CHAPTER 2

Singular Solutions of Semi-Linear Elliptic Problems

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Abstract

We are concerned in this survey with singular solutions to semi-linear elliptic problems. An example of the type of equations we are interested in is the Gelfand–Liouville problem $-\Delta u = \lambda e^u$ on a smooth bounded domain Ω of $\mathbb{R}^N$ with zero Dirichlet boundary condition. We explore up to what degree known results for this problem are valid in other situations with a similar structure, with emphasis on the extremal solution and its properties. Of interest is the question of identifying conditions such that the extremal solution is singular. We find that, in the problems studied, there is a strong link between these conditions and Hardy-type inequalities.

Keywords: Singular solution, Blow-up solution, Stability, Perturbation of singular solutions

Contents

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1. Introduction

In this survey we are interested in singular solutions to semi-linear partial differential equations of the form

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $\lambda > 0$ and $g : [0, \infty) \to \mathbb{R}$ satisfies

$$g \text{ is smooth increasing, convex, } g(0) > 0 \tag{1.2}$$

and superlinear at $+\infty$ in the sense

$$\lim_{u\to+\infty} \frac{g(u)}{u} = +\infty. \tag{1.3}$$

Some typical examples are $g(u) = e^u$ and $g(u) = (1 + u)^p$ with $p > 1$.

We are also interested in some variants of (1.1) such as

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda g(u) & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2, \end{cases} \tag{1.4}$$

where $\lambda > 0$ and $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain and Γ_1, Γ_2 is a partition of $\partial\Omega$ into surfaces separated by a smooth interface, and ν is the exterior unit normal vector.

We shall consider as well the fourth-order equation

$$\begin{cases} \Delta^2 u = \lambda g(u) & \text{in } B \\ u = 0 & \text{on } \partial B \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B, \end{cases} \tag{1.5}$$

where B is the unit ball in $\mathbb{R}^N$.

Equations of the form (1.1) have been studied in various contexts and applications. Liouville [85] considered this equation with $g(u) = e^u$ in connection to surfaces with constant Gauss curvature. The exponential nonlinearity in dimension 3 appears in connection with the equilibrium of gas spheres and the structure of stars, see Emden [53], Fowler [60] and Chandrasekhar [29]. Later Frank-Kamenetskii [61] obtained a model like (1.1) with $g(u) = (1 - \varepsilon u)^m e^{u/(1+\varepsilon u)}$ in combustion theory. Also in connection with combustion theory, Barenblatt, in a volume edited by Gelfand [69], studied the case $g(u) = e^u$ in a ball in dimensions 2 and 3. Since then, this problem has attracted the attention of many researchers [10,19,20,24,34,35,62–64,76,79,83,93,94].

Boundary value problems of the form (1.4) with exponential nonlinearity arise in conformal geometry when prescribing Gaussian curvature of a 2-dimensional domain and curvature of the boundary, see for instance Li, Zhu [84] and the references therein. The study of conformal transformations in manifolds with boundary in higher dimensions also gives rise to nonlinear boundary conditions, see Cherrier [31] and Escobar [54–56]. A related motivation is the study of Sobolev spaces and inequalities, specially the Sobolev trace theorem, see Aubin [6] and the surveys of Rossi [105] and Druet, Hebey [52]. In connection with physical models (1.4), exponential nonlinearity appears in corrosion modelling where there is an exponential relationship between boundary voltages and boundary normal currents. See [21,78,92,107] and [46] for the derivation of this and related corrosion models and references to the applied literature. Nonlinear boundary conditions appear also in some models of heat propagation, where u is the temperature and the normal derivative $\frac{\partial u}{\partial \nu}$ in (1.4) is the heat flux. In [86] the authors derive a similar model in a combustion problem where the reaction happens only at the boundary of the container.

Higher-order equations have attracted the attention of many researchers in the last few years. In particular fourth-order equations with an exponential nonlinearity have been studied in 4 dimensions, in a setting analogous to Liouville's equation by Wei [108], Djadli and Malchiodi [48] and Baraket *et al.* [7]. In higher dimensions Arioli *et al.* [4] considered the bilaplacian together with the exponential nonlinearity in the whole of $\mathbb{R}^N$ and Arioli *et al.* [5] studied (1.5) for $g(u) = e^u$ in ball, which is the natural fourth-order analogue of the classical Gelfand problem (1.1) with $g(u) = e^u$.

A general objective concerning equations (1.1), (1.4) and (1.5) is to study the structure of all solutions (λ, u) and the existence and qualitative properties of singular solutions. These problems share the same basic result:

THEOREM 1.1. *For problems* (1.1), (1.4) *and* (1.5) *there exists a finite parameter* $\lambda^* > 0$ *such that:*

(1) *if* $0 < \lambda < \lambda^*$ *then there exists a minimal bounded solution* u_λ,

(2) *if* $\lambda > \lambda^*$ *then there is no bounded solution.*

We call λ^* the extremal parameter. The branch u_λ with $0 < \lambda < \lambda^*$ is increasing in λ and the linearization of the nonlinear equation around the minimal solution is stable. As $\lambda \to \lambda^*$ the increasing limit $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ exists pointwise and is a solution with parameter λ^* in a weak sense to be given later on (the exact definition depends on the problem). Depending on the situation, u^* maybe bounded or singular.

Some questions that we are interested in are:

- Can one determine in each situation whether u^* is singular or not?
- Are there singular solutions for $\lambda > \lambda^*$?
- What are the singular solutions for $\lambda < \lambda^*$?
- What happens to the singular solutions under perturbations of the equation?

In what follows we shall review in more detail some of the literature related with the previous questions. Then we shall consider in more detail recent works of the author and some collaborators: Dupaigne, Montenegro and Guerra, [43–45].

1.1. *Basic properties*

Theorem 1.1 and the properties mentioned after its statement can be obtained by the method of sub and supersolutions. Indeed, the three problems (1.1), (1.4) and (1.5) have a maximum principle. Partly due to this reason we restrict the analysis of (1.5) to the ball, since the maximum principle for Δ^2 in this domain with Dirichlet boundary conditions $u = \frac{\partial u}{\partial \nu} = 0$ holds [15].

To be more concrete we sketch the argument for equation (1.1). We remark that for λ positive, 0 is a subsolution which is not a solution and for small positive λ one can take as a supersolution the solution to

$$\begin{cases} -\Delta\zeta = 1 & \text{in } \Omega \\ \quad\ \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

Defining λ^* as the supremum of the values such that a classical solution exists, we see that $\lambda^* > 0$ and for any $0 < \lambda < \lambda^*$ there is a bounded solution u_λ, which is minimal among all classical solutions.

To show that λ^* is finite let φ_1 be a positive eigenfunction of $-\Delta$ with Dirichlet boundary condition and eigenvalue $\lambda_1 > 0$. Suppose that u is a classical solution to (1.1) and multiply this equation by φ_1. Integrating and using (1.2), (1.3), which implies $g(u) \geq cu$ for some $c > 0$, we find

$$\lambda_1 \int_\Omega u\varphi_1 = \lambda \int_\Omega g(u)\varphi_1 \geq \lambda c \int_\Omega u\varphi_1 \tag{1.6}$$

which shows that $\lambda \leq \lambda_1/c$. Since there is a constant C such that $g(u) \geq 4\lambda_1 u/\lambda^* - C$ for all $u > 0$, if $\lambda^*/2 < \lambda < \lambda^*$ we have

$$\lambda_1 \int_\Omega u_\lambda\varphi_1 = \lambda \int_\Omega g(u_\lambda)\varphi_1 \geq 2\lambda_1 \int_\Omega u_\lambda\varphi_1 - C' \tag{1.7}$$

for some constant C'. This shows that $\int_\Omega u_\lambda\varphi_1 \leq C$ and implies that $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ exists a.e.

An important property of the minimal branch of solutions is its stability, that is,

$$\mu_1(-\Delta - \lambda g'(u_\lambda)) > 0, \quad \forall\, 0 \leq \lambda < \lambda^*, \tag{1.8}$$

where $\mu_1(-\Delta - \lambda g'(u))$ denotes the first eigenvalue of the operator $-\Delta - \lambda g'(u_\lambda)$ with Dirichlet boundary conditions. We recall that

$$\mu_1 = \inf_{\varphi\in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 - \lambda g'(u_\lambda)\varphi^2}{\int_\Omega \varphi^2} \tag{1.9}$$

and that there exists a first positive eigenfunction of $-\Delta - \lambda g'(u)$, that is,

$$\begin{cases} -\Delta\psi_1 - \lambda g'(u_\lambda)\psi_1 = \mu_1\psi_1 & \text{in } \Omega \\ \qquad\qquad\quad \psi_1 > 0 & \text{in } \Omega \\ \qquad\qquad\quad \psi_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

where we may normalize $\|\psi_1\|_{L^2(\Omega)} = 1$ (see [70]).

Fix $0 \le \lambda < \lambda^*$ and let us show that $\mu_1 > 0$. Since $\lambda < \lambda^*$ we may fix $\lambda < \bar\lambda < \lambda^*$ and write $\bar u = u_{\bar\lambda}$, that is, the minimal solution with parameter $\bar\lambda$. Then by the positivity and convexity of g we have

$$-\Delta(\bar u - u_\lambda) = \bar\lambda g(\bar u) - \lambda g(u_\lambda) > \lambda(g(\bar u) - g(u_\lambda)) \ge \lambda g'(u_\lambda)(\bar u - u_\lambda).$$

Multiplying this inequality by ψ_1 and integrating by parts we find

$$\mu_1 \int_\Omega (\bar u - u_\lambda)\psi_1 > 0.$$

But the integral above is positive because $\psi_1 > 0$ and $\bar u > u_\lambda$ by the strong maximum principle, and we conclude that $\mu_1 > 0$.

Actually the stability characterizes the minimal solution, that is, if (λ, u) is a classical solution to (1.1) such that $\mu_1(-\Delta - \lambda g'(u)) > 0$ then necessarily $u = u_\lambda$. Indeed, since u_λ is the minimal solution we have immediately $u_\lambda \le u$. Now, by convexity of g

$$-\Delta(u_\lambda - u) = \lambda(g(u_\lambda) - g(u)) \ge \lambda g'(u)(u_\lambda - u).$$

Since $\mu_1(-\Delta - \lambda g'(u)) > 0$ the operator $-\Delta - \lambda g'(u)$ satisfies the maximum principle and we deduce that $u_\lambda \ge u$.

The implicit function theorem can also be applied to problems (1.1), (1.4) and (1.5). It implies that starting from the trivial solution $(0, 0)$ there exists a maximal interval $[0, \lambda_0)$ and a C^1 curve of solutions $u(\lambda)$ defined in this interval. Then it is possible to prove that this curve is exactly the branch of minimal solutions u_λ as constructed above and that $\lambda^* = \lambda_0$. For the results here we refer to [69,34,79,35].

1.2. *A second-order semi-linear equation*

In this section we recall some facts related to (1.1), in particular reviewing a few cases where the solution structure is completely known, sufficient conditions for $u^* \in L^\infty$ in general domains, examples where $u^* \notin L^\infty$, and then some properties of the extremal solution such as its stability and uniqueness.

Let us start recalling some of the results for the case $g(u) = e^u$ in the unit ball. In dimension 1 this problem was first studied by Liouville [85]. Bratu [17] found an explicit solution when $N = 2$. Later Chandrasekhar [29] and Frank-Kamenetskii [61] considered $N = 3$ and Barenblatt [69] proved that in dimension 3 for $\lambda = 2$ there are infinitely many solutions. Joseph and Lundgren [76], using phase-plane analysis, gave a complete description of the classical solutions to (1.1) when Ω is the unit ball and $g(u) = e^u$ or $g(u) = (1 + u)^p$, $p > 1$.

THEOREM 1.2 (Joseph and Lundgren [76]). *Let Ω be the unit ball in $\mathbb{R}^N$, $N \ge 1$ and $g(u) = e^u$. Then*

- *If $N = 1, 2$ for any $0 < \lambda < \lambda^*$ there are exactly 2 solutions, while for $\lambda = \lambda^*$ there is a unique solution, which is classical.*

– *If* $3 \leq N \leq 9$ *we have that* u^* *is bounded and* $\lambda^* > \lambda_0$, *where* $\lambda_0 = 2(N-2)$. *For* $\lambda = \lambda_0$ *there are infinitely many solutions that converge to* $U(x) = -2\log|x|$, *which is a singular solution with parameter* λ_0. *For* $|\lambda - \lambda_0| \neq 0$ *but small there are a large number of solutions.*
– *If* $N \geq 10$ *then* $\lambda^* = 2(N-2)$ *and* $u^* = -2\log|x|$. *Moreover for any* $0 < \lambda < \lambda^*$ *there is only one solution.*

When Ω *is the unit ball in* $\mathbb{R}^N$, $N \geq 3$ *and* $g(u) = (1+u)^p$, $p > 1$ *then:*
– *When* $1 < p \leq \frac{N+2}{N-2}$ *there are exactly two solutions for any* $0 < \lambda < \lambda^*$, *while for* $\lambda = \lambda^*$ *there is a unique solution, which is classical.*
– *When* $p > \frac{N+2}{N-2}$ *and* $N < 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$ *we have that* u^* *is bounded and* $\lambda^* > \lambda_p$, *where* $\lambda_p = \frac{2}{p-1}(N - \frac{2}{p-1})$. *For* $\lambda = \lambda_p$ *there are infinitely many solutions that converge to* $U_p = |x|^{-\frac{2}{p-1}} - 1$, *which is a singular solution with parameter* λ_p. *For* $|\lambda - \lambda_p| \neq 0$ *but small there are a large number of solutions.*
– *If* $p > \frac{N+2}{N-2}$ *and* $N \geq 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$ *then* $\lambda^* = \lambda_p$ *and* $u^* = U_p$. *Moreover for any* $0 < \lambda < \lambda^*$ *there is only one solution.*

For general domains Crandall and Rabinowitz [35] showed that if u^* is a classical solution then the branch of minimal solutions (λ, u_λ) can be continued as curve $s \in (-\delta, \delta) \to (\lambda(s), u_s)$ that "bends back", that is, u_s coincides with the minimal branch for $-\delta < s \leq 0$, $\lambda(0) = \lambda^*$, $u_0 = u^*$ and for $0 < s < \delta$ we have $\lambda(s) < \lambda^*$ while u_s is a second solution associated to $\lambda(s)$. These authors and also Mignot and Puel [93, 94] gave sufficient conditions for u^* to be a classical solution in general domains for some nonlinearities.

THEOREM 1.3 (Crandall–Rabinowitz [35], Mignot–Puel [93]). *Let* $\Omega \subseteq \mathbb{R}^N$ *be a bounded smooth domain.*
(1) *If* $g(u) = e^u$ *then* u^* *is classical provided* $N \leq 9$.
(2) *When* $g(u) = (1+u)^p$ *with* $p > 1$, u^* *is classical when*

$$N < 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}.$$

The conditions on p and N in Theorem 1.3 are optimal if Ω is the unit ball by the results of Joseph and Lundgren. A basic fact about the branch of minimal solutions that is important in the proof of this result is that u_λ is stable, in the sense that the first Dirichlet eigenvalue of the linearized operator $-\Delta - \lambda g'(u_\lambda)$ is positive, that is, $\mu_1 > 0$, where μ_1 is given by (1.9). In particular

$$\lambda \int_\Omega g'(u_\lambda)\varphi^2 \leq \int_\Omega |\nabla\varphi|^2 \quad \forall \varphi \in C_0^\infty(\Omega). \tag{1.10}$$

Let us sketch briefly the proof of Theorem 1.3 in the case of the exponential nonlinearity $g(u) = e^u$. The aim is to obtain estimates for the minimal solution u_λ for $0 < \lambda < \lambda^*$ that are independent of λ. Let $j > 0$ and take $\varphi = e^{ju_\lambda} - 1$. Then from (1.10) we have

$$j^2 \int_\Omega e^{2ju_\lambda}|\nabla u_\lambda|^2 \geq \lambda \int_\Omega e^{u_\lambda}(e^{ju_\lambda} - 1)^2. \tag{1.11}$$

Multiplying equation (1.1) by $e^{2ju_\lambda} - 1$ and integrating yields

$$2j \int_\Omega e^{2ju_\lambda} |\nabla u_\lambda|^2 = \lambda \int_\Omega e^{u_\lambda}(e^{2ju_\lambda} - 1). \tag{1.12}$$

Combining (1.11) and (1.12) we see that if $j < 2$ then there is some C independent of λ such that

$$\int_\Omega e^{(2j+1)u_\lambda} \le C.$$

Thus $\|u_\lambda\|_{L^q} \le C$ with C_q independent of λ for any $q < 5$. Hence, if $N \le 9$ by the Sobolev and Morrey embedding theorems we have that $\|u_\lambda\|_{L^\infty} \le C$, and this shows that u^* is bounded, and consequently smooth.

Brezis and Vázquez [20] posed the question of finding whether u^* is bounded for general $g(u)$. The result in this direction that holds for the most general nonlinearity and domain is:

THEOREM 1.4 (Cabré [22]). *Let Ω be a smooth, bounded, strictly convex domain in $\mathbb{R}^N$ with $N \le 4$. If g satisfies* (1.2), (1.3) *then the extremal solution u^* to* (1.1) *is bounded.*

Before this result, Nedev [96] had proved that u^* is bounded if $N \le 3$, without any restriction on the domain. It is not known if the extremal solution u^* is singular for some domains and nonlinearities in dimension $5 \le N \le 9$. Cabré and Capella [24] settled this question in the radial case (see [23] for a related result in the entire space):

THEOREM 1.5 (Cabré–Capella [24]). *Suppose g satisfies* (1.2), (1.3) *and let $\Omega = B_1$ be the unit ball in $\mathbb{R}^N$, $N \le 9$. Then u^* is bounded.*

The proof of [24] is based on a rewriting of the stability inequality (1.10) in a form that makes it independent of g. Indeed, let u_λ denote the minimal solution in $\Omega = B_1$, which is radial, and let us write u_λ' for the radial derivative $\frac{du_\lambda}{dr}$. Let $\eta \in C_0^\infty(B_1)$ and consider $\varphi = \eta u_\lambda'$ in (1.10). Then

$$\int_{B_1} \nabla u_\lambda' \nabla(u_\lambda' \eta^2) + (u_\lambda')^2 |\nabla \eta|^2 \ge \lambda \int_{B_1} g'(u_\lambda)(u_\lambda')^2 \eta^2. \tag{1.13}$$

But u_λ' satisfies

$$-\Delta u_\lambda' + \frac{N-1}{r^2} u_\lambda' = \lambda g'(u_\lambda) u_\lambda'.$$

Multiplying this equation by $u_\lambda' \eta^2$ and integrating by parts we find

$$\int_{B_1} \nabla u_\lambda' \nabla(u_\lambda' \eta^2) + \int_{B_1} \frac{N-1}{r^2}(u_\lambda')^2 \eta^2 = \lambda \int_{B_1} g'(u_\lambda)(u_\lambda')^2 \eta^2. \tag{1.14}$$

Combining (1.13) and (1.14) we obtain

$$\int_{B_1} (u_\lambda')^2 \left(|\nabla \eta|^2 - \frac{N-1}{r^2} \eta^2 \right) \ge 0 \quad \forall \eta \in C_0^\infty(B_1). \tag{1.15}$$

This form of the stability can be used to deduce from it weighted integrability for u'_λ. Indeed, by density we can argue that it holds for $\eta = r^{-a}$ for $a < \frac{N-2}{2}$, but it is only useful to choose a such that $|\nabla\eta|^2 - \frac{N-1}{r^2}\eta^2 \geq 0$. Now, if $\eta = r^{-a}$ then

$$|\nabla\eta|^2 - \frac{N-1}{r^2}\eta^2 = \left(a^2 - N - 1\right) r^{-2a-2}.$$

Then for any $0 < a < \sqrt{N-1}$, from (1.15) we deduce

$$\int_0^1 (u'_\lambda)^2 r^{N-2a-3}\, dr \leq C. \tag{1.16}$$

We note that C depends on a but not on λ. From (1.16) we can deduce now that if $N < 10$ then $\|u_\lambda\|_{L^\infty} \leq C$ with a constant independent of λ. Indeed, let $\beta > 0$ to be fixed later on and $0 < r < 1$. Since $u_\lambda(1) = 0$

$$u_\lambda(r) = -\int_r^1 u'_\lambda(s)\, ds \leq \left(\int_r^1 u'_\lambda(s)^2 s^\beta\right)^{1/2} \left(\int_r^1 s^{-\beta}\, ds\right)^{1/2}.$$

Observe that $N - 2\sqrt{N-1} - 3 < 1$ whenever $N < 10$. Thus for $N < 10$, we may choose $N - 2\sqrt{N-1} - 3 < \beta < 1$ and it follows that

$$u_\lambda(r) \leq \left(\int_0^1 u'_\lambda(s)^2 s^\beta\, ds\right)^{1/2} \left(\int_0^1 s^{-\beta}\, ds\right)^{1/2} \leq C$$

with C independent of r and λ. This shows that u^* is bounded and hence a classical solution.

The argument of [22] for a general strictly convex domain in $\mathbb{R}^N$, $N \leq 4$ follows the same idea as for the radial case, but this time the role u'_λ is taken by $|\nabla u_\lambda|$. The proof is more involved because the equation satisfied by $|\nabla u_\lambda|$ is more complicated.

To continue the discussion of the properties of u^* we shall define precisely the notion of weak solution we will use when dealing with (1.1), and we adopt the one introduced by Brezis et al. [19]:

DEFINITION 1.6. A function $u \in L^1(\Omega)$ is a weak solution to (1.1) if $g(u)\delta(x) \in L^1(\Omega)$ and

$$-\int_\Omega u\Delta\zeta = \lambda \int_\Omega g(u)\zeta \quad \text{for all } \zeta \in C^2(\overline{\Omega}),\ \zeta = 0 \text{ on } \partial\Omega,$$

where

$$\delta(x) = dist(x, \partial\Omega).$$

It is not difficult to show that $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is a weak solution in the above sense. Moreover Nedev [96] proved that in any dimension $u^* \in L^p(\Omega)$ for any $p < \frac{N}{N-4}$ if $N > 4$, for any $p < +\infty$ if $N = 4$ and $u^* \in L^\infty$ for $N \leq 3$.

A question of interest is whether weak solutions may exist for $\lambda > \lambda^*$. Brezis *et al.* [19] showed that this is not the case for (1.1):

THEOREM 1.7 (Brezis–Cazenave–Martel–Ramiandrisoa [19]). *If* $\lambda > \lambda^*$ *then* (1.1) *has no weak solution.*

This result can be restated as follows: if (1.1) has a weak solution for some $\lambda > 0$ then for any $0 < \lambda' < \lambda$, equation (1.1) has a classical solution. The proof of this assertion in [19] is based on a truncation method specially adapted to the nonlinearity. Suppose u is a weak supersolution of (1.1) with parameter λ. In [19] they consider a C^2 concave function $\phi : [0, \infty) \to [0, \infty)$ and set

$$v = \phi(u).$$

Assuming for a moment that u is smooth we can compute

$$\Delta v = \Delta \phi(u) = \phi'(u)\Delta u + \phi''(u)|\nabla u|^2 \leq \phi'(u)\Delta u.$$

If ϕ' is bounded, the inequality

$$\Delta v \leq \phi'(u)\Delta u$$

can be proved in the sense of distributions when $u, \Delta u \in L^1(\Omega)$. Then, given $0 < \lambda' < \lambda$ we seek a concave, bounded ϕ such that v becomes a supersolution to (1.1) with parameter λ'. If u is a weak solution, then

$$-\Delta v \geq -\phi'(u)\Delta u = \lambda\phi'(u)g(u)$$

and we would like to have

$$\lambda\phi'(u)g(u) \geq \lambda' g(\phi(u)).$$

In particular it is sufficient to achieve equality and directly integrating the ODE yields

$$\phi(u) = H^{-1}\left(\frac{\lambda}{\lambda'}H(u)\right), \tag{1.17}$$

where

$$H(t) = \int_0^t \frac{ds}{g(s)}.$$

It can be checked that ϕ defined by (1.17) is concave, increasing with a bounded derivative. Moreover it is bounded if $\int_0^\infty \frac{ds}{g(s)} < +\infty$ and this leads to a proof of the statement in this case. If on the contrary, $\int_0^\infty \frac{ds}{g(s)} = +\infty$, then still v has better regularity that u, and repeating this construction a finite number of times shows that for $\lambda'' < \lambda'$ a bounded supersolution exists, see the details in [19].

Using the same truncation method and a delicate argument Martel [87] was able to prove the uniqueness of u^*.

THEOREM 1.8 (Martel [87]). *If* $\lambda = \lambda^*$ *then* (1.1) *has a unique weak solution.*

Going back to the discussion of whether u^* is bounded or not, we have seen some ideas to prove that under certain conditions u^* is bounded. But there are few situations where it is known that u^* is singular. One of these examples is the case when Ω is the unit ball in $\mathbb{R}^N$, $N \geq 10$ and $g(u) = e^u$. In [76] it is shown through phase-plane analysis that $u^* = -2\log|x|$. Brezis–Vázquez [20] found a new proof of this fact, showing a connection with Hardy's inequality which we recall:

$$\frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2} \leq \int_{\mathbb{R}^N}|\nabla\varphi|^2 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N), \quad (N \geq 3). \tag{1.18}$$

This connection is a characterization of singular energy solutions.

THEOREM 1.9 (Brezis–Vázquez [20]). *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded smooth domain. Suppose $u \in H_0^1(\Omega)$ is a singular weak solution to* (1.1) *for some $\lambda > 0$ such that*

$$\lambda\int_\Omega g'(u)\varphi^2 \leq \int_\Omega |\nabla\varphi|^2 \quad \textit{for all } \varphi \in C_0^\infty(\Omega). \tag{1.19}$$

Then $u = u^*$ and $\lambda = \lambda^*$.

When $\Omega = B_1(0)$ in $\mathbb{R}^N$ with $N \geq 10$ and $g(u) = e^u$ the explicit solution $U = -2\log|x|$ with parameter $\lambda_0 = 2(N-2)$ satisfies condition (1.19) thanks to Hardy's inequality (1.18). Thus the previous result immediately yields $u^* = U$ and $\lambda^* = \lambda_0$. The same idea applies when $g(u) = (1+u)^p$, $p > 1$ in the unit ball: the solution $u = |x|^{-\frac{2}{p-1}} - 1$ satisfies (1.19) when $N \geq 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$.

The idea of the proof of Theorem 1.9 is as follows. First we remark that $\lambda \leq \lambda^*$ by Theorem 1.7. If $\lambda = \lambda^*$ then the uniqueness result Theorem 1.8 implies that $u = u^*$. So we have to rule out the case $\lambda < \lambda^*$, which we do by contradiction. By density we see that (1.19) holds for $\varphi \in H_0^1(\Omega)$. Since by hypothesis $u \in H_0^1(\Omega)$ we are allowed to take $\varphi = u - u_\lambda$, where u_λ denotes the minimal solution. We obtain, after integration by parts and using the equations for u and u_λ,

$$\int_\Omega (g(u_\lambda) - (g(u) + g'(u)(u_\lambda - u)))(u - u_\lambda) \leq 0.$$

But the integrand is nonnegative since $u > u_\lambda$ a.e. and g is convex. This implies

$$g(u_\lambda) = g(u) + g'(u)(u_\lambda - u) \quad \text{a.e. in } \Omega.$$

It follows that g is linear in intervals of the form $[u_\lambda(x), u(x)]$ for a.e. $x \in \Omega$. The union of such intervals is an interval and coincides with $[0, \infty)$ because $u_\lambda = 0$ on $\partial\Omega$ and u is unbounded, contradicting (1.3).

1.3. *Perturbation of singular solutions*

In the search for nonradial examples where the extremal solution is singular, a natural approach is to consider perturbations of the radial case. Let us consider the Gelfand problem in dimension $N \geq 3$, that is

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.20}$$

In dimension $N = 3$ and when $\Omega = B$ is the unit ball, there are infinitely many singular solutions, with a unique singular point which can be prescribed near the origin. This result was announced by H. Matano and proved by Rébaï [101]. Similar results hold when the nonlinearity is $g(u) = (1+u)^p$.

THEOREM 1.10 (Rébaï [101]). *Let B be the unit ball in $\mathbb{R}^3$. Then there exists $\varepsilon > 0$ such that for any $\xi \in B_\varepsilon$ there is a solution (λ, u) of*

$$\begin{cases} \Delta u = \lambda e^u & \text{in } B \setminus \{\xi\} \\ u = 0 & \text{on } \partial B \end{cases} \tag{1.21}$$

which has a nonremovable singularity at ξ.

The solution in the above result has the behavior $u(x) \sim -2\log|x-\xi|$ and it can be seen that (1.21) holds in the sense of distributions.

Pacard [98] proved that for $N > 10$, there exist a dumbbell shaped domain Ω and a positive solution u of $-\Delta u = e^u$ in Ω having prescribed singularities at finitely many points, but $u = 0$ may not hold on $\partial\Omega$. Rébaï [102] extended this result to the case $N = 3$. When the exponential nonlinearity is replaced by $g(u) = u^\alpha$, Mazzeo and Pacard [90] proved that for any exponent α lying in a certain range and for any bounded domain Ω, there exist solutions of $-\Delta u = u^\alpha$ in Ω with $u = 0$ on $\partial\Omega$, with a nonremovable singularity on a finite union of smooth manifolds without boundary. Further results in this direction can be found in [103,99] and their references.

We are interested in the existence of singular solutions to (1.21) in domains in $\mathbb{R}^N$, $N \geq 4$ which are perturbations of the unit ball. Given a C^2 map $\psi : \overline{B_1} \to \mathbb{R}^N$ and $t \in \mathbb{R}$ define

$$\Omega_t = \{x + t\psi(x) : x \in B_1\}.$$

We work with $|t|$ sufficiently small in order that Ω_t is a smooth bounded domain diffeomorphic to B_1 and we consider the Gelfand problem in Ω_t:

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega_t \\ u = 0 & \text{on } \partial\Omega_t. \end{cases} \tag{1.22}$$

Our main result is:

THEOREM 1.11. *Let $N \geq 4$. Then there exists $\delta > 0$ (depending on N and ψ) and a curve $t \in (-\delta, \delta) \mapsto (\lambda(t), u(t))$ such that $(\lambda(t), u(t))$ is a solution to* (1.22) *and $\lambda(0) = 2(N-2)$, $u(0) = \log \frac{1}{|x|^2}$. Moreover there exists $\xi(t) \in B_1$ such that*

$$\left\| u(x,t) - \log \frac{1}{|x-\xi(t)|^2} \right\|_{L^\infty(\Omega_t)} + |\lambda(t) - 2(N-2)| \to 0 \quad \textit{as } t \to 0. \tag{1.23}$$

The behavior of the singular solution at the origin is characterized as follows:

$$u(x,t) = \ln \frac{1}{|x-\xi(t)|^2} + \log\left(\frac{\lambda(0)}{\lambda(t)}\right) + \varepsilon(|x-\xi(t)|),$$

where $\lim_{s\to 0} \varepsilon(s) = 0$ (see [43, Corollary 1.4]).

Once Theorem 1.11 is established it implies that for small t the extremal solution is singular in dimension $N \geq 11$.

COROLLARY 1.12. *Let $N \geq 11$ and $(\lambda(t), u(t))$ be the singular solution of Theorem* 1.11. *Then $u(t)$ is the extremal solution in Ω_t and $\lambda(t)$ the extremal parameter.*

Indeed, let $u = u(t)$ denote the solution of (1.22) obtained in Theorem 1.11. Since $N \geq 11$ we have $2(N-2) < (N-2)^2/4$ and it follows from (1.23) that if $|t|$ is chosen small enough,

$$\lambda(t) e^{\left\| u - \log \frac{1}{|x-\xi(t)|^2} \right\|_{L^\infty(\Omega_t)}} < \frac{(N-2)^2}{4}.$$

Hence for $\varphi \in C_0^\infty(\Omega_t)$,

$$\lambda(t) \int_{\Omega_t} e^u \varphi^2 \leq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x-\xi(t)|^2} \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2,$$

by Hardy's inequality (1.18) and thanks to Theorem 1.9, $u(t)$ is the extremal solution of (1.22).

The proof of Theorem 1.11 is by linearization around the singular solution $-2\log|x|$. First we change variables to replace (1.22) with a problem in the unit ball. The map $id + t\psi$ is invertible for t small and we write the inverse of $y = x + t\psi(x)$ as $x = y + t\tilde{\psi}(t,y)$. Define v by

$$u(y) = v(y + t\tilde{\psi}(t,y)).$$

Then

$$\Delta_y u = \Delta_x v + L_t v,$$

where L_t is a second-order operator given by

$$L_t v = 2t \sum_{i,k} v_{x_i x_k} \frac{\partial \tilde{\psi}_k}{\partial y_i} + t \sum_{i,k} v_{x_k} \frac{\partial^2 \tilde{\psi}_k}{\partial y_i^2} + t^2 \sum_{i,j,k} v_{x_j x_k} \frac{\partial \tilde{\psi}_j}{\partial y_i} \frac{\partial \tilde{\psi}_k}{\partial y_i}.$$

We look for a solution of the form

$$v(x) = \log \frac{1}{|x-\xi|^2} + \phi, \quad \lambda = c^* + \mu, \tag{1.24}$$

where $c^* = 2(N-2)$. Then (1.22) is equivalent to

$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c^*}{|x-\xi|^2}\phi = \dfrac{c^*}{|x-\xi|^2}(e^\phi - 1 - \phi) + \dfrac{\mu}{|x-\xi|^2}e^\phi \\ \qquad + L_t\Big(\log \dfrac{1}{|x-\xi|^2}\Big) \quad \text{in } B \\ \phi = -\log \dfrac{1}{|x-\xi|^2} \quad \text{on } \partial B. \end{cases} \tag{1.25}$$

Here the unknowns are ϕ, ξ and μ. From Hardy's inequality (1.18) we see that whenever $c^* < \frac{(N-2)^2}{4}$, which holds if $N \geq 11$, if the right-hand side of (1.25) belongs to $L^2(B)$ then there is a unique solution in $H_0^1(B)$. But typically solutions are singular at the origin, with a behavior $|x-\xi|^{-\alpha}$ for some $\alpha > 0$ (see Baras and Goldstein [9], Dupaigne [50]). Thus, although the linear operator $-\Delta - \frac{c^*}{|x-\xi|^2}$ may be coercive in $H_0^1(B)$, this functional setting is not useful since the nonlinear term that appears on the right-hand side of (1.25), namely $\frac{c^*}{|x-\xi|^2}(e^\phi - 1 - \phi)$, is too strong. Our approach is to consider other functional spaces, more precisely, weighted Hölder spaces specially adapted to the singularity. It turns out that the singular linear operator has a right inverse in these spaces if the data satisfies some orthogonality conditions. More precisely, if one wants solutions such that $|\phi(x)| \leq C|x-\xi|^\nu$, the number and type of orthogonality conditions that appear depend on ν and the value c^*. In our case we would like $\nu = 0$ and c^* is given, and as we will see, this requires $N+1$ orthogonality conditions (if $N \geq 4$). Fortunately we have $N+1$ free parameters: μ and ξ in (1.24), and this is the reason not to force the position of the singularity of v. If $N = 3$ then only one orthogonality condition is required. This explains that in Theorem 1.10 the position of the singularity can be prescribed arbitrarily near the origin, while μ or equivalently λ has to be adjusted.

The proof of Theorem 1.11, which is presented in Section 2 is divided into the following steps. First, in Section 2.1 we study the Laplacian with a potential which is the inverse square to a point ξ. The main result is the solvability of the associated linear equation in weighted Hölder spaces. The analysis in this section is related to the work of Mazzeo and Pacard [90], see also [28,89]. We also study the differentiability properties of the solution with respect to ξ and we show that the previous results hold for perturbations of the Laplacian with the same singular potential. Then the proof itself of Theorem 1.11 is in Section 2.2.

A similar result can be obtained for power-type nonlinearities: given $p > 1$, consider the problem

$$\begin{cases} -\Delta u = \lambda(1+u)^p & \text{in } \Omega_t \\ u = 0 & \text{on } \partial\Omega_t. \end{cases} \tag{1.26}$$

When $t = 0$, i.e. when the domain is the unit ball, it is known (see Theorem 1.2 or [76,20]) that the extremal solution is unbounded and given by $u^* = |x|^{-2/(p-1)} - 1$ if and only if $N \geq 11$ and

$$N \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}.$$

THEOREM 1.13. *Let $N \geq 11$ and $p > 1$ such that $N > 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}$. Given t small, let $u^*(t)$ denote the extremal solution to* (1.26)*. Then there exists $t_0 = t_0(N, \psi, p) > 0$ such that if $|t| < t_0$, $u^*(t)$ is singular.*

Going back to (1.20) naturally the question arises whether if $N \geq 10$ for any convex smooth, bounded domain $\Omega \subseteq \mathbb{R}^N$ the extremal solution u^* is singular. The restriction of convexity is reasonable since if Ω is an annulus it is easily seen that with no restriction on N the extremal solution u^* is smooth. This question, which appears in [20], was considered by Dancer [36, p. 54–56] who showed that in any dimension there are thin convex domains such that the extremal solution is bounded. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary. We assume furthermore that Ω is convex and $\partial\Omega$ is *uniformly convex*, i.e. its principal curvatures are bounded away from zero. Write $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $x = (x_1, x_2) \in \mathbb{R}^N$ with $x_1 \in \mathbb{R}^{N_1}$, $x_2 \in \mathbb{R}^{N_2}$. For $\varepsilon > 0$ set

$$\Omega_\varepsilon = \{x = (y_1, \varepsilon y_2) : (y_1, y_2) \in \Omega\} \tag{1.27}$$

and consider the Gelfand problem in Ω_ε:

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega_\varepsilon \\ \quad u = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1.28}$$

THEOREM 1.14. *Given $\varepsilon > 0$, let u_ε^* be the extremal solution to* (1.28)*. If $N_2 \leq 9$ then there exists $\varepsilon_0 = \varepsilon_0(N, \Omega) > 0$ such that if $\varepsilon < \varepsilon_0$, u_ε^* is smooth.*

The idea of the proof is to fix the domain by setting

$$v_\varepsilon(y_1, y_2) = u(y_1, \varepsilon y_2).$$

Then v_ε is defined in $\overline{\Omega}$ and satisfies

$$\begin{cases} -(\varepsilon^2 \Delta_{y_1} + \Delta_{y_2}) v_\varepsilon = \varepsilon_j^2 \lambda e^{v_\varepsilon} & \text{in } \Omega \\ \qquad v_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.29}$$

where Δ_{y_i} denotes the Laplacian with respect to the variables y_i, $i = 1, 2$. After taking $\varepsilon \to 0$ one obtains an equation in each "slice" $\Omega_\alpha = \{y_2 : (\alpha, y_2) \in \Omega\}$ which lives in $\mathbb{R}^{N_2}$ with $N_2 \leq 9$. For all these equations there is an a priori bound for stable solutions as seen, for instance, from the proof of Theorem 1.3. We get a contradiction with this a priori bound, and at the same time manage to prove the convergence as $\varepsilon \to 0$ by selecting for each $\varepsilon > 0$ small a value λ_ε such that the minimal solution u_ε of (1.28) with parameter λ_ε satisfies

$$\max_{\Omega_\varepsilon} u_\varepsilon = M, \tag{1.30}$$

where M is a suitably large fixed number. This is possible, if we argue by contradiction, that is, assuming there is a sequence of $\varepsilon \to 0$ such that $u_\varepsilon^* \notin L^\infty(\Omega_\varepsilon)$. For the purpose of proving convergence of v_ε it is important to establish: for some constant C_0 we have

$$\lambda_\varepsilon^* \leq \frac{C_0}{\varepsilon^2} \tag{1.31}$$

and for some constant C independent of ε

$$\|\nabla v_\varepsilon\|_{L^\infty(\Omega)} \leq C. \tag{1.32}$$

For the last property we use the uniform convexity of Ω, which allows us to find $R > 0$ large enough so that for any $y_0 \in \partial\Omega$ there exists $z_0 \in \mathbb{R}^N$ such that the ball $B_R(z_0)$ satisfies $\Omega \subset B_R(z_0)$ and $y_0 \in \partial B_R(z_0)$. For convenience write for $\varepsilon > 0$

$$L_\varepsilon = \varepsilon^2 \Delta_{y_1} + \Delta_{y_2}.$$

Define $\zeta(y) = R^2 - |y - z_0|^2$ so that $\zeta \geq 0$ in Ω and $-L_\varepsilon \zeta = 2\varepsilon N_1 + 2N_2$. From (1.31) we have the uniform bound $\varepsilon^2 \lambda_\varepsilon \leq C$. It follows from (1.29) and the maximum principle that $v_\varepsilon \leq C\zeta$ with C independent of ε and y_0. Since $v_\varepsilon(y_0) = \zeta(y_0) = 0$, this in turn implies that

$$|\nabla v_\varepsilon(y_0)| \leq C \quad \forall y_0 \in \partial\Omega. \tag{1.33}$$

Then, since the linearization of (1.29) around v_ε has a positive first eigenvalue, we deduce (1.32). A complete proof can be found in [43], see also [36].

1.4. *Reaction on the boundary*

We consider the problem (1.4), that is,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda g(u) & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2, \end{cases} \tag{1.34}$$

where $\lambda > 0$ is a parameter, $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain and Γ_1, Γ_2 is a partition of $\partial\Omega$ into surfaces separated by a smooth interface. We will assume that

$$g \text{ is smooth, nondecreasing, convex, } g(0) > 0, \tag{1.35}$$

$$\liminf_{t \to +\infty} \frac{g'(t)t}{g(t)} > 1. \tag{1.36}$$

We recall that the branch of minimal solutions is stable in the sense that for $0 \le \lambda < \lambda^*$:

$$\inf_{\varphi \in C^1(\overline{\Omega}), \varphi = 0 \text{ on } \Gamma_2} \frac{\int_\Omega |\nabla \varphi|^2 \, dx - \lambda \int_{\Gamma_1} f'(u_\lambda)\varphi^2 \, ds}{\int_{\Gamma_1} \varphi^2 \, ds} > 0. \tag{1.37}$$

Assumption (1.36) is not essential, but it simplifies some of the arguments and holds for the examples $g(u) = e^u$, $g(u) = (1+u)^p$, $p > 1$. It allows us to say immediately that u^* is an energy solution in the following sense.

DEFINITION 1.15. We say that u is an energy solution to (1.34) if $u \in H^1(\Omega)$, $g(u) \in L^1(\Gamma_1)$ and

$$\int_\Omega \nabla u \nabla \varphi = \lambda \int_{\Gamma_1} g(u)\varphi \quad \forall \varphi \in C^1(\overline{\Omega}).$$

Indeed, from the stability of the minimal solutions u_λ

$$\lambda \int_{\Gamma_1} g'(u_\lambda) u_\lambda^2 \le \int_\Omega |\nabla u_\lambda|^2 = \lambda \int_{\Gamma_1} g(u_\lambda) u_\lambda.$$

By the hypothesis (1.36) for some $\sigma > 0$ and $C > 0$

$$(1+\sigma) g(u) u \le g'(u) u^2 + C \quad \forall u \ge 0.$$

It follows that there exists C independent of λ such that

$$\lambda \int_{\Gamma_1} g(u_\lambda) u_\lambda \le C$$

and hence

$$\int_\Omega |\nabla u_\lambda|^2 \le C. \tag{1.38}$$

This shows that $u^* \in H^1(\Omega)$. Moreover $g(u^*) \in L^1(\Gamma_1)$. Indeed, let φ be the solution to

$$\begin{cases} \Delta \varphi = 0 & \text{in } \Omega \\ \dfrac{\partial \varphi}{\partial \nu} = 1 & \text{on } \Gamma_1 \\ \varphi = 0 & \text{on } \Gamma_2. \end{cases}$$

Then

$$\int_\Omega \nabla u_\lambda \nabla \varphi = \lambda \int_{\Gamma_1} g(u_\lambda).$$

From (1.38) we deduce $\|g(u_\lambda)\|_{L^1(\Gamma_1)} \le C$ with C independent of λ and the assertion follows.

We are interested in determining whether the extremal solution u^* is bounded or singular in the cases $g(u) = e^u$ and $g(u) = (1+u)^p$, $p > 1$. For this purpose we remark that, as for (1.1) (cf. Theorem 1.9), the stability of a singular energy solution implies that it is the extremal one.

LEMMA 1.16. *Suppose g satisfies* (1.35), (1.36). *Assume that* $v \in H^1(\Omega)$ *is an unbounded solution of* (1.34) *for some* $\lambda > 0$ *such that*

$$\lambda \int_{\Gamma_1} g'(v)\varphi^2 \le \int_{\Omega} |\nabla\varphi|^2 \quad \forall\varphi \in C^1(\overline{\Omega}),\ \varphi = 0 \text{ on } \Gamma_2. \tag{1.39}$$

Then $\lambda = \lambda^*$ *and* $v = u^*$.

We shall give in Section 3.1 a proof of this fact under hypothesis (1.36). We note here, though, that the argument is simpler than for Theorem 1.9 because we know immediately that $u^* \in H^1(\Omega)$ and we do not need to rely on a uniqueness result for u^* similar to Theorem 1.8. The advantage of this approach is that Lemma 1.16 holds also under more general conditions, which include the case that Ω has a corner at the interface $\Gamma_1 \cap \Gamma_2$.

For smooth domains the uniqueness of u^* holds only assuming that g satisfies (1.2) and (1.3) and in a more general class of weak solutions. We will discuss this in Section 3.2. In fact, in that section we will develop some tools and results in the context of problem (1.34), that are now classical for (1.1). These are basically the notion of weak solution and the nonexistence of weak solutions for $\lambda > \lambda^*$ as in Brezis *et al.* [19], the regularity results for u^* in low dimensions of Nedev [96] and the uniqueness of u^* in the class of weak solutions, see Martel [87]. Throughout that section we will assume that g satisfies only (1.2) and (1.3).

We would like to construct singular solutions for some nonlinearities, and as a model case we consider first $g(u) = e^u$. Probably the simplest singular solution one may construct is

$$u_0(x) = \int_{\partial\mathbb{R}^N_+} K(x,y)\log\frac{1}{|y|}\,dy \quad \text{for } x \in \mathbb{R}^N_+, \tag{1.40}$$

where

$$K(x,y) = \frac{2x_N}{N\omega_N}|x-y|^{-N} \tag{1.41}$$

is the Green's function for the Dirichlet problem in $\mathbb{R}^N_+$ on the half space $\mathbb{R}^N_+ = \{(x', x_N)\,/\,x_N > 0\}$. Then u_0 is harmonic in $\mathbb{R}^N_+$ and

$$u_0(x) = \log\frac{1}{|x|} \quad \text{for } x \in \partial\mathbb{R}^N_+,\ x \ne 0.$$

A calculation, see [45], shows the following:

LEMMA 1.17.

$$\frac{\partial u_0}{\partial \nu} = \lambda_{0,N} e^{u_0} \quad \text{on } \partial\mathbb{R}^N_+,$$

where

$$\lambda_{0,N} = \begin{cases} (N-3)\dfrac{\sqrt{\pi}\,\Gamma(\frac{N}{2}-\frac{3}{2})}{2\Gamma(\frac{N}{2}-1)} & \text{if } N \ge 4, \\ 1 & \text{if } N = 3. \end{cases} \tag{1.42}$$

Let

$$\Omega_0 = \{x \in \mathbb{R}^N_+ : u_0(x) > 0\} \quad \Gamma_1 = \partial\Omega \cap \partial\mathbb{R}^N_+ \quad \Gamma_2 = \partial\Omega \setminus \partial\mathbb{R}^N_+.$$

The boundary $\partial\Omega_0$ is not smooth itself but Γ_1, Γ_2 are, and it can be checked that Theorem 1.1 still holds in this case.

Since the singular solution has the form $u_0(x) = -\log|x|$ for $x \in \partial\mathbb{R}^N_+$ its linearized stability is equivalent, by scaling, to

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 \geq \lambda_{0,N} \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|}, \qquad \forall\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+}).$$

Let us recall here Kato's inequality: for $N \geq 3$

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 \geq H_N \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|}, \qquad \forall\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+}), \tag{1.43}$$

where the best constant

$$H_N := \inf\left\{ \frac{\int_{\mathbb{R}^N_+} |\nabla\varphi|^2}{\int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|}} : \varphi \in H^1(\mathbb{R}^N_+), \varphi\Big|_{\partial\mathbb{R}^N_+} \not\equiv 0 \right\} \tag{1.44}$$

is given by

$$H_N = 2\frac{\Gamma(\frac{N}{4})^2}{\Gamma(\frac{N-2}{4})^2} \qquad \forall N \geq 3, \tag{1.45}$$

and Γ is the Gamma function. A proof of it was given by Herbst [73] and we will give later on in Section 3.3 a self-contained proof of (1.43). Actually we are able to improve this inequality in a similar fashion as was done by Brezis and Vázquez [20] or Vázquez and Zuazua [106] for (1.18) (see also [11,20,42,68,106] for other improved versions of Hardy's inequality).

It is not difficult to verify that $\lambda_{0,N} \leq H_N$ if and only if $N \geq 10$ (a proof can be found in [45]). Thus we have:

THEOREM 1.18. *Let $f(u) = e^u$. In any dimension $N \geq 10$ there exists a domain $\Omega \subset \mathbb{R}^N$ and a partition in smooth sets Γ_1, Γ_2 of $\partial\Omega$ such that $u^* \notin L^\infty(\Omega)$.*

Naturally the question becomes whether for all $N \leq 9$ and all domains $\Omega \subseteq \mathbb{R}^N$ one has $u^* \in L^\infty(\Omega)$. A first attempt using the ideas of Crandall–Rabinowitz [35] does not yield the optimal condition on the dimension. For convenience, let $u = u_\lambda$ be the minimal solution of (1.34). Working as in [35] we take $\varphi = e^{ju} - 1$, $j > 0$ in (1.37) and multiply (1.34) by $\psi = e^{2ju} - 1$. We obtain

$$\frac{\lambda}{j^2}\int_{\Gamma_1} e^u\left(e^{ju}-1\right)^2 ds \le \frac{\lambda}{2j}\int_{\Gamma_1} e^u\left(e^{2ju}-1\right) ds.$$

It follows that

$$\begin{aligned}\left(\frac{1}{j}-\frac{1}{2}\right)\int_{\Gamma_1} e^{(2j+1)u}\,ds &\le \frac{2}{j}\int_{\Gamma_1} e^{(j+1)u}\,ds\\ &\le \frac{2}{j}\int_{\Gamma_1\cap A} e^{(j+1)u}\,ds + \frac{2}{j}\int_{\Gamma_1\cap B} e^{(j+1)u}\,ds,\end{aligned}$$

where $A = [(1/j - 1/2)e^{(2j+1)u} < \frac{4}{j}e^{(j+1)u}]$ and $B = [(1/j - 1/2)e^{(2j+1)u} \ge \frac{4}{j}e^{(j+1)u}]$. Given $j \in (0, 2)$, we see that u remains uniformly bounded on A, while

$$\frac{2}{j}\int_{\Gamma_1\cap B} e^{(j+1)u}\,ds \le \frac{1}{2}\left(\frac{1}{j}-\frac{1}{2}\right)\int_{\Gamma_1} e^{(j+1)u}\,ds.$$

We conclude that e^u is bounded in $L^{2j+1}(\partial\Omega)$ independently of λ. If $2j+1 > N-1$ we obtain by elliptic estimates a bound for u in $C^\alpha(\overline{\Omega})$, for some $\alpha \in (0, 1)$. Thus if $N < 6$ we can choose $j \in (0, 2)$ such that $N-1 < 2j+1 < 5$ and obtain a bound for u in $C^\alpha(\overline{\Omega})$ independent of λ.

The above argument proves

PROPOSITION 1.19. *Let $g(u) = e^u$ and assume $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}^N_+$ and $\Gamma_2 \subset \mathbb{R}^N_+$. Assume further that $N < 6$. Then the extremal solution u^* of* (1.34) *belongs to $L^\infty(\Omega)$.*

We are able to overcome this difficulty under some assumptions on the domain, showing that the method used to prove Proposition 1.19 is not suitable for problem (1.34). In Section 3.4 we will give a proof of:

THEOREM 1.20. *Let $g(u) = e^u$, $N \le 9$ and suppose $\Omega \subset \mathbb{R}^N_+$ is an open, bounded set such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}^N_+$ and $\Gamma_2 \subset \mathbb{R}^N_+$, Ω is symmetric with respect to the hyperplanes $x_1 = 0, \dots, x_{N-1} = 0$, and Ω is convex with respect to all directions $x_1, \dots, x_{N-1}$. Then the extremal solution u^* of* (1.34) *belongs to $L^\infty(\Omega)$.*

Our proof is based on a lower bound of the form:

$$\liminf_{x\to 0, x\in\Gamma_1} \frac{u^*(x)}{\log(1/|x|)} \ge 1. \tag{1.46}$$

Then we show that this behavior is too singular in low dimensions $N \le 9$ for the extremal solution to be weakly stable. Our proof of (1.46) is a simple blow-up argument, but is limited to the exponential nonlinearity.

Next we look at (1.34) in the case $g(u) = (1+u)^p$, $p > 1$. Given $0 < \alpha < N-1$ define

$$w_\alpha(x) = \int_{\partial \mathbb{R}^N_+} K(x,y)|y|^{-\alpha} dy \quad \text{for } x \in \mathbb{R}^N_+, \tag{1.47}$$

where K is defined by (1.41). Clearly, $w_\alpha > 0$ in $\mathbb{R}^N_+$. Moreover w_α is harmonic in $\mathbb{R}^N_+$ and w_α extends to a function belonging to $C^\infty(\overline{\mathbb{R}^N_+} \setminus \{0\})$ with

$$w_\alpha(x) = |x|^{-\alpha} \quad \text{for all } x \in \partial \mathbb{R}^N_+ \setminus \{0\}. \tag{1.48}$$

It is not difficult to verify that for some constant $C(N,\alpha)$ we have

$$\frac{\partial w_\alpha}{\partial \nu}(x) = C(N,\alpha)|x|^{-\alpha-1} \qquad \forall x \in \partial \mathbb{R}^N_+ \setminus \{0\}.$$

In Section 3.5 we shall prove

LEMMA 1.21. *For $0 < \alpha < N-1$ we have:*

$$C(N,\alpha) = 2\frac{\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)\Gamma\left(\frac{N-1}{2}-\frac{\alpha}{2}\right)}{\Gamma(\frac{\alpha}{2})\Gamma\left(\frac{N-2}{2}-\frac{\alpha}{2}\right)}. \tag{1.49}$$

An heuristic calculation shows that for (1.34) with nonlinearity $g(u) = (1+u)^p$, the expected behavior of a solution u which is singular at $0 \in \partial\Omega$ should be $u(x) \sim |x|^{\frac{1}{p-1}}$. The boundedness of u^* is then related to the value of $C(N, \frac{1}{p-1})$. Observe that $C(N, \frac{1}{p-1})$ is defined for $p > \frac{N}{N-1}$. In the sequel, when writing $C(N, \frac{1}{p-1})$ we will implicitly assume that this condition holds.

Let us write $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$. For the next result we will assume that Ω is convex with respect to x', that is, $(tx', x_N) + ((1-t)y', x_N) \in \Omega$ whenever $t \in [0,1]$, $x = (x', x_N) \in \Omega$ and $y = (y', x_N) \in \Omega$. We shall also denote by Π_N the projection on $\partial\mathbb{R}^N_+$, namely $\Pi_N(x', x_N) = x'$ for all $x = (x', x_N) \in \mathbb{R}^N_+$.

THEOREM 1.22. *Consider* (1.34) *with* $g(u) = (1+u)^p$. *Assume $\Omega \subset \mathbb{R}^N_+$ is a bounded domain such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}^N_+$ and $\Gamma_2 \subset \mathbb{R}^N_+$, Ω is convex with respect to x' and $\Pi_N(\Omega) = \Gamma_1$. If $p\,C(N, \frac{1}{p-1}) > H_N$ or $1 < p < \frac{N}{N-2}$ then u^* is bounded.*

The same result holds if Ω is convex with respect to all directions $x_1, \dots, x_{N-1}$ and Ω is symmetric with respect to the hyperplanes $x_1 = 0, \dots, x_{N-1} = 0$. The proof (see [45]) of this result is also through a blow-up argument, but this time we do not prove a lower bound such as (1.46).

As a converse to the previous result we have:

THEOREM 1.23. *Consider* (1.34) *with* $g(u) = (1+u)^p$. *If $p\,C(N, \frac{1}{p-1}) \leq H_N$ and $p \geq \frac{N}{N-2}$ there exists a domain Ω such that u^* is singular.*

We shall not give the details here but just mention that $u = w_{\frac{1}{p-1}} - 1$ considered in $\Omega = \{x \in \mathbb{R}^N_+ \mid u(x) > 0\}$, with $\Gamma_1 = \partial\Omega \cap \partial\mathbb{R}^N_+$, $\Gamma_2 = \partial\Omega \setminus \partial\mathbb{R}^N_+$ is a singular solution to (1.34). It satisfies the stability condition (1.39) by Kato's inequality (1.43).

The condition $p\, C(N, \frac{1}{p-1}) \leq H_N$ is not enough to guarantee that the extremal solution is singular for some domain. Actually this condition can hold for some values of p in the range $\frac{N}{N-1} < p < \frac{N}{N-2}$. In this case a singular solution exists in some domains, but it does not correspond to the extremal one. This is similar to what happens to (1.1) with $g(u) = (1+u)^p$ and p in the range $\frac{N}{N-2} < p < \frac{N+2}{N-2}$. For that problem in the unit ball B_1 there exists a weak solution $u = |x|^{-\frac{2}{p-1}} - 1$ which is not the extremal solution (since it is not in H^1), but for p in the smaller range $\frac{N}{N-2} < p \leq \frac{N+2\sqrt{N-1}}{N-4+2\sqrt{N-1}}$ it satisfies condition (1.19), see Theorem 6.2 in [20].

1.5. *A fourth-order variant of the Gelfand problem*

In this section we turn our attention to (1.5) with exponential nonlinearity, that is,

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B \\ u = a & \text{on } \partial B \\ \dfrac{\partial u}{\partial \nu} = b & \text{on } \partial B, \end{cases} \tag{1.50}$$

where $a, b \in \mathbb{R}$. One of the reasons to consider this equation in the unit ball $B = B_1(0)$ is that the maximum principle for Δ^2 with Dirichlet boundary condition ($u = \frac{\partial u}{\partial \nu} = 0$) holds in this domain, see [15], a situation that is not true for general domains [5]. But also most our arguments require the radial symmetry of the solutions. As a consequence u_λ, $0 \leq \lambda < \lambda^*$ and u^* are radially symmetric.

Equation (1.50) with $a = b = 0$ was considered recently by Arioli *et al.* [5]. They give a proof of Theorem 1.1 for this problem and show that the minimal solutions of (1.50) are stable in the sense that

$$\int_B (\Delta\varphi)^2 \geq \lambda \int_B e^{u_\lambda} \varphi^2, \quad \forall \varphi \in C_0^\infty(B), \tag{1.51}$$

see [5, Proposition 37]. These authors work with the following class of weak solutions, which we will adopt here: $u \in H^2(B)$ is a weak solution to (1.50) if $e^u \in L^1(B)$, $u = a$ on ∂B, $\frac{\partial u}{\partial \nu} = b$ on ∂B and

$$\int_B \Delta u \Delta\varphi = \lambda \int_B e^u \varphi, \quad \text{for all } \varphi \in C_0^\infty(B).$$

They also show that if $\lambda > \lambda^*$ then (1.50) has no weak solution, but it does not seem to be possible to adapt their proof for problems like (1.5) with a general nonlinearity. The problem stems from the fact that the truncation method, as described after Theorem 1.7 seems not well suited for the fourth-order equation.

Regarding the regularity of u^*, the authors in [5] find a radial singular solution U_σ to (1.50) with $a = b = 0$ associated to a parameter $\lambda_\sigma > 8(N-2)(N-4)$ for dimensions $N = 5, \ldots, 16$. Their construction is computer assisted. They show that $\lambda_\sigma < \lambda^*$ if $N \leq 10$ and claim to have numerical evidence that this holds for $N \leq 12$.

We start here by establishing the fact that the extremal solution u^* is the unique solution to (1.50) in the class of weak solutions. Actually the statement is stronger:

THEOREM 1.24. *If*

$$v \in H^2(B),\ e^v \in L^1(B),\ v|_{\partial B} = a,\ \tfrac{\partial v}{\partial n}|_{\partial B} \leq b \tag{1.52}$$

and

$$\int_B \Delta v \Delta \varphi \geq \lambda^* \int_B e^v \varphi \quad \forall\, \varphi \in C_0^\infty(B), \varphi \geq 0, \tag{1.53}$$

then $v = u^*$. In particular for $\lambda = \lambda^*$ problem (1.50) has a unique weak solution.

The proof of this result can be found in Section 4.2, while in Section 4.1 we describe the comparison principles that are useful for the arguments. It is analogous to Theorem 1.8 of Martel [87] for (1.1) but our proof does not seem useful for the general version of this problem (1.5). Again, the reason for this limitation is that truncation method developed in [19] is not well adapted to this fourth-order equation.

The results of [5] are an indication that u^* maybe bounded up to dimension $N \leq 12$. We have

THEOREM 1.25. *For any a and b, if $N \leq 12$ then the extremal solution u^* of (1.50) is smooth.*

Our method of proof is different to the one leading to Theorem 1.3 and is similar to the scheme we used for the problem with reaction on the boundary. Indeed, using the same blow-up argument as for the proof of Theorem 1.20 in Section 3.4 it is possible to show that if u^* is singular then

$$\liminf_{r \to 0} \frac{u^*(r)}{\log(1/r^4)} \geq 1 \tag{1.54}$$

(a complete proof can be found in [44]). Now, if $N \leq 4$ the problem is subcritical, and the boundedness of u^* can be proved by other means: no singular solutions exist for positive λ (see [5]) but in dimension $N = 4$ they can blow up as $\lambda \to 0$, see [108].

So assume $5 \leq N \leq 12$ and that u^* is unbounded. Fix $\sigma > 0$. By (1.54), multiplication of (1.50) by $\varphi = |x|^{4-N+2\varepsilon}$ and integration by parts gives

$$\lambda \int_B e^u |x|^{4-N+2\varepsilon} \geq 4(N-2)(N-4)\omega_N(1-\sigma)\frac{1}{\varepsilon} + O(1), \tag{1.55}$$

where ω_N is the surface area of the unit $N-1$-dimensional sphere S^{N-1} and $O(1)$ represents boundary terms, which are bounded as $\varepsilon \to 0$. Using the weak stability of

u^* (1.60) with $\psi = |x|^{\frac{4-N}{2}+\varepsilon}$ multiplied by an appropriate cut-off function yields

$$\begin{aligned}
&\lambda \int_B e^u |x|^{4-N+2\varepsilon} \\
&\quad \le \left(\frac{N^2(N-4)^2}{16} + O(\varepsilon) \right) \int_B |x|^{-N+2\varepsilon} = \omega_N \frac{\frac{N^2(N-4)^2}{16}}{2\varepsilon} + O(1),
\end{aligned} \tag{1.56}$$

since $(\Delta\psi)^2 = (N^2(N-4)^2/16 + O(\varepsilon))|x|^{-N+2\varepsilon}$. From (1.55) and (1.56), and letting $\varepsilon \to 0$ and then $\sigma \to 0$, we find

$$8(N-2)(N-4) \le \frac{N^2(N-4)^2}{16}.$$

This is valid only if $N \ge 13$, a contradiction.

The constant $N^2(N-4)^2/16$ appears in Rellich's inequality [104], which states that if $N \ge 5$ then

$$\int_{\mathbb{R}^N} (\Delta\varphi)^2 \ge \frac{N^2(N-4)^2}{16} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^4} \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \tag{1.57}$$

The constant $N^2(N-4)^2/16$ is known to be optimal as seen from functions such that $\psi = |x|^{\frac{4-N}{2}+\varepsilon}$. This inequality will play an important role in proving that u^* is singular if $N \ge 13$ and $b = 0$.

Going back to Theorem 1.24 we mention that it can be used to deduce properties of the extremal solution in case it is singular. In [5] the authors say that a radial weak solution u to (1.50) is weakly singular if $\lim_{r\to 0} ru'(r)$ exists. For example, the singular solutions U_σ of [5] verify this condition. As a corollary of Theorem 1.24 we show

COROLLARY 1.26. *The extremal solution u^* to* (1.50) *with $b \ge -4$ is always weakly singular.*

We prove this corollary in Section 4.2. A weakly singular solution either is smooth or exhibits a log-type singularity at the origin. More precisely, if u is a non-smooth weakly singular solution of (1.50) with parameter λ then (see [5]) the following refinement of (1.54) holds:

$$\lim_{r\to 0} u(r) + 4\log r = \log \frac{8(N-2)(N-4)}{\lambda},$$
$$\lim_{r\to 0} ru'(r) = -4.$$

In view of Theorem 1.25, it is natural to ask whether u^* is singular in dimension $N \ge 13$. We show that this is true in the case $a = b = 0$.

THEOREM 1.27. *Let $N \ge 13$ and $a = b = 0$. Then the extremal solution u^* to* (1.50) *is unbounded.*

The proof of Theorem 1.27 is related to Theorem 1.9 and a similar result holds for (1.50):

PROPOSITION 1.28. *Assume that $u \in H^2(B)$ is an unbounded weak solution of* (1.50) *satisfying the stability condition*

$$\lambda \int_B e^u \varphi^2 \leq \int_B (\Delta\varphi)^2, \quad \forall \varphi \in C_0^\infty(B). \tag{1.58}$$

Then $\lambda = \lambda^*$ and $u = u^*$.

See the proof in Section 4.2. When $a = 0$ and $b = -4$ we have an explicit solution

$$\bar{u}(x) = -4 \log |x|$$

associated to $\bar{\lambda} = 8(N-2)(N-4)$. Thanks to Rellich's inequality (1.57) the solution $\bar{u}$ satisfies condition (1.58) when $N \geq 13$. Therefore, by Theorem 1.25 and a direct application of Proposition 1.28 we obtain Theorem 1.27 in the case $b = -4$.

For general values of b we do not know any explicit singular solution to the equation (1.50) and Proposition 1.28 is not useful. We instead find a suitable variant of it (see a proof in Section 4.1):

LEMMA 1.29. (a) *Let $u_1, u_2 \in H^2(B_R)$ with $e^{u_1}, e^{u_2} \in L^1(B_R)$. Assume that*

$$\Delta^2 u_1 \leq \lambda e^{u_1} \quad \text{in } B_R$$

in the sense

$$\int_{B_R} \Delta u_1 \Delta\varphi \leq \lambda \int_{B_R} e^{u_1}\varphi \quad \forall \varphi \in C_0^\infty(B_R),\ \varphi \geq 0, \tag{1.59}$$

and $\Delta^2 u_2 \geq \lambda e^{u_2}$ in B_R in the similar weak sense. Suppose also

$$u_1|_{\partial B_R} = u_2|_{\partial B_R} \quad \text{and} \quad \frac{\partial u_1}{\partial n}|_{\partial B_R} = \frac{\partial u_2}{\partial n}|_{\partial B_R}.$$

Assume furthermore that u_1 is stable in the sense that

$$\lambda \int_{B_R} e^{u_1}\varphi^2 \leq \int_{B_R} (\Delta\varphi)^2, \quad \forall \varphi \in C_0^\infty(B_R). \tag{1.60}$$

Then

$$u_1 \leq u_2 \quad \text{in } B_R.$$

(b) *Let $u_1, u_2 \in H^2(B_R)$ be* **radial** *with $e^{u_1}, e^{u_2} \in L^1(B_R)$. Assume $\Delta^2 u_1 \leq \lambda e^{u_1}$ in B_R in the sense of* (1.59) *and $\Delta^2 u_2 \geq \lambda e^{u_2}$ in B_R. Suppose $u_1|_{\partial B_R} \leq u_2|_{\partial B_R}$ and $\frac{\partial u_1}{\partial n}|_{\partial B_R} \geq \frac{\partial u_2}{\partial n}|_{\partial B_R}$ and that the stability condition* (1.60) *holds. Then $u_1 \leq u_2$ in B_R.*

The idea of the proof of Theorem 1.27 consists in estimating accurately from above the function $\lambda^* e^{u^*}$, and to deduce that the operator $\Delta^2 - \lambda^* e^{u^*}$ has a strictly positive first eigenvalue (in the $H_0^2(B)$ sense). Then, necessarily, u^* is singular. Upper bounds for both λ^* and u^* are obtained by finding suitable sub and supersolutions. For example, if for some λ_1 there exists a supersolution then $\lambda^* \geq \lambda_1$. If for some λ_2 one can exhibit a stable singular subsolution u, then $\lambda^* \le \lambda_2$. Otherwise $\lambda_2 < \lambda^*$ and one can then prove that the minimal solution u_{λ_2} is above u, which is impossible. The bound for u^* also requires a stable singular subsolution.

It turns out that in dimension $N \ge 32$ we can construct the necessary subsolutions and verify their stability by hand. Indeed, assume $a = b = 0$, $N \ge 13$ and let us show

$$u^* \le \bar{u} = -4 \log |x| \quad \text{in } B_1. \tag{1.61}$$

For this define $\bar{u}(x) = -4 \log |x|$. Then $\bar{u}$ satisfies

$$\begin{cases} \Delta^2 \bar{u} = 8(N-2)(N-4)e^{\bar{u}} & \text{in } \mathbb{R}^N \\ \bar{u} = 0 & \text{on } \partial B_1 \\ \dfrac{\partial \bar{u}}{\partial n} = -4 & \text{on } \partial B_1. \end{cases}$$

Observe that since $\bar{u}$ is a supersolution to (1.50) with $a = b = 0$ we deduce immediately that $\lambda^* \ge 8(N-2)(N-4)$.

In the case $\lambda^* = 8(N-2)(N-4)$ we have $u_\lambda \le \bar{u}$ for all $0 \le \lambda < \lambda^*$ because $\bar{u}$ is a supersolution, and therefore $u^* \le \bar{u}$ holds.

Suppose now that $\lambda^* > 8(N-2)(N-4)$. We prove that $u_\lambda \le \bar{u}$ for all $8(N-2)(N-4) < \lambda < \lambda^*$. Fix such λ and assume by contradiction that $u_\lambda \le \bar{u}$ is not true. Note that for $r < 1$ and sufficiently close to 1 we have $u_\lambda(r) < \bar{u}(r)$ because $u'_\lambda(1) = 0$ while $\bar{u}'(1) = -4$. Let

$$R_1 = \inf\{0 \le R \le 1 \mid u_\lambda < \bar{u} \text{ in } (R, 1)\}.$$

Then $0 < R_1 < 1$, $u_\lambda(R_1) = \bar{u}(R_1)$ and $u'_\lambda(R_1) \le \bar{u}'(R_1)$. So u_λ is a solution to the problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B_{R_1} \\ u = u_\lambda(R_1) & \text{on } \partial B_{R_1} \\ \dfrac{\partial u}{\partial n} = u'_\lambda(R_1) & \text{on } \partial B_{R_1} \end{cases}$$

while $\bar{u}$ is a stable subsolution to the same problem, because of (1.57) and $8(N-2)(N-4) \le N^2(N-4)^2/16$ for $N \ge 13$. By Lemma 1.29 part (b) we deduce $\bar{u} \le u_\lambda$ in B_{R_1} which is impossible.

An upper bound for λ^* is obtained by considering again a stable, singular subsolution to the problem but with another parameter:

LEMMA 1.30. *For $N \ge 32$ we have*

$$\lambda^* \le 8(N-2)(N-4)e^2. \tag{1.62}$$

PROOF. Consider $w = 2(1 - r^2)$ and define

$$u = \bar{u} - w,$$

where $\bar{u}(x) = -4 \log |x|$. Then

$$\begin{aligned}\Delta^2 u = 8(N-2)(N-4)\frac{1}{r^4} = 8(N-2)(N-4)e^{\bar{u}} &= 8(N-2)(N-4)e^{u+w} \\ &\le 8(N-2)(N-4)e^2 e^u.\end{aligned}$$

Also $u(1) = u'(1) = 0$, so u is a subsolution to (1.50) with parameter $\lambda_0 = 8(N-2)(N-4)e^2$.

For $N \ge 32$ we have $\lambda_0 \le N^2(N-4)^2/16$. Then by (1.57) u is a stable subsolution of (1.50) with $\lambda = \lambda_0$. If $\lambda^* > \lambda_0 = 8(N-2)(N-4)e^2$ the minimal solution u_{λ_0} to (1.50) with parameter λ_0 exists and is smooth. From Lemma 1.29 part (a) we find $u \le u_{\lambda_0}$ which is impossible because u is singular and u_{λ_0} is bounded. Thus we have proved (1.62) for $N \ge 32$. □

With the above remarks we can now prove Theorem 1.27 in the case $N \ge 32$. Combining (1.61) and (1.62) we have that if $N \ge 32$ then $\lambda^* e^{u^*} \le r^{-4} 8(N-2)(N-4)e^2 \le r^{-4} N^2 (N-4)^2/16$. This and (1.57) show that

$$\inf_{\varphi \in C_0^\infty(B)} \frac{\int_B (\Delta\varphi)^2 - \lambda^* \int_B e^{u^*}\varphi^2}{\int_B \varphi^2} > 0$$

which is not possible if u^* is bounded.

For dimensions $13 \le N \le 31$ it seems difficult to find subsolutions as before explicitly. We adopt then an approach that involves a computer-assisted construction and verification of the desired inequalities. More precisely, first we solve numerically (1.50) by following a branch of singular solutions to

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B \\ u = 0 & \text{on } \partial B \\ \dfrac{\partial u}{\partial \nu} = t & \text{on } \partial B. \end{cases} \tag{1.63}$$

We start with $t = -4$, where an explicit solution is known, and follow this branch to $t = 0$, transforming first (1.63) with an Emden–Fowler-type change of variables, which allows us to work with smooth solutions. This numerical solution, which is represented as a piecewise polynomial function with coefficients in $\mathbb{Q}$ that are kept explicitly, serves as the desired subsolution. The verification of the conditions mentioned before is done with a program in Maple, and in such a way that it guarantees a rigorous proof of the inequalities. This and the proof of Theorem 1.27 for $13 \le N \le 31$ is described in Section 4.3.

For general constant boundary values, it seems more difficult to determine the dimensions for which the extremal solution is singular. Observe that u^* is the extremal solution of (1.50) if and only if $u^* - a$ is the extremal solution of the same equation with boundary condition $u = 0$ on ∂B and so we may assume $a = 0$. But one may ask if Theorem 1.27 still holds for any $N \geq 13$ and any b. Here the situation becomes interesting, because the critical dimension for the boundedness of u^* depends on b and is not always equal to 13.

THEOREM 1.31. (a) *Let $N \geq 13$ and $b \geq -4$. There exists a critical parameter $b^{\max} > 0$ such that the extremal solution u^* is singular if and only if $b \leq b^{\max}$.*

(b) *Let $b \geq -4$. There exists a critical dimension $N^{\min} \geq 13$ such that the extremal solution u^* to* (1.50) *is singular if $N \geq N^{\min}$.*

The proof of this result can be found in [44]. Let us remark that it follows from Theorem 1.31, part (a), that for $b \in [-4, 0]$, the extremal solution is singular if and only if $N \geq 13$. We also deduce from this result that there exist values of b for which $N^{\min} > 13$. We do not know whether u^* remains bounded for $13 \leq N < N^{\min}$.

Finally let us mention that it remains open to describe fully the bifurcation diagram of (1.50), in the spirit if the work of Joseph and Lundgren (Theorem 1.2) for the second-order problem with exponential nonlinearity.

1.6. *Other directions*

The literature on the kind of problems we have mentioned is extensive. Nevertheless we would like mention other related directions which have been the matter of recent studies.

In general domains there are few results on the structure of solutions to (1.1). Let us mention here the results of Dancer [37–39]. For analytic nonlinearities g such that $g(u) \sim u^q e^u$ as $u \to +\infty$ in a bounded smooth domain Ω in $\mathbb{R}^3$ he shows that there is an unbounded connected curve of solutions $\widehat{T} = \{(\lambda(s), u(s)) : s \geq 0\}$ starting from $(0, 0)$ such that $\|u(s)\| + |\lambda(s)| \to +\infty$ as $s \to +\infty$ and $-\Delta - \lambda(s) g'(u(s))$ is invertible except at isolated singularities. This curve has infinitely many bifurcation points outside any compact subset, which include the possibility that the curve "bends back" at some of these points. In [37] Dancer also shows that a sequence of solutions to (1.1) with $g(u) = e^u$ in a bounded smooth domain in three dimensions, remains bounded if and only if their Morse indices are uniformly bounded. This is a consequence of a related result that asserts that any solution to

$$-\Delta u = e^u, \quad u < 0 \quad \text{in } \mathbb{R}^3$$

has infinite Morse index. The proof of [37] uses a result of Bidaut-Verón and Verón [14], that characterizes solutions to

$$-\Delta u = \lambda e^u \quad \text{in } \mathbb{R}^3 \setminus B_1 \tag{1.64}$$

such that

$$e^u \leq \frac{C}{|x|^2} \quad \text{in } \mathbb{R}^3 \setminus B_1. \tag{1.65}$$

In [14] it is proved that any solution to (1.64), (1.65) satisfies

$$\lim_{r\to+\infty}\left(u(r,\theta)-\log\frac{1}{r^2}\right)=2\omega(\theta)+\log\frac{2}{\lambda}\quad \text{in } C^k \text{ of } S^2$$

for any $k \geq 1$, where r, θ are spherical coordinates and ω is a smooth solution to

$$\Delta_{S^2}\omega + e^{2\omega} - 1 = 0 \quad \text{on } S^2. \tag{1.66}$$

Here Δ_{S^2} is the Laplace–Beltrami operator on S^2 with the standard metric. It is known that all continuous solutions to (1.66) arise from a single solution and the conformal transformations of S^2, see Chang and Yang [30].

We would like to mention some results for problems similar to (1.1) but where the Laplacian is replaced by a nonlinear operator. For example Clément *et al.* [33] considered the p-Laplacian and k-Hessian operators $S_k(D^2u)$ defined as the sum of all principal $k \times k$ minors of D^2u. Their results were extended by Jacobsen and Schmitt [74,75] and we shall describe them next. Consider

$$\begin{cases} r^{-\gamma}(r^{\alpha}|u'|^{\beta}u')' + \lambda e^u = 0 & 0 < r < 1 \\ u > 0 \quad 0 < r < 1 \\ u'(0) = u(1) = 0, \end{cases} \tag{1.67}$$

where α, β, γ satisfy

$$\begin{cases} \alpha \geq 0 \\ \gamma + 1 > \alpha \\ \beta + 1 > 0. \end{cases} \tag{1.68}$$

This includes the case of the Laplacian ($\alpha = N-1$, $\beta = 0$, $\gamma = N-1$), the p-Laplacian with $p > 1$ ($\alpha = N-1$, $\beta = p-2$, $\gamma = N-1$) and the k-Hessian operator ($\alpha = N-k$, $\beta = k-1$, $\gamma = N-1$). The main result in [74] characterizes in terms of α, β and γ the multiplicity of solutions as a function of λ.

THEOREM 1.32. *Suppose α, β, γ satisfy* (1.68) *and define*

$$\xi = \gamma + 1 - \alpha$$
$$\delta = \frac{\gamma + \beta - \alpha + 2}{\xi}.$$

Case 1. If $\alpha - \beta - 1 \leq 0$ there exists a unique $\lambda^* > 0$ such that (1.67) *has a unique solution for $\lambda = \lambda^*$, and exactly 2 solutions for $0 < \lambda < \lambda^*$.*

Case 2. If $0 < \alpha - \beta - 1 < \frac{4\delta\xi}{\beta+1}$ then (1.67) *has continuum of solutions (λ, u) with $u(0) \to +\infty$ and λ oscillating around $(\alpha - \beta - 1)(\delta\xi)^{\beta+1}$.*

Case 3. If $\frac{4\delta\xi}{\beta+1} \leq \alpha - \beta - 1$ then the equation has a unique solution for $0 < \lambda < (\alpha - \beta - 1)(\delta\xi)^{\beta+1}$ and no solution for $\lambda \geq (\alpha - \beta - 1)(\delta\xi)^{\beta+1}$. Moreover $u(0) \to +\infty$ as $\lambda \to (\alpha - \beta - 1)(\delta\xi)^{\beta+1}$.

The problem (1.1) for the p-Laplacian operator in general smooth, bounded domains, that is,

$$\begin{cases} -\Delta_p u = \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has also been the subject of study. We mention the case $g(u) = e^u$ considered by García-Azorero and Peral [65] and García-Azorero *et al.* [66] who showed that the extremal solution is bounded if $N < p + 4p/(p-1)$ and that this condition is optimal. Recently Cabré and Sanchón [27] (see also [25]) also considered this problem for general g, extending the ideas of [19,20] to this setting.

Another direction of interest is the parabolic counterpart of (1.1). Consider

$$\begin{cases} u_t - \Delta u = \lambda g(u) & \text{in } (0,T) \times \Omega \\ u = 0 & \text{on } \partial\Omega \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.69}$$

where g is a nonlinear function, $\lambda > 0$ and $u_0 \geq 0$, $u_0 \in L^\infty(\Omega)$.

It is well known that if $u_0 \in L^\infty(\Omega)$ and g is Lipschitz, then (1.69) has a classical solution defined on a maximal time interval.

Problem (1.69) with exponential nonlinearity was considered by Fujita [62,63]. Lacey [80] and also Bellout [12] proved, under certain extra conditions, that the solution of (1.69) blows up in finite time for $\lambda > \lambda^*$, see also [81]. In this direction we would like to mention the following results due to Brezis *et al.* [19]. Roughly speaking they imply that with initial condition $u_0 = 0$, the solution to the parabolic problem (1.69) is global if and only $\lambda \leq \lambda^*$, that is, if and only if the stationary problem has a weak solution.

THEOREM 1.33 (Brezis *et al.* [19]). *Assume $g : [0,\infty) \to \infty$ is a C^1 convex nondecreasing function such that there exists $x_0 \geq 0$ with $g(x_0) > 0$ and*

$$\int_{x_0}^{\infty} \frac{du}{g(u)} < +\infty. \tag{1.70}$$

Then if (1.69) *has a global solution for some $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ then there is a weak solution to the elliptic problem* (1.1).

This result has also a converse.

THEOREM 1.34 (Brezis *et al.* [19]). *Assume $g : [0,\infty) \to \infty$ is a C^1 convex nondecreasing function. If* (1.1) *has a weak solution w then for any initial condition $u_0 \in L^\infty(\Omega)$, $0 \leq u_0 \leq w$ the solution to* (1.69) *is global in time.*

Peral and Vázquez [100] considered also the parabolic problem (1.69) with the exponential nonlinearity in $\Omega = B_1$ and with $\lambda = 2(N-2)$, since for this parameter $U(x) = -2\log|x|$ is a weak solution of the stationary problem. They are interested in singular initial conditions and hence they work with the following notion of weak solution: $u \in C((0,\infty); W_0^{1,2}(B_1))$ such that u_t, Δu, $e^u \in L^1([\tau, T] \times B_1)$ for all $0 < \tau < T < +\infty$, equation (1.69) holds a.e. and $u(t,\cdot) \to u_0$ in $L^2(B_1)$ as $t \to 0$. First they take an initial condition u_0 satisfying $0 \le u_0(x) \le U(x)$. They show that (1.69) possesses a minimal and a maximal solution u satisfying $0 \le u(t,x) \le U(x)$. Moreover it becomes classical for $t > 0$. They show that if $3 \le N \le 9$ then any solution satisfying the previous conditions converges to the minimal solution u_λ as $t \to +\infty$. If $N \ge 10$ then $u(t,\cdot) \to U$ as $t \to +\infty$. These authors also study the possibility of having solutions of the parabolic problem above the singular solution U and establish the following

THEOREM 1.35. *Consider* (1.69) *with* $g(u) = e^u$, $\lambda = 2(N-2)$ *and* $\Omega = B_1$. *Then there is no weak solution defined on* $(0,T) \times B_1$ *such that* $u(t,x) \ge U(x)$, *and* $u_0 \not\equiv U$.

The solutions in the above result are shown to blow up completely (such as in Brezis and Cabré [18]) and instantaneously. Dold *et al.* [49] studied the blow-up rate of (1.69) with $\Omega = B_1$ and $g(u) = e^p$ or $g(u) = u^p$, $p > \frac{N+2}{N-2}$. Martel [88] showed that if the initial condition u_0 satisfies $u_0 \in L^\infty(\Omega) \cap W_0^{1,1}(\Omega)$, $u_0 \ge 0$ and $\Delta u_0 + \lambda g(u_0) \ge 0$, then the solution u to (1.69), which is defined on a maximal time interval $[0, T_m)$, blows up completely after T_m if $T_m < +\infty$. This means that for any sequence g_n of bounded approximations of g such that

$$g_n \in C([0,\infty), [0,n)) \quad \text{for all } x \ge 0,\ g_n(x) \uparrow g(x), \text{ as } n \to +\infty$$

the sequence of solutions u_n of (1.69) with g replaced by g_n satisfies

$$\frac{u_n(x,t)}{dist(x,\partial\Omega)} \to +\infty \quad \text{as } n \to +\infty \text{ uniformly for } t \in [T_m + \varepsilon, \infty)$$

for any $\varepsilon > 0$. The hypothesis on the initial condition says, roughly speaking, that $u_t(0) \ge 0$ and hence u is monotone nondecreasing in time, which is seen to be necessary (see below and [59]).

An interesting result of Fila and Poláčik [59] is the following. Consider (1.69) with $g(u) = e^u$ in the unit ball $\Omega = B_1$ and with a radial initial condition $u_0 \in C(\overline{B}_1)$. If $N \le 9$ and the solution u to (1.69) is global, i.e. is a classical solution defined for all times, then u is uniformly bounded, that is,

$$\sup_{t>0, r\in[0,1]} |u(r,t)| < +\infty.$$

In dimensions $N = 1, 2$ this holds for general domains and initial conditions, see [57].

In [59] the authors also show that for $g(u) = e^u$ and also in the radial setting in dimension $3 \le N \le 9$, certain stationary solutions can be connected by solutions that blow up in finite time but can be continued in an L^1 sense. An L^1 solution of the parabolic equation (1.69) is a function $u \in C([0,T]; L^1(\Omega))$ such that $g(u) \in L^1((0,T) \times \Omega)$ and

$$\int_\Omega u\varphi \Big|_\tau^t dx - \int_\tau^t \int_\Omega u\varphi_t \, dx\, ds = \int_\tau^t \int_\Omega (u\Delta\varphi + \lambda g(u)\varphi)\, dx\, ds$$

for all $0 \le \tau < t < T$ and $\varphi \in C^2([0,T] \times \overline{\Omega})$ with $\varphi = 0$ on $[0,T] \times \partial\Omega$. To describe the result [59] we use the notation, following [58]. The solutions to (1.1) with $g(u) = e^u$

in the unit ball B_1 with $3 \le N \le 9$ can be written as a smooth curve

$$(\lambda(s), u(s)), \quad s > 0$$

such that

$$\max_{\overline{B}_1} u(s) = u(s)(0) = s.$$

This curve satisfies

(a) $\lim_{s\to 0} \lambda(s) = 0$, $\lim_{s\to +\infty} \lambda(s) = 2(N-2)$

(b) the critical points of $\lambda(s)$ form a sequence $0 < s_1 < s_2 < \dots$ and the critical values $\lambda(s_j) = \lambda_j$ satisfy

$$\lambda_1 > \lambda_3 \dots > \lambda_{2j+1} \downarrow 2(N-2),$$
$$\lambda_2 < \lambda_4 < \dots \uparrow 2(N-2).$$

For $0 < \lambda < \lambda^*$ let $\tilde{s}_0(\lambda) < \tilde{s}_1(\lambda) < \dots$ denote the sequence of points s such that $\lambda(s) = \lambda$. This sequence is finite if $\lambda \ne 2(N-2)$ and infinite if $\lambda = 2(N-2)$. Write $u_\lambda^j = u(\tilde{s}_j)$. The minimal solution corresponds to $u_\lambda = u_\lambda^0$.

Fila and Poáčik [59] showed that if $\lambda \in (\lambda_2, \lambda_3)$ there exists a smooth initial condition u_0 such that the solution u to (1.69) satisfies:

(1) $u(\cdot, t)$ blows up in finite time T_m,

(2) $u(\cdot, t)$ can be extended to an L^1 global solution (i.e. define on $(0, T)$ for all $T > 0$),

(3) $u(\cdot, t) \to u_\lambda$ as $t \to +\infty$, where $u_\lambda = u_\lambda^0$ is the minimal solution (the convergence is $C^1_{\mathrm{loc}}((0, 1]))$

(4) $u(\cdot, t)$ is defined and smooth for all $t \in (-\infty, T_m)$ and $u(\cdot, t) \to u_\lambda^2$.

This solution is called an L^1 connection between the equilibria u_λ^2 and u_λ^0.

Later Fila and Matano [58] extended the results of [59] showing that for any $k \ge 2$ there is an L^1 connection from u_k^λ to u_0^λ. They also show that if an L^1 connection from u_k^λ to u_j^λ exists then $k \ge j + 2$. See also previous work by Ni *et al.* [97], Lacey and Tzanetis [82].

Nonlinear elliptic and parabolic equations such as (1.1) and (1.69) but with explicit singular terms in them have also been a matter of recent studies. Let us mention Brezis and Cabré [18], who showed that if $u \ge 0$ and

$$-\Delta u \ge \frac{u^2}{|x|^2} \quad \text{in } \Omega$$

in the sense of distributions (assuming $u, u^2/|x|^2 \in L^1_{\mathrm{loc}}(\Omega)$), in a domain Ω containing the origin, then $u \equiv 0$. Dupaigne [50], Dupaigne and Nedev [51] have studied elliptic equations with a singular potential of the form:

$$\begin{cases} -\Delta u - a(x)u = f(u) + \lambda b(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a, b, f, \lambda \ge 0$. They characterize, under some assumptions, in terms of the linear operator $-\Delta - a(x)$ and the nonlinearity $f(u)$ the cases where there are solutions for some $\lambda > 0$ or not. For instance if $a(x) = c/|x|^2$, $f(u) = u^p$, $p > 1$, then there is a solution for some $\lambda > 0$ if and only if $c \le (N-2)^2/4$ and $p < p_0 = 1 + 2/a$, where $a = \frac{N-2-\sqrt{(N-2)^2-4c}}{2}$. See also Kalton and Verbitsky [77].

We have mentioned already that the analysis of singular operators such as the Laplacian with a potential given by the inverse square distance to a point has been used to construct singular solutions to a variety of nonlinear problems, [28,89–91,98,99,101–103]. But in fact the same techniques can be applied to construct solutions in exterior domains which in some sense are singular at infinity, or in other words, that decay slowly at infinity. A model equation is

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}}, \tag{1.71}$$

$$u = 0 \quad \text{on } \partial\mathcal{D}, \qquad \lim_{|x| \to +\infty} u(x) = 0 \tag{1.72}$$

with supercritical p, namely $p > \frac{N+2}{N-2}$.

THEOREM 1.36 ([40,41]). *Let $\mathcal{D}$ be a bounded domain with smooth boundary such that $\mathbb{R}^N \setminus \overline{\mathcal{D}}$ is connected. For any $p > \frac{N+2}{N-2}$ there is a continuum of solutions u_λ, $\lambda > 0$, to* (1.71), (1.72) *such that*

$$u_\lambda(x) = \beta^{\frac{1}{p-1}} |x|^{-\frac{2}{p-1}} (1 + o(1)) \quad \textit{as } |x| \to \infty \tag{1.73}$$

and $u_\lambda(x) \to 0$ as $\lambda \to 0$, uniformly in $\mathbb{R}^N \setminus \mathcal{D}$, where

$$\beta = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right).$$

The idea of the proof is by linearization around $w(x)$, the unique positive radial solution

$$\Delta w + w^p = 0 \quad \text{in } \mathbb{R}^N, \qquad w(0) = 1. \tag{1.74}$$

Note that all radial solutions of $\Delta u + u^p = 0$ defined in all $\mathbb{R}^N$ have the form

$$w_\lambda(x) = \lambda^{\frac{2}{p-1}} w(\lambda|x|), \quad \lambda > 0. \tag{1.75}$$

We look for a solution u_λ in the form of a small perturbation of w_λ. This naturally leads us to study the linearized operator $\Delta + p w_\lambda^{p-1}$ in $\mathbb{R}^N \setminus \mathcal{D}$ under Dirichlet boundary conditions. Since w_λ is small on bounded sets for small λ, an inverse can be found as a small perturbation of an inverse of this operator in the whole $\mathbb{R}^N$ and then, by scaling, it suffices to analyze the case $\lambda = 1$. Thus we need to study $\Delta + p w^{p-1}$ in $\mathbb{R}^N$. Note that at main order one has $w(r) = \beta^{\frac{1}{p-1}} r^{-\frac{2}{p-1}}(1 + o(1))$ as $r \to +\infty$ [72], and hence the singular potential has the form $p\beta/r^2(1 + o(1))$. We construct an inverse in weighted L^∞ norms for $p \ge \frac{N+1}{N-3}$, however if $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ the linearized operator is not surjective, having a range orthogonal to the generators of translations. We overcome this difficulty by adjusting the location of the origin. The invertibility analysis for $p \geq \frac{N+1}{N-3}$ is strongly related to one of Mazzeo and Pacard [90] in the construction of singular solutions with prescribed singularities for $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ in bounded domains. At the radial level, supercritical and subcritical in this range are completely dual.

Problems (1.71)–(1.72) has also a fast decay solution, that is a solution u such that $\limsup_{|x|\to+\infty} |x|^{2-N} u(x) < +\infty$, provided $\frac{N+2}{N-2} < p$ and $p - \frac{N+2}{N-2}$ is small, see [41].

A related result for supercritical problems in bounded domains is the following. Consider

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{in } \mathcal{D} \setminus B_\delta(Q), \tag{1.76}$$

$$u = 0 \quad \text{on } \partial\mathcal{D} \cup \partial B_\delta(Q), \tag{1.77}$$

where $\mathcal{D}$ is a bounded domain with smooth boundary, $B_\delta(Q) \subset \mathcal{D}$ and $\delta > 0$ is to be taken small.

THEOREM 1.37 (del Pino and Wei [47]). *There exists a sequence*

$$\frac{N+2}{N-2} < p_1 < p_2 < p_3 < \dots, \quad \textit{with} \lim_{k\to+\infty} p_k = +\infty \tag{1.78}$$

such that if $p > \frac{N+2}{N-2}$ and $p \neq p_j$ for all j, then there is a $\delta_0 > 0$ such that for any $\delta < \delta_0$, Problems (1.76), (1.77) *possess at least one solution.*

2. Perturbation of singular solutions

2.1. *The Laplacian with the inverse square potential*

We consider the linear problem

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x-\xi|^2}\phi = g & \text{in } B \\ \phi = h & \text{on } \partial B, \end{cases} \tag{2.1}$$

where $B = B_1(0)$, $\xi \in B$ and c is any real number. The main results are Propositions 2.1 and 2.3 below, which assert the solvability of (2.1) in weighted Hölder spaces assuming that the right-hand side verifies certain orthogonality conditions, provided ξ is close to the origin. We use the weighted Hölder spaces that appear in [101,8,28], which are defined as follows. Given Ω a smooth domain, $\xi \in \Omega$, $k \geq 0$, $0 < \alpha < 1$, $0 < r \leq dist(x, \partial\Omega)/2$ and $u \in C^{k,\alpha}_{\mathrm{loc}}(\overline{B} \setminus \{\xi\})$ we define :

$$|u|_{k,\alpha,r,\xi} = \sup_{r\leq|x-\xi|\leq 2r} \sum_{j=0}^{k} r^j |\nabla^j u(x)| + r^{k+\alpha}\left[\sup_{r\leq|x-\xi|,|y-\xi|\leq 2r} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha}\right].$$

Let $d = dist(\xi, \partial\Omega)$ and for any $\nu \in \mathbb{R}$ let

$$\|u\|_{k,\alpha,\nu,\xi;\Omega} = \|u\|_{C^{k,\alpha}(\overline{\Omega}\setminus B_{d/2}(\xi))} + \sup_{0<r\le\frac{d}{2}} r^{-\nu}|u|_{k,\alpha,r,\xi}.$$

Define the Banach space

$$C^{k,\alpha}_{\nu,\xi}(\Omega) = \{u \in C^{k,\alpha}_{\mathrm{loc}}(\overline{\Omega}\setminus\{\xi\}) \,:\, \|u\|_{k,\alpha,\nu,\xi;\Omega} < \infty\}.$$

It embeds continuously in the space of bounded functions if $\nu \ge 0$.

For the analysis of (2.1) when $\xi = 0$ it is convenient to decompose all functions in Fourier series. So we recall that the eigenvalues of the Laplace–Beltrami operator $-\Delta$ on S^{N-1} are given by (see [13])

$$\lambda_k = k(N+k-2), \quad k \ge 0.$$

Let m_k denote the multiplicity of λ_k and $\varphi_{k,l}$, $l = 1,\dots,m_k$ the eigenfunctions associated to λ_k. We normalize these eigenfunctions so that $\{\varphi_{k,l} : k \ge 0, l = 1,\dots,m_k\}$ is an orthonormal system in $L^2(S^{N-1})$. We choose the first functions to be

$$\varphi_{0,1} = \frac{1}{|S^{N-1}|^{1/2}}, \quad \varphi_{1,l} = \frac{x_l}{(\int_{S^{N-1}} x_l^2)^{1/2}} = \left(\frac{N}{|S^{N-1}|}\right)^{1/2} x_l, \quad l = 1,\dots,N.$$

Let $r = |x|$ and $\theta = x/|x|$ denote polar coordinates in $\mathbb{R}^N$.

First we study the kernel of the operator $\Delta + c/|x|^2$. Thus we look for solutions to

$$-\Delta w - \frac{c}{|x|^2} w = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\} \tag{2.2}$$

of the form $w(x) = f(r)\varphi_{k,l}(\theta)$ which yields the ODE:

$$f'' + \frac{N-1}{r} f' + \frac{c-\lambda_k}{r^2} f = 0, \quad \text{for } r > 0. \tag{2.3}$$

Equation (2.3) is of Euler-type and it admits a basis of solutions of the form $f(r) = r^{-\alpha_k^{\pm}}$, where $\alpha_k^{\pm}$ are the roots of the associated characteristic equation, i.e.

$$\alpha_k^{\pm} = \frac{N-2}{2} \pm \sqrt{\left(\frac{N-2}{2}\right)^2 - c + \lambda_k}. \tag{2.4}$$

Note that $\alpha_k^{\pm}$ may have a nonzero imaginary part only for finitely many k's. If k_0 is the first integer k such that $\alpha_k^{\pm} \in \mathbb{R}$ then

$$\dots < \alpha_{k_0+1}^- < \alpha_{k_0}^- \le \frac{N-2}{2} \le \alpha_{k_0}^+ < \alpha_{k_0+1}^+ < \dots,$$

whereas, if $k < k_0$, we denote the imaginary part of α_k^+ by

$$b_k = \sqrt{c - \left(\frac{N-2}{2}\right)^2 - \lambda_k}.$$

For $k \ge 0$, $l = 1,\dots,m_k$, we have a family of real-valued solutions of (2.2), denoted by $w^1 = w^1_{k,l}$, $w_2 = w^2_{k,l}$ and defined on $\mathbb{R}^N \setminus \{0\}$ by:

if $(\frac{N-2}{2})^2 - c + \lambda_k > 0$

$$w^1 = r^{-\alpha_k^+}\varphi_{k,l}(\theta), \quad w^2 = r^{-\alpha_k^-}\varphi_{k,l}(\theta), \tag{2.5}$$

if $(\frac{N-2}{2})^2 - c + \lambda_k = 0$

$$w^1 = r^{-\frac{N-2}{2}}\log r\varphi_{k,l}(\theta), \quad w^2 = r^{-\frac{N-2}{2}}\varphi_{k,l}(\theta), \tag{2.6}$$

if $\frac{N-2}{2})^2 - c + \lambda_k < 0$

$$w^1 = r^{-\frac{N-2}{2}}\sin(b_k\log r)\varphi_{k,l}(\theta), \quad w^2(x) = r^{-\frac{N-2}{2}}\cos(b_k\log r)\varphi_{k,l}(\theta). \tag{2.7}$$

Then the functions $W_{k,l}$ defined by

$$\begin{cases} \text{if } (\frac{N-2}{2})^2 - c + \lambda_k > 0: & W_{k,l}(x) = w^1(x) - w^2(x), \\ \text{if } (\frac{N-2}{2})^2 - c + \lambda_k \le 0: & W_{k,l}(x) = w^1(x), \end{cases} \tag{2.8}$$

solve (2.2) and satisfy

$$W_{k,l}|_{\partial B} = 0.$$

The main result in this section for the case $\xi = 0$ is

PROPOSITION 2.1. *Let $c, \nu \in \mathbb{R}$ and assume*

$$\exists k_1 \textit{such that} \quad \alpha_{k_1}^- \in \mathbb{R} \quad \textit{and} \quad -\alpha_{k_1}^- < \nu < -\alpha_{k_1+1}^-. \tag{2.9}$$

Let $g \in C^{0,\alpha}_{\nu-2,0}(B)$ and $h \in C^{2,\alpha}(\partial B)$ and consider

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x|^2}\phi = g & \text{in } B \\ \qquad\qquad \phi = h & \text{on } \partial B. \end{cases} \tag{2.10}$$

Then (2.10) *has a solution in $C^{2,\alpha}_{\nu,0}(B)$ if and only if*

$$\int_B gW_{k,l} = \int_{\partial B} h\frac{\partial W_{k,l}}{\partial n}, \quad \forall k = 0,\dots,k_1, \forall l = 1,\dots,m_k. \tag{2.11}$$

Under this condition the solution $\phi \in C^{2,\alpha}_{\nu,0}(B)$ to (2.10) *is unique and it satisfies*

$$\|\phi\|_{2,\alpha,\nu,0;B} \le C(\|g\|_{0,\alpha,\nu-2,0;B} + \|h\|_{C^{2,\alpha}(\partial B)}), \tag{2.12}$$

where C is independent of g and h.

Note that with the hypotheses of Lemma 2.1 we have

$$\nu > -\alpha_{k_1}^- \ge -\frac{N-2}{2}. \tag{2.13}$$

This implies that the integrals on the left-hand side of (2.11) exist.

PROOF OF PROPOSITION 2.1. Write ϕ as

$$\phi(x)=\sum_{k=0}^{\infty}\sum_{l=1}^{m_k}\phi_{k,l}(r)\varphi_{k,l}(\theta),\qquad x=r\theta,0<r<1,\theta\in S^{N-1}.$$

Then ϕ solves $-\Delta\phi-\frac{c}{|x|^2}\phi=g$ in $B\setminus\{0\}$ if and only if $\phi_{k,l}$ satisfies the ODE

$$\phi_{k,l}''+\frac{N-1}{r}\phi_{k,l}'+\frac{c-\lambda_k}{r^2}\phi_{k,l}=-g_{k,l}\quad 0<r<1, \tag{2.14}$$

for all $k\geq 0$ and $l=1,\dots,m_k$, where

$$g_{k,l}(r)=\int_{S^{N-1}}g(r\theta)\varphi_{k,l}(\theta)\,d\theta,\qquad 0<r<1,\theta\in S^{N-1}.$$

Note that if $\phi\in L^\infty_\nu(B)$ then there exists a constant $C>0$ independent of r such that

$$|\phi_{k,l}(r)|\leq Cr^\nu. \tag{2.15}$$

Furthermore, $\phi=h$ on ∂B if and only if $\phi_{k,l}(1)=h_{k,l}$ for all k,l, where

$$h_{k,l}=\int_{S^{N-1}}h(\theta)\varphi_{k,l}(\theta)\,d\theta.$$

Step 1. Clearly, $\sup_{0\leq t\leq 1}t^{2-\nu}|g_{k,l}(t)|<\infty$ and observe that (2.11) still holds when g is replaced by $g_{k,l}\varphi_{k,l}$ and h by $h_{k,l}\varphi_{k,l}$. We claim that there is a unique $\phi_{k,l}$ that satisfies (2.14), (2.15) and

$$\phi_{k,l}(1)=h_{k,l}. \tag{2.16}$$

We also have

$$|\phi_{k,l}(r)|\leq C_k r^\nu\left(\sup_{0\leq t\leq 1}t^{2-\nu}|g_{k,l}(t)|+|h_{k,l}|\right),\qquad 0<r<1. \tag{2.17}$$

Case $k=0,\dots,k_1$. A solution to (2.14) is given by:

- if $\alpha^\pm_{k,l}\notin\mathbb{R}$

$$\phi_{k,l}(r)=\frac{1}{b}\int_0^r s\left(\frac{s}{r}\right)^{\frac{N-2}{2}}\sin\left(b_k\log\frac{s}{r}\right)g_{k,l}(s)\,ds, \tag{2.18}$$

- if $\alpha^+_{k,l}=\alpha^-_{k,l}=\frac{N-2}{2}$:

$$\phi_{k,l}(r)=\int_0^r s\left(\frac{s}{r}\right)^{\frac{N-2}{2}}\log\left(\frac{s}{r}\right)g_{k,l}(s)\,ds, \tag{2.19}$$

- if $\alpha^\pm_{k,l}\in\mathbb{R}$, $\alpha^\pm_{k,l}\neq\frac{N-2}{2}$:

$$\phi_{k,l}(r)=\frac{1}{\alpha^+_k-\alpha^-_k}\int_0^r s\left(\left(\frac{s}{r}\right)^{\alpha^+_k}-\left(\frac{s}{r}\right)^{\alpha^-_k}\right)g_{k,l}(s)\,ds. \tag{2.20}$$

In each case, (2.17) holds and (2.16) follows from (2.11).

Concerning uniqueness, suppose that $\phi_{k,l}$ satisfies (2.14) with $g_{k,l} = 0$ and (2.16) with $h_{k,l} = 0$. Then $\phi_{k,l}$ is a linear combination of the functions w^1, w^2 defined in (2.5)–(2.7). By (2.9), (2.13) and (2.17), $\phi_{k,l}$ has to be zero.

Case $k \ge k_1 + 1$. Observe that (2.14) is equivalent to

$$-\Delta\tilde{\phi}_{k,l} + \frac{\lambda_k - c}{|x|^2}\tilde{\phi}_{k,l} = \tilde{g}_{k,l} \quad \text{in } B \setminus \{0\},$$

where $\tilde{\phi}_{k,l}(x) = \phi_{k,l}(|x|)$ and $\tilde{g}_{k,l}(x) = g_{k,l}(|x|)$. Since $\alpha_k^{\pm} \in \mathbb{R}$ we must have $\lambda_k - c \ge -(\frac{N-2}{2})^2$ and hence the equation

$$\begin{cases} -\Delta\tilde{\phi}_{k,l} + \dfrac{\lambda_k - c}{|x|^2}\tilde{\phi}_{k,l} = \tilde{g}_{k,l} & \text{in } B \\ \tilde{\phi}_{k,l} = h_{k,l} & \text{on } \partial B, \end{cases} \tag{2.21}$$

has a unique solution $\tilde{\phi}_{k,l} \in H$, where H is the completion of $C_0^\infty(B)$ with the norm

$$\|\varphi\|_H^2 = \int_B |\nabla\varphi|^2 + \frac{\lambda_k - c}{|x|^2}\varphi^2,$$

see [106].

To show (2.17), observe that for some constant C depending only on N, λ_k and ν,

$$A_{k,l}(r) = r^\nu C\left(\sup_{0<t\le 1} t^{2-\nu}|\tilde{g}_{k,l}(t)| + |h_{k,l}|\right)$$

is a supersolution to (2.21) and $-A_{k,l}$ is a subsolution. To see this, we emphasize that the condition $-\alpha_k^- > \nu > -(N-2)/2$ implies $\nu^2 + (N-2)\nu + c - \lambda_k < 0$. It follows that $|\tilde{\phi}_{k,l}(x)| \le A_{k,l}(|x|)$ for $0 < |x| \le 1$.

We note that $\tilde{\phi}_{k,l}$ is uniquely determined. Indeed, any solution w of (2.21) such that $|w(x) \le C|x|^\nu$ satisfies, by a scaling argument, $|\nabla w(x)| \le C|x|^{\nu-1}$ and this together with (2.13) implies $w \in H^1(B)$, which is contained in H. Uniqueness for (2.21) in $H^1(B)$ can then be proved by an improved Hardy inequality (see [20]).

The computations above also yield the necessity of condition (2.11). Indeed, assuming a solution $\phi \in L_\nu^\infty(B)$ exists, since $\phi_{k,l}$ satisfies the ODE (2.14) we see that for $k = 0, \ldots, k_1$ the difference between $\phi_{k,l}$ and one of the particular solutions (2.18), (2.19) or (2.20) can be written in the form $c_{k,l}r^{-\alpha_k^+} + d_{k,l}r^{-\alpha_k^-}$. Since $|\phi_{k,l}(r)| \le Cr^\nu$ and $\nu > -\alpha_{k_1}^-$ we have $c_{k,l} = d_{k,l} = 0$ and this implies (2.11).

Step 2. Define for $m \ge 1$

$$\mathcal{G}_m = \left\{ g = \sum_{k=0}^m \sum_l g_{k,l}(r)\varphi_{k,l}(\theta) : |x|^{2-\nu}g(x) \in L^\infty(B) \right\}$$

and

$$\mathcal{H}_m = \left\{ h = \sum_{k=0}^m \sum_l h_{k,l}\varphi_{k,l}(\theta) : h_{k,l} \in \mathbb{R} \right\}.$$

Let $g_m \in \mathcal{G}_m$, $h_m \in \mathcal{H}_m$ be such that (2.11) holds. Write

$$g_m(x) = \sum_{k=0}^{m} \sum_{l} g_{k,l}(r)\varphi_{k,l}(\theta), \qquad h_m(\theta) = \sum_{k=0}^{m} h_{k,l}\varphi_{k,l}(\theta).$$

Let $\phi_{k,l}$ be the unique solution to (2.14), (2.15) and (2.16) associated to $g_{k,l}$, $h_{k,l}$ and define $\phi_m(x) = \sum_{k=0}^{m} \sum_l \phi_{k,l}(r)\varphi_{k,l}(\theta)$. We claim that there exists C independent of m such that

$$|\phi_m(x)| \le C|x|^\nu \left(\sup_B |y|^{2-\nu}|g_m(y)| + \sup_{\partial B} |h_m| \right), \quad 0 < |x| < 1. \tag{2.22}$$

By the previous step, (2.22) holds for some constant C which may depend on m. In particular, choosing $m = k_1$, we obtain a bound on the first components $\phi_{k,l}$, $k = 0 \dots k_1$. Hence, it suffices to prove (2.22) in the case $g_{k,l} \equiv 0$ and $h_{k,l} = 0$, $k = 0, \dots, k_1$. Working as in [101], we argue by contradiction assuming that

$$\|\phi_m |x|^{-\nu}\|_{L^\infty(B)} \ge C_m(\|g_m|x|^{2-\nu}\|_{L^\infty(B)} + \|h_m\|_{L^\infty(\partial B)}),$$

where $C_m \to \infty$ as $m \to \infty$ (this argument also appears in [28]). Replacing ϕ_m by $\phi_m / \|\phi_m|x|^{-\nu}\|_{L^\infty(B)}$ if necessary, we may assume

$$\begin{aligned} &\|\phi_m |x|^{-\nu}\|_{L^\infty(B)} = 1, \\ &\|g_m|x|^{2-\nu}\|_{L^\infty(B)} + \|h_m\|_{L^\infty(\partial B)} \to 0 \quad \text{as } m \to \infty. \end{aligned} \tag{2.23}$$

Let $x_m \in B \setminus \{0\}$ be such that $|\phi_m(x_m)||x_m|^{-\nu} \in [\frac{1}{2}, 1]$. Let us show that $x_m \to 0$ as $m \to \infty$. Otherwise, up to a subsequence $x_m \to x_0 \neq 0$. By standard elliptic regularity, up to another subsequence, $\phi_m \to \phi$ uniformly on compact sets of $\overline{B} \setminus \{0\}$ and hence

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x|^2}\phi = 0 & \text{in } B \setminus \{0\} \\ \phi = 0 & \text{on } \partial B. \end{cases}$$

Moreover ϕ satisfies $|\phi(x_0)||x_0|^{-\nu} \in [\frac{1}{2}, 1]$ and $|\phi(x)| \le |x|^\nu$ in B. Writing

$$\phi(x) = \sum_{k \ge k_1+1} \sum_l \phi_{k,l}(r)\varphi_{k,l}(\theta),$$

we see that $\phi_{k,l}$ solves (2.3). The growth restriction $|\phi_{k,l}(r)| \le Cr^\nu$ and the explicit functions w^1, w^2 given by (2.5)–(2.7) rule out the cases $\alpha_k^\pm \notin \mathbb{R}$, $\alpha_k^- = \alpha_k^+$ and force $\phi_{k,l} = a_{k,l}r^{-\alpha_k^-}$. But $\phi_{k,l}(1) = 0$ so we deduce $\phi_{k,l} \equiv 0$ and hence $\phi \equiv 0$, contradicting $|\phi(x_0)||x_0|^{-\nu} \neq 0$.

The above argument shows that $x_m \to 0$. Define $r_m = |x_m|$ and

$$v_m(x) = r_m^{-\nu}\phi_m(r_m x), \quad x \in B_{1/r_m}.$$

Then $|v_m(x)| \le |x|^\nu$ in B_{1/r_m}, $|v_m(\frac{x_m}{r_m})| \in [\frac{1}{2}, 1]$ and

$$-\Delta v_m(x) - \frac{c}{|x|^2}v_m(x) = r_m^{2-\nu}g(r_m x) \quad \text{in } B_{1/r_m}\setminus\{0\}.$$

But

$$r_m^{2-\nu}|g(r_m x)| \le \|g_m(y)|y|^{2-\nu}\|_{L^\infty(B)}|x|^{\nu-2} \to 0, \quad \text{as } m\to\infty$$

by (2.23). Passing to a subsequence, we have that $\frac{x_m}{r_m} \to x_0$ with $|x_0| = 1$, $v_m \to v$ uniformly on compact sets of $\mathbb{R}^N\setminus\{0\}$ and v satisfies

$$-\Delta v - \frac{c}{|x|^2}v = 0 \quad \text{in } \mathbb{R}^N\setminus\{0\}.$$

Furthermore, $|v(x)| \le |x|^\nu$ in $\mathbb{R}^N\setminus\{0\}$ and $|v(x_0)| \ne 0$. Write

$$v(x) = \sum_{k=0}^{\infty}\sum_{l} v_{k,l}(r)\varphi_{k,l}(\theta).$$

Then $|v_{k,l}(r)| \le C_k r^\nu$ for $r > 0$. But $v_{k,l}$ has to be a linear combination of the functions w^1, w^2 given in (2.5)–(2.7), and none of these is bounded by Cr^ν for all $r > 0$. Thus $v \equiv 0$ yielding a contradiction. This establishes (2.22).

Step 3. Finally, a density argument shows that if h, g satisfy (2.11) then there exists a solution ϕ to (2.10) and satisfies (2.22). From (2.22) if we assume that $g \in C^{0,\alpha}_{\nu-2,0}(B)$ and $h \in C^{2,\alpha}(\partial B)$, using Schauder estimates and a scaling argument it is possible to show that the solution ϕ found above satisfies (2.12). □

COROLLARY 2.2. *Assume* (2.9), (2.10), (2.11) *and that* $\nu \ge 0$. *If* $|x|^2 g$ *is continuous at the origin, then so is* ϕ.

PROOF. Let (α_n) denote an arbitrary sequence of real numbers converging to zero, $\tilde g(x) = |x|^2 g(x)$ and $\phi_n(x) = \phi(\alpha_n x)$ for $x \in B_{1/\alpha_n}(0)$. Then ϕ_n solves

$$-\Delta\phi_n - \frac{c}{|x|^2}\phi_n = \frac{\tilde g(\alpha_n x)}{|x|^2} \quad \text{in } B_{1/\alpha_n}(0).$$

Also, (ϕ_n) is uniformly bounded so that up to a subsequence, it converges in the topology of $C^{1,\alpha}(\mathbb{R}^N\setminus\{0\})$ to a bounded solution Φ of

$$-\Delta\Phi - \frac{c}{|x|^2}\Phi = \frac{\tilde g(0)}{|x|^2} \quad \text{in } \mathbb{R}^N\setminus\{0\}.$$

Now $\Phi + \tilde g(0)/c$ is bounded and solves (2.2), so it must be identically zero. It follows that the whole sequence (ϕ_n) converges to $-\tilde g(0)/c$. Let now (x_n) denote an arbitrary sequence of points in $\mathbb{R}^N$ converging to 0 and $\alpha_n = |x_n|$. Then, $\phi(x_n) = \phi_n(\frac{x_n}{|x_n|})$ and up to a subsequence, $\phi(x_n) \to -\tilde{g}(0)/c$. Again, since the limit of such a subsequence is unique, the whole sequence converges. $\square$

Now we would like to consider a potential which is the inverse square to a point $\xi \in B_{1/2}$, that is, we consider the problem

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x-\xi|^2}\phi = \dfrac{g}{|x-\xi|^2} + \mu_0 \dfrac{g_0}{|x-\xi|^2} + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \mu_{k,l} V_{k,l,\xi} & \text{in } B \\ \phi = h & \text{on } \partial B, \end{cases} \tag{2.24}$$

where

$$V_{k,l,\xi}(x) = \eta(|x-\xi|) W_{k,l}\left(\frac{x-\xi}{1-2\varepsilon_0}\right) \quad \text{for } k \geq 1, l = 1, \ldots, m_k, \tag{2.25}$$

$\eta \in C^\infty(\mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta \not\equiv 0$ and $\mathrm{supp}(\eta) \subset [\frac{1}{4}, \frac{1}{2}]$ and $\varepsilon_0 > 0$ is fixed (suitably small).

We have:

PROPOSITION 2.3. *Assume*

$$\exists k_1 \text{ such that } \quad \alpha_{k_1}^- \in \mathbb{R} \quad \text{and} \quad -\alpha_{k_1}^- < \nu < -\alpha_{k_1+1}^-. \tag{2.26}$$

Then there exists $\varepsilon_0 > 0$ such that if $|\xi| < \varepsilon_0$ and $g_0 \in C^{0,\alpha}_{\nu,\xi}(B)$ satisfies

$$\|g_0 - 1\|_{L^\infty(B)} < \varepsilon_0,$$

then given any $g \in C^{0,\alpha}_{\nu,\xi}(B)$ and $h \in C^{2,\alpha}(\partial B)$, there exist unique $\phi \in C^{2,\alpha}_{\nu,\xi}(B)$ and $\mu_0, \mu_{k,l} \in \mathbb{R}$ ($k = 1, \ldots, k_1$, $l = 1, \ldots, m_k$) solution to (2.24). *Moreover we have for some constant $C > 0$ independent of g and h*

$$\|\phi\|_{2,\alpha,\nu,\xi;B} + |\mu_0| + \sum_{k=1}^{k_1}\sum_{l=1}^{m_k} |\mu_{k,l}| \leq C(\|g\|_{0,\alpha,\nu,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)}). \tag{2.27}$$

PROOF. We work with $0 < |\xi| < \varepsilon_0$, where $\varepsilon_0 \in (0, 1/2)$ is going to be fixed later on, small enough. Let $R = 1 - 2\varepsilon_0$. This implies in particular that $B_R(\xi) \subset B$.

We define an operator $T_1 : C^{2,\alpha}(\partial B_R(\xi)) \to C^{1,\alpha}(\partial B_R(\xi)) \times \mathbb{R}$ as follows: given $\phi_0 \in C^{2,\alpha}(\partial B_R(\xi))$, find $\phi \in C^{2,\alpha}_{\nu,\xi}(B_R(\xi))$ and γ_0, $\gamma_{k,l}$ the unique solution to

$$\begin{cases} -\Delta\phi_1 - \dfrac{c}{|x-\xi|^2}\phi_1 = \gamma_0 \dfrac{g_0}{|x-\xi|^2} + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \gamma_{k,l} V_{k,l,\xi} & \text{in } B_R(\xi) \\ \phi_1 = \phi_0 & \text{on } \partial B_R(\xi), \end{cases} \tag{2.28}$$

and set $T_1(\phi_0) = (\frac{\partial \phi_1}{\partial n}, \gamma_0)$. This can be done (see Step 1 below) by adjusting the constants γ_0 and $\gamma_{k,l}$ in such a way that the orthogonality relations (2.11) in Lemma 2.1 are satisfied. Similarly, there is a unique $\tilde\phi_1 \in C^{2,\alpha}_{\nu,\xi}(B_R(\xi))$ and $\tilde\gamma_0$, $\tilde\gamma_{k,l}$ such that

$$\begin{cases} -\Delta\tilde\phi_1 - \dfrac{c}{|x-\xi|^2}\tilde\phi_1 = \dfrac{g}{|x-\xi|^2} + \tilde\gamma_0 \dfrac{g_0}{|x-\xi|^2} & \\ \qquad\qquad + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \tilde\gamma_{k,l} V_{k,l,\xi} & \text{in } B_R(\xi) \\ \tilde\phi_1 = 0 & \text{on } \partial B_R(\xi). \end{cases} \tag{2.29}$$

Given $\tilde\phi_1$, $\tilde\gamma_0$ as in (2.29), we define $\tilde\phi_2$ by

$$\begin{cases} -\Delta\tilde\phi_2 - \dfrac{c}{|x-\xi|^2}\tilde\phi_2 = \dfrac{g}{|x-\xi|^2} + \tilde\gamma_0 \dfrac{g_0}{|x-\xi|^2} & \text{in } B \setminus B_R(\xi) \\ \dfrac{\partial\tilde\phi_2}{\partial n} = \dfrac{\partial\tilde\phi_1}{\partial n} & \text{on } \partial B_R(\xi) \\ \tilde\phi_2 = h & \text{on } \partial B. \end{cases} \tag{2.30}$$

We also define an operator $T_2 : C^{1,\alpha}(\partial B_R(\xi)) \times \mathbb{R} \to C^{2,\alpha}(\partial B_R(\xi))$ by

$$T_2(\Psi, \gamma_0) = \phi_2|_{\partial B_R(\xi)},$$

where ϕ_2 is the solution to

$$\begin{cases} -\Delta\phi_2 - \dfrac{c}{|x-\xi|^2}\phi_2 = \gamma_0 \dfrac{g_0}{|x-\xi|^2} & \text{in } B \setminus B_R(\xi) \\ \dfrac{\partial\phi_2}{\partial n} = \Psi & \text{on } \partial B_R(\xi) \\ \phi_2 = 0 & \text{on } \partial B. \end{cases} \tag{2.31}$$

As we shall see later (see Step 2), equations (2.30) and (2.31) possess indeed a unique solution if ξ is sufficiently small, because the domain $B \setminus B_R(\xi)$ is small.

We construct a solution ϕ of (2.24) as follows: choose $\phi_0 \in C^{2,\alpha}(\partial B_R(\xi))$, let ϕ_1 be the solution to (2.28) and let ϕ_2 be the solution to (2.31) with $\Psi = \frac{\partial\phi_1}{\partial n}$ and γ_0 from problem (2.28). Then set

$$\phi = \begin{cases} \phi_1 + \tilde\phi_1 & \text{in } B_R(\xi) \\ \phi_2 + \tilde\phi_2 & \text{in } B \setminus B_R(\xi), \end{cases}$$

and $\mu_0 = \gamma_0 + \tilde\gamma_0$, $\mu_{k,l} = \gamma_{k,l} + \tilde\gamma_{k,l}$. If we have in addition

$$\phi_1 + \tilde\phi_1 = \phi_2 + \tilde\phi_2 \qquad \text{on } \partial B_R(\xi), \tag{2.32}$$

then ϕ, μ_0 and $\mu_{k,l}$ form a solution to (2.24).

With this notation, solving equation (2.24) thus reduces to finding $\phi_0 \in C^{2,\alpha}(\partial B_R(\xi))$ such that (2.32) holds i.e.

$$T_2 \circ T_1(\phi_0) + \widetilde{\phi}_2 = \phi_0 \quad \text{in } \partial B_R(\xi).$$

The fact that this equation is uniquely solvable (when ξ is small) will follow once we show that $\|T_2\| \to 0$ as $\varepsilon_0 \to 0$, while $\|T_1\|$ remains bounded.

Step 1. Given $\phi_0 \in C^{2,\alpha}(\partial B_R(\xi))$ there exist γ_0 and $\gamma_{k,l}$ such that (2.28) has a unique solution ϕ_1 in $C^{2,\alpha}_{\nu,\xi}(\partial B_R(\xi))$.

In this step we change variables $y = x - \xi$ and work in $B_R(0)$. Solving for γ_0 in the orthogonality relations (2.11) yields

$$\gamma_0 = \frac{\frac{1}{R}\int_{\partial B_R} \phi_0 \frac{\partial W_{0,0}}{\partial n}(\frac{y}{R})}{\int_{B_R} g_0(y+\xi)|y|^{-2} W_{0,0}(\frac{y}{R})} \tag{2.33}$$

and a computation, using $\|g_0 - 1\|_{L^\infty(B_R)} < \varepsilon_0$ shows that

$$\int_{B_R} g_0(y+\xi)|y|^{-2} W_{0,0}\left(\frac{y}{R}\right) = R^{\nu+N-2} C(N,c) + O(\varepsilon_0),$$

where $C(N,c) \neq 0$. In particular this integral remains bounded away from zero as $R \to 1$ ($R = 1 - 2\varepsilon_0$ and $\varepsilon_0 \to 0$) and hence γ_0 stays bounded.

Regarding $\gamma_{k,l}$ we have

$$\gamma_{k,l} = \frac{\frac{1}{R}\int_{\partial B_R(0)} \phi_0 \frac{\partial W_{k,l}}{\partial n}(\frac{y}{R}) - \gamma_0 \int_{B_R} g_0(y+\xi)|y|^{-2} W_{k,l}(\frac{y}{R})}{\int_{B_R} \eta(|y|) W_{k,l}(\frac{y}{R})^2}, \tag{2.34}$$

and we observe that $\int_{B_R} \eta(|y|) W_{k,l}(\frac{y}{R})^2$ is a positive constant depending on k, l and R (which stays bounded away from zero as $R \to 1$). Using Lemma 2.1, it follows that $\|T_1\|$ remains bounded as $R \to 1$ i.e. when $\varepsilon_0 \to 0$.

Step 2. For ξ small enough equation (2.31) is uniquely solvable and $\|T_2\| \le C|\xi|$. Let $z_0 = 1 - |x|^2$. Then $z_0(|\gamma_0| \sup_{B \setminus B_R(\xi)} \frac{|g_0|}{|x-\xi|^2} + \sup_{\partial B_R(\xi)} |\Psi|))$ is a positive supersolution of (2.31). This shows that this equation is solvable and that for its solution ϕ_2 we have the estimate $|\phi_2| \le C|\xi|(|\gamma_0| + \sup_{\partial B_R(\xi)} |\Psi|)$. This and Schauder estimates yield $\|\phi_2\|_{C^{2,\alpha}(\partial B_R(\xi))} \le C|\xi|(|\gamma_0| + \|\Psi\|_{C^{2,\alpha}(\partial B_R(\xi))})$, which is the desired estimate.

Finally, estimate (2.27) follows from (2.12) and formulas (2.33), (2.34). $\square$

Consider each $\xi \in B_{\varepsilon_0}$ functions $g_0(\cdot,\xi)$, $g(\cdot,\xi) \in C^{0,\alpha}_{\nu,\xi}(B)$ and $h(\cdot,\xi) \in C^{2,\alpha}(\partial B)$. By Proposition 2.3 there is a unique $\phi(\cdot,\xi) \in C^{2,\alpha}_{\nu,\xi}(B)$ solution to (2.24). We want to investigate the differentiability properties of the map $\xi \mapsto \phi(\cdot,\xi)$.

PROPOSITION 2.4. *Assume the following conditions:*

$$\exists k_1 \text{ such that } \quad \alpha^-_{k_1} \in \mathbb{R} \quad \text{and} \quad -\alpha^-_{k_1} < \nu < -\alpha^-_{k_1+1},$$

$$\nu > -\frac{N}{2} + 2 \tag{2.35}$$

and

$$\nu - 1 \neq -\alpha_{k_1}^-.$$

Let $\varepsilon_0 > 0$ *and for* $\xi \in B_{\varepsilon_0}$, *let* $g_0(\cdot, \xi)$, $g(\cdot, \xi)$ *be such that*

$$A_0 \equiv \sup_{\xi \in B_{\varepsilon_0}} (\|g_0(\cdot, \xi)\|_{1,\alpha,\nu,\xi;B} + \|D_\xi g_0(\cdot, \xi)\|_{0,\alpha,\nu-1,\xi;B}) < \infty \tag{2.36}$$

and

$$A \equiv \sup_{\xi \in B_{\varepsilon_0}} (\|g(\cdot, \xi)\|_{1,\alpha,\nu,\xi;B} + \|D_\xi g(\cdot, \xi)\|_{0,\alpha,\nu-1,\xi;B}) < \infty.$$

Let $h(\cdot, \xi) \in C^{3,\alpha}(\partial B)$ *with*

$$\sup_{\xi \in B_{\varepsilon_0}} (\|h(\cdot, \xi)\|_{C^3(\partial B)} + \|D_\xi h(\cdot, \xi)\|_{C^{2,\alpha}(\partial B)}) < \infty.$$

Let $\phi(\cdot, \xi)$ *denote the solution to* (2.24). *Then there exists* $\bar{\varepsilon}_0 > 0$ *and a constant* C *such that if* $\varepsilon_0 < \bar{\varepsilon}_0$ *and if* $\|g_0(\cdot, \xi) - 1\|_{L^\infty(B)} < \varepsilon_0$, $|t| < \varepsilon_0$ *and* $\xi_1, \xi_2 \in B_{\varepsilon_0}$ *then*

$$\|\phi(\cdot + \xi_2, \xi_2) - \phi(\cdot + \xi_1, \xi_1)\|_{2,\alpha,\nu-1,0;B_{1/2}} \le C|\xi_2 - \xi_1|. \tag{2.37}$$

Moreover the map $\xi \in B_{\varepsilon_0} \mapsto \phi(\cdot; \xi)$ *is differentiable in the sense that*

$$D_\xi \phi(x, \xi)\eta = \lim_{\tau \to 0} \frac{1}{\tau}(\phi(x, \xi + \tau\eta) - \phi(x, \xi)) \quad \textit{exists for all } x \in B \setminus \{\xi\} \tag{2.38}$$

and $\eta \in \mathbb{R}^N$. *Furthermore* $D_\xi \phi(\cdot, \xi) \in C^{2,\alpha}_{\nu-1,\xi}(B)$, *the maps* $\xi \in B_{\varepsilon_0} \mapsto \mu_0, \mu_{k,l} \in \mathbb{R}$ *are differentiable and*

$$\begin{aligned}
&\|D_\xi \phi(\cdot, \xi)\|_{2,\alpha,\nu-1,\xi;B} + |D_\xi \mu_0| + \sum_{k=1}^{k_1} \sum_{l=1}^{m_k} |D_\xi \mu_{k,l}| \\
&\quad \le C(\|g(\cdot, \xi)\|_{0,\alpha,\nu,\xi;B} + \|D_\xi g(\cdot, \xi)\|_{0,\alpha,\nu-1,\xi;B} \\
&\qquad + \|h(\cdot, \xi)\|_{C^{2,\alpha}(\partial B)} + \|D_\xi h(\cdot, \xi)\|_{C^{2,\alpha}(\partial B)}).
\end{aligned} \tag{2.39}$$

The proof of this result can be found in [43] and we omit it. For simplicity we have stated Proposition 2.4 under the assumption $\nu - 1 \neq -\alpha_{k_1}^-$. A similar result also holds if $\nu - 1 = -\alpha_{k_1}^-$, but estimate (2.37) has to be replaced by:

$$\|\phi(\cdot + \xi_2, \xi_2) - \phi(\cdot + \xi_1, \xi_1)\|_{2,\alpha,\bar{\nu}-1,0;B_{1/2}} \le C|\xi_2 - \xi_1|,$$

where $\nu - \delta < \bar{\nu} < \nu$ for some $\delta > 0$ and with the constant C now depending on $\bar{\nu}$. Similarly, (2.39) is replaced by

$$\begin{aligned} &\|D_\xi \phi(\cdot,\xi)\|_{2,\alpha,\bar{\nu}-1,\xi;B} + |D_\xi \mu_0| + \sum_{k=1}^{k_1}\sum_{l=1}^{m_k} |D_\xi \mu_{k,l}| \\ &\quad \le C(\|g(\cdot,\xi)\|_{0,\alpha,\bar{\nu},\xi;B} + \|D_\xi g(\cdot,\xi)\|_{0,\alpha,\bar{\nu}-1,\xi;B} \\ &\qquad + \|h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}). \end{aligned} \tag{2.40}$$

Next we extend Proposition 2.3 to an operator of the form $-\Delta - L_t - \frac{c}{|x-\xi|^2}$, where L_t is a suitably small second-order differential operator. We will take L_t of the form

$$L_t w = a_{ij}(x,t) D_{ij} w + b_i(x,t) D_i w + c(x,t) w. \tag{2.41}$$

LEMMA 2.5. *Suppose that the coefficients of L_t satisfy: $a_{ij}(\cdot,t)$, $b_i(\cdot,t)$, $c_i(\cdot,t)$ are $C^\alpha(\overline{B})$ and for some C it holds*

$$\|a_{ij}(\cdot,t)\|_{C^\alpha(\overline{B})} + \|b_i(\cdot,t)\|_{C^\alpha(\overline{B})} + \|c(\cdot,t)\|_{C^\alpha(\overline{B})} \le C|t|.$$

Assume

$$\exists k_1 \text{ such that } \quad \alpha^-_{k_1} \in \mathbb{R} \quad \text{and} \quad -\alpha^-_{k_1} < \nu < -\alpha^-_{k_1+1}.$$

Then there exists $\varepsilon_0 > 0$ such that if $|\xi| < \varepsilon_0$, $|t| < \varepsilon_0$ and $g_0 \in C^{0,\alpha}_{\nu,\xi}(B)$ satisfies $\|g_0 - 1\|_{L^\infty(B)} < \varepsilon_0$, then given any $g \in C^{0,\alpha}_{\nu,\xi}(B)$ and $h \in C^{2,\alpha}(\partial B)$, there exist unique $\phi \in C^{2,\alpha}_{\nu,\xi}(B)$ and $\mu_0, \mu_{k,l} \in \mathbb{R}$ ($k = 1,\dots,k_1$, $l = 1,\dots,m_k$) solution to

$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c}{|x-\xi|^2}\phi = \dfrac{g}{|x-\xi|^2} + \mu_0 \dfrac{g_0}{|x-\xi|^2} \\ \qquad\qquad + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \mu_{k,l} V_{k,l,\xi} & \text{in } B \\ \phi = h & \text{on } \partial B. \end{cases} \tag{2.42}$$

Moreover

$$\|\phi\|_{2,\alpha,\nu,\xi;B} + |\mu_0| + \sum_{k=1}^{k_1}\sum_{l=1}^{m_k} |\mu_{k,l}| \le C(\|g\|_{0,\alpha,\nu,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)}). \tag{2.43}$$

PROOF. Fix $h \in C^{2,\alpha}(\partial B)$ and $|\xi| < \varepsilon_0$, where ε_0 is the constant appearing in Proposition 2.3. For $g \in C^{0,\alpha}_{\nu,\xi}(B)$ let $\phi = T(g/|x-\xi|^2)$ be the solution to (2.24) as defined in Proposition 2.3. Then (2.42) is equivalent to $\phi = T(g/|x-\xi|^2 + L_t\phi)$. Define

$$\widetilde{T}(\phi) = T(g/|x-\xi|^2 + L_t\phi).$$

We apply the fixed point theorem to the operator $\tilde{T}$ in a ball $\mathcal{B}_R$ of the Banach space $C^{2,\alpha}_{\nu,\xi}(B)$ equipped with the norm $\|\cdot\|_{2,\alpha,\nu,\xi;B}$.

Note that by Proposition 2.3 we have $\|T(g/|x-\xi|^2)\|_{2,\alpha,\nu,\xi;B} \le C(\|g\|_{0,\alpha,\nu,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)})$. Using this inequality, for $\|\phi\|_{2,\alpha,\nu,\xi;B} \le R$ we have

$$\begin{aligned}\|\widetilde{T}(\phi)\|_{2,\alpha,\nu,\xi;B} &\le C(\|g\|_{0,\alpha,\nu,\xi;B} + \|L_t\phi\|_{0,\alpha,\nu-2,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)})\\ &\le C(\|g\|_{0,\alpha,\nu,\xi} + |t|R + \|h\|_{C^{2,\alpha}(\partial B)}) \le R,\end{aligned}$$

if we first take t so small that $C|t| \le \frac{1}{2}$, and then choose R so large that $C(\|g\|_{0,\alpha,\nu,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)}) \le \frac{R}{2}$.

For $\|\phi_1\|_{2,\alpha,\nu,\xi;B} \le R$, $\|\phi_2\|_{2,\alpha,\nu,\xi;B} \le R$ we have

$$\begin{aligned}\|\widetilde{T}(\phi_1) - \widetilde{T}(\phi_2)\|_{2,\alpha,\nu,\xi;B} &\le C\|L_t(\phi_1-\phi_2)\|_{0,\alpha,\nu-2,\xi;B}\\ &\le C|t|\,\|\phi_1-\phi_2\|_{2,\alpha,\nu,\xi;B},\end{aligned}$$

and we see that $\widetilde{T}$ is a contraction on the ball $\mathcal{B}_R$ of $C^{2,\alpha}_{\nu,\xi}(B)$ if t is chosen small enough. □

The previous results on differentiability also hold for perturbed operators of the form $-\Delta - L_t - \frac{c}{|x-\xi|^2}$.

PROPOSITION 2.6. *Assume the following conditions:*

$$\exists k_1 \text{ such that } \quad \alpha^-_{k_1} \in \mathbb{R} \quad \text{and} \quad -\alpha^-_{k_1} < \nu < -\alpha^-_{k_1+1}$$

$$\nu > -\frac{N}{2} + 2,$$

and

$$\nu - 1 \ne -\alpha^-_{k_1}. \tag{2.44}$$

Let $\varepsilon_0 > 0$ and for $\xi \in B_{\varepsilon_0}$ let $g_0(\cdot,\xi)$, $g(\cdot,\xi) \in C^{1,\alpha}_{\nu,\xi}(B)$ be such that

$$A_0 \equiv \sup_{\xi\in B_{\varepsilon_0}} (\|g_0(\cdot,\xi)\|_{1,\alpha,\nu,\xi;B} + \|D_\xi g_0(\cdot,\xi)\|_{0,\alpha,\nu-1,\xi;B}) < \infty$$

and

$$A \equiv \sup_{\xi\in B_{\varepsilon_0}} (\|g(\cdot,\xi)\|_{1,\alpha,\nu,\xi;B} + \|D_\xi g(\cdot,\xi)\|_{0,\alpha,\nu-1,\xi;B}) < \infty.$$

On the operator L_t we assume

$$\|a_{ij}(\cdot,t)\|_{C^{1,\alpha}(\overline{B})} + \|b_i(\cdot,t)\|_{C^{1,\alpha}(\overline{B})} + \|c(\cdot,t)\|_{C^{1,\alpha}(\overline{B})} \le C|t|.$$

Let $h(\cdot,\xi) \in C^{3,\alpha}(\partial B)$ with

$$\sup_{\xi\in B_{\varepsilon_0}} (\|h(\cdot,\xi)\|_{C^3(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}) < \infty$$

and let $\phi(\cdot,\xi)$ *denote the solution to* (2.42). *Then there exist* $\bar{\varepsilon}_0 > 0$, $C > 0$ *such that if* $\varepsilon_0 < \bar{\varepsilon}_0$, $\|g_0(\cdot,\xi) - 1\|_{L^\infty(B)} < \varepsilon_0$, $|t| < \varepsilon_0$ *and* $\xi_1, \xi_2 \in B_{\varepsilon_0}$, *we have*

$$\|\phi(\cdot + \xi_2, \xi_2) - \phi(\cdot + \xi_1, \xi_1)\|_{2,\alpha,\nu-1,0;B_{1/2}} \le C|\xi_2 - \xi_1|. \tag{2.45}$$

Furthermore,

$$D_\xi\phi(x;\xi)\eta = \lim_{t\to 0}\frac{1}{t}(\phi(x;\xi + t\eta) - \phi(x;\xi)) \quad \textit{exists}\ \forall x \in B\setminus\{\xi\},\ \forall \eta \in \mathbb{R}^N,$$

the maps $\xi \in B_{\varepsilon_0}(0) \mapsto \mu_0, \mu_{k,l} \in \mathbb{R}$ *are differentiable and*

$$\begin{aligned}\|D_\xi\phi(x;\xi)\|_{2,\alpha,\nu-1,\xi;B} \le C(&\|g(\cdot,\xi)\|_{0,\alpha,\nu,\xi;B} + \|D_\xi g(\cdot,\xi)\|_{0,\alpha,\nu-1,\xi;B}\\ &+ \|h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}).\end{aligned} \tag{2.46}$$

The argument uses again the fixed point theorem. Details can be found in [43].

2.2. *Perturbation of singular solutions*

Recall that $c^* = 2(N-2)$. Hence, if $N \ge 4$ then $N - 1 < c^* < 2N$ and therefore $\alpha_1^- > 0$, $\alpha_2^- < 0$ (cf. (2.4)). As mentioned before we choose $\nu = 0$. We see that (2.26) holds now with $k_1 = 1$. We may thus apply Proposition 2.3 and Lemma 2.5. In dimension $N \ge 5$, since (2.35) and (2.44) hold, we may also apply Propositions 2.4 and 2.6.

Write

$$V_{\ell,\xi} := V_{1,\ell,\xi} \quad \ell = 1, \ldots, N,$$

where $V_{1,\ell,\xi}$ is defined in (2.25), and set

$$\tilde{f}(x,t) = L_t\left(\log\frac{1}{|x-\xi|^2}\right)$$

and note that

$$\|\tilde{f}(x,t)|x-\xi|^2\|_{0,\alpha,-2,\xi} \le C|t|. \tag{2.47}$$

Concerning (1.25) we prove:

LEMMA 2.7. *Write* $c = c^* = 2(N-2)$. *Then there exists* $\varepsilon_0 > 0$ *such that if* $|\xi| < \varepsilon_0$, $|t| < \varepsilon_0$, *there exist* $\phi \in C^{2,\alpha}_{0,\xi}(B)$ *and* $\mu_0, \ldots, \mu_N \in \mathbb{R}$ *such that*

$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c}{|x-\xi|^2}\phi = \dfrac{c}{|x-\xi|^2}(e^\phi - 1 - \phi) + \mu_0\dfrac{1}{|x-\xi|^2}e^\phi \\ \qquad\qquad + \tilde{f}(x,t) + \displaystyle\sum_{i=1}^N \mu_i V_{i,\xi} \quad \textit{in } B \\ \phi = -\log\dfrac{1}{|x-\xi|^2} \quad \textit{on } \partial B. \end{cases} \tag{2.48}$$

If $N \geq 5$, we have in addition that:

• *the map $\xi \in B_{\varepsilon_0} \mapsto \phi(\cdot, \xi)$ is differentiable in the sense that*

$$D_\xi \phi(x, \xi)\eta = \lim_{\tau \to 0} \frac{1}{\tau}(\phi(x, \xi + \tau\eta) - \phi(x, \xi)) \quad \text{exists for all } x \in B \setminus \{\xi\}$$

and $\eta \in \mathbb{R}^N$.

• *for $\bar{\nu} < 0$ small, $D_\xi \phi(\cdot, \xi) \in C^{2,\alpha}_{\bar{\nu}-1,\xi}(B)$, the maps $\xi \in B_{\varepsilon_0} \mapsto \mu_0, \mu_i \in \mathbb{R}$ are differentiable and there exists a constant C independent of ξ such that*

$$\|D_\xi \phi(\cdot, \xi)\|_{2,\alpha,\bar{\nu}-1,\xi;B} + |D_\xi \mu_0| + \sum_{k=1}^{k_1} \sum_{l=1}^{m_k} |D_\xi \mu_{k,l}| \leq C. \tag{2.49}$$

PROOF. *Case $N \geq 5$.*

Let ε_0 be as in Lemma 2.5. Consider the Banach space X of functions $\phi(x, \xi)$ defined for $x \in B$, $\xi \in B_{\varepsilon_0}$, which are twice continuously differentiable with respect to x and once with respect to ξ for $x \neq \xi$ for which the following norm is finite

$$\|\phi\|_X = \sup_{\xi \in B_{\varepsilon_0}} \|\phi(\cdot, \xi)\|_{2,\alpha,0,\xi;B} + \lambda \|D_\xi \phi(\cdot, \xi)\|_{2,\alpha,\bar{\nu}-1,\xi;B},$$

where $\lambda > 0$ is a parameter to be fixed later on and $\bar{\nu} < 0$ is close to zero.

Let $\mathcal{B}_R = \{\phi \in X \mid \|\phi\|_X \leq R\}$. Using Lemma 2.5 we may define a nonlinear map $F : \mathcal{B}_R \to X$ by $F(\psi) = \phi$, where $\phi(\cdot, \xi)$ is the solution to (2.42) with

$$g = c(e^\psi - 1 - \psi) + |x - \xi|^2 \tilde{f}(x, t), \quad g_0 = e^\psi, \quad h = -\log \frac{1}{|x - \xi|^2}. \tag{2.50}$$

We shall choose later on $R > 0$ small. Observe that in Lemma 2.5 the constants C in (2.43) and ε_0 associated to $g_0 = e^\psi$, stay bounded and bounded away from zero respectively as we make R smaller, since $e^{-R} \leq e^\psi \leq e^R$ for $\psi \in \mathcal{B}_R$.

Let us show that if t is small then one can choose R small and $\lambda > 0$ small so that $F : \mathcal{B}_R \to \mathcal{B}_R$. Indeed, let $\psi \in \mathcal{B}_R$ and $\phi = F(\psi)$. Then by (2.43), (2.47) we have

$$\begin{aligned} \|\phi\|_{2,\alpha,0,\xi;B} &\leq C(\|c(e^\psi - 1 - \psi) + |x - \xi|^2 \tilde{f}(x, t)\|_{0,\alpha,0,\xi;B} + |\xi|) \\ &\leq C(R^2 + |t| + |\xi|) < \frac{R}{2}, \end{aligned} \tag{2.51}$$

provided R is first taken small enough and then $|t|$ and $|\xi| < \varepsilon_0$ are chosen small. Similarly, recalling (2.40),

$$\begin{aligned} &\|D_\xi \phi\|_{2,\alpha,\bar{\nu}-1,\xi;B} \\ &\quad \leq C(\|c(e^\psi - 1 - \psi) + |x - \xi|^2 \tilde{f}(x, t)\|_{0,\alpha,0,\xi;B} \\ &\qquad + \|c D_\xi(e^\psi - 1 - \psi) + D_\xi(|x - \xi|^2 \tilde{f}(x, t))\|_{0,\alpha,\bar{\nu}-1,\xi;B} + 1) \\ &\quad \leq C\left(R^2 + t + \frac{R^2}{\lambda} + 1\right) \leq \frac{R}{2\lambda}, \end{aligned}$$

if we choose now λ small enough.

Next we show that F is a contraction on $\mathcal{B}_R$. Let $\psi_1, \psi_2 \in \mathcal{B}_R$ and $\phi_\ell = F(\psi_\ell)$, $\ell = 1, 2$. Let $\mu_i^{(\ell)}$, $i = 0, \dots, N$ be the constants in (2.42) associated with ψ_ℓ. By (2.43) and repeating the calculation in (2.51)

$$\sum_{i=0}^{N} |\mu_i^{(\ell)}| \leq R. \tag{2.52}$$

Let $\phi = \phi_1 - \phi_2$. Then ϕ satisfies

$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c}{|x-\xi|^2}\phi = c\left(\dfrac{e^{\psi_1} - 1 - \psi_1}{|x-\xi|^2} - \dfrac{e^{\psi_2} - 1 - \psi_2}{|x-\xi|^2}\right) \\ \qquad\qquad + \mu_0^{(2)} \dfrac{e^{\psi_1} - e^{\psi_2}}{|x-\xi|^2} + (\mu_0^{(1)} - \mu_0^{(2)}) \dfrac{e^{\psi_1}}{|x-\xi|^2} \\ \qquad\qquad + \displaystyle\sum_{i=1}^{N} (\mu_i^{(1)} - \mu_i^{(2)}) V_{i,\xi} \qquad \text{in } B \\ \phi = 0 \qquad \text{on } \partial B. \end{cases} \tag{2.53}$$

Apply (2.43) with $g_0 = \frac{e^{\psi_1}}{|x-\xi|^2}$, $h = 0$ and

$$g := c\left(\frac{e^{\psi_1} - 1 - \psi_1}{|x-\xi|^2} - \frac{e^{\psi_2} - 1 - \psi_2}{|x-\xi|^2}\right) + \mu_0^{(2)} \frac{e^{\psi_1} - e^{\psi_2}}{|x-\xi|^2}, \tag{2.54}$$

to conclude that

$$\|\phi\|_{2,\alpha,0,\xi} + \sum_{i=0}^{N} |\mu_i^{(1)} - \mu_i^{(2)}| \leq C\|g\|_{0,\alpha,0,\xi}. \tag{2.55}$$

Using (2.52), we have in particular that $|\mu_0^{(2)}| \leq R$ and it follows from (2.54) and (2.55) that

$$\|\phi_1 - \phi_2\|_{2,\alpha,0,\xi} \leq CR\|\psi_1 - \psi_2\|_{2,\alpha,0,\xi}. \tag{2.56}$$

Thanks to (2.46) we also have the bound

$$\begin{aligned} \|D_\xi(\phi_1 - \phi_2)\|_{1,\alpha,\bar{\nu}-1,\xi;B} &\leq C\big(\|e^{\psi_1} - \psi_1 - (e^{\psi_1} - \psi_2)\|_{0,\alpha,0,\xi;B} \\ &\quad + \|D_\xi(e^{\psi_1} - \psi_1 - (e^{\psi_1} - \psi_2))\|_{0,\alpha,\bar{\nu}-1,\xi;B}\big) \\ &\leq CR\|\psi_1 - \psi_2\|_{2,\alpha,0,\nu;B} \\ &\quad + CR\|D_\xi(\psi_1 - \psi_2)\|_{0,\alpha,\bar{\nu}-1,\xi;B}. \end{aligned} \tag{2.57}$$

Combining (2.56), (2.57) we obtain

$$\|F(\psi_1) - F(\psi_2)\|_X \leq CR\|\psi_1 - \psi_2\|_X.$$

This shows that F is a contraction if R is taken small enough.

Case $N = 4$. In this case (2.35) fails for $\nu = 0$ and estimates like (2.45) or (2.46) may not hold. So we work with the Banach space X of functions $\phi(x, \xi)$ which are twice continuously differentiable with respect to x and continuous with respect to ξ for $x \neq \xi$, for which the norm

$$\|\phi\|_X = \sup_{\xi \in B_{\varepsilon_0}} \|\phi(\cdot, \xi)\|_{2,\alpha,0,\xi;B}$$

is finite. Working as in the previous case, we easily obtain that F is a contraction on some ball $\mathcal{B}_R$ of X. □

PROOF OF THEOREM 1.11. We define the map $(\xi, t) \mapsto \phi(\xi, t)$ as the small solution to (2.48) constructed in Lemma 2.7 for t, ξ small. We need to show that for t small enough there is a choice of ξ such that $\mu_i = 0$ for $i = 1, \dots, N$. Let

$$\widehat{V}_j(x; \xi) = W_{1,j}(x - \xi)\eta_1(|x - \xi|), \quad j = 0, \dots, N, \tag{2.58}$$

where $\eta_1 \in C^\infty(\mathbb{R})$ is a cut-off function such that $0 \le \eta_1 \le 1$,

$$\begin{cases} \eta_1(r) = 0 & \text{for } r \le \frac{1}{8}, \\ \eta_1(r) = 1 & \text{for } r \ge \frac{1}{4}. \end{cases} \tag{2.59}$$

Multiplication of (2.48) by $\widehat{V}_j(x; \xi)$ and integration in B gives

$$\begin{aligned} &\int_B \left(-\Delta \widehat{V}_j(x; \xi) - L_t \widehat{V}_j(x; \xi) - \frac{c}{|x - \xi|^2} \widehat{V}_j(x; \xi) \right) \phi \\ &\quad + \int_{\partial B} \log \frac{1}{|x - \xi|^2} \frac{\partial \widehat{V}_j(x; \xi)}{\partial n} - \int_{\partial B} \frac{\partial \phi}{\partial n} \widehat{V}_j(x; \xi) \\ &\quad = \int_B \frac{c}{|x - \xi|^2} (e^\phi - 1 - \phi) \widehat{V}_j(x; \xi) + \mu_0 \int_B \frac{e^\phi}{|x - \xi|^2} \widehat{V}_j(x; \xi) \\ &\qquad + \int_B \tilde{f}(x, t) \widehat{V}_j(x; \xi) + \sum_{i=1}^N \mu_i \int_B V_{i,\xi} \widehat{V}_j(x; \xi). \end{aligned}$$

When $\xi = 0$ the matrix $A = A(\xi)$ defined by

$$A_{i,j}(\xi) = \int_B V_{i,\xi} \widehat{V}_j(x; \xi) \quad \text{for } i, j = 1 \dots N$$

is diagonal and invertible and by continuity it is still invertible for small ξ. Thus, we see that $\mu_i = 0$ for $i = 1, \dots, N$ if and only if

$$H_j(\xi, t) = 0, \quad \forall j = 1, \dots, N, \tag{2.60}$$

where, given $j = 1, \dots, N$,

$$\begin{aligned} H_j(\xi, t) = & \int_B \frac{c}{|x-\xi|^2}(e^\phi - 1 - \phi)\widehat{V}_j(x;\xi) + \mu_0 \int_B \frac{e^\phi}{|x-\xi|^2}\widehat{V}_j(x;\xi) \\ & + \int_B \tilde{f}(x,t)\widehat{V}_j(x;\xi) - \int_{\partial B} \log\frac{1}{|x-\xi|^2}\frac{\partial \widehat{V}_j(x;\xi)}{\partial n} \\ & + \int_{\partial B} \frac{\partial\phi}{\partial n}\widehat{V}_j(x;\xi) - \int_B \Big(-\Delta\widehat{V}_j(x;\xi) - L_t\widehat{V}_j(x;\xi) \\ & - \frac{c}{|x-\xi|^2}\widehat{V}_j(x;\xi)\Big)\phi. \end{aligned}$$

If this holds, then $\mu_1(\xi, t) = \cdots = \mu_N(\xi, t) = 0$ and $\phi(\xi, t)$ is the desired solution to (1.25) (with μ in (1.25) equal to $\mu_0(\xi, t)$).

Observe that

$$\begin{aligned} & \frac{\partial}{\partial \xi_k}\left[\int_{\partial B} \log\frac{1}{|x-\xi|^2}\frac{\partial\widehat{V}_j(x;\xi)}{\partial n}\right]_{\xi=0} \\ & = 2\int_{\partial B} x_k \frac{\partial\widehat{V}_j(x;0)}{\partial n} + \int_{\partial B}\log\frac{1}{|x-\xi|^2}\frac{\partial}{\partial\xi_k}\frac{\partial\widehat{V}_j(x;\xi)}{\partial n}\bigg|_{\xi=0} \\ & = 2\int_{\partial B} x_k\frac{\partial\widehat{V}_j(x;0)}{\partial n}. \end{aligned} \tag{2.61}$$

For $j = 1, \dots, N$ we have $W_{1,j}(x) = (|x|^{-\alpha_j^+} - |x|^{-\alpha_j^-})\varphi_j(\frac{x}{|x|})$ for $x \in \partial B$, and hence $\frac{\partial W_{1,j}}{\partial n}(x) = (\alpha_j^- - \alpha_j^+)\varphi_j(x) = \frac{\alpha_j^- - \alpha_j^+}{(\int_{S^{N-1}} x_j^2)^{1/2}} x_j$.

Case $N \geq 5$. By Lemma 2.7, $\phi(\cdot, \xi)$ is differentiable with respect to ξ. We may then compute the derivatives of the other terms of H_j. For instance

$$\frac{\partial}{\partial\xi_k}\int_B \frac{c}{|x-\xi|^2}(e^\phi - 1 - \phi)\widehat{V}_j(x;\xi)\bigg|_{\xi=0,t=0} = 0$$

because the expression above is quadratic in ϕ and the computation can be justified using estimate (2.49).

Similarly

$$\frac{\partial}{\partial\xi_k}\left[\mu_0\int_B \frac{e^\phi}{|x-\xi|^2}\widehat{V}_j(x;\xi)\right]_{\xi=0} = 0.$$

Finally, using that $\phi|_{\xi=0} \equiv 0$ and integration by parts, we find

$$\begin{aligned} & \frac{\partial}{\partial\xi_k}\left[\int_{\partial B_1}\frac{\partial\phi}{\partial n}\widehat{V}_j - \int_B\Big(-\Delta\widehat{V}_j - L_t\widehat{V}_j - \frac{c}{|x|^2}\widehat{V}_j\Big)\phi\right]_{\xi=0,t=0} \\ & \quad = \int_{\partial B}\frac{\partial\widehat{V}_j}{\partial n}\frac{\partial\phi}{\partial\xi_k} - \int_B\Big(-\Delta\frac{\partial\phi}{\partial\xi_k} - \frac{c}{|x|^2}\frac{\partial\phi}{\partial\xi_k}\Big)\widehat{V}_j. \end{aligned} \tag{2.62}$$

But when $\xi = 0$, $\frac{\partial \phi}{\partial \xi_k}$ satisfies

$$\begin{cases} -\Delta \dfrac{\partial \phi}{\partial \xi_k} - \dfrac{c}{|x|^2} \dfrac{\partial \phi}{\partial \xi_k} = \dfrac{\partial \mu_0}{\partial \xi_k} \dfrac{1}{|x|^2} + \displaystyle\sum_{i=1}^{N} \dfrac{\partial \mu_i}{\partial \xi_k} V_{i,0} & \text{in } B \\ \dfrac{\partial \phi}{\partial \xi_k} = 2x_k & \text{on } \partial B \end{cases} \tag{2.63}$$

since at $\xi = 0$, $\phi = 0$ and $\mu_i = 0$ for $0 \le i \le N$. By the conditions (2.11) we find $\frac{\partial \mu_0}{\partial \xi_k} = 0$ and

$$\frac{\partial \mu_i}{\partial \xi_k} = 2 \frac{\int_{\partial B} x_k \frac{\partial W_{1,i}}{\partial \nu}}{\int_B V_{i,0} W_{1,i}}, \quad 1 \le i \le N. \tag{2.64}$$

The integral above is zero whenever $i \neq k$ and thus, using (2.63), (2.64) in (2.62) we obtain

$$\begin{aligned} &\frac{\partial}{\partial \xi_k} \left[\int_{\partial B_1} \frac{\partial \phi}{\partial n} \widehat{V}_j - \int_B \left(-\Delta \widehat{V}_j - L_t \widehat{V}_j - \frac{c}{|x|^2} \widehat{V}_j \right) \phi \right]_{\xi=0, t=0} \\ &\quad = 2 \int_{\partial B} x_k \frac{\partial \widehat{V}_j}{\partial \nu} - 2 \frac{\int_{\partial B} x_k \frac{\partial W_{1,k}}{\partial \nu}}{\int_B V_{k,0} W_{1,k}} \int_B V_{k,0} \widehat{V}_j = 0 \end{aligned}$$

thanks to (2.59). This and (2.61) imply that the matrix $\left(\frac{\partial H_j}{\partial \xi_k}(0,0) \right)_{ij}$ is invertible.

We may then apply the Implicit Function Theorem, to conclude that there exists a differentiable curve $t \to \xi(t)$ defined for $|t|$ small, such that (2.60) holds for $\xi = \xi(t)$. Letting $v(x) = \log \frac{1}{|x - \xi(t)|^2} + \phi(x, \xi(t))$ for $x \in B$ and $u(y) = v(y + t\tilde{\psi}(y))$ for $y \in \Omega_t$, we conclude that u is the desired solution of (1.22).

Case $N = 4$. We use the Brouwer Fixed Point Theorem as follows. Define $H = (H_1, \ldots, H_N)$ and

$$B(\xi) = (B_1, \ldots, B_N) \quad \text{with} \quad B_j(\xi) = \int_{\partial B} \log \frac{1}{|x - \xi|^2} \frac{\partial W_{j,\xi}}{\partial n}.$$

By (2.61), B is differentiable and $DB(0)$ is invertible. (2.60) is then equivalent to

$$\xi = G(\xi),$$

where

$$G(\xi) = DB(0)^{-1} \left(DB(0)\xi - H(\xi, t) \right).$$

To apply the Brouwer Fixed Point Theorem it suffices to prove that for t, ρ small, G is a continuous function of ξ and $G : \overline{B_\rho} \to \overline{B_\rho}$. The following two lemmas are proved in [43].

LEMMA 2.8. *G is continuous for t, ξ small.*

LEMMA 2.9. *If $\rho > 0$ and $|t|$ are small enough then $G : \overline{B_\rho} \to \overline{B_\rho}$.*

3. Reaction on the boundary

3.1. *Characterization and uniqueness of the extremal solution*

In this section we are interested in the characterization of the extremal solution presented in Lemma 1.16. As mentioned in Section 1.4 we shall prove this characterization under the assumptions that g satisfies (1.35) and (1.36), since the argument is simpler and works in the case that Γ_1 and Γ_2 form an angle. Later on in Section 3.2 we shall prove the uniqueness of the extremal solution for the problem with reaction on the boundary, which is the analog of Theorem 1.8 for g satisfying (1.2) and (1.3), assuming that $\partial\Omega$ is smooth.

LEMMA 3.1. *Suppose that* $u \in H^1(\Omega)$ *is a weak solution to* (1.34). *Then for any* $0 < \lambda < \lambda^*$ (1.34) *has a bounded solution.*

PROOF. Let u be an energy solution to (1.34). We basically use the truncation method of [19]. For this the first step is to show that if $\Phi : [0, \infty) \to [0, \infty)$ is a concave C^2 function such that $\Phi' \in L^\infty$ then $\Phi(u)$ is a supersolution, in the sense that

$$\int_\Omega \nabla\Phi(u)\nabla\varphi \geq \lambda \int_{\Gamma_1} \Phi'(u)g(u)\varphi \quad \forall\varphi \in C^1(\overline{\Omega}), \varphi \geq 0. \tag{3.1}$$

Indeed, let $h = \lambda g(u)$ and for $m > 0$ let

$$h_m = \begin{cases} h_m = h & \text{if } |h| \leq m \\ h_m = -m & \text{if } h < -m \\ h_m = m & \text{if } h > m. \end{cases}$$

Let u_m denote the H^1 solution of

$$\begin{cases} \Delta u_m = 0 & \text{in } \Omega \\ \dfrac{\partial u_m}{\partial \nu} = h_m & \text{on } \Gamma_1 \\ u_m = 0 & \text{on } \Gamma_2. \end{cases}$$

Note that $u_m \to u$ in $H^1(\Omega)$ and in $L^1(\Gamma_1)$. Let $\varphi \in C^1(\overline{\Omega})$, $\varphi \geq 0$. Using $\Phi'(u_m)\varphi$ as a test function we find that

$$\int_\Omega \nabla u_m \big(\Phi''(u_m)\nabla u_m\varphi + \Phi'(u_m)\nabla\varphi\big)\,dx - \int_{\Gamma_1} \Phi'(u_m)h_m\varphi = 0.$$

Using that $\Phi'' \leq 0$ and $\varphi \geq 0$ we have

$$\int_\Omega \nabla(\Phi(u_m))\nabla\varphi\,dx \geq \int_{\Gamma_1} h_m\Phi'(u_m)\varphi\,dx.$$

Now we let $m \to \infty$. Since $\Phi' \in L^\infty$ it is not difficult to verify that

$$\int_\Omega \nabla(\Phi(u_m))\nabla\varphi\,dx \to \int_\Omega \nabla(\Phi(u))\nabla\varphi\,dx$$

and

$$\int_{\Gamma_1} h_m \Phi'(u_m)\varphi\, dx \to \int_{\Gamma_1} h\Phi'(u)\varphi\, dx$$

since we have convergence a.e. for a subsequence and

$$|h_m \Phi'(u_m)\varphi| \le \|\Phi'\|_\infty \|\varphi\|_{L^\infty} |h| \in L^1(\Gamma_1)$$

since $h = \lambda g(u) \in L^1(\Gamma)$.

Now note that under (1.36) we have $g(t) \ge ct^\alpha$ for some $\alpha > 1$ and $c > 0$ and hence

$$\int_0^\infty \frac{ds}{g(s)} < +\infty. \tag{3.2}$$

Let $0 < \lambda' < \lambda$ and define

$$\Phi(t) = H^{-1}(\lambda' H(u)/\lambda), \tag{3.3}$$

where

$$H(u) = \int_0^u \frac{ds}{g(s)}. \tag{3.4}$$

Then it is possible to verify that Φ is a C^2 concave function with bounded derivative. Since $\lambda\Phi'(u)g(u) = \lambda' g(\Phi(u))$ it follows from (3.1) that $v = \Phi(u)$ satisfies

$$\int_\Omega \nabla v \nabla\varphi \ge \lambda' \int_{\Gamma_1} g(v)\varphi \quad \forall \varphi \in C^1(\overline{\Omega}), \quad \varphi \ge 0$$

and is thus a supersolution to (1.34) with parameter λ'. Now, condition (3.2) implies that $v = \Phi(u)$ is bounded. By the method of sub and supersolutions (1.34) with parameter λ' has a bounded solution. $\square$

PROOF OF LEMMA 1.16. Under hypothesis (1.36) the argument to prove Lemma 1.16 is similar to that of Theorem 1.9 but simpler because we can immediately say that $u^* \in H^1(\Omega)$ and we do not need to rely on a uniqueness result for u^* similar to Theorem 1.8. By Lemma 3.1 $\lambda \le \lambda^*$. Now, if $\lambda < \lambda^*$ then exactly the same argument as in Theorem 1.9 leads to a contradiction. Thus $\lambda = \lambda^*$. We wish to show that $v = u^*$. Since v is a supersolution to (1.34) we see that $u_\lambda \le v$ for all $0 < \lambda < \lambda^*$ and taking $\lambda \to \lambda^*$ we conclude $u^* \le v$. For the opposite inequality observe that by density (1.39) holds for $\varphi \in H^1(\Omega)$ such that $\varphi = 0$ on Γ_2. By hypothesis $v \in H^1(\Omega)$ and since g satisfies (1.36) we have $u^* \in H^1(\Omega)$. Thus we may choose $\varphi = v - u^*$. We obtain

$$\int_{\Gamma_1} (g(u^*) - (g(v) + g'(v)(u^* - v)))(v - u^*) \le 0.$$

But the integrand is nonnegative since $v \ge u^*$ a.e. and g is convex. This implies

$$g(u^*) = g(v) + g'(v)(u^* - v) \quad \text{a.e. on } \Gamma_1.$$

It follows that g is linear in intervals of the form $[u^*(x), v(x)]$ for a.e. $x \in \Gamma_1$. The union of such intervals is an interval of the form $[a, \infty)$ for some $a \ge 0$. Assuming this property for a moment we reach a contradiction with (1.36).

To prove the claim above we follow the argument of Dupaigne and Nedev [51]. First we show that $u^*(\Gamma_1)$ is dense in $[\operatorname{ess\,inf}_{\Gamma_1} u^*, \operatorname{ess\,sup}_{\Gamma_1} u^*]$. Indeed, if not, then there exists a nontrivial interval (a, b) such that $\{x \in \Gamma_1 : u^*(x) \le a\}$ and $\{x \in \Gamma_1 : u^*(x) \ge b\}$ both have positive measure in Γ_1. Hence there is a smooth function $\eta : \mathbb{R} \to \mathbb{R}$ with $0 \le \eta \le 1$ such that $\eta(u^*)$, is either 0 or 1, but such that $\{x \in \Gamma_1 : \eta(u^*(x)) = 0\}$ and $\{x \in \Gamma_1 : \eta(u^*(x)) = 1\}$ have positive measure. Since $u^* \in H^1(\Omega)$ we have $\eta(u^*) \in H^1(\Omega)$ and therefore $\eta(u^*) \in H^{1/2}(\Gamma_1)$ and has values 0 and 1. But it is known, see for instance Bourgain *et al.* [16], that a function in $W^{s,p}(\Gamma_1; \mathbb{Z})$ with $sp \ge 1$ is constant. This contradiction shows that indeed $u^*(\Gamma_1)$ is dense in $[\operatorname{ess\,inf}_{\Gamma_1} u^*, \operatorname{ess\,sup}_{\Gamma_1} u^*]$. Let $S \subset \Gamma_1$ by a compact set with $dist(S, I) > 0$. By the strong maximum principle $\operatorname{ess\,inf}_S(v - u^*) > 0$. It follows that $\cup_{x\in S}[u^*(x), v(x)] \supseteq \cup_{x\in S}[u^*(x), u^*(x) + \varepsilon]$ and hence is an interval $[a, \infty)$, because $\operatorname{ess\,sup}_{\Gamma_1} u^* = +\infty$ as u^* is unbounded. □

3.2. *Weak solutions and uniqueness of the extremal solution*

Throughout this section we will assume that g satisfies (1.2) and (1.3).

An important tool in the proofs in [19,87] is Hopf's lemma, so before adapting their arguments we need to find a suitable statement that replaces this lemma for problems with mixed boundary condition. Let us recall a form of Hopf's lemma combined with the strong maximum principle which will be our model. Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. If u satisfies

$$\begin{cases} -\Delta u = h & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

with $h \in L^\infty(\Omega)$, $h \ge 0$, $h \not\equiv 0$ then there exists $c_1 > 0$ such that

$$c_1\delta \le u \quad \text{in } \Omega, \tag{3.6}$$

where

$$\delta(x) = dist(x, \partial\Omega).$$

The bound is sharp in the sense that $u \le c_2\delta$ for some $c_2 > 0$ by Schauder's estimates. The constant c_1 in the lower bound of (3.6) above can be made more precise in its dependence on h

$$c\delta(x)\Big(\int_\Omega \delta h\Big) \le u(x) \quad \forall x \in \Omega,$$

where $c > 0$ depends only on Ω. This estimate was proved by Morel and Oswald (unpublished) and can also be found in [18].

Let us consider the following linear problem with mixed boundary condition

$$\begin{cases} \Delta u = h_1 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = h_2 & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2, \end{cases} \tag{3.7}$$

where h_1, h_2 are smooth functions defined on $\overline{\Omega}$ and Γ_1 respectively. Here Γ_1, Γ_2 is a partition of $\partial\Omega$ into surfaces separated by a smooth interface. More precisely $\Gamma_1, \Gamma_2 \subset \partial\Omega$ are smooth $N-1$-dimensional manifolds with a common boundary $\Gamma_1 \cap \Gamma_2 = I$ which is a smooth $N-2$-dimensional manifold.

We shall define next a function which will play the role of δ for (3.5). The definition is motivated by the fact that the function

$$v = Im(z^{1/2}) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{x^2+y^2} - x}, \qquad z = x + iy$$

is harmonic in the upper half of the complex plane $\{z \in \mathbb{C} \mid Re(z) > 0\}$, and satisfies the mixed boundary condition

$$v(x,0) = 0 \quad x > 0, \qquad \frac{\partial v}{\partial y}(x,0) = 0 \quad x < 0.$$

For x in a small fixed neighborhood of $\partial\Omega$ we write $\hat{x}$ for the projection of x on $\partial\Omega$, that is, $\hat{x}$ is the point in Ω closest to x. We let $\nu(x)$ denote the outer unit normal vector to $\partial\Omega$ at $\hat{x}$. Given $x \in \partial\Omega$ in a fixed small neighborhood of I we write $I(x)$ for the point in I with smallest geodesic distance on $\partial\Omega$ to x. Then there exists a neighborhood $\mathcal{U}$ of I in Ω and $r > 0$ such that

$$x \in \mathcal{U} \to (I(\hat{x}), d_I(\hat{x}), \delta(x)) \in I \times (-r, r) \times (0, r) \tag{3.8}$$

is a diffeomorphism, where $d_I(\hat{x})$ denotes the signed geodesic distance on $\partial\Omega$ from $\hat{x}$ to $I(\hat{x})$ with the sign such that

$$d_I(\hat{x}) \le 0 \quad \text{if } \hat{x} \in \Gamma_1, \qquad d_I(\hat{x}) \ge 0 \quad \text{if } \hat{x} \in \Gamma_2.$$

We define $\zeta(x)$ for $x \in \mathcal{U}$ as:

$$\zeta(x) = \sqrt{\sqrt{s^2+t^2} - s}, \quad \text{where } t = \delta(x),\ s = d_I(\hat{x}),$$

and we extend ζ to $\Omega \setminus \mathcal{U}$ as a smooth function such that

$$\inf_{\Gamma_1 \setminus \mathcal{U}} \zeta > 0 \quad \text{and} \quad \zeta = 0 \quad \text{on } \Gamma_2 \setminus \mathcal{U}.$$

The next result is the analog of (3.6) for (3.5).

PROPOSITION 3.2. *Let $h_1 \in L^\infty(\Omega)$, $h_1 \geq 0$ and $h_2 \in L^\infty(\Gamma_1)$, $h_2 \geq 0$ and assume $h_1 \not\equiv 0$ or $h_2 \not\equiv 0$. Let u be the solution to* (3.7). *Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1\zeta \leq u \leq c_2\zeta \quad \text{in } \Omega. \tag{3.9}$$

PROOF. For convenience we write $\mathcal{U}_r$ as the neighborhood of I in Ω introduced in (3.8). We will show, using suitable barriers, that (3.9) holds in $\mathcal{U}_r$ for some $r > 0$ small. Using then the strong maximum principle and the usual Hopf's lemma we will establish the desired inequality in Ω.

Recall that $t = \delta(x)$ and $s = d_I(\hat{x})$ are well-defined smooth functions on $\mathcal{U}_r$. For a function $v(s,t)$ its Laplacian can be expressed as

$$\Delta v = \frac{\partial^2 v}{\partial s^2} + \frac{\partial^2 v}{\partial t^2} + O(t + |s|)|D^2 v| + O(1)|Dv|, \tag{3.10}$$

where $O(t+|s|)$ denotes a function bounded by $t + |s|$ in $\mathcal{U}_r$, $O(1)$ a bounded function, $|D^2v|$ and $|Dv|$ are the norms of the Hessian and gradient of v respectively. Indeed, let us consider a smooth change of variables of a neighborhood of x_0 in I onto an open set in $\mathbb{R}^{N-2}$, that is $\phi : B_r(x_0) \cap I \to V \subset \mathbb{R}^{N-2}$. Define the map

$$\psi(x) = (\phi(I(\hat{x})), d_I(\hat{x}), \delta(x)) = (z, s, t) \in V \times (-r, r) \times (0, r) \subset \mathbb{R}^N.$$

We shall write $y = (z, s, t)$, that is $z = (y_1, \ldots, y_{N-2})$, $s = y_{N-1}$, $t = y_N$. Then $y_1, \ldots, y_N$ are local coordinates of a neighborhood of x_0, and

$$\Delta v = \frac{1}{\sqrt{g}} \partial_{y_k} \left(\sqrt{g} g^{kl} \partial_{y_l} v \right), \tag{3.11}$$

where $g_{ij} = \langle \frac{\partial \psi^{-1}}{\partial y_i}, \frac{\partial \psi^{-1}}{\partial y_j} \rangle$ is the Euclidean metric tensor in the coordinates $y_1, \ldots, y_N$, $g = \det(g_{i,j})$ and g^{kl} is the inverse matrix of g_{ij}. By construction of ψ, when $t = s = 0$ (which corresponds to the interface I) the coefficients g_{ij} are 0 whenever $i = N-1, N$ or $j = N-1, N$, since at I $D\psi$ maps the normal vector $\nu = \frac{\partial}{\partial t}$ to the vector e_N, the vector $\frac{\partial}{\partial s}$ perpendicular to I and tangent to $\partial\Omega$ to e_{N-1} and vectors in the tangent space to I to vectors of $\mathbb{R}^N$ with the last two components equal to 0. Hence if $k = N-1, N$ or $l = N-1, N$ we have $g^{kl} = O(t + |s|)$ and formula (3.10) follows from (3.11).

Let us introduce polar coordinates for s, t:

$$s = r\cos(\theta), \qquad t = r\sin(\theta).$$

As a first term for the subsolution we take

$$u_1 = r^{1/2}\sin(\theta/2).$$

Then according to (3.10) and since $|D^2u_1| = O(r^{-3/2})$, $|Du_1| = O(r^{-1/2})$ we have

$$\Delta u_1 = O(r^{-1/2}). \tag{3.12}$$

Let $1/2 < \gamma < \alpha < 1$, $b > 0$ and define

$$u_2 = r^\alpha(\sin(\gamma\theta) + b\theta^2).$$

Using (3.10) again we find

$$\begin{aligned}\Delta u_2 &= r^{\alpha-2}\Big((\alpha^2-\gamma^2)\sin(\gamma\theta) + \alpha^2 b\theta^2 + 2b\Big) + O(r^{\alpha-1})\\ &\geq cr^{\alpha-2} + O(r^{\alpha-1}),\end{aligned} \tag{3.13}$$

for some positive constant c. Set

$$\underline{u} = u_1 + u_2.$$

By (3.12) and (3.13) there exists $r_0 > 0$ but small such that

$$\Delta\underline{u} \geq cr^{2-\alpha} \quad \text{in the region } r < r_0$$

for some $c > 0$. Let us compute the normal derivative:

$$\frac{\partial \underline{u}}{\partial \nu} = -\frac{\partial \underline{u}}{\partial t}\Big|_{t=0} = \frac{1}{r}\frac{\partial \underline{u}}{\partial \theta}\Big|_{\theta=\pi} = r^{\alpha-1}(\gamma\cos(\gamma\pi) + 2b\pi) \leq -cr^{\alpha-1},$$

where $c > 0$, if b is taken sufficiently small.

We use the maximum principle in the region D contained in $\mathcal{U}_{r_0}$, which in terms of the polar coordinates is given by

$$D = \{r < r_0, 0 < \theta < \pi\}.$$

The boundary of D consists of

$$\partial D = M_0 \cup M_1 \cup M_2,$$

where

$$\begin{aligned} M_1 &= \{0 \leq r \leq r_0, \theta = \pi\} = \partial D \cap \Gamma_1\\ M_2 &= \{0 \leq r \leq r_0, \theta = 0\} = \partial D \cap \Gamma_2\\ M_3 &= \{r = r_0, 0 < \theta < \pi\} = \partial D \cap \Omega.\end{aligned}$$

We have

$$\begin{aligned} \Delta u \leq 0, &\quad \Delta\underline{u} > 0 \quad \text{in } D\\ \frac{\partial u}{\partial \nu} \geq 0, &\quad \frac{\partial \underline{u}}{\partial \nu} < 0 \quad \text{on } M_1\\ u = 0, &\quad \underline{u} = 0 \quad \text{on } M_2\end{aligned}$$

and

$$u \geq c\underline{u} \quad \text{on } M_3,$$

for some $c > 0$. This follows from the standard strong maximum principle and Hopf's lemma applied to u, since the distance from M_3 to the interface I is strictly positive. It follows that

$$u \geq c\underline{u} \quad \text{in } D.$$

This yields the lower bound for u.

To obtain the upper bound for u in (3.9) choose

$$\overline{u} = u_1 - u_2,$$

where α, γ, b are as before, that is $1/2 < \gamma < \alpha < 1, b > 0$. By (3.12) and (3.13)

$$\Delta \overline{u} \leq -cr^{\alpha-2} + O(r^{\alpha-1}) + O(r^{-1/2}) \leq -cr^{\alpha-2} \tag{3.14}$$

for small r for some positive fixed c. Similarly

$$\frac{\partial \overline{u}}{\partial \nu} = -\frac{\partial \overline{u}}{\partial t}\Big|_{t=0} = \frac{1}{r}\frac{\partial \overline{u}}{\partial \theta}\Big|_{\theta=\pi} = -r^{\alpha-1}(\gamma \cos(\gamma\pi) + 2b\pi) \geq cr^{\alpha-1}, \tag{3.15}$$

where $c > 0$, if b is taken sufficiently small. Applying the maximum principle in the same region D as before we find $u \leq C\overline{u}$ in D. □

One consequence of (3.9) is that even if h_1, h_2 are smooth the solution u to (3.7) is in general not smooth, having at worst a behavior of the form $u(x) \sim dist(x, I)^{1/2}$ and $|\nabla u(x)| \sim dist(x, I)^{-1/2}$.

We need to define the notion of weak solution to (1.4), and before this, we need to define what we understand as weak solution to a linear problem. Define the space $L^1_\zeta(\Gamma_1)$ as the space of measurable functions $h : \Gamma_1 \to \mathbb{R}$ such that $\int_{\Gamma_1} |h|\zeta < +\infty$. We define the class of test functions $\mathcal{T}$ as the collection of $\varphi \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\varphi = 0$ on Γ_2, $\Delta\varphi$ can be extended to a continuous function in $\overline{\Omega}$, for any $x \in \partial\Omega \setminus I$ there is $r > 0$ such that $\nabla\varphi$ admits a continuous extension to $\overline{\Omega} \cap B_r(x)$ and $\frac{\partial\varphi}{\partial\nu}$, which is now well defined in $\Gamma_1 \setminus I$ and can be extended as a continuous function on Γ_1. In particular, given $\eta_1 \in C(\overline{\Omega})$, $\eta_2 \in C(\Gamma_1)$ the solution φ to

$$\begin{cases} -\Delta\varphi = \eta_1 & \text{in } \Omega \\ \dfrac{\partial\varphi}{\partial\nu} = \eta_2 & \text{on } \Gamma_1 \\ \varphi = 0 & \text{on } \Gamma_2, \end{cases} \tag{3.16}$$

is in $\mathcal{T}$. Moreover by Proposition 3.2 we see that φ satisfies

$$|\varphi| \leq C\zeta \quad \text{in } \Omega. \tag{3.17}$$

LEMMA 3.3. *Given $h \in L^1_\zeta(\Gamma_1)$ there is a unique $u_1 \in L^1(\Omega)$, $u_2 \in L^1(\Gamma_1)$ such that*

$$\int_\Omega u_1(-\Delta\varphi) + \int_{\Gamma_1}\left(h\varphi - u_2\frac{\partial\varphi}{\partial\nu}\right) = 0 \quad \forall\varphi \in \mathcal{T}. \tag{3.18}$$

Moreover

$$\|u_1\|_{L^1(\Omega)} + \|u_2\|_{L^1(\Gamma_1)} \leq C\|h\|_{L^1_\zeta(\Gamma_1)}, \tag{3.19}$$

and if $h \geq 0$ then $u_1, u_2 \geq 0$.

PROOF. We deal with uniqueness first. Suppose $u_1 \in L^1(\Omega)$, $u_2 \in L^1(\Gamma_1)$ satisfy (3.18) with $h = 0$. Given $\eta \in C_0^\infty(\Gamma_1)$ let φ be the solution to (3.16) with $\eta_1 = 0$, $\eta_2 = \eta$. Then $\varphi \in \mathcal{T}$ and by (3.18)

$$\int_{\Gamma_1} u_2 \eta = 0.$$

Hence $u_2 \equiv 0$. Then given $\eta \in C_0^\infty(\Omega)$, setting φ as the solution to

$$\begin{cases} -\Delta\varphi = \eta & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{3.20}$$

we deduce $\int_\Omega u_1 \eta = 0$. It follows that $u_1 = 0$.

We prove (3.19) in the case $u_1 \geq 0$, $u_2 \geq 0$. For this we may take $\eta_1 = 1$ and $\eta_2 = 1$ in (3.16). Then from (3.18) and (3.17) we see that (3.19) holds.

For the existence part we take $h \in L^1_\zeta(\Gamma_1)$, $h \geq 0$ and let $h_m = \min(m, h)$. Then

$$\begin{cases} \Delta u_m = 0 & \text{in } \Omega \\ \dfrac{\partial u_m}{\partial \nu} = h_m & \text{on } \Gamma_1 \\ u_m = 0 & \text{on } \Gamma_2 \end{cases}$$

has a solution $u_m \in H^1(\Omega)$ and we have the bound

$$\|u_m - u_n\|_{L^1(\Omega)} + \|u_m - u_n\|_{L^1(\Gamma_1)} \leq \|h_n - h_m\|_{L^1_\zeta(\Gamma_1)}.$$

Thus $u_m \to u_1$ in $L^1(\Omega)$ and $u_m|_{\Gamma_1} \to u_2$ in $L^1(\Gamma_1)$. For $\varphi \in \mathcal{T}$ we have

$$\int_\Omega u_m(-\Delta\varphi) + \int_{\Gamma_1} u_m \frac{\partial\varphi}{\partial\nu} - h_m\varphi = 0.$$

Passing to the limit shows that u_1, u_2 satisfies condition (3.18). We see also that $u_1 \geq 0$, $u_2 \geq 0$. For general h we may rewrite it as the difference of two nonnegative functions. □

If h is smooth then we may find a solution $u \in \mathcal{T}$ to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = h & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2 \end{cases} \tag{3.21}$$

and u_1, u_2 in Lemma 3.3 correspond to u restricted to Ω and Γ_1 respectively.

DEFINITION 3.4. We say that $u_1 \in L^1(\Omega)$, $u_2 \in L^1(\Gamma_1)$ is a weak solution to (3.21) if they satisfy (3.18). In the sequel, when referring to a weak solution $u_1 \in L^1(\Omega)$, $u_2 \in L^1(\Gamma)$ to (3.21) we will identify u_1 and u_2 as just u, and according to the context we write $u \in L^1(\Omega)$ or $u \in L^1(\Gamma_1)$.

Weak supersolutions are defined as:

DEFINITION 3.5. We say that $u \in L^1(\Gamma_1)$ is a weak supersolution to (3.21) if

$$\int_\Omega u_1(-\Delta\varphi) + \int_{\Gamma_1}\left(h\varphi - u_2\frac{\partial\varphi}{\partial\nu}\right) \geq 0 \quad \forall\varphi \in \mathcal{T}, \varphi \geq 0. \tag{3.22}$$

We consider the problem (1.4), that is,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial\nu} = \lambda g(u) & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2. \end{cases} \tag{3.23}$$

DEFINITION 3.6. We say that $u \in L^1(\Gamma_1)$ is a weak solution to (3.23) if $g(u) \in L^1_\zeta(\Gamma_1)$ and (3.23) holds in the sense of Definition 3.4 with $h = \lambda g(u)$.

Let us remark that only with hypotheses (1.2) and (1.3) the extremal solution u^* is a weak solution in the sense of Definition 3.6. Indeed, the same calculations as in (1.6) and (1.7) with $\varphi_1 > 0$ the first eigenfunction for

$$\begin{cases} \Delta\varphi_1 = 0 & \text{in } \Omega \\ \dfrac{\partial\varphi_1}{\partial\nu} = \lambda_1\varphi_1 & \text{on } \Gamma_1 \\ \varphi_1 = 0 & \text{on } \Gamma_2 \end{cases}$$

show that

$$\int_{\Gamma_1} g(u_\lambda)\varphi_1 \leq C$$

with C independent of λ. Note that by Proposition 3.2 we have $\zeta \leq C\varphi_1$ and it follows that

$$\int_{\Gamma_1} g(u_\lambda)\zeta \leq C. \tag{3.24}$$

To show that $u^* \in L^1(\Omega)$ let χ solve

$$\begin{cases} -\Delta\chi = 1 & \text{in } \Omega \\ \dfrac{\partial\chi}{\partial\nu} = 0 & \text{on } \Gamma_1 \\ \chi = 0 & \text{on } \Gamma_2. \end{cases}$$

By Proposition 3.2 we have $\chi \leq C\zeta$ and hence, after multiplying (3.23) by χ and integrating by parts we have

$$\int_\Omega u_\lambda = \lambda\int_{\Gamma_1} u_\lambda\chi \leq C\int_{\Gamma_1} g(u_\lambda)\zeta \leq C$$

by (3.24). Hence $u^* \in L^1(\Omega)$, $g(u^*) \in L^1_\zeta(\Gamma_1)$ and it is not difficult to verify that it satisfies Definition 3.6.

Our next result is an adaptation of a result of Nedev [96] for (1.1), that shows that u^* is bounded in dimensions $N \le 3$ for that problem. It also provides some estimates of the form $g(u^*)$ in L^p for some $p > 1$ in any dimension. The argument is the same as in [96] except that some of the exponents change slightly.

THEOREM 3.7. *Assume g satisfies* (1.2) *and* (1.3). *Then if $N \le 2$ we have $u^* \in L^\infty(\Omega)$. If $N \ge 3$ then $g(u^*) \in L^p(\Gamma_1)$ for $1 \le p < \frac{N-1}{2(N-2)}$ and $u^* \in L^p(\Gamma_1)$ for $1 \le p < \frac{N-1}{N-3}$.*

PROOF. We estimate the minimal solution u_λ for $0 < \lambda < \lambda^*$. Let

$$\psi(t) = \int_0^t g'(s)^2\, ds$$

and multiply (3.23) by $\psi(u_\lambda)$ to obtain

$$\int_\Omega g'(u_\lambda)^2 |\nabla u_\lambda|^2 = \lambda \int_{\Gamma_1} g(u_\lambda)\psi(u_\lambda). \tag{3.25}$$

We shall use the notation $\tilde{g}(u) = g(u) - g(0)$. Using the weak stability of u_λ with $\tilde{g}(u_\lambda)$ we have

$$\lambda \int_{\Gamma_1} g'(u_\lambda)\tilde{g}(u_\lambda)^2 \le \int_\Omega g'(u_\lambda)^2 |\nabla u_\lambda|^2.$$

Hence, by (3.25) we have

$$\int_{\Gamma_1} g'(u_\lambda)\tilde{g}(u_\lambda)^2 = \int_{\Gamma_1} g(u_\lambda)\psi(u_\lambda) = \int_{\Gamma_1} \tilde{g}(u_\lambda)\psi(u_\lambda) + g(0)\int_{\Gamma_1} \psi(u_\lambda). \tag{3.26}$$

As in [96] let

$$h(t) = \int_0^t g'(s)(g'(t) - g'(s))\, ds.$$

Then from (3.26) we have

$$\int_{\Gamma_1} \tilde{g}(u_\lambda) h(u_\lambda) \le g(0) \int_{\Gamma_1} \psi(u_\lambda). \tag{3.27}$$

But

$$\lim_{t\to+\infty} \frac{h(t)}{g'(t)} = +\infty. \tag{3.28}$$

Indeed, for any $M > 0$, by the convexity of g we have

$$\begin{aligned} h(t) \ge \int_0^M g'(s)(g'(t) - g'(s))\, ds &\ge \int_0^M g'(s)(g'(t) - g'(M))\, ds \\ &= (g(M) - g(0))(g'(t) - g'(M)). \end{aligned}$$

Dividing by $g'(t)$ we have

$$\liminf_{t\to+\infty} \frac{h(t)}{g'(t)} \geq (g(M) - g(0))$$

(by (1.3) $\lim_{t\to+\infty} g'(t) = +\infty$). Since M is arbitrary we deduce (3.28).

On the other hand

$$\psi(t) = \int_0^t g'(s)^2\, ds \leq g'(t)\tilde{g}(t). \tag{3.29}$$

Thus, by (3.27), (3.28) and (3.29) we find

$$\int_{\Gamma_1} \tilde{g}(u_\lambda)h(u_\lambda) \leq C \quad \text{and} \quad \int_{\Gamma_1} \psi(u_\lambda) \leq C$$

with C independent of λ and also

$$\int_{\Gamma_1} \tilde{g}(u_\lambda)g'(u_\lambda) \leq C.$$

The convexity of g implies $g'(t) \geq \tilde{g}(t)/t$, and hence

$$\int_{\Gamma_1} \frac{\tilde{g}(u_\lambda)^2}{u_\lambda} \leq C. \tag{3.30}$$

It follows that $g(u_\lambda) \in L^1(\Gamma_1)$ since, one needs to control $\int g(u_\lambda)$ in the region where $u_\lambda \geq M$, and there, $\frac{\tilde{g}(u_\lambda)^2}{u_\lambda} \geq u_\lambda$ if M is large enough. By regularity theory

$$\|u_\lambda\|_{L^p(\Gamma_1)} \leq C \quad \text{for } 1 \leq p < \tfrac{N-1}{N-2} \text{ (any } p < \infty \text{ if } N = 2).$$

Let $0 < \alpha < 1$ and

$$A = \{x \in \Gamma_1 : \tilde{g}(u_\lambda) < u_\lambda^{1/\alpha}\}, \qquad B = \{x \in \Gamma_1 : \tilde{g}(u_\lambda)^2/u_\lambda \geq \tilde{g}(u_\lambda)^{2-\alpha}\}.$$

Then A, B cover all Γ_1. By (3.30)

$$\int_B \tilde{g}(u_\lambda)^{2-\alpha} \leq C$$

and

$$\int_A \tilde{g}(u_\lambda)^p \leq \int_A \tilde{u}_\lambda^{p/\alpha} \leq C$$

if $p/\alpha < \frac{N-2}{N-1}$. Choosing $\alpha = \frac{2(N-2)}{2N-3}$ we see that

$$\|g(u_\lambda)\|_{L^p(\Gamma_1)} \leq C \quad \text{for } 1 \leq p < \tfrac{2(N-1)}{2N-3}.$$

Repeating this process yields the desired conclusion. □

Next we show, following the argument of Brezis, Cazenave, Martel and Ramiandrisoa, that there are no weak solutions for $\lambda > \lambda^*$.

THEOREM 3.8. *Assume* g *satisfies* (1.2) *and* (1.3). *Then, for* $\lambda > \lambda^*$, *problem* (3.23) *has no weak solutions.*

For the proof we need the following:

LEMMA 3.9. *Let* $h \in L^1_\zeta(\Gamma_1)$ *and* $u \in L^1(\Gamma_1)$ *be weak solutions of* (3.21). *Let* $\Phi : \mathbb{R} \to \mathbb{R}$ *be a* C^2 *concave function with* $\Phi' \in L^\infty$ *and* $\Phi(0) = 0$. *Then* $\Phi(u)$ *is a weak supersolution to* (3.21) *with* h *replaced by* $\Phi'(u)h$.

PROOF. For $m > 0$ let $h_m = h$ if $|h| \le m$, $h_m = -m$ if $h < -m$ and $h_m = m$ if $h > m$, and let u_m denote the H^1 solution of (3.21) with h replaced by h_m. Note that $u_m \to u$ in $L^1(\Omega)$ and in $L^1(\Gamma_1)$ by (3.19). Let $\varphi \in \mathcal{T}$, $\varphi \ge 0$. Using $\Phi'(u_m)\varphi$ as a test function we find that

$$\int_\Omega \nabla u_m \big(\Phi''(u_m)\nabla u_m \varphi + \Phi'(u_m)\nabla\varphi\big)\,dx - \int_{\Gamma_1} \Phi'(u_m)h_m\varphi = 0.$$

Using that $\Phi'' \le 0$ and $\varphi \ge 0$ we have

$$\int_\Omega \nabla(\Phi(u_m))\nabla\varphi\,dx - \int_{\Gamma_1} h_m\Phi'(u_m)\varphi\,dx \ge 0 \tag{3.31}$$

and integrating by parts

$$\int_\Omega \Phi(u_m)(-\Delta\varphi) + \int_{\Gamma_1} \Phi(u_m)\frac{\partial\varphi}{\partial\nu} - h_m\Phi'(u_m)\varphi \ge 0.$$

Now we let $m \to \infty$. We have

$$\int_\Omega |\Phi(u_m) - \Phi(u)||\Delta\varphi|\,dx \le \|\Delta\varphi\|_\infty\|\Phi'\|_\infty \int_\Omega |u_m - u|\,dx \to 0$$

$$\int_{\Gamma_1} |\Phi(u_m) - \Phi(u)|\left|\frac{\partial\varphi}{\partial\nu}\right|\,dx \le \left\|\frac{\partial\varphi}{\partial\nu}\right\|_\infty \|\Phi'\|_\infty \int_{\Gamma_1} |u_m - u|\,dx \to 0$$

and

$$\int_{\Gamma_1} h_m\Phi'(u_m)\varphi\,dx \to \int_{\Gamma_1} h\Phi'(u)\varphi\,dx$$

since we have convergence a.e. (at least for a subsequence) and

$$|h_m\Phi'(u_m)\varphi| \le \|\Phi'\|_\infty|h|\zeta \in L^1(\Gamma_1)$$

by the assumption $h \in L^1_\zeta(\Gamma_1)$. □

LEMMA 3.10. *If* (3.23) *has a weak supersolution* $w \ge 0$ *then it has a weak solution.*

PROOF. The proof is by the standard iteration method: set $u_0 = 0$ and u_{k+1} as the solution to

$$\begin{cases} \Delta u_{k+1} = 0 & \text{in } \Omega \\ \dfrac{\partial u_{k+1}}{\partial \nu} = \lambda g(u_k) & \text{on } \Gamma_1 \\ u_{k+1} = 0 & \text{on } \Gamma_2. \end{cases}$$

The u_k is an increasing sequence bounded above by w which belongs to $L^1(\Omega)$ and $L^1(\Gamma_1)$, and $g(u_k)$ is increasing, bounded above by $g(w) \in L^1_\zeta(\Gamma_1)$. The limit $u = \lim_{k\to+\infty} u_k$ thus exists and is a weak solution. □

PROOF OF THEOREM 3.8. Assume that (λ, u) is a weak supersolution to (3.23). Let $0 < \lambda' < \lambda$ and Φ be defined as in (3.3), (3.4). By Lemma 3.9 we see that $\Phi(u)$ is a supersolution to (3.23) with parameter λ'. Suppose first that g satisfies $\int_0^\infty ds/g(s) < +\infty$. Then $\Phi(u)$ is also bounded and hence (3.23) with parameter λ' has a bounded solution.

Next we consider the case $\int_0^\infty ds/g(s) = +\infty$. As in [19] let $\varepsilon > 0$ be small and let $\lambda' = (1-\varepsilon)\lambda$. Let $v_1 = \Phi(u)$. Then $0 \le v_1 \le w$. But H is concave, so

$$H(u) \le H(v_1) + (u - v_1)H'(v_1) = H(v_1) + \frac{u - v_1}{g(v_1)}.$$

Recall that by definition of Φ and H (3.3), (3.4) we have $H(v_1) = (1-\varepsilon)H(u)$. Hence

$$\varepsilon H(u) \le \frac{u - v_1}{g(v_1)}$$

and therefore

$$g(v_1) \le C\frac{u}{H(u)} \le C(1+u) \in L^1(\Gamma_1). \tag{3.32}$$

Then by Lemma 3.10 there exists a weak solution u_1 to (3.23) with parameter $(1-\varepsilon)\lambda$ such that $u_1 \le v_1$ and by (3.32) we have $g(u_1) \in L^1(\Gamma_1)$. Thus $u_1 \in L^p(\Gamma_1)$ for any $p < \frac{N-1}{N-2}$ ($p < \infty$ if $N = 2$). Repeating this process, we define $v_2 = \Phi(u_1)$ and as before obtain $g(v_2) \le C(1+u_1) \in L^p(\Gamma_1)$ for any $p < \frac{N-1}{N-2}$ ($p < \infty$ if $N = 2$). Then there is a solution $u_2 \le v_2$ to (3.23) with parameter $(1-\varepsilon)^2\lambda$ and it satisfies $g(u_2) \in L^p(\Gamma_1)$ for $p < \frac{N-1}{N-2}$. By induction there is a solution u_k to (3.23) with parameter $(1-\varepsilon)^k\lambda$ and satisfying $g(u_k) \in L^p(\Gamma_1)$ for any $\frac{1}{p} > 1 - \frac{k-1}{N-1}$ provided $1 - \frac{k-1}{N-1} > 0$. For $k > N$ we find $u_k \in L^\infty(\Gamma_1)$. □

Finally this is the uniqueness result of [87] in the context of problem (3.23).

THEOREM 3.11. *Suppose that g satisfies* (1.2) *and* (1.3). *Then for $\lambda = \lambda^*$, problem* (3.23) *has a unique weak solution.*

PROOF. Let u_1, u_2 be different solutions to (3.23), and without loss of generality we may assume that $u_1 = u^*$ is the minimal one, so that $u_2 > u_1$ in Ω.

First we show that (3.23) has a strict supersolution v. For this we note that any convex combination $v_t = tu_1 + (1-t)u_2$, $t \in (0,1)$ is a supersolution, by the convexity of g. Suppose v_t is still a solution for all $0 < t < 1$. Then

$$g\,(tu_1(x) + (1-t)u_2(x)) = tg(u_1(x)) + (1-t)g(u_2(x)) \quad \text{a.e. on } \Gamma_1$$

and for all $t \in (0,1)$. Then there is a set E of full measure in Γ_1 such that $g\,(tu_1(x) + (1-t)u_2(x)) = tg(u_1(x)) + (1-t)g(u_2(x))$ holds for $t \in (0,1) \cap \mathbb{Q}$ and $x \in E$. This means g is linear in $[u_1(x), u_2(x)]$ for a.e. $x \in \Gamma_1$. The union of the intervals $[u_1(x), u_2(x)]$ with x in a set of full measure in Γ_1 is an interval. The argument is the same as in the end of the proof of Lemma 1.16, with the only difference that in this case, we do not have the information that $u_1 = u^*$ is in $H^1(\Omega)$. But now, thanks to Theorem 3.7 we know that $g(u^*) \in L^p(\Gamma_1)$ for some $p > 1$. Then by L^p theory [2,3] we also have $\nabla u^* \in L^p(\Gamma_1)$ for some $p > 1$ and therefore $u^* \in W^{1,p}(\Gamma)$. As in the proof of Lemma 1.16, this is sufficient to guarantee that $u^*(\Gamma_1)$ is dense in $[\operatorname{ess\,inf}_{\Gamma_1} u^*, \operatorname{ess\,sup}_{\Gamma_1} u^*]$. The conclusion from the previous argument is that u_1, u_2 solve a problem with a linear g, say $g(t) = a + bt$. By a bootstrap argument, u_1, u_2 are bounded solutions. Recall that by the implicit function theorem the first eigenvalue of the linearized operator at u^* is zero. Let $\varphi_1 > 0$ denote the first eigenfunction of the linearized operator, that is,

$$\begin{cases} \Delta\varphi_1 = 0 & \text{in } \Omega \\ \dfrac{\partial\varphi_1}{\partial\nu} = \lambda^* b\varphi_1 & \text{on } \Gamma_1 \\ \varphi_1 = 0 & \text{on } \Gamma_2. \end{cases}$$

Since u^* solves (3.23) with $g(t) = a + bt$, multiplying that equation by φ_1 and integrating by parts yields

$$\int_{\Gamma_1} \lambda^*(a + bu)\varphi_1 = \int_{\Gamma_1} \lambda^* bu\varphi_1.$$

Then $a = 0$ and we reach a contradiction.

We claim that there is some $\varepsilon > 0$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial\nu} = \lambda^* g(u) + \varepsilon & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2 \end{cases} \tag{3.33}$$

has a weak supersolution. Indeed, there is a strict supersolution v to (3.23). Let V be the solution of

$$\begin{cases} \Delta V = 0 & \text{in } \Omega \\ \dfrac{\partial V}{\partial\nu} = \lambda^* g(v) & \text{on } \Gamma_1 \\ V = 0 & \text{on } \Gamma_2 \end{cases}$$

and let χ solve

$$\begin{cases} \Delta\chi = 0 & \text{in } \Omega \\ \dfrac{\partial\chi}{\partial\nu} = 1 & \text{on } \Gamma_1 \\ \chi = 0 & \text{on } \Gamma_2. \end{cases} \tag{3.34}$$

By Proposition 3.2 there is a constant $\varepsilon > 0$ so that $v - V \geq \varepsilon\chi$. Hence $w = V + \varepsilon\chi \leq v$ and

$$\frac{\partial w}{\partial \nu} = \lambda^* g(v) + \varepsilon \geq \lambda^* g(w) + \varepsilon$$

and thus w is the desired supersolution.

Let $0 < \varepsilon_1 < \varepsilon$. Then there exists a bounded supersolution to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial\nu} = \lambda^* g(u) + \varepsilon_1 & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2. \end{cases} \tag{3.35}$$

To see this, define $\Phi : [0, \infty) \to [0, \infty)$ so that

$$\int_0^{\Phi(t)} \frac{ds}{\lambda^* g(s) + \varepsilon_1} = \int_0^t \frac{ds}{\lambda^* g(s) + \varepsilon} \qquad \text{for all } u \geq 0.$$

A calculation as in [19] shows that Φ satisfies the hypothesis of Lemma 3.9. If $\int_0^\infty ds/g(s) < +\infty$ then Φ is bounded. Let w be a supersolution of (3.33). Then by Lemma 3.9 $\Phi(w)$ is a bounded supersolution for (3.35). By the method of sub and supersolutions there is a bounded solution to (3.35).

If $\int_0^\infty ds/g(s) = +\infty$ then an iteration with Φ as in the proof of Theorem 3.8 still yields a bounded solution to (3.35). In fact, let

$$H_\varepsilon(t) = \int_0^t \frac{ds}{\lambda^* g(s) + \varepsilon}$$

and let $0 < \varepsilon_1 < \varepsilon$. Then we may restate the definition of Φ as $\Phi = H_{\varepsilon_1}^{-1} \circ H_\varepsilon$ or

$$H_{\varepsilon_1}(\Phi(t)) = H_\varepsilon(t) \quad \text{for all } t \geq 0.$$

Denoting by w the supersolution to (3.33) and $v = \Phi(w)$ we thus have

$$H_{\varepsilon_1}(v) = H_\varepsilon(w). \tag{3.36}$$

The function H_ε is concave and $v \leq w$, so

$$\frac{H_\varepsilon(w) - H_\varepsilon(v)}{w - v} \leq H_{\varepsilon_1}(v) = \frac{1}{\lambda^* g(v) + \varepsilon}. \tag{3.37}$$

But, thanks to (3.36)

$$
\begin{aligned}
H_\varepsilon(w) - H_\varepsilon(v) = H_{\varepsilon_1}(v) - H_\varepsilon(v) &= \int_0^v \left(\frac{1}{\lambda^* g(s) + \varepsilon_1} - \frac{1}{\lambda^* g(s) + \varepsilon} \right) ds \\
&\geq (\varepsilon - \varepsilon_1) \int_0^v \frac{1}{(\lambda^* g(s))^2} \, ds. \qquad (3.38)
\end{aligned}
$$

From (3.37) and (3.38) we see that

$$
g(v) \leq \frac{C(1+w)}{\varepsilon - \varepsilon_1}.
$$

The rest of the argument proceeds as in the proof of Theorem 3.8.

Since (3.35) has a bounded supersolution it also has a bounded solution w. Let $\lambda' > \lambda^*$ to be chosen later, and set

$$
W = \frac{\lambda'}{\lambda^*} w - \varepsilon_1 \chi,
$$

where χ is the solution of (3.34). Then observe that

$$
\frac{\partial W}{\partial \nu} = \lambda' g(w) + \frac{\lambda'}{\lambda^*} \varepsilon_1 - \varepsilon_1 \geq \lambda' g(w). \qquad (3.39)
$$

We now choose λ'/λ^* close to 1, so that

$$
\left(\frac{\lambda'}{\lambda^*} - 1 \right) w \leq \varepsilon_1 \chi,
$$

and therefore

$$
w \geq W. \qquad (3.40)
$$

This is possible because $w \in L^\infty$ and therefore $w \leq C\chi$ for some constant $C > 0$, by Proposition 3.2. Then (3.39) combined with (3.40) implies that W is a supersolution of (3.23) with λ^* replaced by λ'. This is in contradiction with λ^* being the maximal parameter for (3.23). □

3.3. *Kato's inequality*

In this section we will prove

THEOREM 3.12. *Let $B = B_1(0)$ be the unit ball in $\mathbb{R}^N$, $N \geq 3$. Then for any $1 \leq q < 2$ there exists $c = c(N, q) > 0$ such that*

$$
\int_{\mathbb{R}^N_+ \cap B} |\nabla \varphi|^2 \geq H_N \int_{\partial \mathbb{R}^N_+ \cap B} \frac{\varphi^2}{|x|} + c \|\varphi\|^2_{W^{1,q}(\mathbb{R}^N_+ \cap B)}, \quad \forall \varphi \in C_0^\infty(\overline{\mathbb{R}^N_+ \cap B}).
$$

REMARK 3.13. (a) The singular weight $\frac{1}{|x|}$ on the right-hand side of (1.43) is optimal, in the sense that it may not be replaced by $\frac{1}{|x|^\alpha}$ with $\alpha > 1$. This can be easily seen by choosing $\varphi \in H^1(\mathbb{R}^N_+)$ such that $\varphi(x) = |x|^{-\frac{N-2}{2}+\frac{\alpha-1}{2}}$ in a neighborhood of the origin. Moreover, the infimum in (1.44) is not achieved.
(b) In dimension $N = 2$ the infimum (1.44) is zero.
(c) Using Stirling's formula it is possible to verify that

$$H_N = \frac{N-3}{2} + O\left(\frac{1}{N}\right) \qquad \text{as } N \to \infty. \tag{3.41}$$

Let us turn our attention to the proof of Theorem 3.12. Following an idea of Brezis and Vázquez (equation (4.6) on page 453 of [20]) it turns out to be useful to replace φ in (1.43) by $v = \varphi/w$, where $w = w_\alpha$ with $\alpha = \frac{N-2}{2}$ as defined in (1.47). Observe that $C(N, \frac{N-2}{2}) = H_N$ by (3.57) and hence w is harmonic in the half space $\mathbb{R}^N_+$ and satisfies

$$\frac{\partial w}{\partial \nu} = H_N \frac{w}{|x|} \quad \text{on } \partial\mathbb{R}^N_+.$$

PROOF OF THEOREM 3.12. When $N \geq 3$, $C_0^\infty(\mathbb{R}^N_+ \setminus \{0\})$ is dense in $H^1(\mathbb{R}^N_+)$. So it suffices to prove (1.43) for $\varphi \in C_0^\infty(\mathbb{R}^N_+ \setminus \{0\})$. Fix such a $\varphi \not\equiv 0$ and let w be the function defined by (1.47). Notice that, on supp φ, w is smooth and bounded from above and from below by some positive constants. Hence $v := \frac{\varphi}{w} \in C_0^\infty(\overline{\mathbb{R}^N_+})$ is well defined. Now, $\varphi = vw$, $\nabla\varphi = v\nabla w + w\nabla v$ and

$$|\nabla\varphi|^2 = v^2|\nabla w|^2 + w^2|\nabla v|^2 + 2vw\nabla v\nabla w.$$

Integrating

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 = \int_{\mathbb{R}^N_+} v^2|\nabla w|^2 + \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 + 2\int_{\mathbb{R}^N_+} vw\nabla v\nabla w$$

and by Green's formula

$$\int_{\mathbb{R}^N_+} v^2|\nabla w|^2 = \int_{\partial\mathbb{R}^N_+} v^2 w\frac{\partial w}{\partial \nu} - \int_{\mathbb{R}^N_+} w\nabla(v^2\nabla w)$$

$$= \int_{\partial\mathbb{R}^N_+} v^2 w\frac{\partial w}{\partial \nu} - 2\int_{\mathbb{R}^N_+} wv\nabla w\nabla v,$$

since w is harmonic in $\mathbb{R}^N_+$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 &= \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 + \int_{\partial\mathbb{R}^N_+} v^2 w\frac{\partial w}{\partial \nu} \\ &= \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 + \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{w}\frac{\partial w}{\partial \nu}.\end{aligned} \tag{3.42}$$

But by (3.57) $\frac{\frac{\partial w}{\partial \nu}(x)}{w(x)} = \frac{H_N}{|x|}$ for $x \in \partial\mathbb{R}^N_+$ and hence,

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 \geq H_N \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|} + \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 \quad \forall\varphi \in H^1(\mathbb{R}^N_+). \tag{3.43}$$

The second term on the right-hand side of the above inequality yields the improvement of Kato's inequality when φ has support in the unit ball.

Now we assume $\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+} \setminus \{0\} \cap B)$ and, as before, set $v = \frac{\varphi}{w}$. Our aim is to prove that given $1 \leq q < 2$ there exists $C > 0$ such that

$$I := \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 \geq \frac{1}{C}\|\varphi\|_{W^{1,q}}. \tag{3.44}$$

In spherical coordinates

$$I = \int_0^1 r^{N-1} \int_{S_1^+} w^2(r\theta)|\nabla v(r\theta)|^2 d\theta\, dr,$$

where $S_1^+ = S_1 \cap \mathbb{R}^N_+$ and $S_1 = \{x \in \mathbb{R}^N \,/\, |x| = 1\}$ is the sphere of radius 1. From (3.56) we have $w(x) \geq \frac{1}{C}|x|^{-\frac{N-2}{2}}$ for some $C > 0$ and all $x \in B \cap \mathbb{R}^N_+$. Hence

$$I \geq \frac{1}{C}\int_0^1 r \int_{S_1^+} |\nabla v(r\theta)|^2 d\theta\; dr.$$

Let us compute the Sobolev norm of φ:

$$\begin{aligned}
\|\varphi\|^q_{W^{1,q}} &= \int_{\mathbb{R}^N_+\cap B} |\nabla\varphi|^q dx = \int_0^1 r^{N-1} \int_{S_1^+} |\nabla\varphi(r\theta)|^q d\theta\; dr \\
&= \int_0^1 r^{N-1} \int_{S_1^+} |\nabla v(r\theta)\; w(r\theta) + \nabla w(r\theta)\; v(r\theta)|^q d\theta\; dr \\
&\leq C_q \int_0^1 r^{N-1} \int_{S_1^+} |\nabla v(r\theta)|^q |w(r\theta)|^q + |\nabla w(r\theta)|^q |v(r\theta)|^q d\theta\; dr.
\end{aligned}$$

Define

$$\begin{aligned}
I_1 &:= \int_0^1 r^{N-1} \int_{S_1^+} |\nabla v(r\theta)|^q |w(r\theta)|^q d\theta\; dr \\
I_2 &:= \int_0^1 r^{N-1} \int_{S_1^+} |\nabla w(r\theta)|^q |v(r\theta)|^q d\theta\; dr.
\end{aligned}$$

Since $w(x) \leq C|x|^{-\frac{N-2}{2}}$ we have by Hölder's inequality

$$\begin{aligned}
I_1 &\leq C\int_0^1 r^{N-1-\frac{(N-2)q}{2}} \int_{S_1^+} |\nabla v(r\theta)|^q d\theta\; dr \\
&\leq C\left[\int_0^1 r \int_{S_1^+} |\nabla v(r\theta)|^2 d\theta dr\right]^{\frac{q}{2}} \left[\int_0^1 r^{(N-1-\frac{Nq}{2}+\frac{q}{2})\frac{2}{2-q}} dr\right]^{\frac{2-q}{2}} = CI^{\frac{q}{2}},
\end{aligned} \tag{3.45}$$

since $q < 2$.

Using $|\nabla w(x)| \le C|x|^{-\frac{N}{2}}$ we estimate I_2:

$$I_2 \le C \int_{S_1^+} \int_0^1 r^{N-1-\frac{Nq}{2}} |v(r\theta)|^q \, dr \, d\theta.$$

From the classical Hardy inequality

$$\int_0^1 r^\gamma |f(r)|^p \, dr \le \left(\frac{p}{\gamma+1}\right)^p \int_0^1 r^{\gamma+p} |f'(r)|^p \, dr$$

$(p \ge 1, \gamma > -1, f \in C_0^\infty(0,1))$ we deduce

$$\int_0^1 r^{N-1-\frac{Nq}{2}} |v(r\theta)|^q \, dr \le C \int_0^1 r^{N-1-\frac{Nq}{2}+q} |\nabla v(r\theta)|^q \, dr$$

and therefore

$$I_2 \le C \int_{S_1^+} \int_0^1 r^{N-1-\frac{Nq}{2}+q} |\nabla v(r\theta)|^q \, dr \, d\theta.$$

Hölder's inequality yields

$$\begin{aligned} I_2 &\le C \left[\int_{S_1^+} \int_0^1 r |\nabla v(r\theta)|^2 \, dr \, d\theta \right]^{\frac{q}{2}} \left[\int_{S_1^+} \int_0^1 r^{(N-1-\frac{Nq}{2}+\frac{q}{2})\frac{2}{2-q}} \, dr \, d\theta \right]^{1-\frac{q}{2}} \\ &= C I^{\frac{q}{2}}, \end{aligned} \tag{3.46}$$

where we have used $q < 2$. Gathering (3.45) and (3.46) we conclude that (3.44) holds. □

3.4. *Boundedness of the extremal solution in the exponential case*

In this section we shall give a proof of Theorem 1.20. We proceed by contradiction, assuming that u^* is unbounded. A central point in the argument is to obtain some information of the singularity that u^* should have at the origin. More precisely, we claim that for any $0 < \sigma < 1$ there exists $r > 0$ such that

$$u^*(x) \ge (1-\sigma) \log \frac{1}{|x|} \quad \forall x \in \Gamma_1, \, |x| \le r. \tag{3.47}$$

Observe first that for all $0 < \lambda < \lambda^*$ the minimal solution u_λ is symmetric in the variables $x_1, \ldots, x_{N-1}$ by uniqueness of the minimal solution and the symmetry of Ω. Moreover, using the symmetry and convexity assumptions on Ω combined with the moving-plane method (see Proposition 5.2 in [32]) we deduce that u_λ achieves its maximum at the origin.

Assume by contradiction that (3.47) is false. Then there exists $\sigma > 0$ and a sequence $x_k \in \Gamma_1$ with $x_k \to 0$ such that

$$u^*(x_k) < (1-\sigma)\log\frac{1}{|x_k|}. \tag{3.48}$$

Let $s_k = |x_k|$ and choose $0 < \lambda_k < \lambda^*$ such that

$$\max_{\overline{\Omega}} u_{\lambda_k} = u_{\lambda_k}(0) = \log\frac{1}{s_k}. \tag{3.49}$$

Note that $\lambda_k \to \lambda^*$, otherwise u_{λ_k} would remain bounded. Let

$$v_k(x) = \frac{u_{\lambda_k}(s_k x)}{\log\frac{1}{s_k}} \qquad x \in \Omega_k \equiv \frac{1}{s_k}\Omega.$$

Then $0 \le v_k \le 1$, $v_k(0) = 1$, $\Delta v_k = 0$ in Ω_k and

$$\begin{aligned}\frac{\partial v_k}{\partial \nu}(x) &= \frac{1}{\log\frac{1}{s_k}} s_k\lambda_k \exp(u_{\lambda_k}(s_k x))\\ &\le \frac{\lambda_k}{\log\frac{1}{s_k}} \to 0,\end{aligned}$$

by (3.49). By elliptic regularity $v_k \to v$ uniformly on compact sets of $\overline{\mathbb{R}^N_+}$ to a function v satisfying $0 \le v \le 1$, $v(0) = 1$, $\Delta v = 0$ in $\mathbb{R}^N_+$, $\frac{\partial v}{\partial \nu} = 0$ on $\partial\mathbb{R}^N_+$. Extending v evenly to $\mathbb{R}^N$ we deduce that $v \equiv 1$. Since $|x_k| = s_k$ we deduce that

$$\frac{u_{\lambda_k}(x_k)}{\log\frac{1}{s_k}} \to 1,$$

which contradicts (3.48).

Now we use (3.47) to obtain a contradiction with the stability property of u^*. Let $\phi(x) = \int_{\partial\mathbb{R}^N_+} K(x,y)|y|^{2-N+\varepsilon}dy$ and $\psi(x) = \int_{\partial\mathbb{R}^N_+} K(x,y)|y|^{\frac{2-N+\varepsilon}{2}}dy$. Then,

$$\frac{\partial\phi}{\partial\nu} = K_\phi|x|^{1-N+\varepsilon} \qquad \frac{\partial\psi}{\partial\nu} = K_\psi|x|^{\frac{-N+\varepsilon}{2}}, \tag{3.50}$$

where the constants K_ϕ, K_ψ are given by

$$K_\phi = \lambda_{0,N}\varepsilon + O(\varepsilon^2) \quad \text{and} \quad K_\psi = H_N + O(\varepsilon).$$

Indeed, since u_0 and ϕ are harmonic in Ω,

$$\int_{\partial\Omega} u_0\frac{\partial\phi}{\partial\nu} = \int_{\partial\Omega}\phi\frac{\partial u_0}{\partial\nu}.$$

Clearly, $\int_{\Gamma_2} \left|\phi \frac{\partial u_0}{\partial \nu}\right| \leq C$, for some constant C independent of ε. So

$$K_\phi \int_0^1 \log\left(\frac{1}{r}\right) \frac{1}{r} r^{2-N+\varepsilon} r^{N-2} dr = \lambda_{0,N} \int_0^1 \frac{1}{r} r^{2-N+\varepsilon} r^{N-2} dr + O(1)$$
$$= \frac{\lambda_{0,N}}{\varepsilon} + O(1).$$

Now, $\int_0^1 \log \frac{1}{r} r^{-1+\varepsilon} dr = \frac{1}{\varepsilon^2}$ so we end up with

$$K_\phi = \lambda_{0,N}\varepsilon + O(\varepsilon^2).$$

Similarly, since ψ and w (defined in (1.47)) are harmonic in Ω, we have

$$\int_{\partial\Omega} w \frac{\partial \psi}{\partial \nu} = \int_{\partial\Omega} \psi \frac{\partial w}{\partial \nu}.$$

As before the boundary terms on Γ_2 are bounded independently of ε so

$$K_\psi \int_0^1 r^{-1+\varepsilon} dr = H_N \int_0^1 r^{-1+\varepsilon} dr + O(1).$$

Hence,

$$K_\psi = H_N + O(\varepsilon).$$

Multiplying (1.34) by ϕ and integrating by parts twice yields

$$\int_{\partial\Omega} u_\lambda \frac{\partial \phi}{\partial \nu} = \lambda \int_{\partial\Omega} \phi \frac{\partial u_\lambda}{\partial \nu} = \lambda \int_{\Gamma_1} \phi e^{u_\lambda} + \lambda \int_{\Gamma_2} \phi \frac{\partial u_\lambda}{\partial \nu} \leq \lambda \int_{\Gamma_1} \phi e^{u_\lambda}. \tag{3.51}$$

Let $\eta \in C^\infty(\mathbb{R}^N)$ be such that $\eta \equiv 1$ in $B_R(0)$, where $R > 0$ is small and fixed, and $\eta = 0$ on Γ_2. Using the stability condition (1.37) with $\eta\psi$ yields

$$\lambda \int_{\Gamma_1 \cap B_R(0)} e^{u_\lambda} \psi^2 \leq \int_\Omega |\nabla(\eta\psi)|^2 = \int_{\partial\Omega} \frac{\partial}{\partial \nu}(\eta\psi)(\eta\psi) - \int_\Omega (\eta\psi)\Delta(\eta\psi)$$
$$\leq \int_{\Gamma_1 \cap B_R(0)} \frac{\partial \psi}{\partial \nu} \psi + C, \tag{3.52}$$

where the constant C does not depend on ε and λ. Since $\psi^2 = \phi$ on $\partial\mathbb{R}^N_+$ combining (3.51) and (3.52) we obtain

$$\int_{\partial\Omega} u_\lambda \frac{\partial \phi}{\partial \nu} \leq \int_{\Gamma_1 \cap B_R(0)} \frac{\partial \psi}{\partial \nu} \psi + C$$

and letting $\lambda \nearrow \lambda^*$ we find

$$\int_{\partial\Omega} u^* \frac{\partial \phi}{\partial \nu} \leq \int_{\Gamma_1 \cap B_R(0)} \frac{\partial \psi}{\partial \nu} \psi + C.$$

Using (3.50) we arrive at

$$K_\phi \int_{\Gamma_1 \cap B_R(0)} u^* |x|^{1-N+\varepsilon} \le K_\psi \int_{\Gamma_1 \cap B_R(0)} |x|^{1-N+\varepsilon} + C$$

and thus

$$\int_{\Gamma_1 \cap B_R(0)} u^* |x|^{1-N+\varepsilon} \le \frac{1}{\varepsilon^2} \omega_{N-1} \frac{H_N}{\lambda_{0,N}} + O\left(\frac{1}{\varepsilon}\right),$$

where ω_{N-1} is the area of the $N-1$-dimensional sphere. We rewrite this inequality as

$$\int_0^R r^{-1+\varepsilon} u^*(r)\, dr \le \frac{1}{\varepsilon^2} \frac{H_N}{\lambda_{0,N}} + O\left(\frac{1}{\varepsilon}\right). \tag{3.53}$$

Let $\sigma > 0$ and $r(\sigma) > 0$ be such that (3.47) holds for $|x| \le r(\sigma)$. Then using (3.47) and (3.53) we find

$$(1-\sigma) \int_0^{r(\sigma)} \log \frac{1}{r} r^{\varepsilon - 1}\, dr \le \frac{1}{\varepsilon} \frac{K_\psi}{K_\phi} + C = \frac{1}{\varepsilon^2} \frac{H_N}{\lambda_{0,N}} + O\left(\frac{1}{\varepsilon}\right). \tag{3.54}$$

Integrating

$$(1-\sigma)\left(\frac{1}{\varepsilon^2} r(\sigma)^\varepsilon + \frac{1}{\varepsilon} r(\sigma)^\varepsilon \log \frac{1}{r(\sigma)}\right) \le \frac{1}{\varepsilon^2} \frac{H_N}{\lambda_{0,N}} + O\left(\frac{1}{\varepsilon}\right).$$

Letting $\varepsilon \to 0$ yields

$$(1-\sigma) \le \frac{H_N}{\lambda_{0,N}}.$$

As σ is arbitrarily small we deduce $H_N \ge \lambda_{0,N}$. But

$$H_N \ge \lambda_{0,N} \quad \text{if and only if} \quad N \ge 10 \tag{3.55}$$

(see [45]). This proves the theorem. □

3.5. *Auxiliary computations*

PROOF OF LEMMA 1.21. We write $x = (x', x_N) \in \mathbb{R}^N_+$ with $x' \in \mathbb{R}^{N-1}$, $x_N > 0$. It follows from (1.47) and a simple change of variables that

$$w_\alpha(x', x_N) = w_\alpha(e(x'), x_N) \quad \text{for all rotations } e \in \mathcal{O}(N-1),$$

and similarly

$$w_\alpha(Rx', Rx_N) = R^{-\alpha} w_\alpha(x', x_N). \tag{3.56}$$

Differentiating with respect to x_N yields

$$\frac{\partial w_\alpha}{\partial x_N}(Rx', Rx_N) = R^{-\alpha-1}\frac{\partial w_\alpha}{\partial x_N}(x', x_N).$$

Let $x \in \partial\mathbb{R}^N_+$, $x = (x', 0)$ and plug $R = \frac{1}{|x|} = \frac{1}{|x'|}$ in the previous formula to find

$$\frac{\partial w_\alpha}{\partial \nu}(x) = -\frac{\partial w_\alpha}{\partial x_N}(x', 0) = |x|^{-\alpha-1}\left(-\frac{\partial w_\alpha}{\partial x_N}\left(\frac{x'}{|x'|}, 0\right)\right).$$

Define

$$C(N, \alpha) = -\frac{\partial w_\alpha}{\partial x_N}\left(\frac{x'}{|x'|}, 0\right) \tag{3.57}$$

and observe that it is independent of $x' \in \mathbb{R}^{N-1}$.

Using (3.56) and the radial symmetry of w in the variables x', there exists a function $v : [0, \infty) \to \mathbb{R}$ such that

$$w_\alpha(x', x_N) = |x'|^{-\alpha} w_\alpha\left(\frac{x'}{|x'|}, \frac{x_N}{|x'|}\right) = |x'|^{-\alpha} v\left(\frac{x_N}{|x'|}\right). \tag{3.58}$$

Writing $r = |x'|$, $t = \frac{x_N}{|x'|}$, we have

$$r^{-\alpha} v(t) = w_\alpha(x', rt), \quad \forall x' \in \mathbb{R}^{N-1}, \ |x'| = r.$$

The equation $\Delta w = 0$ is equivalent to

$$(1 + t^2)v''(t) + (2\alpha + 4 - N)t v'(t) + \alpha(\alpha - N + 3)v(t) = 0, \quad t > 0, \tag{3.59}$$

while (1.48) implies

$$v(0) = 1.$$

The initial condition for v' is related to (3.57)

$$v'(0) = -C(N, \alpha).$$

In addition to these initial conditions we remark that w_α is a smooth function in $\mathbb{R}^N_+$ and this together with (3.58) implies that

$$\lim_{t\to\infty} v(t)t^\alpha \text{ exists.} \tag{3.60}$$

Using the change of variables $z = it$ with i the imaginary unit and defining the new unknown $h(z) := v(-iz)$ equation (3.59) becomes

$$(1 - z^2)h''(z) - (2\alpha + 4 - N)zh'(z) - \alpha(\alpha - N + 3)h(z) = 0, \tag{3.61}$$

with initial conditions

$$\lim_{t>0,t\to 0} h(it) = 1, \qquad \lim_{t>0,t\to 0} h'(it) = iC(N,\alpha). \tag{3.62}$$

On the other hand (3.60) implies

$$\lim_{t\in\mathbb{R},t\to\infty} h(it)t^{\alpha} \text{ exists.} \tag{3.63}$$

The substitution

$$g(z) = (1-z^2)^{\frac{\alpha}{2}+\frac{1}{2}-\frac{N}{4}} h(z) \tag{3.64}$$

transforms equation (3.61) into

$$(1-z^2)g''(z) - 2zg'(z) + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right) g(z) = 0, \tag{3.65}$$

with

$$\mu = \alpha + \frac{2-N}{2}, \qquad \nu = \frac{N-4}{2}. \tag{3.66}$$

The general solution to (3.65) is well known. Indeed, equation (3.65) belongs to the class of Legendre's equations. Two linearly independent solutions of (3.65) are given by the Legendre functions $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ (see [1]), which are defined in $\mathbb{C}\setminus\{-1,1\}$ and analytic in $\mathbb{C}\setminus(-\infty,1]$ (see [1, Formulas 8.1.2 – 8.1.6]). Moreover the limits of $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ on both sides of $(-1,1)$ exist and we shall use the notation

$$\begin{aligned} P_\nu^\mu(x+i0) &= \lim_{z\to x,\mathrm{Im}(z)>0} P_\nu^\mu(z), \quad -1<x<1,\\ P_\nu^\mu(x-i0) &= \lim_{z\to x,\mathrm{Im}(z)<0} P_\nu^\mu(z), \quad -1<x<1, \end{aligned} \tag{3.67}$$

and a similar notation for Q_ν^μ.

The solution g of (3.65) is therefore given by

$$g(z) = c_1 P_\nu^\mu(z) + c_2 Q_\nu^\mu(z),$$

for appropriate constants c_1, c_2. These constants are determined by the initial conditions (3.62), which imply:

$$c_1 P_\nu^\mu(0+i0) + c_2 Q_\nu^\mu(0+i0) = 1, \tag{3.68}$$

$$c_1 \frac{d}{dz} P_\nu^\mu(0+i0) + c_2 \frac{d}{dz} Q_\nu^\mu(0+i0) = iC(N,\alpha). \tag{3.69}$$

In order to evaluate $C(N,\alpha)$, we also use condition (3.63), which is equivalent to

$$\lim_{t\to\infty,\ t\in\mathbb{R}} (c_1 P_\nu^\mu(it) + c_2 Q_\nu^\mu(it)) t^{\frac{N}{2}-1} \text{ exists.} \tag{3.70}$$

But according to [1, Formulas 8.1.3, 8.1.5]

$$P_\nu^\mu(z) \sim z^\nu \quad \text{as } |z| \to \infty$$
$$Q_\nu^\mu(z) \sim z^{-\nu-1} \quad \text{as } |z| \to \infty.$$

This and (3.64), (3.70) imply that $c_1 = 0$ and we obtain from (3.68), (3.69)

$$C(N,\alpha) = -i\,\frac{\frac{d}{dz}Q_\nu^\mu(0+i0)}{Q_\nu^\mu(0+i0)}. \tag{3.71}$$

From the properties and formulas in [1] the following values can be deduced:

$$Q_\nu^\mu(0+i0) = -i2^{\mu-1}\pi^{\frac{1}{2}}e^{i\mu\pi - i\nu\frac{\pi}{2}}\frac{\Gamma(\frac{\nu}{2}+\frac{\mu}{2}+\frac{1}{2})}{\Gamma(\frac{\nu}{2}-\frac{\mu}{2}+1)} \tag{3.72}$$

$$\frac{d}{dz}Q_\nu^\mu(0+i0) = 2^\mu\pi^{\frac{1}{2}}e^{i\mu\pi - i\nu\frac{\pi}{2}}\frac{\Gamma(\frac{\nu}{2}+\frac{\mu}{2}+1)}{\Gamma(\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{2})}. \tag{3.73}$$

The relations (3.71), (3.72), (3.73) and the values (3.66) yield formula (1.49). □

4. A fourth-order variant of the Gelfand problem

4.1. *Comparison principles*

LEMMA 4.1 (Boggio's principle, [15]). *If $u \in C^4(\overline{B}_R)$ satisfies*

$$\begin{cases} \Delta^2 u \ge 0 & \text{in } B_R \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B_R \end{cases}$$

then $u \ge 0$ in B_R.

LEMMA 4.2. *Let $u \in L^1(B_R)$ and suppose that*

$$\int_{B_R} u\Delta^2\varphi \ge 0$$

for all $\varphi \in C^4(\overline{B}_R)$ such that $\varphi \ge 0$ in B_R, $\varphi|_{\partial B_R} = 0 = \frac{\partial\varphi}{\partial n}|_{\partial B_R}$. Then $u \ge 0$ in B_R.

PROOF. Let $\zeta \in C_0^\infty(\Omega)$, $\zeta \ge 0$ and solve

$$\begin{cases} \Delta^2 \varphi = \zeta & \text{in } B_R \\ \varphi = \dfrac{\partial \varphi}{\partial n} = 0 & \text{on } \partial B_R. \end{cases}$$

By Boggio's principle $\varphi \ge 0$ in B_R and we deduce that $\int_{B_R} u\zeta \ge 0$. Since $\zeta \in C_0^\infty(\Omega)$, $\zeta \ge 0$ is arbitrary we deduce $u \ge 0$. □

LEMMA 4.3. *If $u \in H^2(B_R)$ is radial, $\Delta^2 u \geq 0$ in B_R in the weak sense, that is*

$$\int_{B_R} \Delta u \Delta \varphi \geq 0 \quad \forall \varphi \in C_0^\infty(B_R),\ \varphi \geq 0$$

and $u|_{\partial B_R} \geq 0$, $\frac{\partial u}{\partial n}|_{\partial B_R} \leq 0$ then $u \geq 0$ in B_R.

PROOF. We only deal with the case $R = 1$ for simplicity. Solve

$$\begin{cases} \Delta^2 u_1 = \Delta^2 u & \text{in } B_1 \\ u_1 = \dfrac{\partial u_1}{\partial n} = 0 & \text{on } \partial B_1 \end{cases}$$

in the sense $u_1 \in H_0^2(B_1)$ and $\int_{B_1} \Delta u_1 \Delta \varphi = \int_{B_1} \Delta u \Delta \varphi$ for all $\varphi \in C_0^\infty(B_1)$. Then $u_1 \geq 0$ in B_1 by Lemma 4.2.

Let $u_2 = u - u_1$ so that $\Delta^2 u_2 = 0$ in B_1. Define $f = \Delta u_2$. Then $\Delta f = 0$ in B_1 and since f is radial we find that f is constant. It follows that $u_2 = ar^2 + b$. Using the boundary conditions we deduce $a + b \geq 0$ and $a \leq 0$, which imply $u_2 \geq 0$. □

Similarly we have

LEMMA 4.4. *If $u \in H^2(B_R)$ and $\Delta^2 u \geq 0$ in B_R in the weak sense, that is*

$$\int_{B_R} \Delta u \Delta \varphi \geq 0 \quad \forall \varphi \in C_0^\infty(B_R), \varphi \geq 0$$

and $u|_{\partial B_R} = 0$, $\frac{\partial u}{\partial n}|_{\partial B_R} \leq 0$ then $u \geq 0$ in B_R.

The next lemma is a consequence of a decomposition lemma of Moreau [95]. For a proof see [67] or [68].

LEMMA 4.5. *Let $u \in H_0^2(B_R)$. Then there exist unique $w, v \in H_0^2(B_R)$ such that $u = w + v$, $w \geq 0$, $\Delta^2 v \leq 0$ in B_R and $\int_{B_R} \Delta w \Delta v = 0$.*

PROOF OF LEMMA 1.29. (a) Let $u = u_1 - u_2$. By Lemma 4.5 there exist $w, v \in H_0^2(B_R)$ such that $u = w + v$, $w \geq 0$ and $\Delta^2 v \leq 0$. Observe that $v \leq 0$ so $w \geq u_1 - u_2$. By hypothesis we have

$$\int_{B_R} \Delta(u_1 - u_2)\Delta\varphi \leq \lambda \int_{B_R} (e^{u_1} - e^{u_2})\varphi \quad \forall \varphi \in C_0^\infty(B_R), \varphi \geq 0,$$

and by density this holds also for w:

$$\int_{B_R} (\Delta w)^2 = \int_{B_R} \Delta(u_1 - u_2)\Delta w \leq \lambda \int_{B_R} (e^{u_1} - e^{u_2})w, \tag{4.1}$$

where the equality holds because $\int_{B_R} \Delta w \Delta v = 0$. By density we deduce from (1.39)

$$\lambda \int_{B_R} e^{u_1} w^2 \leq \int_{B_R} (\Delta w)^2. \tag{4.2}$$

Combining (4.1) and (4.2) we obtain

$$\int_{B_R} e^{u_1} w^2 \le \int_{B_R} (e^{u_1} - e^{u_2}) w.$$

Since $u_1 - u_2 \le w$ the previous inequality implies

$$0 \le \int_{B_R} (e^{u_1} - e^{u_2} - e^{u_1}(u_1 - u_2)) w. \tag{4.3}$$

But by convexity of the exponential function $e^{u_1} - e^{u_2} - e^{u_1}(u_1 - u_2) \le 0$ and we deduce from (4.3) that $(e^{u_1} - e^{u_2} - e^{u_1}(u_1 - u_2))w = 0$. Recalling that $u_1 - u_2 \le w$ we deduce that $u_1 \le u_2$.

(b) We solve for $\tilde{u} \in H_0^2(B_R)$ such that

$$\int_{B_R} \Delta\tilde{u}\Delta\varphi = \int_{B_R} \Delta(u_1 - u_2)\Delta\varphi \quad \forall\varphi \in C_0^\infty(B_R).$$

By Lemma 4.3 it follows that $\tilde{u} \ge u_1 - u_2$. Next we apply the decomposition of Lemma 4.5 to $\tilde{u}$, that is $\tilde{u} = w + v$ with $w, v \in H_0^2(B_R)$, $w \ge 0$, $\Delta^2 v \le 0$ in B_R and $\int_{B_R} \Delta w \Delta v = 0$. Then the argument follows that of Lemma 1.29. □

4.2. *Uniqueness of the extremal solution*

PROOF OF THEOREM 1.24. Suppose that $v \in H^2(B)$ satisfies (1.52), (1.53) and $v \not\equiv u^*$. Notice that we do not need v to be radial.

The idea of the proof is as follows:

Step 1. The function

$$u_0 = \frac{1}{2}(u^* + v)$$

is a supersolution to the following problem

$$\begin{cases} \Delta^2 u = \lambda^* e^u + \mu\eta e^u & \text{in } B \\ u = a & \text{on } \partial B \\ \dfrac{\partial u}{\partial n} = b & \text{on } \partial B \end{cases} \tag{4.4}$$

for some $\mu = \mu_0 > 0$, where $\eta \in C_0^\infty(B)$, $0 \le \eta \le 1$ is a fixed radial cut-off function such that

$$\eta(x) = 1 \quad \text{for } |x| \le \tfrac{1}{2}, \qquad \eta(x) = 0 \quad \text{for } |x| \ge \tfrac{3}{4}.$$

Step 2. Using a solution to (4.4) we construct, for some $\lambda > \lambda^*$, a supersolution to (1.50). This provides a solution u_λ for some $\lambda > \lambda^*$, which is a contradiction.

Proof of Step 1. Observe that given $0 < R < 1$ we must have for some $c_0 = c_0(R) > 0$

$$v(x) \geq u^*(x) + c_0 \quad |x| \leq R. \tag{4.5}$$

To prove this we recall the Green's function for Δ^2 with Dirichlet boundary conditions

$$\begin{cases} \Delta_x^2 G(x, y) = \delta_y & x \in B \\ G(x, y) = 0 & x \in \partial B \\ \dfrac{\partial G}{\partial n}(x, y) = 0 & x \in \partial B, \end{cases}$$

where δ_y is the Dirac mass at $y \in B$. Boggio gave an explicit formula for $G(x, y)$ which was used in [71] to prove that in dimension $N \geq 5$ (the case $1 \leq N \leq 4$ can be treated similarly)

$$G(x, y) \sim |x - y|^{4-N} \min\left(1, \frac{d(x)^2 d(y)^2}{|x - y|^4}\right), \tag{4.6}$$

where

$$d(x) = dist(x, \partial B) = 1 - |x|,$$

and $a \sim b$ means that for some constant $C > 0$ we have $C^{-1}a \leq b \leq Ca$ (uniformly for $x, y \in B$). Formula (4.6) yields

$$G(x, y) \geq c d(x)^2 d(y)^2 \tag{4.7}$$

for some $c > 0$ and this in turn implies that for smooth functions $\tilde{v}$ and $\tilde{u}$ such that $\tilde{v} - \tilde{u} \in H_0^2(B)$ and $\Delta^2(\tilde{v} - \tilde{u}) \geq 0$,

$$\begin{aligned} \tilde{v}(y) - \tilde{u}(y) &= \int_{\partial B} \left(\frac{\partial \Delta_x G}{\partial n_x}(x, y)(\tilde{v} - \tilde{u}) - \Delta_x G(x, y) \frac{\partial(\tilde{v} - \tilde{u})}{\partial n} \right) dx \\ &\quad + \int_B G(x, y) \Delta^2(\tilde{v} - \tilde{u})\, dx \\ &\geq c d(y)^2 \int_B (\Delta^2 \tilde{v} - \Delta^2 \tilde{u}) d(x)^2\, dx. \end{aligned}$$

Using a standard approximation procedure, we conclude that

$$v(y) - u^*(y) \geq c d(y)^2 \lambda^* \int_B (e^v - e^{u^*}) d(x)^2\, dx.$$

Since $v \geq u^*$, $v \not\equiv u^*$ we deduce (4.5).

Let $u_0 = (u^* + v)/2$. Then by Taylor's theorem

$$e^v = e^{u_0} + (v - u_0)e^{u_0} + \frac{1}{2}(v - u_0)^2 e^{u_0} + \frac{1}{6}(v - u_0)^3 e^{u_0} + \frac{1}{24}(v - u_0)^4 e^{\xi_2} \tag{4.8}$$

for some $u_0 \le \xi_2 \le v$ and

$$e^{u^*} = e^{u_0} + (u^* - u_0)e^{u_0} + \frac{1}{2}(u^* - u_0)^2 e^{u_0} + \frac{1}{6}(u^* - u_0)^3 e^{u_0} + \frac{1}{24}(u^* - u_0)^4 e^{\xi_1} \tag{4.9}$$

for some $u^* \le \xi_1 \le u_0$. Adding (4.8) and (4.9) yields

$$\frac{1}{2}(e^v + e^{u^*}) \ge e^{u_0} + \frac{1}{8}(v - u^*)^2 e^{u_0}. \tag{4.10}$$

From (4.5) with $R = 3/4$ and (4.10) we see that $u_0 = (u^* + v)/2$ is a supersolution of (4.4) with $\mu_0 := c_0/8$.

Proof of Step 2. Let us now show how to obtain a weak supersolution of (1.50) for some $\lambda > \lambda^*$. Given $\mu > 0$, let u denote the minimal solution to (4.4). Define φ_1 as the solution to

$$\begin{cases} \Delta^2 \varphi_1 = \mu \eta e^u & \text{in } B \\ \varphi_1 = 0 & \text{on } \partial B \\ \dfrac{\partial \varphi_1}{\partial n} = 0 & \text{on } \partial B, \end{cases}$$

and φ_2 be the solution of

$$\begin{cases} \Delta^2 \varphi_2 = 0 & \text{in } B \\ \varphi_2 = a & \text{on } \partial B \\ \dfrac{\partial \varphi_2}{\partial n} = b & \text{on } \partial B. \end{cases}$$

If $N \ge 5$ (the case $1 \le N \le 4$ can be treated similarly), relation (4.7) yields

$$\varphi_1(x) \ge c_1 d(x)^2 \quad \text{for all } x \in B, \tag{4.11}$$

for some $c_1 > 0$. But u is a radial solution of (4.4) and therefore it is smooth in $B \setminus B_{1/4}$. Thus

$$u(x) \le M\varphi_1 + \varphi_2 \quad \text{for all } x \in B_{1/2}, \tag{4.12}$$

for some $M > 0$. Therefore, from (4.11) and (4.12), for $\lambda > \lambda^*$ with $\lambda - \lambda^*$ sufficiently small we have

$$\left(\frac{\lambda}{\lambda^*} - 1\right) u \le \varphi_1 + \left(\frac{\lambda}{\lambda^*} - 1\right) \varphi_2 \quad \text{in } B.$$

Let $w = \frac{\lambda}{\lambda^*} u - \varphi_1 - (\frac{\lambda}{\lambda^*} - 1)\varphi_2$. The inequality just stated guarantees that $w \le u$. Moreover

$$\Delta^2 w = \lambda e^u + \frac{\lambda \mu}{\lambda^*} \eta e^u - \mu \eta e^u \ge \lambda e^u \ge \lambda e^w \quad \text{in } B$$

and

$$w = a \quad \frac{\partial w}{\partial n} = b \quad \text{on } \partial B.$$

Therefore w is a supersolution to (1.50) for λ. By the method of sub and supersolutions a solution to (1.50) exists for some $\lambda > \lambda^*$, which is a contradiction. $\square$

PROOF OF COROLLARY 1.26. Let u denote the extremal solution of (1.50) with $b \geq -4$. We may also assume that $a = 0$. If u is smooth, then the result is trivial. So we restrict to the case where u is singular. By Theorem 1.25 we have in particular that $N \geq 13$. If $b = -4$ by Theorem 1.24 we know that if $N \geq 13$ then $u = -4 \log |x|$ so that the desired conclusion holds. Hence we assume $b > -4$ in this section.

For $\rho > 0$ define

$$u_\rho(r) = u(\rho r) + 4 \log \rho,$$

so that

$$\Delta^2 u_\rho = \lambda^* e^{u_\rho} \quad \text{in } B_{1/\rho}.$$

Then

$$\left.\frac{du_\rho}{d\rho}\right|_{\rho=1, r=1} = u'(1) + 4 > 0.$$

Hence, there is $\delta > 0$ such that

$$u_\rho(r) < u(r) \quad \text{for all } 1 - \delta < r \leq 1,\ 1 - \delta < \rho \leq 1.$$

This implies

$$u_\rho(r) < u(r) \quad \text{for all } 0 < r \leq 1,\ 1 - \delta < \rho \leq 1. \tag{4.13}$$

Otherwise set

$$r_0 = \sup\{0 < r < 1 | u_\rho(r) \geq u(r)\}.$$

This definition yields

$$u_\rho(r_0) = u(r_0) \quad \text{and} \quad u'_\rho(r_0) \leq u'(r_0). \tag{4.14}$$

Write $\alpha = u(r_0)$, $\beta = u'(r_0)$. Then u satisfies

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{on } B_{r_0} \\ u(r_0) = \alpha & \\ u'(r_0) = \beta & \end{cases}$$

while u_ρ is a supersolution to this problem, since $u'_\rho(r_0) \leq \beta$ by (4.14). But this problem does not have a strict supersolution, and we conclude that

$$u(r) = u_\rho(r) \quad \text{for all } 0 < r \leq r_0,$$

which in turn implies by standard ODE theory that

$$u(r) = u_\rho(r) \quad \text{for all } 0 < r \leq 1,$$

a contradiction. This proves estimate (4.13).

From (4.13) we see that

$$\left.\frac{du_\rho}{d\rho}\right|_{\rho=1}(r) \geq 0 \quad \text{for all } 0 < r \leq 1. \tag{4.15}$$

But

$$\left.\frac{du_\rho}{d\rho}\right|_{\rho=1}(r) = u'(r)r + 4 \quad \text{for all } 0 < r \leq 1$$

and this together with (4.15) implies

$$\frac{du_\rho}{d\rho}(r) = \frac{1}{\rho}(u'(\rho r)\rho r + 4) \geq 0 \quad \text{for all } 0 < r \leq \frac{1}{\rho}, 0 < \rho \leq 1, \tag{4.16}$$

which means that $u_\rho(r)$ is nondecreasing in ρ. We wish to show that $\lim_{\rho\to 0} u_\rho(r)$ exists for all $0 < r \leq 1$. For this we shall show

$$u_\rho(r) \geq -4\log(r) + \log\left(\frac{8(N-2)(N-4)}{\lambda^*}\right) \quad \text{for all } 0 < r \leq \frac{1}{\rho}, 0 < \rho \leq 1. \tag{4.17}$$

Set

$$u_0(r) = -4\log(r) + \log\left(\frac{8(N-2)(N-4)}{\lambda^*}\right)$$

and suppose that (4.17) is not true for some $0 < \rho < 1$. Let

$$r_1 = \sup\{0 < r < 1/\rho | u_\rho(r) < u_0(r)\}.$$

Observe that $\lambda^* > 8(N-2)(N-4)$. Otherwise $w = -4\log r$ would be a strict supersolution of the equation satisfied by u, which is not possible by Theorem 1.24. In particular, $r_1 < 1/\rho$ and

$$u_\rho(r_1) = u_0(r_1) \quad \text{and} \quad u'_\rho(r_1) \geq u'_0(r_1).$$

It follows that u_0 is a supersolution of

$$\begin{cases} \Delta^2 u = \lambda^* e^u & \text{in } B_{r_1} \\ u = A & \text{on } \partial B_{r_1} \\ \dfrac{\partial u}{\partial n} = B & \text{on } \partial B_{r_1}, \end{cases} \tag{4.18}$$

with $A = u_\rho(r_1)$ and $B = u'_\rho(r_1)$. Since u_ρ is a singular stable solution of (4.18), it is the extremal solution of the problem by Proposition 1.28. By Theorem 1.24, there is no strict supersolution of (4.18) and we conclude that $u_\rho \equiv u_0$ first for $0 < r < r_1$ and then for $0 < r \leq 1/\rho$. This is impossible for $\rho > 0$ because $u_\rho(1/\rho) = 0$ and $u_0(1/\rho) < 0$. This proves (4.17).

By (4.16) and (4.17) we see that

$$v(r) = \lim_{\rho \to 0} u_\rho(r) \quad \text{exists for all } 0 < r < +\infty,$$

where the convergence is uniform (even in C^k for any k) on compact sets of $\mathbb{R}^N \setminus \{0\}$. Moreover v satisfies

$$\Delta^2 v = \lambda^* e^v \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{4.19}$$

Then for any $r > 0$

$$v(r) = \lim_{\rho \to 0} u_\rho(r) = \lim_{\rho \to 0} u(\rho r) + 4\log(\rho r) - 4\log(r) = v(1) - 4\log(r).$$

Hence, using equation (4.19) we obtain

$$v(r) = -4\log r + \log\left(\frac{8(N-2)(N-4)}{\lambda^*}\right) = u_0(r).$$

But then

$$u'_\rho(r) = u'(\rho r)\rho \to -4, \quad \text{as } \rho \to 0,$$

and therefore, with $r = 1$

$$\rho u'(\rho) \to -4 \quad \text{as } \rho \to 0. \quad \square \tag{4.20}$$

PROOF OF PROPOSITION 1.28. Let $u \in H^2(B)$, $\lambda > 0$ be a weak unbounded solution of (1.50). If $\lambda < \lambda^*$ from Lemma 1.29 we find that $u \le u_\lambda$, where u_λ is the minimal solution. This is impossible because u_λ is smooth and u unbounded. If $\lambda = \lambda^*$ then necessarily $u = u^*$ by Theorem 1.24. $\square$

4.3. *A computer-assisted proof for dimensions* $13 \le N \le 31$

Throughout this section we assume $a = b = 0$. As was mentioned before, the proof of Theorem 1.27 relies on precise estimates for u^* and λ^*. We first present some conditions under which it is possible to find these estimates. Later we show how to meet such conditions with a computer-assisted verification.

The first lemma is analogous to Lemma 1.30.

LEMMA 4.6. *Suppose there exist* $\varepsilon > 0$, $\lambda > 0$ *and a radial function* $u \in H^2(B) \cap W^{4,\infty}_{\mathrm{loc}}(B \setminus \{0\})$ *such that*

$$\Delta^2 u \le \lambda e^u \quad \text{for all } 0 < r < 1$$

$$|u(1)| \le \varepsilon, \quad \left|\frac{\partial u}{\partial n}(1)\right| \le \varepsilon$$

$$u \notin L^\infty(B)$$

$$\lambda e^\varepsilon \int_B e^u \varphi^2 \le \int_B (\Delta\varphi)^2 \quad \textit{for all } \varphi \in C_0^\infty(B). \tag{4.21}$$

Then

$$\lambda^* \le \lambda e^{2\varepsilon}.$$

PROOF. Let

$$\psi(r) = \varepsilon r^2 - 2\varepsilon \tag{4.22}$$

so that

$$\Delta^2 \psi \equiv 0, \qquad \psi(1) = -\varepsilon, \qquad \psi'(1) = 2\varepsilon$$

and

$$-2\varepsilon \le \psi(r) \le -\varepsilon \quad \text{for all } 0 \le r \le 1.$$

It follows that

$$\Delta^2(u+\psi) \le \lambda e^u = \lambda e^{-\psi} e^{u+\psi} \le \lambda e^{2\varepsilon} e^{u+\psi}.$$

On the boundary we have $u(1) + \psi(1) \le 0$, $u'(1) + \psi'(1) \ge 0$. Thus $u + \psi$ is a singular subsolution to the equation with parameter $\lambda e^{2\varepsilon}$. Moreover, since $\psi \le -\varepsilon$ we have $\lambda e^{2\varepsilon} e^{u+\psi} \le \lambda e^\varepsilon e^u$ and hence, from (4.21) we see that $u+\psi$ is stable for the problem with parameter $\lambda e^{2\varepsilon}$. If $\lambda e^{2\varepsilon} < \lambda^*$ then the minimal solution associated with the parameter $\lambda e^{2\varepsilon}$ would be above $u + \psi$, which is impossible because u is singular. □

LEMMA 4.7. *Suppose we can find $\varepsilon > 0$, $\lambda > 0$ and $u \in H^2(B) \cap W^{4,\infty}_{\mathrm{loc}}(B \setminus \{0\})$ such that*

$$\Delta^2 u \ge \lambda e^u \quad \textit{for all } 0 < r < 1$$

$$|u(1)| \le \varepsilon, \qquad \left|\frac{\partial u}{\partial n}(1)\right| \le \varepsilon.$$

Then

$$\lambda e^{-2\varepsilon} \le \lambda^*.$$

PROOF. Let ψ be given by (4.22). Then $u - \psi$ is a supersolution to the problem with parameter $\lambda e^{-2\varepsilon}$. □

The next result is the main tool to guarantee that u^* is singular. The proof, as in (1.61), is based on an upper estimate of u^* by a stable singular subsolution.

LEMMA 4.8. *Suppose there exist $\varepsilon_0, \varepsilon > 0$, $\lambda_a > 0$ and a radial function $u \in H^2(B) \cap W^{4,\infty}_{\mathrm{loc}}(B \setminus \{0\})$ such that*

$$\Delta^2 u \le (\lambda_a + \varepsilon_0)e^u \quad \text{for all } 0 < r < 1 \tag{4.23}$$

$$\Delta^2 u \ge (\lambda_a - \varepsilon_0)e^u \quad \text{for all } 0 < r < 1 \tag{4.24}$$

$$|u(1)| \le \varepsilon, \qquad \left|\frac{\partial u}{\partial n}(1)\right| \le \varepsilon \tag{4.25}$$

$$u \notin L^\infty(B) \tag{4.26}$$

$$\beta_0 \int_B e^u \varphi^2 \le \int_B (\Delta\varphi)^2 \quad \text{for all } \varphi \in C_0^\infty(B), \tag{4.27}$$

where

$$\beta_0 = \frac{(\lambda_a + \varepsilon_0)^3}{(\lambda_a - \varepsilon_0)^2} e^{9\varepsilon}. \tag{4.28}$$

Then u^* is singular and

$$(\lambda_a - \varepsilon_0)e^{-2\varepsilon} \le \lambda^* \le (\lambda_a + \varepsilon_0)e^{2\varepsilon}. \tag{4.29}$$

PROOF. By Lemmas 4.6 and 4.7 we have (4.29). Let

$$\delta = \log\left(\frac{\lambda_a + \varepsilon_0}{\lambda_a - \varepsilon_0}\right) + 3\varepsilon,$$

and define

$$\varphi(r) = -\frac{\delta}{4} r^4 + 2\delta.$$

We claim that

$$u^* \le u + \varphi \quad \text{in } B_1. \tag{4.30}$$

To prove this, we shall show that for $\lambda < \lambda^*$

$$u_\lambda \le u + \varphi \quad \text{in } B_1. \tag{4.31}$$

Indeed, we have

$$\Delta^2 \varphi = -\delta 2N(N+2)$$
$$\varphi(r) \ge \delta \quad \text{for all } 0 \le r \le 1$$
$$\varphi(1) \ge \delta \ge \varepsilon, \quad \varphi'(1) = -\delta \le -\varepsilon$$

and therefore

$$\begin{aligned}\Delta^2(u+\varphi) &\le (\lambda_a + \varepsilon_0)e^u + \Delta^2\varphi \le (\lambda_a + \varepsilon_0)e^u = (\lambda_a + \varepsilon_0)e^{-\varphi}e^{u+\varphi} \\ &\le (\lambda_a + \varepsilon_0)e^{-\delta}e^{u+\varphi}.\end{aligned} \tag{4.32}$$

By (4.29) and the choice of δ

$$(\lambda_a + \varepsilon_0)e^{-\delta} = (\lambda_a - \varepsilon_0)e^{-3\varepsilon} < \lambda^*. \tag{4.33}$$

To prove (4.31) it suffices to consider λ in the interval $(\lambda_a - \varepsilon_0)e^{-3\varepsilon} < \lambda < \lambda^*$. Fix such λ and assume that (4.31) is not true. Write

$$\bar{u} = u + \varphi$$

and let

$$R_1 = \sup\{0 \le R \le 1 \mid u_\lambda(R) = \bar{u}(R)\}.$$

Then $0 < R_1 < 1$ and $u_\lambda(R_1) = \bar{u}(R_1)$. Since $u'_\lambda(1) = 0$ and $\bar{u}'(1) < 0$ we must have $u'_\lambda(R_1) \le \bar{u}'(R_1)$. Then u_λ is a solution to the problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B_{R_1} \\ u = u_\lambda(R_1) & \text{on } \partial B_{R_1} \\ \dfrac{\partial u}{\partial n} = u'_\lambda(R_1) & \text{on } \partial B_{R_1} \end{cases}$$

while, thanks to (4.32) and (4.33), $\bar{u}$ is a subsolution to the same problem. Moreover $\bar{u}$ is stable thanks to (4.27) since, by Lemma 4.6,

$$\lambda < \lambda^* \le (\lambda_a + \varepsilon_0)e^{2\varepsilon} \tag{4.34}$$

and hence

$$\lambda e^{\bar{u}} \le (\lambda_a + \varepsilon_0)e^{2\varepsilon}e^{2\delta}e^u \le \beta_0 e^u.$$

We deduce $\bar{u} \le u_\lambda$ in B_{R_1} which is impossible, since $\bar{u}$ is singular while u_λ is smooth. This establishes (4.30).

From (4.30) and (4.34) we have

$$\lambda^* e^{u^*} \le \beta_0 e^{-\varepsilon} e^u$$

and therefore

$$\inf_{\varphi \in C_0^\infty(B)} \frac{\int_B (\Delta\varphi)^2 - \lambda^* e^{u^*}\varphi^2}{\int_B \varphi^2} > 0.$$

This is not possible if u^* is a smooth solution. □

For each dimension $13 \le N \le 31$ we construct u satisfying (4.23) to (4.27) of the form

$$u(r) = \begin{cases} -4\log r + \log\left(\frac{8(N-2)(N-4)}{\lambda}\right) & \text{for } 0 < r < r_0 \\ \tilde{u}(r) & \text{for } r_0 \le r \le 1, \end{cases} \tag{4.35}$$

where $\tilde{u}$ is explicitly given. Thus u satisfies (4.26) automatically.

Numerically it is better to work with the change of variables

$$w(s) = u(e^s) + 4s, \quad -\infty < s < 0$$

which transforms the equation $\Delta^2 u = \lambda e^u$ into

$$Lw + 8(N-2)(N-4) = \lambda e^w, \quad -\infty < s < 0,$$

where

$$\begin{aligned} Lw = {} & \frac{d^4 w}{ds^4} + 2(N-4)\frac{d^3 w}{ds^3} + (N^2 - 10N + 20)\frac{d^2 w}{ds^2} \\ & - 2(N-2)(N-4)\frac{dw}{ds}. \end{aligned}$$

The boundary conditions $u(1) = 0$, $u'(1) = 0$ then yield

$$w(0) = 0, \quad w'(0) = 4.$$

Regarding the behavior of w as $s \to -\infty$ observe that

$$u(r) = -4 \log r + \log\left(\frac{8(N-2)(N-4)}{\lambda}\right) \quad \text{for } r < r_0$$

if and only if

$$w(s) = \log \frac{8(N-2)(N-4)}{\lambda} \quad \text{for all } s < \log r_0.$$

The steps we perform are the following:

(1) We fix $x_0 < 0$ and using numerical software we follow a branch of solutions to

$$\begin{cases} L\hat{w} + 8(N-2)(N-4) = \lambda e^{\hat{w}}, & x_0 < s < 0 \\ \hat{w}(0) = 0, \quad \hat{w}'(0) = t \\ \hat{w}(x_0) = \log \dfrac{8(N-2)(N-4)}{\lambda}, & \dfrac{d^2\hat{w}}{ds^2}(x_0) = 0, \quad \dfrac{d^3\hat{w}}{ds^3}(x_0) = 0 \end{cases}$$

as t increases from 0 to 4. The numerical solution $(\hat{w}, \hat{\lambda})$ we are interested in corresponds to the case $t = 4$. The five boundary conditions are due to the fact that we are solving a fourth-order equation with an unknown parameter λ.

(2) Based on $\hat{w}$, $\hat{\lambda}$ we construct a C^3 function w which is constant for $s \le x_0$ and piecewise polynomial for $x_0 \le s \le 0$. More precisely, we first divide the interval $[x_0, 0]$ into smaller intervals of length h. Then we generate a cubic spline approximation g_{fl} with floating point coefficients of $\frac{d^4\hat{w}}{ds^4}$. From g_{fl} we generate a piecewise cubic polynomial g_{ra} which uses rational coefficients and we integrate it four times to obtain w, where the constants of integration are such that $\frac{d^j w}{ds^j}(x_0) = 0$, $1 \le j \le 3$ and $w(x_0)$ is a rational approximation of $\log(8(N-2)(N-4)/\lambda)$. Thus w is a piecewise

polynomial function that in each interval is of degree 7 with rational coefficients, and which is globally C^3. We also let λ be a rational approximation of $\hat{\lambda}$. With these choices note that $Lw+8(N-2)(N-4)-\lambda e^w$ is a small constant (not necessarily zero) for $s \le x_0$.

(3) The conditions (4.23) and (4.24) we need to check for u are equivalent to the following inequalities for w

$$Lw + 8(N-2)(N-4) - (\lambda + \varepsilon_0)e^w \le 0, \quad -\infty < s < 0 \tag{4.36}$$

$$Lw + 8(N-2)(N-4) - (\lambda - \varepsilon_0)e^w \ge 0, \quad -\infty < s < 0. \tag{4.37}$$

Using a program in Maple we verify that w satisfies (4.36) and (4.37). This is done evaluating a second-order Taylor approximation of $Lw + 8(N-2)(N-4) - (\lambda + \varepsilon_0)e^w$ at sufficiently close mesh points. All arithmetic computations are done with rational numbers, thus obtaining exact results. The exponential function is approximated by a Taylor polynomial of degree 14 and the difference with the real value is controlled.

(4) We show that the operator $\Delta^2 - \beta e^u$, where $u(r) = w(\log r) - 4\log r$, satisfies condition (4.27) for some $\beta \ge \beta_0$, where β_0 is given by (4.28).

We refer the interested reader to [44], but we shall justify here that, although βe^u is singular, the operator $\Delta^2 - \beta e^u$ has indeed a positive eigenfunction in $H_0^2(B)$ with finite eigenvalue if β is not too large, if $N \ge 13$. The reason is that near the origin

$$\beta e^u = \frac{c}{|x|^4},$$

where c is a number close to $8(N-2)(N-4)\beta/\lambda$. If β is not too large compared to λ then $c < N^2(N-4)^2/16$ and hence, using (1.57), $\Delta^2 - \beta e^u$ is coercive in $H_0^2(B_{r_0})$.

The full information on the Maple files and data used can be found at:
http://www.lamfa.u-picardie.fr/dupaigne/
http://www.ime.unicamp.br/~msm/bilaplace-computations/bilaplace-computations.html

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