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Standing waves for supercritical nonlinear Schrödinger equations ☆

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Abstract

Let $V(x)$ be a non-negative, bounded potential in $\mathbb{R}^N$, $N \geqslant 3$ and p supercritical, $p > \frac{N+2}{N-2}$. We look for positive solutions of the standing-wave nonlinear Schrödinger equation $\Delta u - V(x)u + u^p = 0$ in $\mathbb{R}^N$, with $u(x) \to 0$ as $|x| \to +\infty$. We prove that if $V(x) = o(|x|^{-2})$ as $|x| \to +\infty$, then for $N \geqslant 4$ and $p > \frac{N+1}{N-3}$ this problem admits a continuum of solutions. If in addition we have, for instance, $V(x) = O(|x|^{-\mu})$ with $\mu > N$, then this result still holds provided that $N \geqslant 3$ and $p > \frac{N+2}{N-2}$. Other conditions for solvability, involving behavior of V at ∞, are also provided.

1. Introduction and statement of the main results

We consider standing waves for a nonlinear Schrödinger equation (NLS) in $\mathbb{R}^N$ of the form

$$-i\frac{\partial \psi}{\partial t} = \Delta\psi - Q(y)\psi + |\psi|^{p-1}\psi \tag{1.1}$$

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where $p > 1$, namely solutions of the form $\psi(t, y) = \exp(i\lambda t)u(y)$. Assuming that the amplitude $u(y)$ is positive and vanishes at infinity, we see that ψ satisfies (1.1) if and only if u solves the nonlinear elliptic problem

$$\Delta u - V(x)u + u^p = 0, \quad u > 0, \quad \lim_{|x|\to+\infty} u(x) = 0, \tag{1.2}$$

where $V(y) = Q(y) + \lambda$. In the rest of this paper we will assume that V is a bounded, non-negative function.

Construction of solutions to this problem has been a topic of broad interest in recent years. Most results in the literature deal with the *subcritical case*, $1 < p < \frac{N+2}{N-2}$ and the semiclassical limit,

$$\varepsilon^2 \Delta u - V(x)u + u^p = 0, \quad u > 0, \quad \lim_{|x|\to+\infty} u(x) = 0. \tag{1.3}$$

A typical result, due to Floer and Weinstein [18] for $N = 1$ and to Oh [25] in the general subcritical case reads as follows: if $\inf V > 0$ and V has a non-degenerate critical point x_0, then a solution u_ε exists for all small ε, concentrating near x_0 with a spike shape corresponding to an ε-scaling of the positive, exponentially decaying ground state of

$$\Delta w - V(x_0)w + w^p = 0.$$

Many results on existence of concentrating solutions have been proven, under various assumptions on the potential or the nonlinearity, with the aid of perturbation or variational methods, lifting non-degeneracy and also allowing the potential to vanish in some region or even be negative somewhere, see for instance [1,10,12,15,16,20–22,26,28]. Concentration on higher-dimensional manifolds has been established in the radial case in [3,5,6] and in the general case when $N = 2$ in [17]. It should be noticed that concerning radial solutions, supercriticality is typically not an issue if concentration is searched far away from the origin like in the results in [3,5,6].

Subcriticality is a rather essential constraint in the use of many methods devised in the literature. Very little is known in the supercritical case. In the critical case, a positive solution is established in [7] when $\varepsilon = 1$ and $\|V\|_{L^{N/2}}$ is small. When ε is small and $p = \frac{N+2}{N-2}$, it is proved in [11] that there are no single bubble solutions when $N \geqslant 5$. Results in the nearly critical case from above are contained in [23,24]: setting $\varepsilon = 1$ and letting $p = \frac{N+2}{N-2} + \delta$, they find multiple solutions concentrating as $\delta \to 0^+$, at a critical point of V with negative value for $N \geqslant 7$. $\|V\|_{L^{N/2}}$ is also required to be globally small, so that in particular the maximum principle holds.

The smallness of the potential at infinity is an issue that has been treated in [2,4,8,9,27]. In the subcritical case, with a combination of variational and perturbation techniques it is proven for instance in [2,4] that concentration at a non-degenerate critical point of V still takes place under the requirement that V is positive and

$$\liminf_{|x|\to+\infty} |x|^2 V(x) > 0.$$

In general one does not expect existence of solutions if V decreases faster than this rate.

In this paper we simply let $\varepsilon = 1$ and shall treat the case under the following *dual* assumption on the positive potential V:

$$\lim_{|x|\to+\infty} |x|^2 V(x) = 0. \tag{1.4}$$

We establish a new phenomenon, very different from the subcritical case: one of dispersion. There is a continuum of solutions u_λ of problem (1.2) which asymptotically vanish. This is always the case if the power p is above the critical exponent in one dimension less. This constraint is not needed if further decay on V is required, case in which pure supercriticality suffices.

Theorem 1. *Assume that $V \geqslant 0$, $V \in L^\infty(\mathbb{R}^N)$ and that* (1.4) *holds. Let $N \geqslant 4$, $p > \frac{N+1}{N-3}$. Then problem* (1.2) *has a continuum of solutions $u_\lambda(x)$ such that*

$$\lim_{\lambda\to 0} u_\lambda(x) = 0$$

uniformly in $\mathbb{R}^N$.

In reality the continuum of solutions in this result turns out to be a two-parameter family, dependent not only on all small λ but also on a point $\xi \in \mathbb{R}^N$, see Remark 5.2. The basic obstruction to extend the result to the whole supercritical range is that the linearized operator around some canonical approximation will no longer be onto if $\frac{N+2}{N-2} < p \leqslant \frac{N+1}{N-3}$, certain N solvability conditions becoming needed. This problem can be overcome through a further adjustment of the above mentioned parameter ξ. We do not know if the decay condition (1.4) of V suffices for this adjustment, but this is the case if further conditions on V are imposed. For instance, the result of Theorem 1 is also true if (1.4) holds and V is symmetric with respect to N coordinate axes,

$$V(x_1, \dots, x_i, \dots, x_N) = V(x_1, \dots, -x_i, \dots, x_N) \quad \text{for all } i = 1, \dots, N, \tag{1.5}$$

see Remark 4.1. On the other hand, additional requirements on the behavior at infinity for V are also sufficient. We have the validity of the following result.

Theorem 2. *Assume that $V \geqslant 0$, $V \in L^\infty(\mathbb{R}^N)$ and $\frac{N+2}{N-2} < p \leqslant \frac{N+1}{N-3}$. Then the result of Theorem* 1 *also holds true if either*

(a) *there exist $C > 0$ and $\mu > N$ such that*

$$V(x) \leqslant C|x|^{-\mu}, \quad |x| \geqslant 1; \quad \textit{or}$$

(b) *there exist a bounded non-negative function $f : S^{N-1} \to \mathbb{R}$, not identically* 0*, and $N - \frac{4}{p-1} < \mu \leqslant N$ such that*

$$\lim_{|x|\to+\infty} \left(|x|^\mu V(x) - f\left(\frac{x}{|x|}\right) \right) = 0.$$

The proofs of Theorems 1 and 2 will be based on the construction of a sufficiently good approximation and asymptotic analysis. It is well known that the problem

$$\Delta w + w^p = 0 \quad \text{in } \mathbb{R}^N \tag{1.6}$$

possesses a positive radially symmetric solution $w(|x|)$ whenever $p > \frac{N+2}{N-2}$. We fix in what follows the solution w of (1.6) such that

$$w(0) = 1. \tag{1.7}$$

Then all radial solutions to this problem can be expressed as

$$w_\lambda(x) = \lambda^{\frac{2}{p-1}} w(\lambda x). \tag{1.8}$$

At main order one has

$$w(r) \sim C_{p,N} r^{-\frac{2}{p-1}}, \quad r \to +\infty, \tag{1.9}$$

which implies that this behavior is actually common to all solutions $w_\lambda(r)$. The idea is to consider $w_\lambda(r)$ as a first approximation for a solution of problem (1.2), provided that $\lambda > 0$ is chosen small enough. Needless to mention, a variational approach applicable to the subcritical case is not suitable to the supercritical. The analogy here revealed should be an interesting line to explore in searching for a better understanding of solvability for supercritical problems. In particular, the approach we use here is also applicable to equations in exterior domains, see [13].

2. The operator $\Delta + pw^{p-1}$ in $\mathbb{R}^N$

Our main concern in this section is to prove existence of solution in certain weighted spaces for

$$\Delta\phi + pw^{p-1}\phi = h \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

where w is the radial solution to (1.6), (1.7) and h is a known function having a specific decay at infinity.

We work in weighted L^∞ spaces adjusted to the nonlinear problem (1.2) and in particular taking into account the behavior of w at infinity. We are looking for a solution ϕ to (2.1) that is small compared to w at infinity, thus it is natural to require that it has a decay of the form $O(|x|^{-\frac{2}{p-1}})$ as $|x| \to +\infty$. As a result we shall assume that h behaves like this but with two powers subtracted, that is, $h = O(|x|^{-\frac{2}{p-1}-2})$ at infinity. These remarks motivate the definitions

$$\|\phi\|_* = \sup_{|x|\leqslant 1} |x|^\sigma \left|\phi(x)\right| + \sup_{|x|\geqslant 1} |x|^{\frac{2}{p-1}} \left|\phi(x)\right|, \tag{2.2}$$

and

$$\|h\|_{**} = \sup_{|x|\leqslant 1} |x|^{2+\sigma} \left|h(x)\right| + \sup_{|x|\geqslant 1} |x|^{\frac{2}{p-1}+2} \left|h(x)\right|, \tag{2.3}$$

where $\sigma > 0$ will be fixed later as needed.

For the moment these norms allow a singularity at the origin, but later on we will place this singularity a point $\xi \in \mathbb{R}^N$.

The main result in this section is

Proposition 2.1. *Assume $N \geqslant 4$ and $p > \frac{N+1}{N-3}$. For $0 < \sigma < N-2$ there exists a constant $C > 0$ such that for any h with $\|h\|_{**} < +\infty$, Eq.* (2.1) *has a solution $\phi = T(h)$ such that T defines a linear map and*

$$\|T(h)\|_* \leqslant C\|h\|_{**}$$

where C is independent of λ.

An obstruction arises if $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$, which can be handled by considering suitable orthogonality conditions with respect to translations of w. Let us define

$$Z_i = \eta \frac{\partial w}{\partial x_i}, \tag{2.4}$$

and $\eta \in C_0^\infty(\mathbb{R}^N)$, $0 \leqslant \eta \leqslant 1$,

$$\eta(x) = 1 \quad \text{for } |x| \leqslant R_0, \qquad \eta(x) = 0 \quad \text{for } |x| \geqslant R_0 + 1.$$

We work with $R_0 > 0$ fixed large enough.

Proposition 2.2. *Assume $N \geqslant 3$, $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ and let $0 < \sigma < N-2$. There is a linear map $(\phi, c_1, \ldots, c_N) = T(h)$ defined whenever $\|h\|_{**} < \infty$ such that*

$$\Delta\phi + pw^{p-1}\phi = h + \sum_{i=1}^N c_i Z_i \quad \text{in } \mathbb{R}^N \tag{2.5}$$

and

$$\|\phi\|_* + \sum_{i=1}^N |c_i| \leqslant C\|h\|_{**}.$$

Moreover, $c_i = 0$ for all $1 \leqslant i \leqslant N$ if and only if h satisfies

$$\int_{\mathbb{R}^N} h \frac{\partial w}{\partial x_i} = 0 \quad \forall 1 \leqslant i \leqslant N. \tag{2.6}$$

The above operators are constructed "by hand" decomposing h and ϕ into sums of spherical harmonics where the coefficients are radial functions. The nice property is of course that since w is radial, the problem decouples into an infinite collection of ODEs. The most difficult case is the mode $k = 1$ which corresponds to the translation modes. This analysis is essentially contained in [13] and [14], where supercitical problems on exterior domains are studied. For the reader's convenience we include proofs of Propositions 2.1 and 2.2 in Appendix A.

3. The operator $\Delta - V_\lambda + pw^{p-1}$ in $\mathbb{R}^N$

The nonlinear equation, after a change of variables, involves the linearized problem

$$\begin{cases} \Delta\phi + pw^{p-1}\phi - V_\lambda\phi = h + \sum_{i=1}^{N} c_i Z_i \quad \text{in } \mathbb{R}^N, \\ \lim_{|x|\to+\infty} \phi(x) = 0, \end{cases} \tag{3.1}$$

where Z_i is defined in (2.4) and given $\lambda > 0$ and $\xi \in \mathbb{R}^N$ we define

$$V_\lambda(x) = \lambda^{-2} V\left(\frac{x-\xi}{\lambda}\right).$$

Because of the concentration of V_λ at ξ it is desirable to have a linear theory which allows singularities at ξ. Thus, for $\sigma > 0$ and $\xi \in \mathbb{R}^N$ we define

$$\|\phi\|_{*,\xi} = \sup_{|x-\xi|\leqslant 1} |x-\xi|^{\sigma} \big|\phi(x)\big| + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{\frac{2}{p-1}} \big|\phi(x)\big|,$$

$$\|h\|_{**,\xi} = \sup_{|x-\xi|\leqslant 1} |x-\xi|^{2+\sigma} \big|h(x)\big| + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{2+\frac{2}{p-1}} \big|h(x)\big|.$$

We will consider ξ with a bound

$$|\xi| \leqslant \Lambda$$

and the estimates we present will depend on Λ.

For the linear theory it suffices to assume

$$V \in L^\infty(\mathbb{R}^N), \quad V \geqslant 0, \qquad V(x) = o(|x|^{-2}) \quad \text{as } |x| \to +\infty. \tag{3.2}$$

Proposition 3.1. *Let $|\xi| \leqslant \Lambda$. Suppose V satisfies* (3.2) *and $\|h\|_{**,\xi} < \infty$.*

(a) *If $p > \frac{N+1}{N-3}$ for $\lambda > 0$ sufficiently small Eq.* (3.1) *with $c_i = 0$, $1 \leqslant i \leqslant N$ has a solution $\phi = \mathcal{T}_\lambda(h)$ that depends linearly on h and there is C such that*

$$\big\|\mathcal{T}_\lambda(h)\big\|_{*,\xi} \leqslant C\|h\|_{**,\xi}.$$

(b) *If $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ for $\lambda > 0$ sufficiently small Eq.* (3.1) *has a solution $(\phi, c_1, \dots, c_N) = \mathcal{T}_\lambda(h)$ that depends linearly on h and there is C such that*

$$\|\phi\|_{*,\xi} + \max_{1\leqslant i\leqslant N} |c_i| \leqslant C\|h\|_{**,\xi}.$$

The constant C is independent of λ.

Proof. We shall solve (3.1) by writing $\phi = \varphi + \psi$ where φ, ψ are new unknown functions.

Let $R > 0$, $\delta > 0$ with $2\delta \leqslant R$ be small positive numbers, to be fixed later independently of λ, and consider cut-off functions $\zeta_0, \zeta_1 \in C^\infty(\mathbb{R}^N)$ such that

$$\zeta_0(x) = 0 \quad \text{for } |x-\xi| \leqslant R, \qquad \zeta_0(x) = 1 \quad \text{for } |x-\xi| \geqslant 2R,$$

and

$$\zeta_1(x) = 0 \quad \text{for } |x-\xi| \leqslant \delta, \qquad \zeta_1(x) = 1 \quad \text{for } |x-\xi| \geqslant 2\delta.$$

To find a solution of (3.1) it is sufficient to find a solution φ, ψ of the following system

$$\begin{cases} \Delta\varphi + pw^{p-1}\varphi = -p\zeta_0 w^{p-1}\psi + \zeta_1 V_\lambda \varphi + \zeta_1 h + \displaystyle\sum_{i=1}^N c_i Z_i \quad \text{in } \mathbb{R}^N, \\ \displaystyle\lim_{|x|\to+\infty} \varphi(x) = 0 \end{cases} \tag{3.3}$$

and

$$\begin{cases} \Delta\psi - V_\lambda\psi + p(1-\zeta_0)w^{p-1}\psi = (1-\zeta_1)V_\lambda\varphi + (1-\zeta_1)h \quad \text{in } \mathbb{R}^N, \\ \displaystyle\lim_{|x|\to+\infty} \psi(x) = 0. \end{cases} \tag{3.4}$$

Given φ with $\|\varphi\|_* < +\infty$ Eq. (3.4) has indeed a solution $\psi(\varphi)$ if $R > 0$ is small, because $\|p(1-\zeta_0)w^{p-1}\|_{L^{N/2}} \to 0$ as $R \to 0$. Since $|\psi| \leqslant \frac{C}{|x|^{N-2}}$ for large $|x|$ the right-hand side of (3.3) has finite $\|\ \|_{**}$ norm. Therefore, according to Propositions 2.1 or 2.2, (3.3) has a solution when $\psi = \psi(\varphi)$ which we write as $F(\varphi)$. We shall show that F has a fixed point in the Banach space

$$X = \{\varphi \in L^\infty(\mathbb{R}^N) / \|\varphi\|_* < +\infty\}$$

equipped with the norm

$$\|\varphi\|_X = \sup_{|x|\leqslant 1} |\varphi(x)| + \sup_{|x|\geqslant 1} |x|^{\frac{2}{p-1}} |\varphi(x)|.$$

For $\varphi \in X$ we will first establish a pointwise estimate for the solution $\psi(\varphi)$ of (3.4). With this we will find a bound of the $\|\ \|_{**}$ norm of the right-hand side of (3.3).

Estimate for the solution of (3.4). Assume that $\varphi \in X$. Then the solution ψ to (3.4) satisfies

$$|\psi(x)| \leqslant (C\delta^{N-2}\|\varphi\|_X + C_\delta \|h\|_{**,\xi})|x-\xi|^{2-N} \quad \text{for all } |x-\xi| \geqslant \delta, \tag{3.5}$$

and

$$|\psi(x)| \leqslant C_\delta(\|\varphi\|_X + \|h\|_{**,\xi})|x-\xi|^{-\sigma} \quad \text{for all } |x-\xi| \leqslant \delta, \tag{3.6}$$

where C is independent of δ.

We decompose $\psi = \psi_1 + \psi_2$ where

$$\begin{cases} \Delta\psi_1 - V_\lambda\psi_1 + p(1-\zeta_0)w^{p-1}\psi_1 = (1-\zeta_1)V_\lambda\varphi \quad \text{in } \mathbb{R}^N, \\ \lim_{|x|\to+\infty} \psi_1(x) = 0 \end{cases} \tag{3.7}$$

and

$$\begin{cases} \Delta\psi_2 - V_\lambda\psi_2 + p(1-\zeta_0)w^{p-1}\psi_2 = (1-\zeta_1)h \quad \text{in } \mathbb{R}^N, \\ \lim_{|x|\to+\infty} \psi_1(x) = 0. \end{cases} \tag{3.8}$$

Then the solution ψ_1 to (3.7) satisfies

$$\left|\psi_1(x)\right| \leqslant C\delta^{N-2}\|\varphi\|_X|x-\xi|^{2-N} \quad \text{for all } |x-\xi| \geqslant \delta, \tag{3.9}$$

where C is independent of δ. For this, first we derive a bound for the solution $\tilde{\psi}$ to

$$\begin{cases} -\Delta\tilde{\psi}_1 = \chi_{B_{2\delta}(\xi)}V_\lambda|\varphi| \quad \text{in } \mathbb{R}^N, \\ \lim_{|x|\to+\infty} \tilde{\psi}_1(x) = 0. \end{cases}$$

Let $\bar{\psi}(y) = \tilde{\psi}_1(\xi + \delta y)$, which satisfies the equation

$$-\Delta\bar{\psi} = \delta^2\chi_{B_2}\lambda^{-2}V\left(\frac{\delta y}{\lambda}\right)\left|\varphi(\xi+\delta y)\right| \quad \text{in } \mathbb{R}^N$$

and using that $V(x) \leqslant C|x|^{-2}$ and that $|\varphi| \leqslant C\|\varphi\|_X$ in $B_{2\delta}(\xi)$ we obtain

$$-\Delta\bar{\psi} \leqslant C\chi_{B_2}\|\varphi\|_X|y|^{-2} \quad \text{in } \mathbb{R}^N.$$

Hence

$$\left|\bar{\psi}(y)\right| \leqslant C\|\varphi\|_X|y|^{2-N} \quad \text{for all } |y| \geqslant 1$$

and this yields

$$\left|\tilde{\psi}_1(x)\right| \leqslant C\delta^{N-2}\|\varphi\|_X|x-\xi|^{2-N} \quad \text{for all } |x-\xi| \geqslant \delta.$$

This estimate implies (3.9).

On the other hand, comparison with $v(y) = |y|^{-\sigma}$ shows that

$$\left|\bar{\psi}(y)\right| \leqslant C\|\varphi\|_X|y|^{-\sigma} \quad \text{for all } |y| \leqslant 1$$

which yields

$$\left|\tilde{\psi}_1(x)\right| \leqslant C_\delta\|\varphi\|_X|x-\xi|^{-\sigma} \quad \text{for all } |x-\xi| \leqslant \delta.$$

This inequality implies

$$\left|\psi_1(y)\right| \leqslant C_\delta \|\varphi\|_X |x-\xi|^{-\sigma} \quad \text{for all } |x-\xi| \leqslant \delta.$$

Finally, a similar computation shows that

$$\left|\psi_2(x)\right| \leqslant C_\delta \|h\|_{**,\xi} |x-\xi|^{2-N} \quad \text{for all } |x-\xi| \geqslant \delta$$

and

$$\left|\psi_2(x)\right| \leqslant C_\delta \|h\|_{**,\xi} |x-\xi|^{-\sigma} \quad \text{for all } |x-\xi| \leqslant \delta.$$

Estimate of $\|\zeta_0 w^{p-1}\psi(\varphi)\|_{**}$. We write for simplicity $\psi = \psi(\varphi)$. We have

$$\left\|\zeta_0 w^{p-1}\psi\right\|_{**} \leqslant C\delta^{N-2}\|\varphi\|_X + C_\delta \|h\|_{**,\xi}, \tag{3.10}$$

with C independent of λ and δ.

Indeed,

$$\left\|\zeta_0 w^{p-1}\psi\right\|_{**} = \sup_{|x|\leqslant 1} \zeta_0 w^{p-1}|\psi| + \sup_{|x|\geqslant 1} |x|^{2+\frac{2}{p-1}} \zeta_0 w^{p-1}|\psi|.$$

Since $\zeta_0(x)$ vanishes for $|x-\xi| \leqslant R$ we have by (3.5)

$$\sup_{|x|\leqslant 1} \zeta_0 w^{p-1}|\psi| \leqslant C\delta^{N-2}\|\varphi\|_X + C_\delta \|h\|_{**,\xi}$$

where the constant C does not depend on δ. Similarly by (3.5)

$$\sup_{|x|\geqslant 1} |x|^{2+\frac{2}{p-1}} \zeta_0 w^{p-1}|\psi| \leqslant C\delta^{N-2}\|\varphi\|_X + C_\delta \|h\|_{**,\xi}.$$

Estimate for $\|\zeta_1 V_\lambda \varphi\|_{**}$. Let us consider first

$$\begin{aligned}\sup_{|x|\leqslant 1} |x|^{2+\sigma} \zeta_1 V_\lambda |\varphi| &\leqslant \|\varphi\|_X \lambda^{-2} \sup_{|x|\leqslant 1,\ |x-\xi|\geqslant \delta} V\left(\frac{x-\xi}{\lambda}\right) \\ &\leqslant \|\varphi\|_X a\left(\frac{\delta}{\lambda}\right) \sup_{|x|\leqslant 1,\ |x-\xi|\geqslant \delta} |x-\xi|^{-2} \leqslant \|\varphi\|_X a\left(\frac{\delta}{\lambda}\right)\delta^{-2},\end{aligned}$$

where

$$a(R) = \sup_{|x|\geqslant R} |x|^2 V(x), \quad a(R) \to 0 \quad \text{as } R \to +\infty.$$

Similarly

$$\sup_{|x|\geqslant 1} |x|^{2+\frac{2}{p-1}} \zeta_1 V_\lambda |\varphi| \leqslant \|\varphi\|_X \delta^{-2} a\left(\frac{1}{\lambda}\right).$$

Thus, we find

$$\|\zeta_1 V_\lambda \varphi\|_{**} \leqslant C\|\varphi\|_* \delta^{-2} a\left(\frac{\delta}{\lambda}\right). \tag{3.11}$$

By Propositions 2.1 and 2.2 we know that, given $\varphi \in X$, the solution $F(\varphi)$ to (3.3) where $\psi = \psi(\varphi)$ satisfies

$$\|F(\varphi)\|_* \leqslant C\|\zeta_0 w^{p-1}\psi\|_{**} + C\|\zeta_1 V_\lambda \varphi\|_{**} + C\|\zeta_1 h\|_{**}.$$

But since the right-hand side of (3.3) is bounded near the origin, from standard elliptic estimates we derive

$$\|F(\varphi)\|_X \leqslant C\|\zeta_0 w^{p-1}\psi\|_{**} + C\|\zeta_1 V_\lambda \varphi\|_{**} + C\|\zeta_1 h\|_{**}.$$

From (3.10) and (3.11) we have

$$\|F(\varphi_1) - F(\varphi_2)\|_X \leqslant C\left(\delta^{N-2} + \delta^{-2} a\left(\frac{\delta}{\lambda}\right)\right)\|\varphi_1 - \varphi_2\|_X.$$

By choosing and fixing $\delta > 0$ small we see that for all $\lambda > 0$ sufficiently small F has a unique fixed point $\varphi \in X$. Moreover, letting $\psi = \psi(\varphi)$, we see thanks to Propositions 2.1 and 2.2 and estimates (3.10) and (3.11) that φ satisfies

$$\|\varphi\|_X \leqslant C\left(\delta^{N-2} + \delta^{-2} a\left(\frac{\delta}{\lambda}\right)\right)\|\varphi\|_X + C_\delta \|h\|_{**,\xi},$$

which yields

$$\|\varphi\|_X \leqslant C\|h\|_{**,\xi}$$

for $\lambda > 0$ small. This and (3.5), (3.6) then show that $\phi = \varphi + \psi$ is a solution (3.1) satisfying

$$\|\phi\|_{*,\xi} \leqslant C\|h\|_{**,\xi}. \qquad \square$$

4. Proof of Theorem 1

By the change of variables $\lambda^{-\frac{2}{p-1}} u(\frac{x}{\lambda})$ the equation

$$\Delta u - Vu + u^p = 0 \quad \mathbb{R}^N,$$

is equivalent to

$$\Delta u - V_\lambda u + u^p = 0 \quad \mathbb{R}^N,$$

where

$$V_\lambda(x) = \lambda^{-2} V\left(\frac{x}{\lambda}\right).$$

Thus V_λ is as in the previous section with $\xi = 0$.

Let us look for a solution of the form $u = w + \phi$, which yields the following equation for ϕ

$$\Delta\phi - V_\lambda\phi + pw^{p-1}\phi = N(\phi) + V_\lambda w$$

where

$$N(\phi) = -(w+\phi)^p + w^p + pw^{p-1}\phi. \tag{4.1}$$

Using the operator $\mathcal{T}_\lambda$ defined in Proposition 3.1(a) we are led to solving the fixed point problem

$$\phi = \mathcal{T}_\lambda\big(N(\phi) + V_\lambda w\big). \tag{4.2}$$

We claim that

$$\|V_\lambda w\|_{**,0} \to 0 \quad \text{as } \lambda \to 0. \tag{4.3}$$

Indeed, let $R > 0$ and observe that

$$\begin{aligned}\sup_{|x|\leqslant 1} |x|^{2+\sigma} V_\lambda(x) w(x) &\leqslant \lambda^{-2}\|w\|_{L^\infty} \sup_{|x|\leqslant 1} |x|^{2+\sigma} V\left(\frac{x}{\lambda}\right)\\ &\leqslant \lambda^{-2}\|w\|_{L^\infty}\Big(\sup_{|x|\leqslant R\lambda} \cdots + \sup_{R\lambda\leqslant |x|\leqslant 1} \cdots\Big).\end{aligned}$$

But

$$\lambda^{-2} \sup_{|x|\leqslant R\lambda} |x|^{2+\sigma} V\left(\frac{x}{\lambda}\right) \leqslant \lambda^\sigma R^{2+\sigma}\|V\|_{L^\infty} \tag{4.4}$$

and

$$\lambda^{-2} \sup_{R\lambda\leqslant |x|\leqslant 1} |x|^{2+\sigma} V\left(\frac{x}{\lambda}\right) \leqslant a(R) \sup_{R\lambda\leqslant |x|\leqslant 1} |x|^\sigma \leqslant a(R), \tag{4.5}$$

where

$$a(R) = \sup_{|x|\geqslant R} |x|^2 V(x).$$

On the other hand,

$$\begin{aligned}\sup_{|x|\geqslant 1} |x|^{2+\frac{2}{p-1}} w(x) V_\lambda(x) &\leqslant C\lambda^{-2} \sup_{|x|\geqslant 1} |x|^2 V\left(\frac{x}{\lambda}\right)\\ &\leqslant Ca\left(\frac{1}{\lambda}\right).\end{aligned} \tag{4.6}$$

From (4.4)–(4.6) it follows that

$$\|V_\lambda w\|_{**,0} \leqslant C\left(\lambda^\sigma R^{2+\sigma} + a(R) + a\left(\frac{1}{\lambda}\right)\right).$$

Letting $\lambda \to 0$ we see

$$\limsup_{\lambda\to 0} \|V_\lambda w\|_{**,0} \leqslant Ca(R),$$

and, since $a(R) \to 0$ as $R \to +\infty$, we have established (4.3).

We estimate $N(\phi)$ depending on whether $p \geqslant 2$ or $p < 2$.

Case $p \geqslant 2$. In this case, since w is bounded, we have

$$\left|N(t)\right| \leqslant C\left(t^2 + |t|^p\right) \quad \text{for all } t \in \mathbb{R}.$$

Since

$$\left|\phi(x)\right| \leqslant |x|^{-\sigma} \|\phi\|_{*,0} \quad \text{for all } |x| \leqslant 1$$

and working with $\|\phi\|_{*,0} \leqslant 1$, $0 < \sigma \leqslant \frac{2}{p-1}$, we obtain

$$\begin{aligned} \sup_{|x|\leqslant 1} |x|^{2+\sigma}\left|N\left(\phi(x)\right)\right| &\leqslant C\|\phi\|_{*,0}^2 \sup_{|x|\leqslant 1} |x|^{2-\sigma} + C\|\phi\|_{*,0}^p \sup_{|x|\leqslant 1} |x|^{2-(p-1)\sigma} \\ &\leqslant C\|\phi\|_{*,0}^2. \end{aligned} \tag{4.7}$$

On the other hand,

$$\left|\phi(x)\right| \leqslant C|x|^{-\frac{2}{p-1}} \|\phi\|_{*,0} \quad \text{for all } |x| \geqslant 1$$

and

$$w(x) \leqslant C\left(1 + |x|\right)^{-\frac{2}{p-1}} \quad \text{for all } x \in \mathbb{R}^N,$$

so we have

$$\sup_{|x|\geqslant 1} |x|^{2+\frac{2}{p-1}}\left|N\left(\phi(x)\right)\right| \leqslant C\|\phi\|_{*,0}^2. \tag{4.8}$$

From (4.7) and (4.8) it follows that if $p \geqslant 2$ and $0 < \sigma \leqslant \frac{2}{p-1}$ then

$$\left\|N(\phi)\right\|_{**,0} \leqslant C\|\phi\|_{*,0}^2. \tag{4.9}$$

Case $p < 2$. In this case $|N(\phi)| \leqslant C|\phi|^p$ and hence, if $0 < \sigma \leqslant \frac{2}{p-1}$

$$\sup_{|x|\leqslant 1} |x|^{2+\sigma}\left|N(\phi)\right| \leqslant C \sup_{|x|\leqslant 1} |x|^{2+\sigma}|\phi|^p \leqslant C\|\phi\|_{*,0}^p. \tag{4.10}$$

Similarly

$$\sup_{|x|\geqslant 1} |x|^{2+\frac{2}{p-1}}\left|N(\phi)\right| \leqslant C \sup_{|x|\geqslant 1} |x|^{2+\frac{2}{p-1}}\left|\phi(x)\right|^p$$
$$\leqslant C\|\phi\|_{*,0}^p. \tag{4.11}$$

From (4.10) and (4.11) it follows that for any $1 < p < 2$ and $0 < \sigma \leqslant \frac{2}{p-1}$

$$\left\|N(\phi)\right\|_{**,0} \leqslant C\|\phi\|_{*,0}^p. \tag{4.12}$$

From (4.9) and (4.12) he have

$$\left\|N(\phi)\right\|_{**,0} \leqslant C\left(\|\phi\|_{*,0}^2 + \|\phi\|_{*,0}^p\right). \tag{4.13}$$

We have already observed that $u = w_\lambda + \phi$ is a solution of (1.2) if ϕ satisfies the fixed point problem (4.2). Consider the set

$$\mathcal{F} = \left\{\phi : \mathbb{R}^N \to \mathbb{R} \big/ \|\phi\|_{*,0} \leqslant \rho\right\},$$

where $\rho > 0$ is to be chosen (suitably small) and the operator

$$\mathcal{A}(\phi) = \mathcal{T}_\lambda\left(N(\phi) + V_\lambda w\right).$$

We prove that $\mathcal{A}$ has a fixed point in $\mathcal{F}$. We start with the estimate,

$$\left\|\mathcal{A}(\phi)\right\|_{*,0} \leqslant C\left(\left\|N(\phi)\right\|_{**,0} + \|V_\lambda w\|_{**,0}\right)$$
$$\leqslant C\left(\|\phi\|_{*,0}^2 + \|\phi\|_{*,0}^p + \|V_\lambda w\|_{**,0}\right)$$

by (4.13). We can obtain a right-hand side bounded by ρ by choosing $\rho > 0$ small independent of λ and then using (4.3). This yields $\mathcal{A}(\mathcal{F}) \subset \mathcal{F}$.

Now we show that $\mathcal{A}$ is a contraction mapping in $\mathcal{F}$. Let us take ϕ_1, ϕ_2 in $\mathcal{F}$. Then

$$\left\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\right\|_{*,0} \leqslant C\left\|N(\phi_1) - N(\phi_2)\right\|_{**,0}. \tag{4.14}$$

Write

$$N(\phi_1) - N(\phi_2) = D_{\bar\phi}N(\bar\phi)(\phi_1 - \phi_2),$$

where $\bar\phi$ lies in the segment joining ϕ_1 and ϕ_2. Then, for $|x| \leqslant 1$,

$$|x|^{2+\sigma}\left|N(\phi_1) - N(\phi_2)\right| \leqslant |x|^2\left|D_{\bar\phi}N(\bar\phi)\right|\|\phi_1 - \phi_2\|_{*,0},$$

while, for $|x| \geqslant 1$,

$$|x|^{2+\frac{2}{p-1}}\left|N(\phi_1) - N(\phi_2)\right| \leqslant |x|^2\left|D_{\bar\phi}N(\bar\phi)\right|\|\phi_1 - \phi_2\|_{*,0}.$$

Then we have

$$\big\|N(\phi_1)-N(\phi_2)\big\|_{**,0}\leqslant C\sup_x\big(|x|^2\big|D_{\bar\phi}N(\bar\phi)\big|\big)\|\phi_1-\phi_2\|_{*,0}. \tag{4.15}$$

Directly from the definition of N, we compute

$$D_{\bar\phi}N(\bar\phi)=p\big[(w+\bar\phi)^{p-1}-w^{p-1}\big].$$

If $p\geqslant 2$ and $0<\sigma\leqslant\frac{2}{p-1}$ then

$$\begin{aligned}|x|^2\big|D_{\bar\phi}N(\bar\phi)\big|&\leqslant C|x|^2w^{p-2}\big|\bar\phi(x)\big|\\&\leqslant C\big(\|\phi_1\|_{*,0}+\|\phi_2\|_{*,0}\big)\leqslant C\rho\quad\text{for all } x.\end{aligned} \tag{4.16}$$

Similarly, if $p<2$ and $0<\sigma\leqslant\frac{2}{p-1}$ then

$$\begin{aligned}|x|^2\big|D_{\bar\phi}N(\bar\phi)\big|&\leqslant C|x|^2\big|\bar\phi(x)\big|^{p-1}\\&\leqslant C\lambda^{-2}\big(\|\phi_1\|_{*,0}^{p-1}+\|\phi_2\|_{*,0}^{p-1}\big)\leqslant C\rho^{p-1}\quad\text{for all } x.\end{aligned} \tag{4.17}$$

Estimates (4.16) and (4.17) show that

$$\sup_x\big(|x|^2\big|D_{\bar\phi}N(\bar\phi)\big|\big)\leqslant C\big(\rho+\rho^{p-1}\big). \tag{4.18}$$

Gathering relations (4.14), (4.15) and (4.18) we conclude that $\mathcal{A}$ is a contraction mapping in $\mathcal{F}$, and hence a fixed point in this region indeed exists. This finishes the proof of the theorem.

Remark 4.1. We observe that the above proof actually applies with no changes to the case $\frac{N+2}{N-2}<p<\frac{N+1}{N-3}$ provided that V is symmetric with respect to N coordinate axis, namely

$$V(x_1,\dots,x_i,\dots,x_N)=V(x_1,\dots,-x_i,\dots,x_N)\quad\text{for all } i=1,\dots,N.$$

In this case the problem is invariant with respect to the above reflections, and we can formulate the fixed point problem in the space of functions with these even symmetries with the linear operator defined in Proposition 2.2. Indeed, the orthogonality conditions (2.6) are automatically satisfied, so that the associated numbers c_i are all zero.

5. The case $\frac{N+2}{N-2}<p\leqslant\frac{N+1}{N-3}$

Because of the obstruction in the solvability of the linearized operator for p in this range, it will be necessary to do the rescaling about a point ξ suitably chosen. For this reason we make the change of variables $\lambda^{-\frac{2}{p-1}}u(\frac{x-\xi}{\lambda})$ and look for a solution of the form $u=w+\phi$, leading to the following equation for ϕ:

$$\Delta\phi-V_\lambda\phi+pw^{p-1}\phi=N(\phi)+V_\lambda w,$$

where

$$V_\lambda(x) = \lambda^{-2} V\left(\frac{x-\xi}{\lambda}\right)$$

and N is the same as in the previous section, namely

$$N(\phi) = -(w+\phi)^p + w^p + p w^{p-1}\phi.$$

We will change slightly the previous notation to make the dependence of the norms in σ explicit. Hence we set

$$\|\phi\|_{*,\xi}^{(\sigma)} = \sup_{|x-\xi|\leqslant 1} |x-\xi|^\sigma \left|\phi(x)\right| + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{\frac{2}{p-1}} \left|\phi(x)\right|,$$

$$\|h\|_{**,\xi}^{(\sigma)} = \sup_{|x-\xi|\leqslant 1} |x-\xi|^{2+\sigma} \left|h(x)\right| + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{2+\frac{2}{p-1}} \left|h(x)\right|.$$

In the rest of the section we assume that

$$\frac{N+2}{N-2} < p < \frac{N+1}{N-3}.$$

The case $p = \frac{N+1}{N-3}$ can be handled similarly, with a slight modification of the norms where it is more convenient to define

$$\|\phi\|_{*,\xi}^{(\sigma)} = \sup_{|x-\xi|\leqslant 1} |x-\xi|^\sigma \left|\phi(x)\right| + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{\frac{2}{p-1}+\alpha} \left|\phi(x)\right|,$$

$$\|h\|_{**,\xi}^{(\sigma)} = \sup_{|x-\xi|\leqslant 1} |x-\xi|^{2+\sigma} \left|h(x)\right| + \sup_{|x-\xi|\geqslant 1} |x-\xi|^{2+\frac{2}{p-1}+\alpha} \left|h(x)\right|$$

for some small fixed $\alpha > 0$, see Remarks 5.3 and A.1.

Lemma 5.1. *Let $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ and V satisfy* (3.2) *and $\Lambda > 0$. Then there is $\varepsilon_0 >$ such that for $|\xi| < \Lambda$ and $\lambda < \varepsilon_0$ there exist $\phi_\lambda, c_1(\lambda), \ldots, c_N(\lambda)$ solution to*

$$\begin{cases} \Delta\phi - V_\lambda\phi + p w^{p-1}\phi = N(\phi) + V_\lambda w + \displaystyle\sum_{i=1}^N c_i Z_i, \\ \displaystyle\lim_{|x|\to+\infty} \phi(x) = 0. \end{cases} \tag{5.1}$$

We have in addition

$$\|\phi_\lambda\|_{*,\xi} + \max_{1\leqslant i\leqslant N} \left|c_i(\lambda)\right| \to 0 \quad \text{as } \lambda \to 0.$$

If V satisfies also

$$V(x) \leqslant C|x|^{-\mu} \quad \text{for all } x \tag{5.2}$$

for some $\mu > 2$, *then for* $0 < \sigma \leqslant \mu - 2$, $\sigma < N - 2$

$$\|\phi_\lambda\|_{*,\xi}^{(\sigma)} \leqslant C_\sigma \lambda^\sigma \quad \textit{for all } 0 < \lambda < \varepsilon_0. \tag{5.3}$$

Proof. Similarly as in the proof of Theorem 1 we fix $0 < \sigma < \min(2, \frac{2}{p-1})$ and define for small $\rho > 0$

$$\mathcal{F} = \left\{\phi : \mathbb{R}^N \to \mathbb{R} \big/ \|\phi\|_{*,\xi}^{(\sigma)} \leqslant \rho\right\}$$

and the operator $\phi_1 = \mathcal{A}_\lambda(\phi)$ where $\phi_1, c_1, \ldots, c_N$ is the solution of Proposition 3.1 to

$$\begin{cases} \Delta\phi_1 - V_\lambda\phi_1 + pw^{p-1}\phi_1 = N(\phi) + V_\lambda w + \displaystyle\sum_{i=1}^N c_i Z_i \quad \text{in } \mathbb{R}^N, \\ \displaystyle\lim_{|x|\to+\infty} \left|\phi(x)\right| = 0, \end{cases}$$

where N is given by (4.1).

In the case $p \geqslant 2$ and $0 < \sigma \leqslant \frac{2}{p-1}$ it is not difficult to check that

$$\left\|N(\phi)\right\|_{**,\xi}^{(\sigma)} \leqslant C\left(\|\phi\|_{*,\xi}^{(\sigma)}\right)^2$$

and for $\phi_1, \phi_2 \in \mathcal{F}$ it holds

$$\left\|N(\phi_1) - N(\phi_2)\right\|_{**,\xi}^{(\sigma)} \leqslant C\rho\|\phi_1 - \phi_2\|_{*,\xi}^{(\sigma)}.$$

Similarly, if $p < 2$ and $0 < \sigma \leqslant \frac{2}{p-1}$ then

$$\|N\|_{\xi,**}^{(\sigma)} \leqslant C\left(\|\phi\|_{*,\xi}^{(\sigma)}\right)^p \quad \text{for all } \phi \in \mathcal{F}$$

and if $\phi_1, \phi_2 \in \mathcal{F}$ then

$$\left\|N(\phi_1) - N(\phi_2)\right\|_{**,\xi}^{(\sigma)} \leqslant C\rho^{p-1}\|\phi_1 - \phi_2\|_{*,\xi}^{(\sigma)}.$$

We also have

$$\|V_\lambda w\|_{**,\xi}^{(\sigma)} = o(1) \quad \text{as } \lambda \to 0.$$

Therefore, if $\rho = 2C\|V_\lambda w)\|_{**,\xi}^{(\sigma)}$ then $\mathcal{A}_\lambda$ possesses a unique fixed point ϕ_λ in $\mathcal{F}$ and it satisfies

$$\|\phi_\lambda\|_{*,\xi}^{(\sigma)} \leqslant C\|V_\lambda w\|_{**,\xi}^{(\sigma)} = o(1). \tag{5.4}$$

Under assumption (5.2) and for $0 < \theta \leqslant \mu - 2$ we can estimate $\|V_\lambda w\|_{**,\xi}^{(\theta)}$ as follows:

$$\sup_{|x-\xi|\leqslant 1} |x-\xi|^{2+\theta}\lambda^{-2}V\left(\frac{x-\xi}{\lambda}\right)w(x) \leqslant \sup_{|x-\xi|\leqslant\lambda} \cdots + \sup_{\lambda\leqslant|x-\xi|\leqslant 1} \cdots.$$

But

$$\sup_{|x-\xi|\leqslant\lambda} |x-\xi|^{2+\theta}\lambda^{-2}V\left(\frac{x-\xi}{\lambda}\right)w(x) \leqslant \|V\|_{L^\infty}\|w\|_{L^\infty}\lambda^\theta. \tag{5.5}$$

In the other case

$$\begin{aligned}\sup_{\lambda\leqslant|x-\xi|\leqslant 1} |x-\xi|^{2+\theta}\lambda^{-2}V\left(\frac{x-\xi}{\lambda}\right)w(x) &\leqslant C\|w\|_{L^\infty}\lambda^{\mu-2}\sup_{\lambda\leqslant|x-\xi|\leqslant 1}|x-\xi|^{2+\theta-\mu}\\ &\leqslant C\lambda^\theta. \end{aligned}\tag{5.6}$$

Finally

$$\sup_{|x-\xi|\geqslant 1} |x-\xi|^{2+\frac{2}{p-1}}\lambda^{-2}V\left(\frac{x-\xi}{\lambda}\right)w(x) \leqslant C\lambda^{\mu-2}\sup_{|x-\xi|\geqslant 1}|x|^{2-\mu} = C\lambda^{\mu-2} \tag{5.7}$$

and collecting (5.5), (5.6) and (5.7) yields

$$\|V_\lambda w\|_{**,\xi}^{(\theta)} \leqslant C\lambda^\theta. \tag{5.8}$$

In order to improve the estimate of the fixed point ϕ_λ we need to estimate better $N(\phi_\lambda)$. First we observe that ϕ_λ is uniformly bounded. Indeed, the function $u_\lambda = w + \phi_\lambda$ solves

$$\begin{cases} \Delta u_\lambda - V_\lambda u_\lambda + u_\lambda^p = \sum_{i=1}^N c_i(\lambda)Z_i \quad \text{in } \mathbb{R}^N,\\ \lim_{|x|\to+\infty} u_\lambda(x) = 0. \end{cases}\tag{5.9}$$

For x with $|x-\xi| = 1$ $u_\lambda(x)$ remains bounded because $|\phi_\lambda(x)| \leqslant C$. Then a uniform upper bound for u_λ follows from (5.9) and by observing that $\|u_\lambda^p\|_{L^q(B_1)}$ remains bounded as $\lambda \to 0$ for $q > \frac{N}{2}$. In fact

$$\int_{B_1} u_\lambda^{pq} \leqslant C\int_{B_1} w^{pq} + |\phi_\lambda|^{pq} \leqslant C + C\int_{B_1}|x|^{-\sigma pq}\,dx \leqslant C$$

if we choose $\sigma > 0$ small. Hence

$$\left|u_\lambda(x)\right| \leqslant C \quad \text{for all } |x-\xi| \leqslant 1. \tag{5.10}$$

It follows from then that

$$\left|\phi_\lambda(x)\right| \leqslant C \quad \text{for all } x. \tag{5.11}$$

We shall estimate $\|\phi_\lambda\|_{*,\xi}^{(\theta)}$ for a $\theta > \sigma$. Since ϕ_λ is a fixed point of $\mathcal{A}_\lambda$, if $0 < \theta < N-2$ and $\theta \leqslant \mu - 2$ we have, by (5.8)

$$\|\phi_\lambda\|_{*,\xi}^{(\theta)} = \big\|\mathcal{A}(\phi_\lambda)\big\|_{*,\xi}^{(\theta)} \leqslant C\big(\big\|N(\phi_\lambda)\big\|_{**,\xi}^{(\theta)} + \|V_\lambda w\|_{**,\xi}^{(\theta)}\big) \tag{5.12}$$

$$\leqslant C\big\|N(\phi_\lambda)\big\|_{*,\xi}^{(\theta)} + C\lambda^{\theta}. \tag{5.13}$$

When $p \geqslant 2$

$$\big|N(\phi_\lambda)\big| \leqslant C\big(w^{p-2}|\phi|^2 + |\phi|^p\big). \tag{5.14}$$

Then

$$\sup_{|x-\xi|\leqslant 1} |x-\xi|^{2+\theta}\big|N\big(\phi_\lambda(x)\big)\big| \leqslant \sup_{|x-\xi|\leqslant\lambda} \cdots + \sup_{\lambda\leqslant|x-\xi|\leqslant 1} \cdots .$$

Thanks to (5.11)

$$\sup_{|x-\xi|\leqslant\lambda} |x-\xi|^{2+\theta}\big|N\big(\phi_\lambda(x)\big)\big| \leqslant C\lambda^{2+\theta} \tag{5.15}$$

and by (5.4)

$$\sup_{\lambda\leqslant|x-\xi|\leqslant 1} |x-\xi|^{2+\theta}\big|N\big(\phi(x)\big)\big| \leqslant C\big(\|\phi\|_{*,\xi}^{(\sigma)}\big)^2 \sup_{\lambda\leqslant|x-\xi|\leqslant 1} |x-\xi|^{2+\theta-2\sigma}$$

$$\leqslant C\lambda^{\min(2+\theta,2\sigma)}. \tag{5.16}$$

Using (5.4) again yields

$$\sup_{|x-\xi|\geqslant 1} |x|^{2+\frac{2}{p-1}}\big|N\big(\phi(x)\big)\big| \leqslant C\big(\|\phi_\lambda\|_{*,\xi}^{(\sigma)}\big)^2 \leqslant C\lambda^{2\sigma} \tag{5.17}$$

and from (5.15), (5.16) and (5.17) we deduce

$$\big\|N(\phi_\lambda)\big\|_{**,\xi}^{(\theta)} \leqslant C\lambda^{\min(2+\theta,2\sigma)}.$$

This relation and (5.12) imply

$$\|\phi_\lambda\|_{*,\xi}^{(\theta)} \leqslant C\lambda^{\min(\theta,2\sigma)}$$

provided $0 < \theta < N-2$ and $\theta \leqslant \mu - 2$. Repeating this argument a finite number of times we deduce the validity of (5.3) in the case $p \geqslant 2$.

If $p < 2$ instead of (5.14), using

$$\big|N(\phi)\big| \leqslant C|\phi|^p$$

we infer

$$\big\|N(\phi_\lambda)\big\|_{**,\xi}^{(\theta)} \leqslant C\lambda^{\min(2+\theta,p\sigma)}$$

and the same argument as before yields the conclusion. □

Proof of Theorem 2. We have found a solution $\phi_\lambda, c_1(\lambda), \ldots, c_N(\lambda)$ to (5.1). By Lemma A.4 the solution constructed satisfies for all $1 \leqslant j \leqslant N$:

$$\int_{\mathbb{R}^N} \left(V_\lambda \phi_\lambda + V_\lambda w + N(\phi_\lambda) + \sum_{i=1}^N c_i Z_i \right) \frac{\partial w}{\partial x_j}(y) = 0.$$

Thus, for all λ small, we need to find $\xi = \xi_\lambda$ so that $c_i = 0$, $1 \leqslant i \leqslant N$, that is

$$\int_{\mathbb{R}^N} \big(V_\lambda \phi_\lambda + V_\lambda w + N(\phi_\lambda) \big) \frac{\partial w}{\partial x_j} = 0 \quad \forall 1 \leqslant j \leqslant N. \tag{5.18}$$

Condition (5.18) is actually sufficient under the assumption, which will turn out to be satisfied in our cases, that ξ_λ is bounded as $\lambda \to 0$ because, in this situation, the matrix with coefficients

$$\int_{\mathbb{R}^N} Z_i(y - \xi) \frac{\partial w}{\partial x_j}(y)\, dy$$

is invertible, provided the number R_0 in the definition of Z_i is chosen large enough.

The dominant term in (5.18) is

$$\lambda^{-2} \int_{\mathbb{R}^N} V\left(\frac{y - \xi}{\lambda} \right) w \frac{\partial w}{\partial y_j} = \lambda^{-2} \int_{\mathbb{R}^N} V\left(\frac{x}{\lambda} \right) w(x + \xi) \frac{\partial w}{\partial x_j}(x + \xi) \tag{5.19}$$

whose asymptotic behavior depends on the decay of $V(x)$ as $|x| \to +\infty$.

Part (a). Case $V(x) \leqslant C|x|^{-\mu}$, $\mu > N$. In this case we have

$$\int_{\mathbb{R}^N} V\left(\frac{x}{\lambda} \right) w(x + \xi) \frac{\partial w}{\partial x_j}(x + \xi) = \lambda^N C_V w(\xi) \frac{\partial w}{\partial x_j}(\xi) + o\big(\lambda^N\big) \quad \text{as } \lambda \to 0,$$

where $C_V = \int_{\mathbb{R}^N} V$ and the convergence is uniform with respect to $|\xi| < \varepsilon_0$. We obtain the existence of a solution ξ to (5.18) thanks to the non-degeneracy of 0 as a critical point of $w^2(\xi)$. Furthermore, the point ξ will be close to 0. Before we need to show that the other terms in (5.18) are small compared to (5.19).

Indeed,

$$\int_{\mathbb{R}^N} \left| N(\phi_\lambda) \frac{\partial w}{\partial x_j} \right| = \int_{B_1(\xi)} \cdots + \int_{\mathbb{R}^N \setminus B_1(\xi)} \cdots.$$

In the case $p \geqslant 2$, by (5.3), we have

$$\int_{B_1(\xi)} \left| N(\phi_\lambda) \frac{\partial w}{\partial x_j} \right| \leqslant \big(\|\phi_\lambda\|_{*,\xi}^{(\sigma)} \big)^2 \int_{B_1(\xi)} |x - \xi|^{-2\sigma} \leqslant C\lambda^{2\sigma}$$

and

$$\int_{\mathbb{R}^N\setminus B_1(\xi)} \left|N(\phi_\lambda)\frac{\partial w}{\partial x_j}\right| \leqslant C\big(\|\phi_\lambda\|_{*,\xi}^{(\sigma)}\big)^2 \int_{\mathbb{R}^N\setminus B_1(\xi)} |x-\xi|^{-\frac{4}{p-1}-3} \leqslant C\lambda^{2\sigma}.$$

Choosing $(N-2)/2 < \sigma < \min(N-2, N/2)$ we obtain

$$\int_{\mathbb{R}^N} \left|N(\phi_\lambda)\frac{\partial w}{\partial w_j}\right| = o\big(\lambda^{N-2}\big) \quad \text{as } \lambda \to 0. \tag{5.20}$$

Similarly, if $p < 2$ we have

$$\int_{\mathbb{R}^N} \left|N(\phi_\lambda)\frac{\partial w}{\partial w_j}\right| = O\big(\lambda^{p\sigma}\big) \quad \text{as } \lambda \to 0,$$

and taking $(N-2)/p < \sigma < \min(N-2, N/p)$ we still obtain (5.20).

In order to estimate the last term $\int_{\mathbb{R}^N} V_\lambda \phi_\lambda \frac{\partial w}{\partial x_j}$ in (5.18) we consider it together with (5.19). Let $u_\lambda = w + \phi_\lambda$. We claim that there exist two positive constants $c < C$, independent of λ such that

$$c < u_\lambda(x) < C, \quad x \in B_1(\xi). \tag{5.21}$$

A uniform upper bound for u_λ was already established in the proof of Lemma 5.1 in (5.10). We now show the lower bound in (5.21).

Observe first that u_λ solves

$$\Delta u - V_\lambda u + u^p = \sum_{i=1}^N c_i(\lambda) Z_i \quad \text{in } \mathbb{R}^N. \tag{5.22}$$

Consider the auxiliary function v defined by

$$v(r) = \begin{cases} a(r+\lambda)^q & \text{if } 0 < r < A\lambda \\ 1 + d(1 - r^{-s}) & \text{if } A\lambda < r < 1, \end{cases}$$

where the choice of the parameters A, s, q, a, d, c will be made shortly and $r = |x - \xi|$.

Recall that V satisfies $V \geqslant 0$, $V \in L^\infty(\mathbb{R}^N)$ and $V(x) \leqslant C|x|^{-\mu}$ where $\mu > N$. Actually it will be enough for the next argument that $\mu > 2$.

We take first s so that

$$0 < s < \min(1, \mu - 2).$$

Then choose a number $A > 0$ sufficiently large so that

$$\sup_x |x|^\mu V(x) \leqslant \min\left(\frac{1}{4}, \frac{s(N-2-s)}{8}\right) A^{\mu-2}. \tag{5.23}$$

Next we take $q \geqslant 1$ such that

$$q(q+N-2) = \max\Big(4\|V\|_{L^\infty(\mathbb{R}^N)},\ (A+1)^2 \sup_x |x|^\mu V(x)\Big) \tag{5.24}$$

and then

$$a = \frac{\lambda^{-q}}{(A+1)^{q-1}(A+1+\frac{q}{s}A(1-A^s\lambda^s))},$$
$$d = \frac{\lambda^s A^s}{\frac{s(A+1)}{qA} + 1 - A^s\lambda^s}.$$

We have

$$d \geqslant \frac{\lambda^s A^s}{3}$$

since $s \leqslant 1$, $q \geqslant 1$. Then v is C^1 in B_1, $v = 1$ on ∂B_1 and a calculation shows that v satisfies for $\lambda > 0$ sufficiently small

$$-\Delta v + \lambda^{-2} V\left(\frac{x}{\lambda}\right) v \leqslant 0 \quad \text{in } B_1. \tag{5.25}$$

To see this when $0 < r < \lambda$, using $\lambda^{-2}V(\frac{x}{\lambda}) \leqslant \lambda^{-2}\|V\|_{L^\infty}$, we estimate

$$\begin{aligned}
-\Delta v + \lambda^{-2} V\left(\frac{x}{\lambda}\right) v &= -aq(r+\lambda)^{q-2}(q+N-2) + \lambda^{-2} V\left(\frac{x}{\lambda}\right) a(r+\lambda)^q \\
&\leqslant a(r+\lambda)^{q-2}\big(-q(q+N-2) + \lambda^{-2}\|V\|_{L^\infty}(r+\lambda)^2\big) \\
&\leqslant a(r+\lambda)^{q-2}\big(-q(q+N-2) + 4\|V\|_{L^\infty}\big) \leqslant 0,
\end{aligned}$$

by (5.24). In the case $\lambda < r < A\lambda$ we use $\lambda^{-2}V(\frac{x}{\lambda}) \leqslant C_1 \lambda^{\mu-2}|x|^{-\mu}$ where $C_1 = \sup_x |x|^\mu V(x)$. We obtain

$$\begin{aligned}
-\Delta v + \lambda^{-2} V\left(\frac{x}{\lambda}\right) v &= -aq(r+\lambda)^{q-2}(N-2-q) + \lambda^{-2} V\left(\frac{x}{\lambda}\right) a(r+\lambda)^q \\
&\leqslant a(r+\lambda)^{q-2}\big(-q(q+N-2) + C_1\lambda^{\mu-2}r^{-\mu}(r+\lambda)^2\big) \\
&\leqslant a(r+\lambda)^{q-2}\big(-q(q+N-2) + C_1(A+1)^2\big) \leqslant 0
\end{aligned}$$

thanks to (5.24).

Next, when $A\lambda < r < 1$ we have

$$\begin{aligned}
-\Delta v + \lambda^{-2} V\left(\frac{x}{\lambda}\right) v &= -ds(N-2-s)r^{-2-s} + \lambda^{-2} V\left(\frac{x}{\lambda}\right)\big(1 + d\big(1 - r^{-s}\big)\big) \\
&\leqslant -\frac{\lambda^s A^s}{4} s(N-2-s)r^{-2-s} + C_1\lambda^{\mu-2}r^{-\mu}
\end{aligned}$$

$$= \lambda^s r^{-2-s}\left(-\frac{s(N-2-s)A^s}{4} + C_1\lambda^{\mu-2-s} r^{2+s-\mu}\right)$$

$$\leqslant \lambda^s r^{-2-s}\left(-\frac{s(N-2-s)A^s}{4} + C_1 A^{2+s-\mu}\right) \leqslant 0$$

by the choice of A (5.23).

Let $\chi(r) = \frac{1}{2N}(1-r^2)$, so that

$$-\Delta\chi \equiv 1, \quad \chi = 0 \quad \text{on } \partial B_1,$$

and consider $z = u_\lambda + (\sum_{i=1}^N |c_i(\lambda)| \|Z_i\|_{L^\infty})\chi$. Then from (5.22) (5.25) we deduce that

$$-\Delta z + \lambda^{-2} V\left(\frac{x}{\lambda}\right) z \geqslant 0 \quad \text{in } B_1.$$

The convergence $\phi_\lambda \to 0$ as $\lambda \to 0$ is uniform on compact sets of $\mathbb{R}^N \setminus \{0\}$ and hence $u_\lambda \to w$ uniformly on the sphere ∂B_1. Thus, by the maximum principle applied to the operator $-\Delta + \lambda^{-2} V(\frac{x}{\lambda})$ in B_1 we deduce

$$u_\lambda + \sum_{i=1}^N \left|c_i(\lambda)\right| \|Z_i\|_{L^\infty} \geqslant \frac{w(1)}{2} v \quad \text{in } B_1,$$

for λ small enough. Since v is bounded from below and $c_i(\lambda) \to 0$ we see that

$$u_\lambda \geqslant c \quad \text{in } B_1 \tag{5.26}$$

where $c > 0$ is independent of λ.

Thus we get (5.21). Going back to (5.18) we set

$$F_\lambda^{(j)}(\xi) = \lambda^{-2} \int\limits_{\mathbb{R}^N} V\left(\frac{x}{\lambda}\right) u_\lambda \frac{\partial w}{\partial x_j}(x+\xi) + \int\limits_{\mathbb{R}^N} N(\phi_\lambda) \frac{\partial w}{\partial x_j}(x+\xi)$$

and $F_\lambda = (F_\lambda^{(1)}, \ldots, F_\lambda^{(N)})$. Fix now $\delta > 0$ small and work with $|\xi| = \delta$. Then from (5.20), (5.26) and (5.10) we have for small λ

$$\left\langle F_\lambda(\xi), \xi \right\rangle < 0 \quad \text{for all } |\xi| = \delta.$$

By degree theory we deduce that F_λ has a zero in B_δ.

Part (b.1). Case $\lim_{|x|\to+\infty}(|x|^\mu V(x) - f(\frac{x}{|x|})) = 0$, where $N - \frac{4}{p-1} < \mu < N$, $f \not\equiv 0$.

Remark 5.1. We note that $2 < N - \frac{4}{p-1} < 3$ when $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$. Thus if $\mu \geqslant 3$ this condition is satisfied.

This situation is very different from the previous one. Here the main term of (5.18) behaves, as $\lambda \to 0$,

$$\lambda^{-2} \int_{\mathbb{R}^N} V\left(\frac{x}{\lambda}\right) w(x+\xi) \frac{\partial w}{\partial x_j}(x+\xi) \sim \lambda^{\mu-2} \int_{\mathbb{R}^N} |x|^{-\mu} f\left(\frac{x}{|x|}\right) w(x+\xi) \frac{\partial w}{\partial x_j}(x+\xi).$$

Indeed, we have

$$\begin{aligned} G_j(\xi) &:= \int_{\mathbb{R}^N} \left(\lambda^{-2} V\left(\frac{x}{\lambda}\right) \phi_\lambda(x+\xi) + N(\phi_\lambda) + \lambda^{-2} V\left(\frac{x}{\lambda}\right) w(x+\xi)\right) \frac{\partial w}{\partial x_j}(x+\xi) \\ &= \lambda^{\mu-2} \int_{\mathbb{R}^N} |x|^{-\mu} f\left(\frac{x}{|x|}\right) w(x+\xi) \frac{\partial w}{\partial x_j}(x+\xi) + o\left(\lambda^{\mu-2}\right) \end{aligned} \tag{5.27}$$

uniformly for ξ on compact sets of $\mathbb{R}^N$. This is proved observing first that

$$\int_{\mathbb{R}^N} \left| N(\phi_\lambda) \frac{\partial w}{\partial w_j}(x+\xi) \right| = o\left(\lambda^{\mu-2}\right) \quad \text{as } \lambda \to 0$$

uniformly for ξ on compact sets of $\mathbb{R}^N$, as follows from (5.20), for instance taking $\sigma = \mu - 2$.

Using now (5.21), we have that

$$\left| \lambda^{-2} \int_{\mathbb{R}^N} V\left(\frac{x}{\lambda}\right) \phi_\lambda(x) \frac{\partial w}{\partial x_j}(x+\xi)\, dx \right| \leqslant C \lambda^{\mu-2+\sigma}. \tag{5.28}$$

Indeed we see that

$$\begin{aligned} \left| \lambda^{-2} \int_{B_1(0)} V\left(\frac{x}{\lambda}\right) \phi_\lambda(x+\xi) \frac{\partial w}{\partial x_j}(x+\xi)\, dx \right| &\leqslant C \lambda^{\mu-2} \|\phi_\lambda\|_{*,\xi}^{(\sigma)} \int_{B_1(0)} |x|^{-\mu-\sigma}\, dx \\ &\leqslant C \lambda^{\mu-2+\sigma}; \end{aligned} \tag{5.29}$$

and

$$\begin{aligned} \left| \lambda^{-2} \int_{\mathbb{R}^N \setminus B_1(0)} V\left(\frac{x}{\lambda}\right) \phi_\lambda(x) \frac{\partial w}{\partial x_j}(x+\xi)\, dx \right| &\leqslant C \lambda^{\mu-2} \|\phi_\lambda\|_{*,\xi}^{(\sigma)} \int_{R^N \setminus B_1(0)} |x|^{-\mu} |x|^{-\frac{4}{p-1}-1} \\ &\leqslant C \lambda^{\mu-2+\sigma}. \end{aligned}$$

Define now $\tilde{F}$ to be given by

$$\tilde{F}(\xi) := \frac{1}{2} \int_{\mathbb{R}^N} |x|^{-\mu} f\left(\frac{x}{|x|}\right) w(x+\xi)^2\, dx.$$

By the dominated convergence theorem

$$\tilde{F}(\xi) = \frac{\beta^{2/(p-1)}}{2}|\xi|^{N-\mu-\frac{4}{p-1}} \int\limits_{\mathbb{R}^N} |y|^{-\mu} f\left(\frac{y}{|y|}\right)\left|y + \frac{\xi}{|\xi|}\right|^{-\frac{4}{p-1}} + o\bigl(|\xi|^{N-\mu-\frac{4}{p-1}}\bigr).$$

Similarly

$$\nabla\tilde{F}(\xi)\cdot\xi = \frac{\beta^{2/(p-1)}}{2}\left(N-\mu-\frac{4}{p-1}\right)|\xi|^{N-\mu-\frac{4}{p-1}} \int |y|^{-\mu} f\left(\frac{y}{|y|}\right)\left|y + \frac{\xi}{|\xi|}\right|^{-\frac{4}{p-1}} + o\bigl(|\xi|^{N-\mu-\frac{4}{p-1}}\bigr).$$

Therefore

$$\nabla\tilde{F}(\xi)\cdot\xi < 0 \quad \text{for all } |\xi| = R$$

for large R. Using this and degree theory we obtain the existence of ξ such that $c_i = 0$, $1 \leqslant i \leqslant N$.

Part (b.2). Case $\lim_{|x|\to+\infty}(|x|^N V(x) - f(\frac{x}{|x|})) = 0$, where $f \not\equiv 0$.

In this case, we will have

$$\begin{aligned} G_j(\xi) &:= \int\limits_{\mathbb{R}^N}\left(\lambda^{-2}V\left(\frac{x}{\lambda}\right)\phi_\lambda(x+\xi) + N(\phi_\lambda) + \lambda^{-2}V\left(\frac{x}{\lambda}\right)w(x+\xi)\right)\frac{\partial w}{\partial x_j}(x+\xi) \\ &= \int\limits_{\mathbb{R}^N}\lambda^{-2}V\left(\frac{x}{\lambda}\right)u_\lambda(x+\xi)\frac{\partial w}{\partial x_j}(x+\xi) + O\bigl(\lambda^{N-2}\bigr) \end{aligned} \tag{5.30}$$

uniformly for ξ on compact sets of $\mathbb{R}^N$.

Similar to part (a), we derive that for small fixed ρ

$$\bigl\langle G(\xi), \xi\bigr\rangle < 0 \quad \text{for all } |\xi| = \rho. \tag{5.31}$$

Indeed, for $\rho > 0$ small it holds

$$\bigl\langle \nabla w(\xi), \xi\bigr\rangle < 0 \quad \text{for all } |\xi| = \rho.$$

Thus, for $\delta > 0$ small and fixed

$$\gamma \equiv \sup_{x\in B_\delta}\bigl\langle \nabla w(x+\xi), \xi\bigr\rangle < 0 \quad \text{for all } |\xi| = \rho. \tag{5.32}$$

We decompose

$$\int\limits_{\mathbb{R}^N}\lambda^{-2}V\left(\frac{x}{\lambda}\right)u_\lambda(x)\bigl\langle\nabla w(x+\xi), \xi\bigr\rangle = \int\limits_{B_\delta}\cdots + \int\limits_{\mathbb{R}^N\setminus B_\delta}\cdots,$$

where

$$\left|\lambda^{-2}\int_{\mathbb{R}^N\setminus B_\delta} V\left(\frac{x}{\lambda}\right)u_\lambda(x+\xi)\big\langle\nabla w(x+\xi),\xi\big\rangle\,dx\right| \leqslant C\lambda^{N-2}\int_{|x|\geqslant\delta}|x|^{-N}|x|^{-\frac{4}{p-1}-1}$$
$$\leqslant C\lambda^{N-2}. \tag{5.33}$$

On the other hand, for $R>0$ we may write

$$\int_{B_\delta}\lambda^{-2}V\left(\frac{x}{\lambda}\right)u_\lambda(x+\xi)\big\langle\nabla w(x+\xi),\xi\big\rangle = \int_{B_\delta\setminus B_{\lambda R}}\cdots+\int_{B_{\lambda R}}\cdots.$$

We have

$$\int_{B_{\lambda R}} V\left(\frac{x}{\lambda}\right)u_\lambda(x+\xi)\big\langle\nabla w(x+\xi),\xi\big\rangle = O\big(\lambda^N\big). \tag{5.34}$$

Since, by (5.21), $c_1\leqslant u_\lambda(x)\leqslant c_2$ for all $x\in B_1(\xi)$ where $0<c_1<c_2$, using (5.32) we obtain

$$\int_{B_\delta\setminus B_{\lambda R}} V\left(\frac{x}{\lambda}\right)u_\lambda(x+\xi)\big\langle\nabla w(x+\xi),\xi\big\rangle \leqslant c_1\gamma\int_{B_\delta\setminus B_{\lambda R}} V\left(\frac{x}{\lambda}\right). \tag{5.35}$$

But

$$\int_{B_\delta\setminus B_{\lambda R}} V\left(\frac{x}{\lambda}\right)dx = \int_{B_\delta\setminus B_{\lambda R}}|x|^{-N}f\left(\frac{x}{|x|}\right)dx$$
$$+\int_{B_\delta\setminus B_{\lambda R}}|x|^{-N}\left(V(x)|x|^N-f\left(\frac{x}{|x|}\right)\right)dx$$

and

$$\int_{B_\delta\setminus B_{\lambda R}}|x|^{-N}f\left(\frac{x}{|x|}\right)dx = \log\frac{1}{\lambda}\int_{S^{N-1}}f+O(1) \tag{5.36}$$

while given any $\varepsilon>0$ there is $R>0$ such that

$$\left|\int_{B_\delta\setminus B_{\lambda R}}|x|^{-N}\left(V(x)|x|^N-f\left(\frac{x}{|x|}\right)\right)dx\right| \leqslant \varepsilon\log\frac{1}{\lambda}. \tag{5.37}$$

From (5.33)–(5.37) we deduce the validity of (5.31). Applying again degree theory we conclude that for some $|\xi|<\rho$ we have $G(\xi)=0$. This finishes the proof of the theorem. □

Remark 5.2. We remark that the above functional analytic setting could have also been applied in the proof of Theorem 1, so that the continuum of solutions there found turns out to be a two-parameter family, dependent not only on all small λ but also on a point ξ arbitrary taken to be the origin.

Remark 5.3. The proof of Theorem 2 in the case $p = \frac{N+1}{N-3}$ follows exactly the same lines with the modified norms as explained at the beginning of this section. The argument works because we assume here that V has more decay, which implies that even with the modified norm, the error $\|V_\lambda w\|_{**,\xi}^{(\sigma)}$ converges to 0. Indeed, we have

$$\sup_{|x-\xi|\geqslant 1} |x-\xi|^{2+\frac{2}{p-1}+\alpha}\lambda^{-2}V\left(\frac{x-\xi}{\lambda}\right)w(x) \leqslant C\lambda^{\mu-2}\sup_{|x-\xi|\geqslant 1}|x|^{2+\alpha-\mu} = C\lambda^{\mu-2}$$

provided $\alpha < \mu - 2$.

Appendix A. Proofs of Propositions 2.1 and 2.2

Next we proceed to the proofs of Propositions 2.1 and 2.2.

Let (ϕ, h) satisfy (2.1). We write h as

$$h(x) = \sum_{k=0}^{\infty} h_k(r)\Theta_k(\theta), \quad r > 0,\ \theta \in S^{N-1}, \tag{A.1}$$

where Θ_k, $k \geqslant 0$ are the eigenfunctions of the Laplace–Beltrami operator $-\Delta_{S^{N-1}}$ on the sphere S^{N-1}, normalized so that they constitute an orthonormal system in $L^2(S^{N-1})$. We take Θ_0 to be a positive constant, associated to the eigenvalue 0 and Θ_i, $1 \leqslant i \leqslant N$ is an appropriate multiple of $\frac{x_i}{|x|}$ which has eigenvalue $\lambda_i = N-1$, $1 \leqslant i \leqslant N$. In general, λ_k denotes the eigenvalue associated to Θ_k, we repeat eigenvalues according to their multiplicity and we arrange them in an non-decreasing sequence. We recall that the set of eigenvalues is given by $\{j(N-2+j) \mid j \geqslant 0\}$.

We look for a solution ϕ to (2.1) in the form

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k(r)\Theta_k(\theta).$$

Then ϕ satisfies (2.1) if and only if

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi_k = h_k \quad \text{for all } r > 0,\ \text{for all } k \geqslant 0. \tag{A.2}$$

To construct solutions of this ODE we need to consider two linearly independent solutions $z_{1,k}$, $z_{2,k}$ of the homogeneous equation

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi_k = 0, \quad r \in (0,\infty). \tag{A.3}$$

Once these generators are identified, the general solution of the equation can be written through the variation of parameters formula as

$$\phi(r) = z_{1,k}(r)\int z_{2,k}h_k r^{N-1}\,dr - z_{2,k}(r)\int z_{1,k}h_k r^{N-1}\,dr,$$

where the symbol $\int$ designates arbitrary antiderivatives, which we will specify in the choice of the operators. It is helpful to recall that if one solution $z_{1,k}$ to (A.3) is known, a second, linearly independent solution can be found in any interval where $z_{1,k}$ does not vanish as

$$z_{2,k}(r) = z_{1,k}(r) \int z_{1,k}(r)^{-2} r^{1-N}\, dr. \tag{A.4}$$

One can get the asymptotic behaviors of any solution z as $r \to 0$ and as $r \to +\infty$ by examining the indicial roots of the associated Euler equations. It is known that as $r \to +\infty$ $r^2 w(r)^{p-1} \to \beta$ where

$$\beta = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right).$$

Thus we get the limiting equation, for $r \to \infty$,

$$r^2\phi'' + (N-1)r\phi' + (p\beta - \lambda_k)\phi = 0, \tag{A.5}$$

while as $r \to 0$,

$$r^2\phi'' + (N-1)r\phi' - \lambda_k\phi = 0. \tag{A.6}$$

In this way the respective behaviors will be ruled by $z(r) \sim r^{-\mu}$ as $r \to +\infty$ where μ solves

$$\mu^2 - (N-2)\mu + (p\beta - \lambda_k) = 0$$

while as $r \to 0$ μ satisfies

$$\mu^2 - (N-2)\mu - \lambda_k = 0.$$

The following lemma takes care of mode zero.

Lemma A.1. *Let $k = 0$ and $p > \frac{N+2}{N-2}$. Then Eq.* (A.2) *has a solution ϕ_0 which depends linearly on h_0 and satisfies*

$$\|\phi_0\|_* \leqslant C\|h_0\|_{**}. \tag{A.7}$$

Proof. For $k = 0$ the possible behaviors at 0 for a solution $z(r)$ to (A.3) are simply

$$z(r) \sim 1, \qquad z(r) \sim r^{2-N}$$

while at $+\infty$ this behavior is more complicated. The indicial roots of (A.6) are given by

$$\mu_{0\pm} = \frac{N-2}{2} \pm \frac{1}{2}\sqrt{(N-2)^2 - 4p\beta}.$$

The situation depends of course on the sign of $D = (N-2)^2 - 4p\beta$. It is observed in [19] that $D > 0$ if and only if $N > 10$ and $p > p_c$ where we set

$$p_c = \begin{cases} \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N > 10, \\ \infty & \text{if } N \leqslant 10. \end{cases}$$

Thus when $p < p_c$, $\mu_{0\pm}$ are complex with negative real part, and the behavior of a solution $z(r)$ as $r \to +\infty$ is oscillatory and given by

$$Z(r) = O\big(r^{-\frac{N-2}{2}}\big).$$

When $p > p_c$, we have $\mu_{0+} > \mu_{0-} > \frac{2}{p-1}$.

Independently of the value of p, one can get immediately a solution of the homogeneous problem. Since Eq. (1.6) is invariant under the transformation $\lambda \mapsto \lambda^{\frac{2}{p-1}} w(\lambda r)$ we see by differentiation in λ that the function

$$z_{1,0} = r w' + \frac{2}{p-1} w$$

satisfies Eq. (A.3) for $k = 0$. At this point it is useful to recall asymptotic formulae derived in [19] which yield the asymptotic behavior for w. It is shown that if $p = p_c$,

$$w(r) = \frac{\beta^{\frac{1}{p-1}}}{r^{\frac{2}{p-1}}} + \frac{a_1 \log r}{r^{\mu_{0-}}} + o\left(\frac{\log r}{r^{\mu_{0-}}}\right), \quad r \to +\infty, \tag{A.8}$$

where $a_1 < 0$, and if $p > p_c$

$$w(r) = \frac{\beta^{\frac{1}{p-1}}}{r^{\frac{2}{p-1}}} + \frac{a_1}{r^{\mu_{0-}}} + o\left(\frac{1}{r^{\mu_{0-}}}\right), \quad r \to +\infty. \tag{A.9}$$

Using these estimates, and easily derived ones for w', we get that as $r \to +\infty$

$$\text{if } p < p_c\text{:} \quad \big|z_{1,0}(r)\big| \leqslant C r^{\frac{N-2}{2}}, \tag{A.10}$$

$$\text{if } p = p_c\text{:} \quad z_{1,0}(r) = c r^{-\frac{N-2}{2}} \log r\big(1 + o(1)\big), \tag{A.11}$$

$$\text{if } p > p_c\text{:} \quad z_{1,0}(r) = c r^{-\mu_{0-}}\big(1 + o(1)\big), \tag{A.12}$$

where $c \neq 0$.

Case $p < p_c$. We define $z_{2,0}(r)$ for small $r > 0$ by

$$z_{2,0}(r) = z_{1,0}(r) \int_{r_0}^{r} z_{1,0}(s)^{-2} s^{1-N}\, ds, \tag{A.13}$$

where r_0 is small so that $z_{1,0} > 0$ in $(0, r_0)$ (which is possible because $z_{1,r} \sim 1$ near 0). Then $z_{2,0}$ is extended to $(0, +\infty)$ so that it is a solution to the homogeneous equation (A.3) (with $k = 0$) in this interval. As mentioned earlier $z_{2,0}(r) = O(r^{-\frac{N-2}{2}})$ as $r \to +\infty$.

We define

$$\phi_0(r) = z_{1,0}(r) \int_1^r z_{2,0} h_0 s^{N-1}\, ds - z_{2,0}(r) \int_0^r z_{1,0} h_0 s^{N-1}\, ds,$$

and omit a calculation that shows that this expression satisfies (A.7).

Case $p \geqslant p_c$. The strategy is the same as in the previous case, but this time it is more convenient to rewrite the variation of parameters formula in the form

$$\phi_0(r) = -z_{1,0}(r) \int_1^r z_{1,0}(s)^{-2} s^{1-N} \int_0^s z_{1,0}(\tau) h_0(\tau) \tau^{N-1}\, d\tau\, ds,$$

which is justified because when $p \geqslant p_c$ we have $z_{1,0}(r) > 0$ for all $r > 0$, which follows from the fact that $\lambda \mapsto \lambda^{\frac{2}{p-1}} w(\lambda r)$ is increasing for $\lambda > 0$, see [19]. Again, a calculation using now (A.11) and (A.12) shows that ϕ_0 satisfies the estimate (A.7). □

Next we consider mode $k = 1$.

Lemma A.2.

(a) *Let $k = 1$ and $p > \frac{N+1}{N-3}$. Then Eq.* (A.2) *has a solution ϕ_1 which is linear with respect to h_1 and satisfies*

$$\|\phi_1\|_* \leqslant C\|h_1\|_{**}. \tag{A.14}$$

(b) *Let $N \geqslant 3$ and $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ ($p > \frac{N+2}{N-2}$ if $N = 3$). If $\|h\|_{**} < +\infty$ and*

$$\int_0^\infty h_1(r) w'(r) r^{N-1}\, dr = 0 \tag{A.15}$$

then (A.2) *has a solution ϕ_1 satisfying* (A.14) *and depending linearly on h_1* (*condition* (A.15) *makes sense when $p < \frac{N+1}{N-3}$ and $\|h_1\|_{**} < +\infty$*).

Proof. (a) In this case the indicial roots that govern the behavior of the solutions $z(r)$ as $r \to +\infty$ of the homogeneous equation (A.3) are given by $\mu_1 = \frac{2}{p-1} + 1$ and $\mu_2 = N - 3 - \frac{2}{p-1}$. Since we are looking for solutions that decay at a rate $r^{-\frac{2}{p-1}}$ as $r \to +\infty$ we will need $N - 3 - \frac{2}{p-1} > \frac{2}{p-1}$, which is equivalent to the hypothesis $p > \frac{N+1}{N-3}$. On the other hand the behavior near 0 of $z(r)$ can be $z(r) \sim r$ or $z(r) \sim r^{1-N}$.

Similarly as in the case $k = 0$ we have a solution to (A.3), namely $z_1(r) = -w'(r)$ and luckily enough it is positive in all $(0, +\infty)$. With it we can build

$$\phi_1(r) = -z_1(r) \int_1^r z_1(s)^{-2} s^{1-N} \int_0^s z_1(\tau) h_1(\tau) \tau^{N-1} \, d\tau \, ds. \tag{A.16}$$

From this formula and using $p > \frac{N+1}{N-3}$ we obtain (A.14).

(b) Since $z_1(r) \leqslant C r^{-\frac{2}{p-1}-1}$ and $p < \frac{N+1}{N-3}$ it is not difficult to check that $z_1 h_1 \tau^{N-1}$ is integrable in $(0, +\infty)$ if $\|h_1\|_{**} < +\infty$. Thus, by (A.15) we can rewrite (A.16) as

$$\phi_1(r) = z_1(r) \int_1^r z_1(s)^{-2} s^{1-N} \int_s^\infty z_1(\tau) h_1(\tau) \tau^{N-1} \, d\tau \, ds \tag{A.17}$$

and from this formula (A.14) readily follows. $\square$

Finally we consider mode $k \geqslant 2$.

Lemma A.3. *Let $k \geqslant 2$ and $p > \frac{N+2}{N-2}$. If $\|h_k\|_{**} < \infty$ Eq.* (A.2) *has a unique solution ϕ_k with $\|\phi_k\|_* < \infty$ and there exists $C_k > 0$ such that*

$$\|\phi_k\|_* \leqslant C_k \|h_k\|_{**}. \tag{A.18}$$

Proof. Let us write L_k for the operator in (A.2), that is,

$$L_k \phi = \phi'' + \frac{N-1}{r} \phi' + \left(p w^{p-1} - \frac{\lambda_k}{r^2} \right) \phi.$$

This operator satisfies the maximum principle in any interval of the form $(\delta, \frac{1}{\delta})$, $\delta > 0$. Indeed let $z = -w'$, so that $z > 0$ in $(0, +\infty)$ and it is a supersolution, because

$$L_k z = \frac{N-1-\lambda_k}{r^2} z < 0 \quad \text{in } (0, +\infty), \tag{A.19}$$

since $\lambda_k \geqslant 2N$ for $k \geqslant 2$. To prove solvability of (A.2) in the appropriate space we construct a supersolution ψ of the form

$$\psi = C_1 z + v, \qquad v(r) = \frac{1}{r^\sigma + r^{\frac{2}{p-1}}},$$

where C_1 is going to be fixed later on. A computation shows that

$$L_k v = \left(2N - 4 - \frac{4}{p-1} - \lambda_k \right) r^{-\frac{2}{p-1}-2} \bigl(1 + o(1)\bigr), \quad r \to +\infty,$$

and hence

$$L_k\psi \leqslant -\frac{4p}{p-1}r^{-\frac{2}{p-1}-2} + o\big(r^{-\frac{2}{p-1}}\big), \quad r \to +\infty.$$

Similarly

$$L_k v = \big(\sigma^2 - (N-2)\sigma - \lambda_k\big)r^{-\sigma-2}\big(1 + o(1)\big), \quad r \to 0.$$

Therefore we may find $0 < R_1 < R_2$ (independent of C_1) such that

$$L_k\psi \leqslant -r^{-\sigma-2}, \quad r \leqslant R_1,$$

and

$$L_k\psi \leqslant -\frac{2p}{p-1}r^{-\frac{2}{p-1}-2}, \quad r \geqslant R_2.$$

Using (A.19) we find C_1 large so that

$$L_k\psi \leqslant -c\min\big(r^{-\sigma-2}, r^{-\frac{2}{p-1}-2}\big) \quad \text{in } (0, +\infty),$$

for some $c > 0$.

For h_k with $\|h_k\|_{**} < \infty$ by the method of sub- and supersolutions there exists, for any $\delta > 0$ a solution ϕ_δ of

$$L_k\phi_\delta = h_k \quad \text{in } \left(\delta, \frac{1}{\delta}\right),$$
$$\phi_\delta(\delta) = \phi_\delta\left(\frac{1}{\delta}\right) = 0$$

satisfying the bound

$$|\phi_\delta| \leqslant C\psi\|h_k\|_{**} \quad \text{in } \left(\delta, \frac{1}{\delta}\right).$$

Using standard estimates up to a subsequence we have $\phi_\delta \to \phi_k$ as $\delta \to 0$ uniformly on compact subsets of $(0, +\infty)$, and ϕ_k is a solution of (A.2) which satisfies

$$|\phi_k| \leqslant C\psi\|h_k\|_{**} \quad \text{in } (0, \infty).$$

The maximum principle yields that the solution to (A.2) bounded in this way is actually unique. $\square$

We are ready to complete the proofs of Propositions 2.1 and 2.2.

Proofs of Propositions 2.1 and 2.2. Let $m > 0$ be an integer. By Lemmas A.1, A.2 and A.3 we see that if $\|h\|_{**} < \infty$ and its Fourier series (A.1) has $h_k \equiv 0$ $\forall k \geqslant m$ there exists a solution ϕ to (2.1) that depends linearly with respecto to h and moreover

$$\|\phi\|_* \leqslant C_m \|h\|_{**}$$

where C_m may depend only on m. We shall show that C_m may be taken independent of m. Assume on the contrary that there is sequence of functions h_j such that $\|h_j\|_{**} < \infty$, each h_j has only finitely many nontrivial Fourier modes and that the solution $\phi_j \not\equiv 0$ satisfies

$$\|\phi_j\|_* \geqslant C_j \|h_j\|_{**},$$

where $C_j \to +\infty$ as $j \to +\infty$. Replacing ϕ_j by $\frac{\phi_j}{\|\phi_j\|_*}$ we may assume that $\|\phi_j\|_* = 1$ and $\|h_j\|_{**} \to 0$ as $j \to +\infty$. We may also assume that the Fourier modes associated to $\lambda_0 = 0$ and $\lambda_1 = \cdots = \lambda_N = N - 1$ are zero.

Along a subsequence (which we write the same) we must have

$$\sup_{|x|>1} |x|^{\frac{2}{p-1}} \left|\phi_j(x)\right| \geqslant \frac{1}{2} \tag{A.20}$$

or

$$\sup_{|x|<1} |x|^{\sigma} \left|\phi_j(x)\right| \geqslant \frac{1}{2}. \tag{A.21}$$

Assume first that (A.20) occurs and let $x_j \in \mathbb{R}^N$ with $|x_j| > 1$ be such that

$$|x_j|^{\frac{2}{p-1}} \left|\phi_j(x_j)\right| \geqslant \frac{1}{4}.$$

Then again we have to distinguish two possibilities. Along a new subsequence (denoted the same) $x_j \to x_0 \in \mathbb{R}^N$ or $|x_j| \to +\infty$.

If $x_j \to x_0$ then $|x_0| \geqslant 1$ and by standard elliptic estimates $\phi_j \to \phi$ uniformly on compact sets of $\mathbb{R}^N$. Thus ϕ is a solution to (2.1) with right-hand side equal to zero that also satisfies $\|\phi\|_* < +\infty$ and is such that the Fourier modes $\phi_0 = \cdots = \phi_N$ are zero. But the unique solution to this problem is $\phi \equiv 0$, contradicting $|\phi(x_0)| \geqslant \frac{1}{4}$.

If $|x_j| \to \infty$ consider $\tilde\phi_j(y) = |x_j|^{\frac{2}{p-1}} \phi_j(|x_j| y)$. Then $\tilde\phi_j$ satisfies

$$\Delta\tilde\phi_j + p w_j^{p-1} \tilde\phi_j = \tilde h_j \quad \text{in } \mathbb{R}^N$$

where $w_j(y) = |x_j|^{\frac{2}{p-1}} w(|x_j| y)$ and $\tilde h_j(y) = |x_j|^{2+\frac{2}{p-1}} h(|x_j| y)$. But since $\|\phi_j\|_* = 1$ we have

$$\left|\tilde\phi_j(y)\right| \leqslant |y|^{-\frac{2}{p-1}}, \quad |y| > \frac{1}{|x_j|}, \tag{A.22}$$

so $\tilde{\phi}_j$ is uniformly bounded on compact sets of $\mathbb{R}^N \setminus \{0\}$. Similarly, for $|y| > \frac{1}{|x_j|}$

$$\left|\tilde{h}_j(y)\right| \leqslant |y|^{-2-\frac{2}{p-1}} \|h_j\|_{**}$$

and hence $\tilde{h}_j \to 0$ uniformly on compact sets of $\mathbb{R}^N \setminus \{0\}$ as $j \to +\infty$. Also $w_j(y) \to C_{p,N}|y|^{-\frac{2}{p-1}}$ uniformly on compact sets of $\mathbb{R}^N \setminus \{0\}$. By elliptic estimates $\tilde{\phi}_j \to \phi$ uniformly on compact sets of $\mathbb{R}^N \setminus \{0\}$ and ϕ solves

$$\Delta\phi + C_{p,N}|y|^{-\frac{2}{p-1}}\phi = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Moreover, since $\tilde{\phi}_j(\frac{x_j}{|x_j|}) \geqslant \frac{1}{4}$ we see that ϕ is nontrivial, and from (A.22) we have the bound

$$\left|\phi(y)\right| \leqslant |y|^{-\frac{2}{p-1}}, \quad |y| > 0. \tag{A.23}$$

Expanding ϕ as

$$\phi(x) = \sum_{k=N+1}^{\infty} \phi_k(r)\Theta_k(\theta)$$

(we assumed at the beginning that the first $N+1$ Fourier modes were zero) we see that ϕ_k has to be a solution to

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \frac{\beta p - \lambda_k}{r^2}\phi_k = 0 \quad \forall r > 0,\ \forall k \geqslant N+1.$$

The solutions to this equation are linear combinations of $r^{a_k^\pm}$ where $a_k^+ > 0$ and $a_k^- < 0$. Thus ϕ_k cannot have a bound of the form (A.23) unless it is identically zero, a contradiction.

The analysis of the case (A.21) is similar and this proves our claim. By density, for any h with $\|h\|_{**} < \infty$ a solution ϕ to (2.1) can be constructed and it satisfies $\|\phi\|_* \leqslant C\|h\|_{**}$.

The necessity of condition (2.6) is handled in the following lemma. □

Lemma A.4. *Suppose* $\|h\|_{**} < +\infty$ *and that* ϕ *is a solution to* (2.1) *such that* $\|\phi\|_* < +\infty$. *Then necessarily* h *satisfies* (2.6).

Proof. Let

$$\phi_1(r) = \int_{S^{N-1}} \phi(r\theta)\Theta_1(\theta)\,d\theta, \quad r > 0,$$

and

$$h_1(r) = \int_{S^{N-1}} h(r\theta)\Theta_1(\theta)\,d\theta, \quad r > 0.$$

Then

$$\phi_1'' + \frac{N-1}{r}\phi_1' + \left(pw^{p-1} - \frac{N-1}{r^2}\right)\phi_1 = h_1 \quad \text{for all } r > 0 \tag{A.24}$$

and we know $|\phi_1(r)| \leqslant Cr^{-\frac{2}{p-1}}$ for $r \geqslant 1$, $|\phi_1(r)| \leqslant Cr^{-\sigma}$ for $0 < r \leqslant 1$. From elliptic estimates we also know $|\phi_1'(r)| \leqslant Cr^{-\frac{2}{p-1}-1}$ for $r \geqslant 1$ and $|\phi_1(r)| \leqslant Cr^{-\sigma-1}$ for $0 < r \leqslant 1$. Multiplying (A.24) by w' and integrating by parts in the interval $[\delta, \frac{1}{\delta}]$ where $\delta > 0$ we find

$$\begin{aligned}&\left(-r^{N-1}\phi_1 w'' + r^{N-1}\phi_1' w'\right)\Big|_\delta^{1/\delta} + \int_\delta^{1/\delta}\left(\left(r^{N-1}w''\right)' + r^{N-1}\left(pw^{p-1} - \frac{N-1}{r}\right)w'\right)\phi_1 \\ &\quad = \int_\delta^{1/\delta} r^{N-1}h_1 w'.\end{aligned} \tag{A.25}$$

But w' is a solution of (A.24) with right-hand side equal to 0 and hence, letting $\delta \to 0$ and using $p < \frac{N+1}{N-3}$ we obtain

$$0 = \int_0^\infty h_1 w' r^{N-1}\, dr$$

which is the desired conclusion. □

Remark A.1. If $p = \frac{N+1}{N-3}$ Proposition 2.2 and Lemma A.4 are still valid if one redefines the norms as

$$\|\phi\|_* = \sup_{|x|\leqslant 1} |x|^\sigma \left|\phi(x)\right| + \sup_{|x|\geqslant 1} |x|^{\frac{2}{p-1}+\alpha}\left|\phi(x)\right|,$$

$$\|h\|_{**} = \sup_{|x|\leqslant 1} |x|^{2+\sigma}\left|h(x)\right| + \sup_{|x|\geqslant 1} |x|^{\frac{2}{p-1}+2+\alpha}\left|h(x)\right|,$$

where $\alpha > 0$ is fixed small. Indeed, in relation (A.25) the boundary terms still go away as $\delta \to 0$ if h_1 decays faster than $r^{-\frac{2}{p-1}-2-\alpha}$ because in such a case the solution ϕ_1, a decaying solution of Eq. (A.24), can be re-expressed for large r as

$$\phi_1(r) = cw'(r) + O\left(r^{-\frac{2}{p-1}-\alpha}\right), \qquad \phi_1'(r) = cw''(r) + O\left(r^{-\frac{2}{p-1}-1-\alpha}\right)$$

for a certain constant c. Let us also observe that formula (A.17) has the right mapping property for the above norms provided that the orthogonality condition holds.

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