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Équations aux dérivées partielles/*Partial Differential Equations*

Positive solutions of elliptic equations in $\mathbb{R}^N$ with a super-subcritical nonlinearity

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Abstract. We consider the problem:

$$\begin{cases} \Delta u + u^p + u^q = 0 & \text{in } \mathbb{R}^N, \\ 0 < u(x) \to 0 & \text{as } |x| \to +\infty, \end{cases}$$

where $1 < p < \frac{N+2}{N-2} < q$. We prove that if q is fixed and we let p approach $\frac{N+2}{N-2}$ from below, then this problem has a large number of radial solutions. A similar fact takes place if one fixes $p > \frac{N}{N-2}$ and then lets q approach $\frac{N+2}{N-2}$. If q is fixed and p gets close enough to $\frac{N}{N-2}$, then no solution exists. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Solutions positives d'équations elliptiques dans $\mathbb{R}^N$ avec non-linéarité super-sous-critique

Résumé. *On considère le problème de trouver des solutions de l'équation elliptique* :

$$\begin{cases} \Delta u + u^p + u^q = 0 & \textit{dans } \mathbb{R}^N, \\ 0 < u(x) \to 0 & \textit{lorsque } |x| \to +\infty, \end{cases}$$

où $1 < p < \frac{N+2}{N-2} < q$. Si l'on fixe q et p croît en tendant vers $\frac{N+2}{N-2}$, alors il y a un grand nombre des solutions radiales. On peut obtenir un résultat analogue si l'on fixe $p > \frac{N}{N-2}$ et q approche $\frac{N+2}{N-2}$. De plus, si on fixe q et on prend p assez proche de $\frac{N}{N-2}$, alors il n'existe pas de solution. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

On considère le problème de trouver des solutions de l'equation elliptique semi-linéaire :

Note présentée par Haïm BRÉZIS.

$$\begin{cases} \Delta u + u^p + u^q = 0 & \text{dans } \mathbb{R}^N, \\ 0 < u(x) \to 0 & \text{lorsque } |x| \to +\infty, \end{cases} \tag{1}$$

$$1 < p < \frac{N+2}{N-2} < q. \tag{2}$$

Dans le cas $p = q$ on connaît déjà la relation entre l'exposant $\frac{N+2}{N-2}$ et l'existence des solutions. Si $p < \frac{N+2}{N-2}$ nous n'avons pas des solutions, tandis que si $p \geqslant \frac{N+2}{N-2}$ il existe des solutions radiales. Nous sommes intéresés à l'existence de solutions dans le cas où les puissances de p et q sont sous- et sur-critiques. Une solution explicite a été donnée par Lin et Ni dans le cas où (2) est vérifié et $q = 2p - 1$, mais le cas général reste encore ouvert.

On a le résultat suivant :

THÈORÉME 1. –
(a) *Soit* $q > \frac{N+2}{N-2}$ *et* $k \geqslant 1$. *Alors il existe un nombre* p_k *avec* $p_k < \frac{N+2}{N-2}$ *tel que si* $p_k < p < \frac{N+2}{N-2}$, *alors* (1) *a au moins* k *solutions.*
(b) *Soit* $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ *et* $k \geqslant 1$. *Alors il existe un nombre* q_k *avec* $p_k > \frac{N+2}{N-2}$ *tel que si* $q_k > q > \frac{N+2}{N-2}$, *alors* (1) *a au moins* k *solutions.*
(c) *Soit* $q > \frac{N+2}{N-2}$. *Alors il existe un nombre* $\bar{p}$ *avec* $\frac{N}{N-2} < \bar{p} < \frac{N+2}{N-2}$ *tel que si* $1 < p < \bar{p}$, *alors il n'existe pas de solution de* (1)–(2).

Pour la démonstration de ces résultats on utilise un changement de coordonnées qui ramène l'equation ordinaire des solutions radiales à un système autonome du premier ordre. Les solutions dans ces coordonnées correspondent aux orbites hétérocliniques entre deux points singuliers du flux associé. Pour la démonstration, on fait une analyse exhaustive de la géométrie du flux, en faisant une perturbation quand on prend un exposant critique pur p et q est fixé supercritique.

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1. Introduction

In this work we consider the problem of finding positive solutions of the following semilinear elliptic equation in $\mathbb{R}^N$:

$$\begin{cases} \Delta u + u^p + u^q = 0 & \text{in } \mathbb{R}^N, \\ 0 < u(x) \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{1.1}$$

Here Δ denotes the Laplacian operator in $\mathbb{R}^N$, $N \geqslant 3$. We also assume that the powers p and q are respectively sub- and supercritical, namely

$$1 < p < \frac{N+2}{N-2} < q. \tag{1.2}$$

It is natural to search for radially symmetric solutions $u = u(|x|)$ for (1.1), so that $u(r),\ r = |x|$, satisfies the ordinary differential equation:

$$\begin{cases} u'' + \frac{N-1}{r} u' + u^p + u^q = 0,\ r > 0, \\ u'(0) = 0, \quad 0 < u(r) \to 0 \quad \text{as } r \to +\infty. \end{cases} \tag{1.3}$$

A solution of (1.3) will be called a *radial ground state* of (1.1). In the case of a nonlinearity constituted by a pure power, namely $p = q$, the role of the critical exponent in the problem of existence of positive ground states is well understood. If $p < \frac{N+2}{N-2}$, no positive solutions exist [3], while if $p = \frac{N+2}{N-2}$ all positive solutions are necessarily radial around some point. At this exponent, as well as for $p > \frac{N+2}{N-2}$, radial ground states are constituted by a one-parameter family of functions. More precisely, for every $\alpha > 0$, the solution $u(r)$ of the initial value problem (1.3) with $p = q$, $u'(0) = 0,\ u(0) = \alpha > 0$ is a family of ground states.

We also notice that in case (1.2), it follows from a result in [2] that all solutions of (1.1) which decay at a sufficiently fast rate are necessarily radial around some point; this is however not known for *all* solutions.

A question raised by W.-M. Ni is the following: are there radial ground states of (1.1) under the restriction (1.2)? Given the completely different pictures exhibited by purely subcritical and purely supercritical powers, an answer is not obvious. An interesting example was discovered by Lin and Ni in [4]. If p and q satisfy (1.2) and additionally $q = 2p - 1$, then there is an explicit solution of the form $u(r) = A(B + r^2)^{-1/(p-1)}$, where A and B are positive constants depending on p and N. The question of existence of ground states in the general range (1.2) has remained however widely open.

THEOREM 1. –
(a) *Let* $q > \frac{N+2}{N-2}$ *be fixed. Then, given* $k \geqslant 1$ *there exists a number* $p_k < \frac{N+2}{N-2}$ *such that if* $p_k < p < \frac{N+2}{N-2}$, *then* (1.1) *has at least* k *ground states.*
(b) *Let* $q > \frac{N+2}{N-2}$ *be fixed. Then there is a number* $\bar{p} > \frac{N}{N-2}$ *such that if* $1 < p < \bar{p}$ *then there is no radial ground state of* (1.1).
(c) *Let* $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ *be fixed. Then there exists a number* $q_k > \frac{N+2}{N-2}$ *such that if* $\frac{N+2}{N-2} < q < q_k$, *then* (1.1) *has at least* k *ground states.*

We observe that the nonexistence result (b) is optimal, in the sense that for $q = 2p - 1$ there are ground states, and $\frac{N+2}{N-2} = 2\frac{N}{N-2} - 1$. We should also mention that there are (explicit) numbers $p^*(N),\ q^*(N)$ such that given $q < q^*(N)$, respectively $p > p^*(N)$, the sequences p_k and q_k above can be chosen so that for $p = p_k$, respectively $q = q_k$, there are infinitely many ground states. We will not elaborate about this fact in this Note.

2. Sketch of proofs

The proof of Theorem 1.1 is based on the introduction of the classical Emden–Fowler transformation

$$x(t) = r^{\frac{2}{q-1}} u(r)|_{r=\mathrm{e}^t} \tag{2.1}$$

which transforms equation (1.3) into the equivalent problem:

$$\begin{cases} x'' + \alpha x' + x^q + \mathrm{e}^{\gamma t} x^p - \beta x = 0, & -\infty < t < +\infty, \\ x(t) > 0 \text{ for all } t, \quad x(t) \to 0 \text{ as } t \to \pm\infty, \end{cases} \tag{2.2}$$

where

$$\alpha = N - 2 - \frac{4}{q-1}, \quad \beta = \frac{2}{q-1}\left(N - 2 - \frac{2}{q-1}\right), \quad \gamma = 2\frac{q-p}{q-1}.$$

By definiteness, we mean here $x^a = x_+^a$ where $x_+ = \max\{x, 0\}$. Introducing the variables $y = x'$ and $z = \mathrm{e}^{\gamma t}$, the problem becomes equivalent to the autonomous first order system:

$$x' = y, \quad y' = -\alpha y + \beta x - x^q - z x^p, \quad z' = \gamma z, \quad z \geqslant 0. \tag{2.3}$$

Our task is therefore equivalent to finding a solution $\mathbf{x}(t) = (x(t), y(t), z(t))$ of this system, with $x(t) > 0$, $z(t) > 0$, such that $\mathbf{x}(t) \to (0,0,0)$ as $t \to -\infty$, and $\mathbf{x}(t) \to (0,0,\infty)$ as $t \to +\infty$.

We observe that the plane $z = 0$ is invariant under the flow associated to this system. This plane contains the two singularities of the flow $O_0 = (0,0,0)$ and $P_0 = \left(\beta^{1/(p-1)}, 0, 0\right)$. For the flow restricted to this plane, O_0 is a hyperbolic saddle; P_0 is an attractor, either a focus or a node, and they are connected by a heteroclinic orbit, precisely a branch of the unstable manifold of O_0 restricted to $z=0$. This manifold is transversal to $x=0$. This phase plane analysis (corresponding to the case of a single power) is actually well known, and observed first by Fowler [1].

Now, for the entire flow, an expanding vertical direction is added, in such a way that from standard invariant manifold theory, O_0 has a two-dimensional unstable manifold, $W^u(O_0)$ transversal to the $x=0$ and $z=0$ planes. Its closure turns out to contain the (one-dimensional) unstable manifold of P_0, $W^u(P_0)$.

In order to analyze the behavior of trajectories near $z=+\infty$ it is convenient to introduce the additional transformation,

$$\tilde{x}=xz^{\frac{1}{p-1}}, \quad \tilde{y}=(y+\frac{\gamma}{p-1})z^{\frac{1}{p-1}}, \quad \tilde{z}=\frac{1}{z^{\frac{q-1}{p-1}}}.$$

which makes the system equivalent to

$$\tilde{x}'=\tilde{y}, \quad \tilde{y}'=\tilde{\alpha}\tilde{y}+\tilde{\beta}\tilde{x}-\tilde{x}^p-\tilde{z}\tilde{x}^q, \quad \tilde{z}'=-\tilde{\gamma}\tilde{z}, \quad z\geqslant 0.$$

with

$$\tilde{\alpha}=\frac{4}{p-1}-(N-2), \quad \tilde{\beta}=\frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right), \quad \tilde{\gamma}=2\,\frac{q-p}{p-1}.$$

This transformation corresponds to using the exponent p instead of q in the Emden–Fowler transformation (2.1), which is expected to reflect better the behavior of a ground state at infinity. In fact, the effect of this transformation is to "blow-up" the "singularity" $(0,0,\infty)$ into the plane $\tilde{z}=0$. The system (2.3) extends up to $\tilde{z}=0$, which remains invariant under the associated flow, with singularities at the points $O_\infty=(0,0,0)$ and $P_\infty=\big(\tilde{\beta}^{1/(p-1)},0,0\big)$. Restricted to $\tilde{z}=0$, O_∞ is a hyperbolic saddle and P_∞ is a repulsive node or focus. The stable manifold of O_∞ is transversal to $\tilde{x}=0$ and connects this point with P_∞. For the entire flow, a contracting vertical direction is added, so that O_∞ has a two-dimensional stable manifold $W^s(O_\infty)$, transversal to the planes $\tilde{x}=0$ and $\tilde{z}=0$ and whose closure contains the one-dimensional stable manifold of P_∞, $W^s(P_\infty)$.

A ground state (with fast decay) corresponds precisely to a trajectory (other than the z-axis) which lies simultaneously on $W^u(O_0)$ and on $W^s(O_\infty)$. The proof of parts (a) and (c) of the theorem are thus reduced to establishing that in the situations there described many such trajectories appear.

In order to prove part (a), we fix q and let $p=\frac{N+2}{N-2}$. In such a case, it is a consequence of a Pohozaev type identity, that all trajectories in $W^u(O_\infty)$ which start positive as $t\to-\infty$ remain positive up to $t=+\infty$, and they cannot approach O_∞. The same is true for the "singular solution" whose orbit is $W^u(P_0)$. We observe that the orbit $W^s(P_\infty)$ is such that its $\tilde{x}$ coordinate approaches the constant value $\tilde{\beta}^{1/(p-1)}$. Further, P_∞ restricted to $\tilde{z}=0$ is a center, namely every orbit close to it is periodic. From these facts, it can be shown that the trajectory $W^u(P_0)$ "winds around" $W^s(P_\infty)$ infinitely many times unless they are identically equal (in terms of their $\tilde{x}$ coordinates, they cross each other transversally an infinite number of times). Let us assume they do not coincide. The other case can be treated with arguments slightly different to the ones to follow. Once p is perturbed down from the critical value, one still sees the trajectory in $W^u(P_0)$ winding around $W^s(P_\infty)$ an arbitrarily large number of times. For a given number z_0 let us consider the sections $U(z_0)=W^u(O_0)\cap\{z=z_0\}$ and $S(z_0)=W^s(O_\infty)\cap\{z=z_0\}$. These sections are constituted by curves with a fixed point at the z-axis, and with endpoints precisely in the respective z_0-sections of $W^u(P_0)$ and $W^s(P_\infty)$. Finally, it can be proven that if between heights $z=a$ and $z=b$ the planar vector joining these endpoints has a total winding number equal to k (which corresponds precisely to $W^u(P_0)$ and $W^s(P_\infty)$ winding around each other k times between those heights), then the curves $S(b)$ and $U(b)$ must intersect at least $k-1$ times, thus leading to at least $k-1$ distinct ground states.

The proof of part (c) is analogous. The situation here actually mirrors the one just described. Finally, the proof of (b) is based on the following fact: for $p\leqslant\frac{N}{N-2}$, the singularities at ∞, O_∞ and P_∞ collapse and become a single repelling singularity O_∞ for the flow restricted to $\tilde{z}=0$, thus allowing only the $\tilde{z}$ axis as a trajectory approaching to it. For given q all trajectories in $W^u(O_0)$ which get close to O_∞ must leave the plane $x = 0$ at a certain (uniform) positive distance in the ˜ coordinates from the z-axis. This structure is still preserved if one lets p become slightly bigger than $\frac{N}{N-2}$, thus yielding the result.

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