamma\left(|\xi_i - \xi_j|\right) - \Gamma\left(|\xi_i + \xi_j|\right) - 2 \int_{-\infty}^{0} e^{\lambda s} \frac{\partial \Gamma}{\partial x_2}(\xi_i + \xi_j - e_2 s)\, ds.$$

By a computation we get

$$-\int_{-\infty}^{0} e^{\lambda s} \frac{\partial \Gamma}{\partial x_2}(\xi_i + \xi_j - e_2 s)\, ds \geqslant \frac{c}{\delta\lambda}$$

for some $c > 0$. Also, for $|\xi_i - \xi_j| = \delta$ and $dist(\xi_i, \partial\Omega) = O(1/\lambda)$, since $\xi_i - \xi_j$ is almost perpendicular to $2\xi_j$, we get

$$\left|\Gamma\left(|\xi_i - \xi_j|\right) - \Gamma\left(|\xi_i + \xi_j|\right)\right| \leqslant \frac{C}{\lambda}.$$

Therefore

$$G_\lambda(\xi_i, \xi_j) \geqslant C \frac{1}{\delta\lambda}$$

where $C > 0$ is a universal constant. Choosing $0 < \delta < \bar{\delta}$ small independent of λ we have the conclusion of the claim for λ large enough. □

We will apply the Ljusternik–Schnirelmann theory, see [6], to estimate the number of critical points of the functional $\tilde{F}$ on Ω_0. Let us recall that the Ljusternik–Schnirelmann category of a closed subset A of Ω_0 relative to Ω_0, which we write as $cat_{\Omega_0}(A)$, is the smallest integer ℓ such that A can be covered by ℓ closed contractible sets.

It is easy to see that $cat_{\Omega_0}(\Omega_0)$ is at least 2, which is equivalent to say that Ω_0 is not contractible. For completeness, we give a short proof. It is sufficient to construct continuous functions

$$f : S^1 \to \Omega_0, \qquad P : \Omega_0 \to S^1$$

such that $P \circ f : S^1 \to S^1$ has nonzero winding number. Let Γ denote a connected component of $\partial\Omega$ and $\gamma : S^1 \to \Gamma$ be a parametrization Γ, i.e., a smooth diffeomorphism. Set

$$g(x) = x - \frac{\theta_0}{\lambda}\nu(x), \quad x \in \Gamma,$$

where ν is the exterior unit normal vector of $\partial\Omega$. We represent $S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$. Let $f : S^1 \to \Omega_0$ be the continuous function defined by

$$f(z) = \left(g\left(\gamma(z)\right), g\left(\gamma\left(ze^{i2\pi\frac{1}{m}}\right)\right), \ldots, g\left(\gamma\left(ze^{i2\pi\frac{m-1}{m}}\right)\right)\right), \quad z \in S^1. \tag{7.5}$$

Next we define P as follows. For $\xi \in \Omega$ close to $\partial\Omega$ there is a unique closest point $\hat{\xi} \in \partial\Omega$. In particular, for $(\xi_1, \ldots, \xi_m) \in \Omega_0$, $(\hat{\xi}_1, \ldots, \hat{\xi}_m) \in \Gamma^m$. Let

$$P(\xi_1, \ldots, \xi_m) = \prod_{j=1}^m \gamma^{-1}(\hat{\xi}_j) \in S^1.$$

Note that $P : \Omega_0 \to S^1$ is continuous and

$$P \circ f(z) = z^m e^{\pi i(m-1)}, \quad z \in S^1,$$

so $P \circ f$ has nonzero winding number.

Lemma 7.2. *Let $M > 0$ and $\delta > 0$ small. There is a closed subset $A \subset \Omega_0$ with $cat_{\Omega_0}(A) \geqslant 2$ such that*

$$\sup_{\xi \in A} \varphi_m(\xi) \leqslant mh_\lambda(\theta_0) + \frac{C}{\lambda}$$

for some constant C independent of λ.

Proof. Let f be defined as in (7.5) and let

$$A = \left\{f(z)\colon z \in S^1\right\}.$$

The same argument showing that $cat_{\Omega_0}(\Omega_0) \geqslant 2$ gives that $cat_{\Omega_0}(A) \geqslant 2$. By construction of f, if $\xi = (\xi_1, \ldots, \xi_m) \in A$ then the m coordinates of ξ are uniformly separated, independently of δ and λ. This implies that

$$\varphi_m(\xi) \leqslant mh_\lambda(\theta_0) + \frac{C}{\lambda}$$

for $C > 0$, independent of δ and λ. $\square$

Proof of Theorem 1.2. We take Ω_0 with an initial choice of $\delta > 0$ small and $M > 0$ large so that Ω_0 is not empty.

To prove the theorem we need to show the existence of critical points of $F(\xi)$ where $\xi = (\xi_1, \dots, \xi_m) \in \Omega^m$ with ξ_i satisfying (2.2), (2.3). Finding critical points of F is equivalent to finding critical points of

$$\tilde{F}(\xi) = -\frac{1}{4\pi}\big(F(\xi) + 16m\pi + 16m\pi\log(\epsilon) - 8m\pi\log(8)\big) + m\log(\lambda).$$

By Lemma 5.2 and Proposition 6.1

$$\tilde{F}(\xi) = \varphi_m(\xi) + m\log(\lambda) + \Theta_{\epsilon,\lambda}(\xi),$$

where $\Theta_{\epsilon,\lambda}$ satisfies $|\Theta_{\epsilon,\lambda}(\xi)| \leqslant C_{\delta,M}\epsilon^2\lambda\log(\lambda)$ for $\xi \in \Omega_0$.

Define

$$\mathcal{A}_k = \big\{A \subset \Omega_0\colon\ A \text{ is closed and } cat_{\Omega_0}(A) \geqslant k\big\}, \quad k \in \mathbb{N},$$

and

$$c_k = \inf_{A\in\mathcal{A}_k}\sup_{\xi\in A}\tilde{F}(\xi).$$

Since $\mathcal{A}_{k+1} \subset \mathcal{A}_k$, is immediate that $c_k \leqslant c_{k+1}$, for all k. Moreover, we have

$$c_1 = \inf_{\xi\in\Omega_0}\tilde{F}(\xi)$$

and $c_1 \leqslant c_2 < +\infty$. Note that

$$\begin{aligned} c_1 &\leqslant \inf\big\{\tilde{F}(\xi)\colon\ \xi = (\xi_1, \dots, \xi_m),\ \xi_i \in S^*,\ |\xi_i - \xi_j| \geqslant \delta_0\big\} \\ &\leqslant mh_\lambda(\theta_0) + \frac{C}{\lambda} + C\epsilon^2\lambda\log\lambda \end{aligned}$$

where $\delta_0 >$ is fixed small and C es independent of M and δ.

Now choose $\tilde{M} > M$ and $0 < \tilde{\delta} < \delta$ and set $\tilde{\Omega}_0 = \Omega_0(\tilde{M}, \tilde{\delta})$. Using Lemma 7.1 we can achieve

$$\inf_{\partial\tilde{\Omega}_0}\tilde{F} > c_1$$

for $\lambda \geqslant \lambda_0$ and $\epsilon^2\lambda\log\lambda \leqslant \epsilon_0$. Define now

$$\begin{aligned} \tilde{c}_1 &= \inf_{\xi\in\tilde{\Omega}_0}\tilde{F}(\xi), \\ \tilde{c}_2 &= \inf_{A\in\tilde{\mathcal{A}}_2}\sup_{\xi\in A}\tilde{F}(\xi) \end{aligned}$$

where

$$\tilde{\mathcal{A}}_2 = \big\{A \subset \tilde{\Omega}_0\colon\ \text{is closed and } cat_{\Omega_0}(A) \geqslant 2\big\}.$$

Observe that $\Omega_0 \subset \tilde{\Omega}_0$ and therefore $\tilde{c}_1 = c_1$ and $\tilde{c}_2 \leqslant c_2$. Taking $\tilde{M}$ larger and $\tilde{\delta}$ smaller if necessary, using Lemma 7.1 we have

$$\sup_A \tilde{F} < \inf_{\partial\tilde{\Omega}_0} \tilde{F}$$

where the set A is the set found in Lemma 7.2. This implies the values of $\tilde{F}$ on $\partial\tilde{\Omega}_0$ are strictly larger than $\tilde{c}_2$, and using Ljusternik–Schnirelmann theory we deduce that $\tilde{c}_2$ is a critical value of $\tilde{F}$. If $\tilde{c}_2 > \tilde{c}_1$, then we obtain immediately 2 different critical points of $\tilde{F}$ corresponding to 2 different solutions. If $\tilde{c}_2 = \tilde{c}_1$, then the set of critical points of $\tilde{F}$ with value $\tilde{c}_2 = \tilde{c}_1$ has category at least 2. In this case we conclude that there are infinitely many critical points for $\tilde{F}$ in $\tilde{\Omega}_0$. Since there is a finite number of permutations, we obtain the existence of infinitely different solutions in this situation. □

Remark 7.3. We believe that the assumption on ϵ and λ in Theorem 1.2 can be sharpened to $\lambda\epsilon$ small. This slight improvement can be accomplished by estimating more carefully the error in Lemma 6.3, where it seems possible to improve the error to $\epsilon^2\lambda$.

Proof of Theorem 1.3. As in the proof of Theorem 1.1, to find critical points of the function F we use the expansion (7.2), recalling the error term $\tilde{\Theta}(\epsilon,\lambda,d)$ satisfies

$$\left|\theta(\epsilon,\lambda,d)\right| \leqslant C\epsilon^2\lambda\log(\lambda), \qquad \left|\nabla\Theta(\epsilon,\lambda,d)\right| \leqslant C\epsilon^2\lambda^3.$$

Let $x_0 \in \partial\Omega$ a non-degenerate critical point of the mean curvature κ. For $\gamma \in (0,1)$, we have the following expressions for the derivative of the function $R_\lambda(\xi) := H_\lambda(\xi,\xi)$, see [8]:

$$\nabla_T R_\lambda(x) = \lambda^{-1}\nabla\kappa(\hat{x})\mathrm{v}\big(\lambda d(x)\big) + O\big(\lambda^{-(1+\gamma)}\big), \tag{7.6}$$

$$\big\langle\nabla R_\lambda(x), \nu(\hat{x})\big\rangle = -\lambda h'_\lambda\big(\lambda d(x)\big) - \kappa(\hat{x})\mathrm{v}\big(\lambda d(x)\big) + O\big(\lambda^{-\gamma}\big), \tag{7.7}$$

which hold uniformly for $m \leqslant \lambda d(x) \leqslant M$, for some constants $m, M > 0$. Here, $\hat{x}$ is the (unique) projection of the point x over $\partial\Omega$, ∇_T is the tangential derivative and $\mathrm{v} : (0,+\infty) \to \mathbb{R}$ is the function given in (1.7).

Since x_0 is a non-degenerate critical point of κ, then there exists $\sigma, c > 0$ such that

$$\big|\nabla\kappa(\hat{x})\big| \geqslant c|\hat{x}-x_0|, \quad \forall|\hat{x}-x_0| \leqslant \sigma. \tag{7.8}$$

On the other hand, the function $h_\lambda(\theta)$ has a unique critical point $\theta_0 > 0$ which is non-degenerate. Taking c, σ smaller if it is necessary, we have

$$\big|h'_\lambda(\theta)\big| \geqslant c|\theta-\theta_0|, \quad \forall|\theta-\theta_0| \leqslant \sigma. \tag{7.9}$$

It is known that the function v is continuous and strictly negative, so we can consider σ such that

$$\inf_{\theta\in[\theta_0-\sigma,\theta_0+\sigma]}\big|\mathrm{v}(\theta)\big| > 0. \tag{7.10}$$

We assume $\sigma < \theta_0$ since $\theta_0 > 0$. Consider $0 < \beta < \gamma$ and define the compact set

$$\mathcal{K}_\lambda := \big\{x \in \Omega:\ \big|\lambda d(x) - \theta_0\big| \leqslant \sigma\lambda^{-1/2},\ |\hat{x}-x_0| \leqslant \lambda^{-\beta}\big\}.$$

Define the function

$$R_\lambda^0(x) = h_\lambda\big(\lambda d(x)\big) + \lambda^{-1}\kappa(\hat{x})\mathrm{v}\big(\lambda d(x)\big).$$

Note that this function has a critical point in the interior of $\mathcal{K}_\lambda$. Defining the function

$$\tilde{R}_\lambda(x) = R_\lambda(x) + \Theta(\epsilon, \lambda, d)$$

we can see that the function

$$R_\lambda^t(x) = t\tilde{R}_\lambda(x) + (1-t)R_\lambda^0(x), \quad t \in [0,1]$$

is a homotopy between $\tilde{R}_\lambda$ and R_λ^0. Since

$$\big|\nabla R_\lambda^t(x)\big|^2 = \big|\nabla_T R_\lambda^t(x)\big|^2 + \big\langle \nabla R_\lambda^t(x), \nu(\hat{x})\big\rangle^2$$

then, if $|\lambda d(x) - \theta_0| = \sigma\lambda^{-1/2}$, using (7.7) and (7.9) and taking λ large enough we conclude

$$\big|\big\langle \nabla R_\lambda^t(x), \nu(\hat{x})\big\rangle\big| \geqslant c'\lambda^{1/2} + O\big(\epsilon^2\lambda^3\big). \tag{7.11}$$

If $|\hat{x} - x_0| = \lambda^{-\beta}$, then using (7.6), (7.8) and (7.10), taking λ large enough we conclude

$$\big|\nabla R_\lambda^t(x)\big| \geqslant c'\lambda^{-(1+\beta)} + O\big(\epsilon^2\lambda^3\big) \tag{7.12}$$

with $0 < \beta < \gamma$. This implies that if we set $\lambda < \epsilon^{-\alpha}$ with $\alpha < \frac{1}{2}$, then we can choose $0 < \beta$ suitably small (for example, $\beta < \frac{2-4\alpha}{\alpha}$) we conclude that the term $|\nabla R_\lambda^t(x)|$ in (7.12) remains uniformly positive if $\epsilon\lambda^{-\alpha} < \epsilon_0$ for ϵ_0 is sufficiently small and $\lambda > \lambda_0$, with λ_0 large enough.

Finally, (7.11) and (7.12) imply that we can use degree theory to conclude the existence of a critical point of $\tilde{R}_\lambda$ under the conditions over ϵ and λ given above. □

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Appendix A. Proof of Lemmas 6.2 and 6.3

Let $H^+ = \{(x_1, x_2) \in \mathbb{R}^2\colon\ x_2 > 0\}$ and $\partial H^+ = \{(x_1, 0)\colon\ x_1 \in \mathbb{R}\}$. For $g : \mathbb{R} \to \mathbb{R}$, consider v solution of the problem

$$\begin{cases} -\Delta v = 0, & \text{in } H_+, \\ \dfrac{\partial v}{\partial \nu} + av = g, & \text{on } \partial H_+, \end{cases}$$

for $a > 0$ fixed. If g has some decay at infinity, the solution of this problem is given by

$$v(x_1, x_2) = \int_{-\infty}^{+\infty} k_a(x_1 - y, x_2)g(y)\,dy, \quad \forall (x_1, x_2) \in H^+, \tag{A.1}$$

where

$$k_a(x_1,x_2)=\frac{1}{\pi}\int_0^{+\infty}\frac{e^{-at}(x_2+t)}{x_1^2+(x_2+t)^2}\,dt,\quad \forall (x_1,x_2)\in H^+,$$

see [13,8,7].

Consider $j\in\{1,\dots,m\}$ fixed. After a rotation and translation we can suppose that $\xi_j=(0,d_j)$ whose projection on $\partial\Omega$ is the origin. For later purposes, we denote $\xi_j^*=(0,-d_j)$ the reflection of ξ_j across ∂H_+. Let $\delta>0$ be fixed and $\mathcal{U}$ be a neighborhood of the origin. Consider a conformal mapping

$$F: B(0,\delta)\cap\overline{\Omega}\to\mathcal{U}\cap\overline{H}_+. \tag{A.2}$$

The function F can be taken so that $F(0)=0$ and $F'(0)$ is the identity.

In addition, consider a smooth cut-off function

$$\begin{cases}\eta:\mathbb{R}^2\to\mathbb{R},\\ \eta(x)=1 \quad \text{if } |x-\xi_j|\leqslant\dfrac{\delta}{2},\\ \eta(x)=0\quad \text{if } |x-\xi_j|>\delta.\end{cases} \tag{A.3}$$

Particular properties for this cut-off function will be stated later in each case.

Recall $\rho=\epsilon/\sqrt{\lambda}$. Let $h_j=\tilde{H}_j(x)-H_\lambda(x,\xi_j)$, which solves the equation

$$\begin{cases}-\Delta h_j=0, & \text{in } \Omega,\\ \dfrac{\partial h_j}{\partial\nu}+\lambda h_j=O\left(\dfrac{\rho^2}{|x-\xi_j|^3}\right)+O\left(\dfrac{\lambda\rho^2}{|x-\xi_j|^2}\right), & \text{on } \partial\Omega.\end{cases} \tag{A.4}$$

For the proof of Lemma 6.2 we will need the following lemma.

Lemma A.1. *With the definition of F and η given in* (A.2) *and* (A.3) *respectively, we have*

$$\left|h_j(x)\right|\leqslant C_1\lambda\rho^2+C_2\lambda^2\rho^2\eta(x)\left(\frac{1+\lambda(F(x))_2}{1+\lambda^2((F(x))_1)^2}\right). \tag{A.5}$$

Proof. We change variables, considering the set $\lambda\Omega$ and writing $y\in\lambda\Omega$ as $y=\lambda x$ with $x\in\Omega$. Define $\tilde{h}_j(y)=h_j(y/\lambda)$ for y in $\lambda\Omega$, which satisfies

$$\begin{cases}-\Delta\tilde{h}_j=0, & \text{in } \lambda\Omega,\\ \dfrac{\partial\tilde{h}_j}{\partial\nu}+\tilde{h}_j=O\left(\dfrac{\lambda^2\rho^2}{|y-\lambda\xi_j|^3}\right)+O\left(\dfrac{\lambda^2\rho^2}{|y-\lambda\xi_j|^2}\right), & \text{on } \partial(\lambda\Omega).\end{cases} \tag{A.6}$$

Consider v_1 a solution to

$$\begin{cases}-\Delta v_1=0, & \text{in } H_+,\\ \dfrac{\partial v_1}{\partial\nu}+v_1=\dfrac{\lambda^2\rho^2}{1+y_1^2}, & \text{on } \partial H_+.\end{cases}$$

Using the explicit expression (A.1) for this case, it is possible to get the following bound for v_1,

$$\left|v_1(y_1, y_2)\right| \leqslant C\lambda^2\rho^2 \begin{cases} \frac{1}{1+y_2} & \text{if } |y_1| < y_2, \\ \frac{1}{1+|y_1|^2} + \frac{1+y_2}{(1+|y_1|)^2} & \text{if } |y_1| \geqslant y_2. \end{cases}$$

In particular, we have $|v_1(y_1, 0)| \leqslant C\frac{\lambda^2\rho^2}{1+y_1^2}$. Moreover, we have

$$\left|\nabla v_1(y_1, y_2)\right| \leqslant C\lambda^2\rho^2 \begin{cases} \frac{1}{y_2(1+y_2)} & \text{if } |y_1| < y_2, \\ (\frac{1+y_2}{(1+|y_1|)^3} + \frac{1}{1+y_1^2})1_{\{y_2>1\}} + (\frac{1+y_2}{1+y_1^2})1_{\{y_2<1\}} & \text{if } |y_1| \geqslant y_2 \end{cases}$$

so

$$\nabla v_1 = O\left(\frac{\lambda^2\rho^2}{|(y_1, y_2)|^2}\right) \quad \text{if } \left|(y_1, y_2)\right| > 1. \tag{A.7}$$

Let $F_\lambda(y) = \lambda F(\frac{y}{\lambda})$, with F as in (A.2). This function F_λ is defined on $B(0, \lambda\delta) \cap \overline{\lambda\Omega}$. Denote by $\mu_\lambda(y)$ the conformal factor of F_λ in y, which has an expansion given by $\mu_\lambda(y) = 1 + O(|\frac{y}{\lambda}|)$. Consider $\tilde{Y}$ the solution to

$$\begin{cases} -\Delta\tilde{Y} = \dfrac{\rho^2}{\lambda}, & \text{in } \lambda\Omega, \\ \dfrac{\partial\tilde{Y}}{\partial\nu} + \tilde{Y} = \lambda\rho^2, & \text{in } \partial(\lambda\Omega) \end{cases}$$

and $\tilde{\eta}(y) = \eta(\frac{y}{\lambda})$ for $y \in \lambda\Omega$, η as in (A.3). Then we set

$$\tilde{w} = C_1\tilde{Y} + C_2\tilde{\eta}v_1\big(F_\lambda(y)\big), \tag{A.8}$$

where $C_1 > 0$ and $C_2 > 0$ are constants to be fixed later on. We have for $y \in \partial(\lambda\Omega)$,

$$\begin{aligned} \frac{\partial\tilde{w}}{\partial\nu} + \tilde{w} &= C_1\lambda\rho^2 + C_2\left[\tilde{\eta}\big(\nabla v_1\big(F_\lambda(y)\big)\mu_\lambda(y)\cdot\nu + v_1\big(F_\lambda(y)\big)\big) + \frac{\partial\tilde{\eta}}{\partial\nu}v_1\big(F_\lambda(y)\big)\right] \\ &= C_1\lambda\rho^2 + C_2\left[\tilde{\eta}\left(-\frac{\partial v_1}{\partial y_2}\big(F_\lambda(y)\big) + v_1\big(F_\lambda(y)\big)\right) + \tilde{\eta}\left|\frac{\partial v_1}{\partial y_2}\big(F_\lambda(y)\big)\right| O\left(\frac{|y|}{\lambda}\right)\right. \\ &\quad \left. + O\left(\frac{\rho^2}{\lambda}\right)\right]. \end{aligned}$$

Using the estimates for v_1, we can conclude

$$\frac{\partial\tilde{w}}{\partial\nu} + \tilde{w} \geqslant \frac{\partial\tilde{h}_j}{\partial\nu} + \tilde{h}_j \quad \text{on } \partial(\lambda\Omega)$$

if we take C_1, C_2 large. On the other hand

$$-\Delta\tilde{w} = C_1\frac{\rho^2}{\lambda} - C_2\left(\Delta\tilde{\eta}(y)v_1\big(F_\lambda(y)\big) + 2\nabla\tilde{\eta}(y)\nabla v_1\big(F_\lambda(y)\big)\cdot F'\left(\frac{y}{\lambda}\right)\right), \quad y \in \lambda\Omega.$$

But $|F_\lambda(y)| = O(|y|)$ and $F'(\frac{y}{\lambda})$ is bounded in $B(0,\lambda\delta)\cap\lambda\Omega$, so, using (A.7), we have $-\Delta\tilde{w} \geqslant 0$ in $\lambda\Omega$. This implies that $\tilde{h}_j \leqslant \tilde{w}$ in $\lambda\Omega$. A similar argument tells us that $-\tilde{w} \leqslant \tilde{h}_j$ in $\lambda\Omega$. Then, we get for $y\in\lambda\Omega$,

$$\left|\tilde{h}_j(y)\right| \leqslant C_1\lambda\rho^2 + C_2\lambda^2\rho^2\tilde{\eta}(y)\left(\frac{1+(F_\lambda(y))_2}{1+((F_\lambda(y))_1)^2} + \frac{1}{1+((F_\lambda(y))_1)^2}\right). \tag{A.9}$$

Returning to the x variables we have the statement of the lemma. □

Proof of Lemma 6.2. By definition of h_j,

$$\begin{aligned}\int_{\partial\Omega} H_j(x)\frac{\partial H_j(x)}{\partial\nu}\,d\sigma(x) &= \int_{\partial\Omega} H_\lambda(x,\xi_j)\frac{\partial H_\lambda(x,\xi_j)}{\partial\nu}\,d\sigma(x)\\ &\quad + 2\int_{\partial\Omega} H_\lambda(x,\xi_j)\frac{\partial h_j(x)}{\partial\nu}\,d\sigma(x) + O\big((\lambda\rho)^4\big).\end{aligned}$$

We will need to estimate the middle term of the right-hand side of the last equation. For δ small, we have the following expansion of $H_\lambda(x,\xi_j)$ for $x\in\overline{\Omega}\cap B(0,\delta)$, see [8]

$$H_\lambda(x,\xi_j) = O\left(\frac{1}{\lambda}\right) + \Gamma\big(x-\xi_j^*\big) - 2\lambda\int_{-\infty}^{0} e^{\lambda s}\Gamma\big(x-\big(\xi_j^*+se_2\big)\big)\,ds,$$

where the $O(\frac{1}{\lambda})$ term in the last equation is in the uniform sense in $\overline{\Omega}\cap B(0,\delta)$. Using this estimate for $H(x,\xi_j)$ lead us to get

$$\begin{aligned}&\int_{\partial\Omega} H_\lambda(x,\xi_j)\frac{\partial h_j(x)}{\partial\nu}\,dx\\
&= O\big(\lambda\rho^2\big) + \int_{\partial\Omega\cap B(0,\delta)}\left(\Gamma\big(x-\xi_j^*\big) - 2\lambda\int_{-\infty}^{0} e^{\lambda s}\Gamma\big(x-\big(\xi_j^*+se_2\big)\big)\,ds\right)\frac{\partial h_j}{\partial\nu}\,dx\\
&= O\big(\lambda\rho^2\big) - \int_{\partial\Omega\cap B(0,\delta)}\Gamma\big(x-\xi_j^*\big)\frac{\partial h_j}{\partial\nu}\,dx\\
&\quad + 2\int_{\partial\Omega\cap B(0,\delta)}\left(\int_{-\infty}^{0} e^{t}\left(\Gamma\big(x-\xi_j^*\big) - \Gamma\left(x-\left(\xi_j^*+\frac{t}{\lambda}e_2\right)\right)\right)dt\right)\frac{\partial h_j}{\partial\nu}\,dx\\
&\leqslant O\big(\lambda\rho^2\big) + O\big(\lambda^3\rho^2\big)\int_{-\delta}^{\delta}\frac{\log((\lambda x_1)^2+1)}{1+(\lambda x_1)^2}\,dx_1 + O\big(\lambda^3\rho^2\big)\int_{-\delta}^{\delta}\frac{1}{1+(\lambda x_1)^2}\,dx_1,\end{aligned}$$

where, in the last inequality we have used the boundary condition satisfied by h_j, the estimate (A.9) and the properties of the function F. So, the estimate for the desired term is

$$\int_{\partial\Omega} H_\lambda(x,\xi_j)\frac{\partial h_j(x)}{\partial\nu}\,d\sigma(x) = O\big((\lambda\rho)^2\big).$$

Now we will see the estimate for the derivative of this error term. Differentiating with respect to ξ_j the error term (for simplicity, here ∂_ξ denote the derivative with respect to ξ_{j_k}, with $k = 1$ or 2):

$$\partial_\xi\left(\int_{\partial\Omega} \tilde{H}_j \frac{\partial \tilde{H}_j}{\partial\nu} - H_\lambda \frac{\partial H_\lambda}{\partial\nu}\right) = \int_{\partial\Omega} \partial_\xi H_\lambda \frac{\partial h_j}{\partial\nu} + \partial_\xi h_j \frac{\partial H_\lambda}{\partial\nu} + \partial_\xi h_j \frac{\partial h_j}{\partial\nu} + H_\lambda \frac{\partial(\partial_\xi h_j)}{\partial\nu} + h_j \frac{\partial(\partial_\xi H_\lambda)}{\partial\nu} + h_j \frac{\partial(\partial_\xi h_j)}{\partial\nu}.$$

Using the equation satisfied by H_λ, we can conclude

$$\begin{cases} -\Delta \partial_\xi H_\lambda(x,\xi_j) = 0, & x \in \Omega, \\ \dfrac{\partial \partial_\xi H_\lambda(x,\xi_j)}{\partial\nu} + \lambda \partial_\xi H_\lambda(x,\xi_j) = O\left(\dfrac{1}{|x-\xi_j|^2}\right) + \lambda O\left(\dfrac{1}{|x-\xi_j|}\right), & y \in \partial\Omega. \end{cases}$$

We put $Z_j = \partial_\xi H_\lambda$. Expanding the domain in λ, we can get

$$\begin{cases} -\Delta Z_j = 0, & y \in \lambda\Omega, \\ \dfrac{\partial Z_j}{\partial\nu} + Z_j = O\left(\dfrac{\lambda}{|y-\xi_j'|^2}\right) + O\left(\dfrac{\lambda}{|y-\xi_j'|}\right), & y \in \partial(\lambda\Omega). \end{cases}$$

We use the same method applied in Lemma A.1 on this function Z_j, but now considering $\tilde{Y}$ solution of the problem

$$\begin{cases} -\Delta \tilde{Y} = \dfrac{1}{\lambda}, & y \in \lambda\Omega, \\ \dfrac{\partial \tilde{Y}}{\partial\nu} + \tilde{Y} = 1, & y \in \partial(\lambda\Omega), \end{cases}$$

and v_1 solution of the problem

$$\begin{cases} -\Delta v_1 = 0, & \text{in } H_+, \\ \dfrac{\partial v_1}{\partial\nu} + v_1 = \dfrac{\lambda}{\sqrt{1+y_1^2}}, & \text{on } \partial H_+. \end{cases}$$

In this case, the function v_1 has the following bounds

$$\left|v_2(y_1,y_2)\right| \leqslant C\lambda \begin{cases} \frac{1}{1+|y_1|} + \frac{(1+y_2)\max(1,\log(|y_1|))}{(1+|y_1|)^2} & \text{if } |y_1| \geqslant y_2, \\ \frac{1}{1+|y_2|} & \text{if } |y_1| \leqslant y_2. \end{cases}$$

Using elliptic estimates we have $|\nabla v_1| \leqslant C\frac{1}{y_2}|v_1|$ in the set $y_2 > |y_1|$ and $|\nabla v_1| \leqslant O(1)$ in the set $y_2 \leqslant |y_1|$, $y_2 \geqslant \frac{1}{10}$. We will take η as before, but with the extra property that in the set $\{y \in \lambda\Omega\colon d(y,\partial(\lambda\Omega)) < \frac{1}{10}\}$, $(\nabla_N \eta)(\frac{y}{\lambda}) = 0$, where ∇_N is the derivative in the normal direction relative to the boundary. This can be done due to the regularity of the boundary and taking λ large enough if it is necessary.

Then, using maximum principle as in Lemma A.1 we have the function

$$w(y) = C_1 \tilde{Y}(y) + C_2 \eta\left(\frac{y}{\lambda}\right) v_1\left(\lambda F\left(\frac{y}{\lambda}\right)\right)$$

is a supersolution to Z_j in $\lambda\Omega$ and $-w$ is a subsolution to Z_j in $\lambda\Omega$, with F as in (A.2) and η as in (A.3), provided $C_1, C_2 > 0$ fixed appropriately.

In the same way, we will estimate $\partial_\xi h_j$ noting that this function satisfies the equation

$$\begin{cases} -\Delta(\partial_\xi h_j) = 0, & \text{in } \Omega, \\ \dfrac{\partial(\partial_\xi h_j)}{\partial \nu} + \lambda(\partial_\xi h_j) = O\left(\dfrac{\mu_j^2 \rho^2}{\lambda^2 |x-\xi_j|^4}\right) + O\left(\dfrac{\mu_j^2 \rho^2}{\lambda |x-\xi_j|^3}\right), & \text{on } \partial\Omega. \end{cases}$$

Using the same method as before, we conclude

$$\partial_\xi h_j \leqslant \lambda h_j.$$

With this, we can estimate at main order

$$\begin{aligned} \int_{\partial\Omega} \partial_\xi H_\lambda \frac{\partial h_j}{\partial \nu} &\leqslant \int_{\partial\Omega} \left(\lambda + \lambda \frac{1}{\sqrt{1+(\lambda x_1)^2}}\right)\left(O\left(\frac{\lambda \rho^2}{|x-\xi_j|^2}\right) + \lambda^2 \rho^2 + \lambda^3 \rho^2 \frac{1}{1+((F_\lambda(y))_1)^2}\right) \\ &\leqslant O\left((\epsilon\lambda)^2\right), \end{aligned}$$

$$\begin{aligned} \int_{\partial\Omega} \partial_\xi h_j \frac{\partial H_\lambda}{\partial \nu} &\leqslant \int_{\partial\Omega} \left(\lambda^2 \rho^2 + \lambda^3 \rho^2 \left(\frac{1}{1+((F_\lambda(y))_2)^2}\right)\right)\left(H_\lambda + \frac{1}{|x-\xi_j|} + \lambda \log(|x-\xi_j|)\right) \\ &\leqslant (\lambda\epsilon)^2 + (\lambda\epsilon)^2 \int_{\partial\Omega \cap B(\xi_j, \delta/2)} \left(\Gamma\left(x-\xi_j^*\right) - 2\lambda \int_{-\infty}^{0} e^{\lambda s} \Gamma\left(x - \left(\xi_j^* + s e_2\right)\right) ds\right) \\ &= O\left((\epsilon\lambda)^2 \log(\lambda)\right), \end{aligned}$$

$$\int_{\partial\Omega} H_\lambda \frac{\partial(\partial_\xi h_j)}{\partial \nu} \leqslant O\left((\epsilon\lambda)^2 \log(\lambda)\right),$$

$$\int_{\partial\Omega} h_j \frac{\partial(\partial_\xi H_\lambda)}{\partial \nu} \leqslant O\left((\epsilon\lambda)^2 \log(\lambda)\right).$$

This implies that the derivative of the error has an order $O((\epsilon\lambda)^2 \log(\lambda))$. □

Proof of Lemma 6.3. Let

$$I = \int_{\partial\Omega} U_j \frac{\partial H_j}{\partial \nu} - \int_{\partial\Omega} G_\lambda(x, \xi_j) \frac{\partial H_\lambda(x, \xi_j)}{\partial \nu}.$$

Using that $U_j = u_j + H_j$ and $G_\lambda = \Gamma + H_\lambda$ we have

$$I = \int_{\partial\Omega} u_j \frac{\partial H_j}{\partial \nu} - \int_{\partial\Omega} \Gamma \frac{\partial H_\lambda(x, \xi_j)}{\partial \nu} + O\left(\lambda \epsilon^2\right).$$

Using the definition of u_j:

$$\begin{aligned}
&\int_{\partial\Omega} u_j \frac{\partial H_j}{\partial\nu} - \int_{\partial\Omega} \Gamma \frac{\partial H_\lambda(x,\xi_j)}{\partial\nu} \\
&\quad = \int_{\partial\Omega} \big(\log\big(8\mu_j^2\big) - 2\log\big(\mu_j^2\rho^2 + |x-\xi_j|^2\big) - \log(\lambda)\big)\frac{\partial H_j}{\partial\nu} - \int_{\partial\Omega} \Gamma \frac{\partial H_\lambda}{\partial\nu} \\
&\quad = \int_{\partial\Omega} O\left(\frac{\mu_j^2\rho^2}{|x-\xi_j|^2}\right)\frac{\partial H_j}{\partial\nu} + \int_{\partial\Omega} \Gamma\left(\frac{\partial H_j}{\partial\nu} - \frac{\partial H_\lambda}{\partial\nu}\right) \\
&\quad = \int_{\partial\Omega} O\left(\frac{\mu_j^2\rho^2}{|x-\xi_j|^2}\right)\frac{\partial H_j}{\partial\nu} + \int_{\partial\Omega} \Gamma \frac{\partial h_j}{\partial\nu} \\
&\quad = \int_{\partial\Omega} O\left(\frac{\mu_j^2\rho^2}{|x-\xi_j|^2}\right)\frac{\partial H_j}{\partial\nu} + O\big((\lambda\rho)^2\big)
\end{aligned}$$

where, for the last equality, we have used the boundary condition satisfied by h_j and its bounds found in (A.9). We continue the estimation noting that

$$\begin{aligned}
\int_{\partial\Omega} \frac{\mu_j^2\rho^2}{|x-\xi_j|^2}\frac{\partial H_j}{\partial\nu} &= \int_{\partial\Omega} \frac{\mu_j^2\rho^2}{|x-\xi_j|^2}\left(\frac{\partial H_\lambda(x,\xi_j)}{\partial\nu} + \frac{\partial h_j}{\partial\nu}\right) \\
&= \int_{\partial\Omega\cap B(\xi_j,\frac{\delta}{2})} \frac{\mu_j^2\rho^2}{|x-\xi_j|^2}\frac{\partial H_\lambda(x,\xi_j)}{\partial\nu} + O\big(\lambda^2\epsilon^4\big) := K + O\big(\lambda^2\epsilon^4\big). \qquad \text{(A.10)}
\end{aligned}$$

To prove the estimate (6.24) we will need a more accurate bound for $\frac{\partial H_\lambda(x,\xi_j)}{\partial\nu}$ at least at points $x \in \partial\Omega$ near ξ_j. We will use expanded variables $y = \lambda x \in \lambda\Omega$, where $x \in \Omega$. In these expanded variables, H_λ satisfies

$$\begin{cases}
-\Delta H_\lambda = 0, & y \in \lambda\Omega, \\
\dfrac{\partial H_\lambda}{\partial\nu} + H_\lambda = 4\dfrac{(y-\xi_j')\nu}{|y-\xi_j'|^2} + 4\log\big|y-\xi_j'\big| - 4\log(\lambda), & y \in \partial(\lambda\Omega).
\end{cases}$$

We use the method of Lemma A.1 with $\tilde{Y}$ satisfying

$$\begin{cases}
-\Delta\tilde{Y}(y) = \dfrac{\log(\lambda)}{\lambda^2}, & y \in \lambda\Omega, \\
\dfrac{\partial\tilde{Y}(y)}{\partial\nu} + \tilde{Y}(y) = 1, & y \in \partial(\lambda\Omega),
\end{cases}$$

and v_1 satisfying

$$\begin{cases}
-\Delta v_1(y) = 0, & y \in H_+, \\
\dfrac{\partial v_1(y)}{\partial\nu} + v_1(y) = 2\log\big(1+y_1^2\big) - 4\log(\lambda), & y \in \partial H_+,
\end{cases}$$

and using the explicit expression (A.1), we conclude

$$\left|v_1(y_1, y_2)\right| \leqslant C \begin{cases} 1 + \log(1+y_2) + \log\bigl(1+|y_1|\bigr) - \log(\lambda) & \text{if } |y_1| \geqslant y_2, \\ 1 + \log(1+y_2) - \log(\lambda) & \text{if } |y_1| < y_2. \end{cases}$$

Here we will consider F as in (A.2) and η as in (A.3). We use the same method as in Lemma A.1 to conclude that the function $\tilde{w}$ defined as

$$\tilde{w}(y) = C_1\tilde{Y}(y) + C_2\eta\left(\frac{y}{\lambda}\right)v_1\left(\lambda F\left(\frac{y}{\lambda}\right)\right)$$

is a supersolution to H_λ in $\lambda\Omega$ and $-\tilde{w}$ is a subsolution to H_λ in $\lambda\Omega$, provided $C_1, C_2 > 0$ fixed adequately. This implies

$$\left|H_\lambda(y,\xi_j)\right| \leqslant C_1\log(\lambda) + C_2\eta\left(\frac{y}{\lambda}\right)v_1\left(\lambda F\left(\frac{y}{\lambda}\right)\right).$$

Using the boundary condition of H_λ and returning to the original variables, we have

$$\left|\frac{\partial H_\lambda(x,\xi_j)}{\partial\nu}\right| \leqslant C_1\lambda\log(\lambda) + C_2\lambda v_1\bigl(\lambda F(x)\bigr) + \frac{(x-\xi_j)\nu}{|x-\xi_j|^2} + 4\lambda\left|\log\bigl(|x-\xi_j|\bigr)\right|.$$

We use this to estimate the integral term K defined in (A.10), which, in main order is estimated as

$$K = O\bigl(\epsilon^2\lambda\log(\lambda)\bigr).$$

As in the proof of Lemma 6.2, differentiating with respect to ξ the error term it is possible to conclude the order $O((\lambda\epsilon)^2\log(\lambda))$ for the derivative of the error. This concludes the lemma. □

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