Calc. Var. 15, 519–527 (2002) **Calculus of Variations**
DOI (Digital Object Identifier) 10.1007/s005260100135

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On an open question about functions of bounded variation

Received: 23 April 2001 / Accepted: 27 September 2001 /
Published online: 29 April 2002 –

Mathematics Subject Classification (2000): 46E35

1 Introduction

In this note we address an open question posed by Bourgain, Brezis and Mironescu [BBM]. Their initial motivation was the study of the behavior of the $W^{s,p}$ norm of a function f, when $1 \leq p < \infty$ is fixed and $s \to 1$, $0 < s < 1$. A often used semi-norm in $W^{s,p}(\Omega)$ for this range of s and p, where $\Omega \subset \mathbf{R}^n$ is a bounded smooth domain is

$$|f|^p_{W^{s,p}(\Omega)} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy.$$

Slightly more general is the problem of finding the limit of expressions of the type

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_i(x-y) \, dx \, dy \ ,$$

where ρ_i is a sequence of radial mollifiers, that is, a sequence of functions such that

$$\rho_i \geq 0, \quad \rho_i(x) = \rho_i(|x|), \quad \int_{\mathbf{R}^n} \rho_i = 1 \tag{1}$$

and

$$\lim_{i \to \infty} \int_\delta^\infty \rho_i(r) r^{n-1} \, dr = 0 \quad \text{for all } \delta > 0. \tag{2}$$

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After the completion of this work, the author learned that L. Ambrosio has independently given a proof of the main result in this paper.

One of the results of Bourgain, Brezis and Mironescu (see [BBM, Theorem 3]) asserts that: if $f \in W^{1,1}(\Omega)$ then

$$\lim_{i\to\infty} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_i(x-y)\,dx\,dy = K_{1,n}\int_\Omega |\nabla f|, \tag{3}$$

where $K_{1,n}$ is a constant depending only on n (see an expression in (4) below). They also showed [BBM, Theorem 3'] that if $f \in L^1(\Omega)$ then $f \in \mathrm{BV}(\Omega)$ if and only if

$$\liminf_{i\to\infty} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_i(x-y)\,dx\,dy < \infty,$$

and in this case

$$\begin{aligned} C_1|f|_{\mathrm{BV}(\Omega)} &\le \liminf_{i\to\infty} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_i(x-y)\,dx\,dy \\ &\le \limsup_{i\to\infty} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_i(x-y)\,dx\,dy \le C_2|f|_{\mathrm{BV}(\Omega)}, \end{aligned}$$

where C_1 and C_2 depend only on Ω. The question raised by Bourgain, Brezis and Mironescu was if for $f \in \mathrm{BV}(\Omega)$ we still have (3). In this note we give a proof of this.

Theorem 1 *Let $\Omega \subset \mathbf{R}^n$ be open, bounded with a Lipschitz boundary, and let $f \in \mathrm{BV}(\Omega)$. Consider a sequence ρ_i satisfying (1) and (2). Then*

$$\lim_{i\to\infty} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_i(x-y)\,dx\,dy = K_{1,n}|f|_{\mathrm{BV}(\Omega)}.$$

For the proof we consider the following measures

$$\mu_i = \left(\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_i(x-y)\,dx\right)\,dy.$$

The main observation is the following (which is interesting in its own).

Lemma 2 *Assume the sequence (ρ_i) satisfies (1) and (2) and let $f \in \mathrm{BV}(\Omega)$. Then*

$$\mu_i \rightharpoonup K_{1,n}|\nabla f|$$

weakly in the sense of Radon measures in Ω. This lemma holds if $\Omega \subset \mathbf{R}^n$ is any open set (not necessarily bounded or with a smooth boundary).

2 Notation

The constant $K_{1,n}$ mentioned in the introduction is defined by

$$K_{1,n} = \int_{S^{n-1}} |e \cdot w| \, ds(w), \tag{4}$$

where $e \in \mathbf{R}^n$ is any unit vector. We use the notation $x \cdot y$ for the usual inner product in $\mathbf{R}^n$ and $|x|$ is the Euclidean norm in $\mathbf{R}^n$. S^{n-1} is the unit sphere $\{ x \in \mathbf{R}^n, \ |x| = 1 \}$, and ds is the usual surface measure on S^{n-1}.

The semi-norm $|f|_{\mathrm{BV}(\Omega)}$ is defined by

$$|f|_{\mathrm{BV}(\Omega)} = \sup \left\{ \int_\Omega f \, \mathrm{div}\varphi \mid \varphi \in C_c^1(\Omega, \mathbf{R}^n), \ |\varphi| \leq 1 \text{ in } \Omega \right\}$$

and $\mathrm{BV}(\Omega)$ is defined as

$$\mathrm{BV}(\Omega) = \{ f \in L^1(\Omega) \mid \quad |f|_{\mathrm{BV}(\Omega)} < \infty \}.$$

(see for instance Evans and Gariepy [EG].)

Recall that if $f \in \mathrm{BV}(\Omega)$, ∇f is a vector valued Radon measure on Ω. We denote its total variation by $|\nabla f|$ and note that if $U \subset \Omega$ is open $|\nabla f|(U)$ can be computed by

$$|\nabla f|(U) = |f|_{\mathrm{BV}(U)} = \sup \left\{ \int_\Omega f \, \mathrm{div}\varphi \mid \varphi \in C_c^1(U, \mathbf{R}^n), \ |\varphi| \leq 1 \text{ in } U \right\}.$$

We recall also that the Radon-Nikodym derivative

$$\sigma = \frac{d\nabla f}{d|\nabla f|}$$

is a vector valued $|\nabla f|$-measurable function and $|\sigma(x)| = 1$ $|\nabla f|$-a.e. With this notation the relation

$$\int_\Omega \mathrm{div}\varphi f \, dx = \int_\Omega \varphi \cdot \, d(\nabla f) = \int_\Omega \varphi \cdot \sigma \, d|\nabla f|$$

holds for all $\varphi \in C_c^1(\Omega, \mathbf{R}^n)$.

Let us mention here a density result (see [EG, Theorem 2, p. 172] for example): if $f \in \mathrm{BV}(\Omega)$ there is a sequence $f_j \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that

$$f_j \to f \qquad \text{in } L^1(\Omega)$$

and for any $U \subset \Omega$ open

$$\int_U |\nabla f_j| \, dx \to |\nabla f|(U). \tag{5}$$

In [EG] a sequence f_j is constructed with the property that $|f_j|_{\mathrm{BV}(\Omega)} \to |f|_{\mathrm{BV}(\Omega)}$, but the same sequence satisfies also (5).

Finally we mention that if Ω is bounded and has a Lipschitz boundary, there is an extension operator $E : \mathrm{BV}(\Omega) \to \mathrm{BV}(\mathbf{R}^n)$, i.e. E is a bounded linear operator that satisfies $Ef = f$ a.e in Ω for all $f \in \mathrm{BV}(\Omega)$, with the additional property: if U_δ is the open set

$$U_\delta = \{\, x \in \mathbf{R}^n \mid \mathrm{dist}(x, \partial\Omega) < \delta \,\}$$

then we can control

$$|\nabla(Ef)|(U_\delta) \le C_1 |\nabla f|(U_{(C_2\delta)} \cap \Omega) \tag{6}$$

where $C_1 > 0$, $C_2 > 0$ depend only on Ω. This can be achieved by a standard reflection across the boundary, so that $|\nabla(Ef)|(\partial\Omega) = 0$ (that is, E doesn't create any jump across the boundary of Ω).

3 Proof of the theorem

The main computation that we need to prove Lemma 2 is the following.

Lemma 3 *Let E be a Borel set and $R > 0$. Let*

$$E_R = E + B_R(0) = \{ x + y \mid x \in E,\ y \in B_R(0) \,\}$$

and suppose that $E_R \subset \Omega$. Then

$$\int_E d\mu_i \le K_{1,n} \int_{E_R} |\nabla f| + \frac{2}{R} \|f\|_{L^1(\Omega)} \int_{\mathbf{R}^n - B_R(0)} \rho_i. \tag{7}$$

Proof. We start with the

Case $f \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$. We split

$$\int_E d\mu_i = I_1 + I_2$$

where

$$I_1 = \int\limits_E \int\limits_{|x-y|<R} \frac{|f(x) - f(y)|}{|x-y|} \rho_i(x-y)\, dx\, dy$$

$$I_2 = \int\limits_E \int\limits_{\substack{x\in\Omega \\ |x-y|\ge R}} \frac{|f(x) - f(y)|}{|x-y|} \rho_i(x-y)\, dx\, dy.$$

For $y \in E$ and $|x - y| < R$ we have

$$f(x) - f(y) = \int_0^1 \nabla f(tx + (1-t)y) \cdot (x-y)\, dt$$

so

$$
\begin{aligned}
I_1 &\le \int\limits_E \int\limits_{|x-y|<R} \int_0^1 \Big|\nabla f(tx+(1-t)y)\cdot \frac{(x-y)}{|x-y|}\Big|\rho_i(x-y)\,dt\,dx\,dy \\
&= \int\limits_{|h|<R} \int_0^1 \int\limits_E \Big|\nabla f(y+th)\cdot \frac{h}{|h|}\Big|\,dy\,dt\;\rho_i(h)\,dh \\
&\le \int\limits_{|h|<R} \int\limits_{E_R} \Big|\nabla f(z)\cdot \frac{h}{|h|}\Big|\,dz\;\rho_i(h)\,dh \\
&= \int_0^R \int\limits_{E_R} \int\limits_{S^{n-1}} |\nabla f(z)\cdot w|\,ds(w)\,dz\; r^{n-1}\rho_i(r)\,dr \\
&= \int_0^R \int\limits_{E_R} |S^{n-1}|\;K_{1,n}\;|\nabla f(z)|\,dz\; r^{n-1}\rho_i(r)\,dr \\
&= K_{1,n}\Big(\int_{E_R} |\nabla f|\Big)\Big(\int_{B_R(0)} \rho_i\Big) \\
&\le K_{1,n}\int_{E_R} |\nabla f|.
\end{aligned}
$$

To estimate I_2, let us consider f to be extended by 0 outside Ω. Then

$$
\begin{aligned}
I_2 &\le \int\limits_E \int\limits_{|x-y|\ge R} \frac{|f(x)-f(y)|}{|x-y|}\rho_i(x-y)\,dxdy \\
&\le \frac{1}{R}\int\limits_{|h|\ge R} \int\limits_{\mathbf{R}^n} |f(y+h)-f(y)|\,dy\;\rho_i(h)\,dh \\
&\le \frac{2}{R}\|f\|_{L^1(\Omega)} \int_{|h|\ge R} \rho_i(h)\,dh.
\end{aligned}
$$

This proves the lemma in the case $f \in C^\infty(\Omega)\cap W^{1,1}(\Omega)$.

Case $f \in \mathrm{BV}(\Omega)$.

There is a sequence $(f_j) \in C^\infty(\Omega)\cap W^{1,1}(\Omega)$ such that $f_j \to f$ in $L^1(\Omega)$ and

$$\int_U |\nabla f_j| \to \int_U |\nabla f| \qquad \text{for all } U \subset \Omega \text{ open.}$$

Using (7) with f_j and noting that E_R is an open set, we conclude that (7) is valid for f. □

Proof of Lemma 2. We divide this proof in several steps.

Step 1. There is a subsequence i_j and a Radon measure μ in Ω such that

$$\mu_{i_j} \rightharpoonup \mu.$$

Proof. It is enough to show that for all compact sets $E \subset \Omega$ we have

$$\sup_i \int_E d\mu_i < \infty. \tag{8}$$

For such an E choose $R > 0$ small enough so that $E_R \subset \Omega$. Then (8) follows from (7).

Step 2.

$$\mu(B) \le K_{1,n}|\nabla f|(B) \tag{9}$$

for all Borel sets $B \subset \Omega$. In particular, μ is absolutely continuous with respect to $|\nabla f|$ and therefore we can write

$$\mu(B) = \int_B g\, d|\nabla f|$$

where g is $|\nabla f|$-measurable and

$$g \le K_{1,n} \qquad |\nabla f|\text{–a.e.} \tag{10}$$

Proof. Is enough to show that for all compact sets $E \subset \Omega$ we have

$$\mu(E) \le K_{1,n}|\nabla f|(E). \tag{11}$$

Let then $E \subset \Omega$ be compact and $R > 0$ small enough so that $E_{2R} \subset \Omega$. Then observe that

$$\begin{aligned}\mu(E) &\le \mu(E_R)\\ &\le \liminf_{i\to\infty} \mu_i(E_R), \qquad \text{and using (7) with } E \text{ replaced by } E_R\\ &\le \liminf_{i\to\infty} \quad K_{1,n}|\nabla f|(E_{2R}) + \frac{1}{R}\|f\|_{L^1(\Omega)} \int_{\mathbf{R}^n - B_{2R}} \rho_i\\ &= K_{1,n}|\nabla f|(E_{2R}).\end{aligned}$$

We now let $R \to 0$ and use that since E is compact, we have

$$E = \bigcap_{R>0} E_{2R}$$

so that

$$|\nabla f|(E_{2R}) \searrow |\nabla f|(E) \qquad \text{as } R \searrow 0.$$

Step 3.

$$\mu = K_{1,n}|\nabla f| \tag{12}$$

and the whole initial sequence μ_i converges weakly.

Proof. Here we need two facts proved in [BBM] which we state now: consider $\varphi \in C_c^\infty(\mathbf{R}^n)$ and a unit vector $e \in \mathbf{R}^n$. Then for all $x \in \mathbf{R}^n$ we have

$$\lim_{i\to\infty} \int\limits_{(y-x)\cdot e\geq 0} \frac{\varphi(y)-\varphi(x)}{|y-x|}\rho_i(y-x)\,dy = \frac{1}{2}K_{1,n}\nabla\varphi(x)\cdot e \tag{13}$$

and for any $f \in L^1(\mathbf{R}^n)$

$$\begin{aligned}&\left|\int\limits_{\mathbf{R}^n}\int\limits_{(y-x)\cdot e\geq 0} f(x)\frac{\varphi(y)-\varphi(x)}{|y-x|}\rho_i(y-x)\,dy\,dx\right| \\ &+\left|\int\limits_{\mathbf{R}^n}\int\limits_{(y-x)\cdot e\leq 0} f(x)\frac{\varphi(y)-\varphi(x)}{|y-x|}\rho_i(y-x)\,dy\,dx\right| \\ &\qquad\leq \int\limits_{\mathbf{R}^n}\int\limits_{\mathbf{R}^n} \frac{|f(x)-f(y)|}{|x-y|}|\varphi(y)|\rho_i(x-y)\,dx\,dy.\end{aligned} \tag{14}$$

Here the integrals on the left are well defined because φ is Lipschitz, and the integral on the right is well defined (but may be infinite for a general $f \in L^1(\mathbf{R}^n)$.)

Now we consider $f \in \mathrm{BV}(\Omega)$, extended by 0 outside Ω, and take $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$. We let $i \to \infty$ in (14). By (13) the left hand side of (14) has limit

$$K_{1,n}\left|\int_\Omega f(x)\nabla\varphi(x)\cdot e\,dx\right| = K_{1,n}\left|\int_\Omega \varphi\,d(\nabla f\cdot e)\right|. \tag{15}$$

On the other hand, for the right hand side of (14) we claim that

$$\lim_{i\to\infty}\int\limits_{\mathbf{R}^n}\int\limits_{\mathbf{R}^n} \frac{|f(x)-f(y)|}{|x-y|}\varphi(y)\rho_i(x-y)\,dx\,dy = \int_\Omega \varphi\,d\mu. \tag{16}$$

Indeed, to prove (16) let $R = \mathrm{dist}(\mathrm{supp}(\varphi), \partial\Omega)$ and note that

$$\begin{aligned}&\int\limits_{\mathbf{R}^n}\int\limits_{\mathbf{R}^n} \frac{|f(x)-f(y)|}{|x-y|}\varphi(y)\rho_i(x-y)\,dx\,dy = \\ &\qquad \int_\Omega \varphi\,d\mu_i + \int\limits_{\Omega}\int\limits_{\mathbf{R}^n-\Omega} \frac{|f(x)-f(y)|}{|x-y|}\varphi(y)\rho_i(x-y)\,dx\,dy\end{aligned}$$

by the definition of the measures μ_i. The second term on the right is bounded by

$$\frac{2}{R}\|\varphi\|_\infty\|f\|_{L^1(\Omega)}\int_{\mathbf{R}^n-B_R}\rho_i \;\to 0 \quad \text{as } i\to\infty.$$

Therefore, combining (14), (15) and (16) we find that

$$K_{1,n}\left|\int_\Omega \varphi\,d(\nabla f\cdot e)\right| \leq \int_\Omega \varphi\,d\mu$$

for all unit vectors $e \in \mathbf{R}^n$ and all $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$. This shows that

$$K_{1,n} \nabla f \cdot e \leq \mu \tag{17}$$

(as measures) for all unit vectors $e \in \mathbf{R}^n$.

To conclude recall the densities

$$g = \frac{d\mu}{d|\nabla f|}, \quad \text{and} \quad \sigma = \frac{d\nabla f}{d|\nabla f|}$$

and also recall that by the differentiation theorem for Radon measures, for $|\nabla f|$-a.e. $x \in \Omega$ we have

$$g(x) = \lim_{R\to 0} \frac{\mu(B_R(x))}{|\nabla f|(B_R(x))} \quad \text{and} \quad \sigma(x) = \lim_{R\to 0} \frac{\nabla f(B_R(x))}{|\nabla f|(B_R(x))}.$$

Take such an $x \in \Omega$. By (17), for any $R > 0$ small and unit vector e we have

$$K_{1,n} \frac{\nabla f(B_R(x)) \cdot e}{|\nabla f|(B_R(x))} \leq \frac{\mu(B_R(x))}{|\nabla f|(B_R(x))}.$$

Taking the supremum over all unit vectors e we find

$$K_{1,n} \frac{|\nabla f(B_R(x))|}{|\nabla f|(B_R(x))} \leq \frac{\mu(B_R(x))}{|\nabla f|(B_R(x))}$$

and letting $R \to 0$ we obtain

$$K_{1,n} |\sigma(x)| \leq g(x).$$

But $|\sigma(x)| = 1$ $|\nabla f|$-a.e., so this and (10) prove that

$$\mu = K_{1,n} |\nabla f|.$$

Finally the compactness of the sequence μ_i and the uniqueness of any possible limit show that the whole sequence μ_i converges weakly to $K_{1,n}|\nabla f|$. □

Proof of Theorem 1. For $\delta > 0$ and small let

$$V_\delta = \{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta \}.$$

Then

$$\partial V_\delta = \{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) = \delta \}$$

so $|\nabla f|(\partial V_\delta) = 0$ for all but perhaps countably many $\delta's$ in an interval $(0, \delta_0)$. For any such δ

$$\mu_i(V_\delta) \to K_{1,n} |\nabla f|(V_\delta).$$

To conclude note that $|\nabla f|(\Omega - V_\delta) \to 0$ as $\delta \to 0$ we only need to control

$$\mu_i(\Omega - V_\delta)$$

uniformly as $i \to \infty$. Consider $\tilde{f} = Ef$ where $E : \mathrm{BV}(\Omega) \to \mathrm{BV}(\mathbf{R}^n)$ is an extension operator with the additional property (6) mentioned in Section 2.

Now, by (7) applied to the function $\tilde{f}$ with $E = \Omega - V_\delta$ we have

$$\mu_i(\Omega - V_\delta) \le K_{1,n} |\nabla \tilde{f}|\big((\Omega - V_\delta) + B_R\big) + \frac{2}{R} \|\tilde{f}\|_{L^1(\mathbf{R}^n)} \int_{\mathbf{R}^n - B_R} \rho_i .$$

Letting $i \to \infty$ we see that

$$\limsup_{i\to\infty} \mu_i(\Omega - V_\delta) \le K_{1,n} |\nabla \tilde{f}|\big((\Omega - V_\delta) + B_R\big)$$

and this holds for any $R > 0$. We take $R = \delta$ and use property (6) of the extension $\tilde{f}$:

$$\limsup_{i\to\infty} \mu_i(\Omega - V_\delta) \le K_{1,n} C_1 |\nabla f|\big(\{\, x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < 2C_2\delta \,\}\big)$$

and the right hand side of this inequality has limit 0 as $\delta \to 0$. □

Acknowledgement. The author is deeply grateful to Prof. H. Brezis for suggesting the problem and his support.

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