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**Calculus of Variations**

# Nonlinear elliptic problems with a singular weight on the boundary

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**Abstract** We study existence of solutions to

$$-\Delta u = \frac{u^p}{|x|^2} \quad u > 0 \text{ in } \Omega$$

with $u = 0$ on $\partial\Omega$, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$ with $0 \in \partial\Omega$ and $1 < p < \frac{N+2}{N-2}$. The existence of solutions depends on the geometry of the domain. On one hand, if the domain is starshaped with respect to the origin there are no energy solutions. On the other hand, in *dumbbell domains* via a perturbation argument, the equation has solutions.

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## 1 Introduction

Consider the problem

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2}, & u > 0 \text{ in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $p > 0$ and $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$ is a bounded domain with smooth boundary.

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If $0 < p < 1$ it is simple to prove existence of a solution, independently of the location of the origin. By using Hardy's inequality the energy functional satisfies

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega \frac{u^{p+1}}{|x|^2} \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - C(\Omega, p)\left(\int_\Omega |\nabla u|^2\right)^{\frac{p+1}{2}}$$

and then one can proceed by minimization as in [1].

The related problem

$$\begin{cases} -\Delta u = \dfrac{u^{p^*(\alpha)-1}}{|x|^\alpha}, & u > 0 \text{ in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

with $0 \in \partial\Omega$, $0 < \alpha < 2$ and $p^*(\alpha) = \dfrac{2(N-\alpha)}{N-2}$, has been studied by Ghoussoub–Kang in [17] and by Ghoussoub–Robert in [18], with variational methods. We recall that $p^*(\alpha)$ is the critical exponent in the embedding of $H_0^1(\Omega)$ in the corresponding weighted space (see for instance [7]).

In [17] and [18] the authors give sufficient local conditions on the boundary at 0, precisely a condition of negativity of the curvatures and the mean curvature at 0, respectively, for the best constant in this embedding to be attained, which yields existence of a solution to (1.2).

In this work we are interested in (1.1) with $1 < p < \frac{N+2}{N-2}$ and $0 \in \partial\Omega$. Then (1.1) is supercritical because when $\alpha = 2$ then $p^*(\alpha) - 1 = 1$. Before we discuss the situation $0 \in \partial\Omega$ in more detail, we make a few comments on the case $0 \notin \partial\Omega$.

If $0 \notin \overline{\Omega}$ then using the mountain pass theorem of Ambrosetti and Rabinowitz [4] one can show that (1.1) has a solution. If we assume $0 \in \Omega$ there is no *weak solution* to (1.1), which we define as follows:

**Definition 1.1** We call a function $u \in L^1(\Omega)$, $u \geq 0$ a weak solution of (1.1) if $dist(x, \partial\Omega)\dfrac{u^p}{|x|^2} \in L^1(\Omega)$, $u > 0$ in $\Omega$ and

$$\int_\Omega u(-\Delta\phi) = \int_\Omega \frac{u^p}{|x|^2}\phi \quad \text{for all } \phi \in \mathcal{C}^\infty(\overline{\Omega}) \text{ with } \phi|_{\partial\Omega} = 0.$$

Actually (1.1) has no weak supersolution even locally, as was pointed out by Brezis and Cabré [6]. Indeed, assume that $u \geq 0$ is a super-solution in $\mathcal{D}'(\Omega)$ to problem (1.1), i.e. $u$, $\dfrac{u^p}{|x|^2} \in L^1_{\text{loc}}(\Omega)$ and for all $\phi \in \mathcal{C}_0^\infty(\Omega)$, $\phi \geq 0$ the following inequality holds

$$\int_\Omega u(-\Delta\phi) \geq \int_\Omega \frac{u^p}{|x|^2}\phi.$$

If $u$ is not identically zero, by the maximum principle for the Laplace operator $u(x) \geq c$ in a small ball $B_r(0)$, where $c > 0$. Then $u$ is a super-solution to the problem $-\Delta v(x) = c\dfrac{1}{|x|^2}$ in $B_r(0)$, $v(x) = 0$ on $|x| = r$. Therefore $u(x) \geq c\log(\frac{r}{x})$. By the Picone inequality we obtain

$$\int_{B_r(0)} |\nabla\phi|^2 \geq \int_{B_r(0)} \left(\frac{-\Delta u}{u}\right)\phi^2 \geq \int_{B_r(0)} \frac{\phi^2}{|x|^2} c^{p-1}\left(\log\left(\frac{r}{x}\right)\right)^{p-1}, \quad \forall\phi \in C_0^\infty(B_r(0))$$

which is a contradiction with the classical Hardy–Sobolev inequality. Brezis and Cabré [6] proved that even a weaker type of local solution does not exist in the case $p = 2$. More precisely, they prove that if $u \in L^2_{\mathrm{loc}}(\Omega)$, $u \geq 0$ satisfies

$$-|x|^2 \Delta u \geq u^2 \quad \text{in } \mathcal{D}'(\Omega \setminus \{0\})$$

then $u \equiv 0$. From the above results we see that if $0 \in \Omega$ then there is a local obstruction for the existence of solutions.

Returning to (1.1) with $1 < p < \frac{N+2}{N-2}$ and $0 \in \partial\Omega$, there is no local obstruction for the existence of solutions. In fact we, can exhibit at least 2 types of local solutions. For the first one consider the half space $H = \{(x_1, \dots, x_N) : x_1 > 0\}$. Then one can find a solution of the equation in $H$ of the form $u(x) = \varphi(\theta)$ where $\theta = x/|x|$ is on $S_+^{N-1} = S^{N-1} \cap H$. The equation for $\varphi$ becomes

$$\begin{cases} -\Delta_{S^{N-1}} \varphi = \varphi^p & \text{in } S_+^{N-1} \\ \varphi = 0 & \text{on } \partial S_+^{N-1}. \end{cases} \tag{1.3}$$

If $1 < p < \frac{N+1}{N-3}$ ($p > 1$ if $N = 2, 3$), this problem has a positive solution and yields a solution of (1.1) in $H^1(H \cap B_R(0))$ for any $R > 0$ (if $N \geq 3$).

A second local solution can be obtained as follows. Fix $r_0 > 0$ small and define

$$D_{r_0} = \{x \in \Omega \ : |x| < r_0\},\ \Gamma_1 = \partial\Omega \cap \{|x| < r_0\} \ \text{ and } \ \Gamma_2 = \Omega \cap \{|x| = r_0\}.$$

Let $\lambda > 0$ and let us write $d_{\Gamma_1}(x) = dist(x, \Gamma_1)$. In Sect. 3.4 we construct a supersolution $w$ to the problem

$$\begin{cases} -\Delta w = \dfrac{w^p}{|x|^2} & w > 0 \text{ in } D_{r_0} \\ w = 0 & \text{on } \Gamma_1,\ w = \lambda d_{\Gamma_1} \quad \text{on } \Gamma_2 \end{cases} \tag{1.4}$$

for any $\lambda > 0$ which is small enough, and furthermore $w(x) \leq C d_{\Gamma_1}(x)$ for some constant $C$. Using monotone iterations this implies the existence of a solution to (1.4) for $\lambda > 0$ small, and this solution is also bounded by $C d_{\Gamma_1}(x)$. If $0 \in \Omega$ a local supersolution does not exist, as pointed out before. The fact that (1.4) has a solution for small $\lambda > 0$ is related to general results of Kalton and Verbitsky [20] and Brezis and Cabré [6].

However, the existence of solutions of the full problem (1.1) is not always true if $0 \in \partial\Omega$. Indeed, if $\Omega$ is starshaped with respect to the origin, using Pohozaev's identity we can prove that there are no energy solutions of (1.1), see Sect. 2.

Despite this negative result we will prove that for convenient non-starshaped domains there exists a solution to problem (1.1). Just to motivate our analysis, consider the open set $\Omega = B_1((1, 0, \dots, 0)) \cup B_1((3, 0, \dots, 0))$ and let $v$ be a solution to

$$\begin{cases} -\Delta v = \dfrac{v^p}{|x|^2} & , v > 0 \quad \text{in } B_1((3, 0, \dots, 0)) \\ v = 0 & \text{on } \partial B_1((3, 0, \dots, 0)) \end{cases}$$

obtained, for instance, using the classical Mountain Pass Theorem. Then

$$u(x) = \begin{cases} 0 & \text{if } x \in B_1((1, 0, \dots, 0)) \\ v(x) & \text{if } x \in B_1((3, 0, \dots, 0)) \end{cases}$$

is a solution of (1.1). Notice that the role of the geometry of the two components is irrelevant and that in this case the domain is not connected. The idea is that a perturbation of the domain allows us to obtain connected domains for which problem (1.1) has a solution.

With this idea of perturbation in mind we introduce the following type of domains.

**Definition 1.2** We call $\Omega_\epsilon$ a dumbbell domain if it is a domain with smooth boundary of the form $\Omega_\epsilon = \Omega_1 \cup C_\epsilon \cup \Omega_2$ where $\Omega_1$ and $\Omega_2$ are smooth bounded domains such that $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$, and $C_\epsilon$ is a region contained in a tubular neighborhood of radius less than $\epsilon > 0$ around a curve joining $\Omega_1$ and $\Omega_2$.

In is this type of domains we can solve problem (1.1). Precisely we will prove the following perturbative result.

**Theorem 1.3** *Assume that*

*(1)* $\Omega_\epsilon$ *is a dumbbell domain.*
*(2)* $0 \in \partial\Omega_1 \cap \partial\Omega_\epsilon$
*(3)* $1 < p < \frac{N+2}{N-2}$

*Then there exist* $\epsilon_0 > 0$ *such that if* $0 < \epsilon < \epsilon_0$,

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2} \quad u > 0 & \text{in } \Omega_\epsilon \\ u = 0 & \text{on } \partial\Omega_\epsilon \end{cases} \tag{1.5}$$

*has a solution.*

In this statement the domains $\Omega_1$, $\Omega_2$ and the curve along which runs $C_\epsilon$ in the definition of $\Omega_\epsilon$ are fixed as $\epsilon$ varies. The solution that we find is $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$.

Perturbations of the domain have been used before to construct solutions to semilinear equations. For example, Dancer [10, p. 140] showed that the problem

$$\begin{cases} -\Delta u = u^p \quad u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with $p > 1$ subcritical and $\Omega$ a dumbell domain consisting of 2 or more balls joined by a thin tube, has multiple solutions. These solutions are obtained by gluing positive and zero solutions on different balls. A crucial ingredient in [10] is that the positive and zero solutions in the ball are nondegenerate. For (1.6) in a ball this was known [9,12], but for the equation (1.1) this seems not known, although probably it holds for generic domains. Our approach does not require nondegeneracy.

Other perturbations of the domain have been applied to (1.6) with the critical exponent $p = \frac{N+2}{N-2}$. We recall that a first result with a sufficient topological condition for existence is contained in Bahri, Coron [5]. They proved that if there exists a positive integer $d$ such that $H_d(\Omega; Z_2) \neq 0$, the critical problem has a solution. Here $H_d(\Omega; Z_2)$ is the homology group of dimension $d$ with $Z_2$-coefficients. The idea that the obstruction to the problem is topological is not exact as was established by Dancer [11], Ding [13], and extended by Passaseo [21]. In these references the construction of the domains also involves a perturbation by removing some small sets. A sufficient geometrical condition to have solution in the critical problem is yet unknown.

Problems with singular potentials have been largely studied in a variational setting during the past years. See for instance [2,3,15,19,22], and the references therein. However for (1.5) a standard variational formulation is not possible since it is a supercritical problem.

Notice that in our problem, if $0 \in \Omega$ the nonexistence result is independent of the topology of the domain. If we consider $0 \in \partial\Omega$, Theorem 1.3 is independent of the geometry of the domain in a neighborhood of the singularity, in contrast to the results in [17] and [18]. A sufficient geometrical condition for existence, as in the critical problem, is far from being clear at the moment. The perturbative argument only allows us to say, in a rough way, that there is a solution in domains in which we can glue a local small solution close to the singularity with some modification of a solution given by a mountain pass argument.

The paper is organized as follows. In Sect. 2 we deduce the nonexistence of energy solutions in starshaped domains by a Pohozaev identity. In Sect. 3 we prove Theorem 1.3.

We use a variational method applied to a functional that is a perturbation of the *formal energy functional* associated to problem (1.5), and that also contains a penalty term that forces the solution to be small in integral sense near the singularity. Once this is achieved, using a local supersolution near the singularity, we obtain a uniform control of the solution near this point. We point out that the local supersolution is *formally* related to the results in [20].

Finally in Sect. 4 we make some remarks concerning other powers in $u$ and $|x|$.

## 2 Nonexistence in starshaped domains

We say that $u$ is an energy solution of (1.1) if $u \in H_0^1(\Omega)$, $u > 0$ such that

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega \frac{u^p}{|x|^2}\varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

**Proposition 2.1** *Assume $N \geq 3$ and $1 < p < \frac{N+2}{N-2}$. Then (1.1) has no energy solutions if $0 \in \partial\Omega$ and $\Omega$ is starshaped with respect to the origin.*

For the proof we need the following estimates.

**Lemma 2.2** *Assume $N \geq 2$ and $1 < p < \frac{N+2}{N-2}$ ($p > 1$ if $N = 2$). If $u$ is an energy solution of (1.1) then $u \in L^\infty(\Omega)$, and there is some $C > 0$ such that*

$$|\nabla u(x)| \leq \frac{C}{|x|}, \quad |D^2 u(x)| \leq \frac{C}{|x|^2} \quad \textit{for all } x \in \Omega.$$

*Proof* By standard elliptic estimates $u$ is smooth away from the origin. The uniform bound is a consequence of the apriori estimates of Gidas and Spruck [16], after a suitable scaling. More precisely, suppose $x_0 \in \Omega$, $x_0 \neq 0$. Let $r = |x_0|/2$ and define $v(y) = u(x_0 + ry)$ for $y \in (\Omega - x_0)/r$. Then $v$ satisfies

$$-\Delta v = \frac{r^2}{|x_0 + ry|^2} v^p \quad \text{in } (\Omega - x_0)/r.$$

We restrict this equation to $(\Omega - x_0)/r \cap B_1(0)$. In this region the weight $\frac{r^2}{|x_0+ry|^2}$ is bounded and smooth. Using the apriori estimates of [16] we deduce $v(0) \leq C$ for some universal $C$.

The estimates for the gradient and second derivatives can be deduced by scaling. □

*Proof of Proposition 2.1* For the proof we use Pohozaev's identity: multiply the equation by $x \cdot \nabla u$ and integrate in $\Omega \setminus B_\rho(0)$ where $\rho > 0$. We have

$$\int_{\Omega\setminus B_\rho(0)} (-\Delta u) x \cdot \nabla u = \frac{2-N}{2} \int_{\Omega\setminus B_\rho(0)} |\nabla u|^2 - \frac{1}{2} \int_{\partial(\Omega\setminus B_\rho(0))} |\nabla u|^2 x \cdot \nu$$

and

$$\begin{aligned} \int_{\Omega\setminus B_\rho(0)} \frac{u^p}{|x|^2} x \cdot \nabla u &= -\frac{1}{p+1} \int_{\Omega\setminus B_\rho(0)} u^{p+1} div\left(\frac{x}{|x|^2}\right) + \frac{1}{p+1} \int_{\partial(\Omega\setminus B_\rho(0))} \frac{u^{p+1}}{|x|^2} \nu \cdot x \\ &= -\frac{N-2}{p+1} \int_{\Omega\setminus B_\rho(0)} \frac{u^{p+1}}{|x|^2} + \frac{1}{p+1} \int_{\partial(\Omega\setminus B_\rho(0))} \frac{u^{p+1}}{|x|^2} \nu \cdot x \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{2-N}{2} \int_{\Omega\setminus B_\rho(0)} |\nabla u|^2 - \frac{1}{2} \int_{\partial(\Omega\setminus B_\rho(0))} |\nabla u|^2 x \cdot \nu \\ &= -\frac{N-2}{p+1} \int_{\Omega\setminus B_\rho(0)} \frac{u^{p+1}}{|x|^2} + \frac{1}{p+1} \int_{\partial(\Omega\setminus B_\rho(0))} \frac{u^{p+1}}{|x|^2} \nu \cdot x. \end{aligned}$$

Multiplying the equation by $u$ and integrating in $\Omega \setminus B_\rho(0)$ yields

$$-\int_{\partial(\Omega\setminus B_\rho(0))} u \frac{\partial u}{\partial \nu} + \int_{\Omega\setminus B_\rho(0)} |\nabla u|^2 = \int_{\Omega\setminus B_\rho(0)} \frac{u^{p+1}}{|x|^2}$$

Hence

$$\begin{aligned} 0 = (N-2)\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\Omega\setminus B_\rho(0)} \frac{u^{p+1}}{|x|^2} + \frac{1}{p+1} \int_{\partial(\Omega\setminus B_\rho(0))} \frac{u^{p+1}}{|x|^2} \nu \cdot x \\ + \frac{1}{2} \int_{\partial(\Omega\setminus B_\rho(0))} |\nabla u|^2 x \cdot \nu + \frac{N-2}{2} \int_{\partial(\Omega\setminus B_\rho(0))} u \frac{\partial u}{\partial \nu}. \end{aligned}$$

Now we let $\rho \to 0$. Since $u = 0$ on $\partial\Omega$ and by Lemma 2.2 we have

$$\int_{\partial(\Omega\setminus B_\rho(0))} \frac{u^{p+1}}{|x|^2} \nu \cdot x = \int_{\partial B_\rho(0)\cap\Omega} \frac{u^{p+1}}{|x|^2} \nu \cdot x = O(\rho^{N-2}) \quad \text{as } \rho \to 0.$$

Similarly one can take the limit in the other terms and arrive at

$$0 = (N-2)\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega \frac{u^{p+1}}{|x|^2} + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 x \cdot \nu$$

In an starshaped domain about the origin this implies $u \equiv 0$ in $\Omega$. □

## 3 Existence in dumbbell domains

In this section we will prove Theorem 1.3. A standard variational formulation does not work directly. Indeed the associated energy functional

$$E(u) = \frac{1}{2}\int_{\Omega_\epsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} \frac{u^{p+1}}{|x|^2}$$

is no a priori well defined in $H_0^1(\Omega_\epsilon)$.

Therefore we will proceed by using some energy functionals that are:

- truncation, by considering the truncated weight $\dfrac{1}{|x|^2+\delta}$,
- penalization of the contribution to the truncated energy in $\Omega_1$, where the singularity of the potential is located.

We proceed in several steps.

### 3.1 The truncated-penalized functional

We start by fixing a function $\eta \in \mathcal{C}^1(\mathbb{R})$ with the properties:

$$\begin{cases} \eta(s) = 0 & s \in [0,1] \\ 0 \le \eta(s) \le 2 \ \text{ and } \ \eta'(s) \ge 0 & s \in [1,2] \\ \eta(s) = s & s \in [2,\infty). \end{cases}$$

Given $\theta > 0$ define

$$\eta_\theta(t) = \eta(t/\theta) \quad t \ge 0. \tag{3.1}$$

Fix $q$ in the range

$$\min\left(2, (p-1)\frac{N}{2}\right) < q < p+1.$$

This is possible because $(p-1)\frac{N}{2} < p+1$ since $p < 2^* - 1$, where $2^* = \frac{2N}{N-2}$. Let $g \in \mathcal{C}^1(\mathbb{R})$ be a function such that

$$\begin{cases} g(s) = 0 & s \le 0 \\ 0 \le g(s) \le 1 \ \text{ and } \ 0 \le g'(s) \le q s^{(p-1)\frac{N}{2}-1} & s \in [0,1] \\ g(s) = s^{(p-1)\frac{N}{2}} & s \ge 1 \end{cases}$$

and given $\delta > 0$ set

$$g_\delta(t) = \delta^{(p-1)\frac{N}{2}} g(t/\delta). \tag{3.2}$$

For $\epsilon, \delta, \theta > 0$ we define the penalized energy functional $E_{\delta,\epsilon,\theta} : H_0^1(\Omega_\epsilon) \to \mathbb{R}$ by

$$E_{\delta,\epsilon,\theta}(u) = \frac{1}{2}\int_{\Omega_\epsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} \frac{(u^+)^{p+1}}{|x|^2+\delta} + \eta_\theta(I_\delta(u)),$$

where $u^+ = \max(u, 0)$ and

$$I_\delta(u) = \int\limits_{\Omega_1} \frac{g_\delta(u+\delta)}{(|x|^2+\delta)^{\frac{N}{2}}}.$$

We will see that this penalization is quite natural.

It is standard to verify that $E_{\delta,\epsilon,\theta} : H_0^1(\Omega_\epsilon) \to \mathbb{R}$ is $\mathcal{C}^1$. If $u$ is a critical point of $E_{\delta,\epsilon,\theta}$ then it satisfies the Euler equation

$$\begin{cases} -\Delta u + a(x,u) g_\delta'(u+\delta) = \dfrac{(u^+)^p}{|x|^2+\delta} & \text{in } \Omega_\epsilon \\ u = 0 & \text{on } \partial\Omega_\epsilon \end{cases} \tag{3.3}$$

where

$$a(x,u) = \eta_\theta'(I_\delta(u)) \frac{\chi_{\Omega_1}(x)}{(|x|^2+\delta)^{\frac{N}{2}}}$$

and $\chi_{\Omega_1}$ is the characteristic function of $\Omega_1$. A solution $u \in H_0^1(\Omega_\epsilon)$ to (3.3) is not necessarily positive, but satisfies $u \geq -\delta$ in $\Omega_\epsilon$.

### 3.2 Study of the functional $E_{\delta,\epsilon,\theta}$

We study in this paragraph the functional $E_{\delta,\epsilon,\theta}$ for each $\delta$, $\epsilon$, $\theta$ fixed with respect to the application of the Mountain Pass Theorem and we will obtain uniform estimates from above and from below away from zero of the mini-max *mountain pass* critical values.

**Lemma 3.1** *Fix $\epsilon, \theta, \delta > 0$. Then $E_{\delta,\epsilon,\theta} : H_0^1(\Omega_\epsilon) \to \mathbb{R}$ is $C^1$ and satisfies the Palais–Smale condition.*

*Proof* Let us verify that $E_{\delta,\epsilon,\theta}$ satisfies the Palais–Smale condition. Let $(u_n)$ be a sequence in $H_0^1(\Omega_\epsilon)$ such that $E_{\delta,\epsilon,\theta}(u_n) \leq C$ and $E_{\delta,\epsilon,\theta}'(u_n) \to 0$ in $H^{-1}(\Omega_\epsilon)$. Then

$$\begin{aligned} C + o(1)\|u_n\|_{H_0^1} &\geq E_{\delta,\epsilon,\theta}(u_n) - \frac{1}{p+1} E_{\delta,\epsilon,\theta}'(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \|u_n\|_{H_0^1}^2 + \eta_\theta(I_\delta(u_n)) - \frac{1}{p+1}\eta_\theta'(I_\delta(u_n)) \int\limits_{\Omega_1} \frac{g_\delta'(u_n+\delta)u_n}{(|x|^2+\delta)^{\frac{N}{2}}}. \end{aligned}$$

Assume first $I_\delta(u_n) \geq 2\theta$. Then

$$\eta_\theta(I_\delta(u_n)) - \frac{1}{p+1}\eta_\theta'(I_\delta(u_n)) \int\limits_{\Omega_1} \frac{g_\delta'(u_n+\delta)u_n}{(|x|^2+\delta)^{\frac{N}{2}}} = \frac{1}{\theta}\left( I_\delta(u) - \frac{1}{p+1} \int\limits_{\Omega_1} \frac{g_\delta'(u_n+\delta)u_n}{(|x|^2+\delta)^{\frac{N}{2}}} \right).$$

We claim that

$$g_\delta'(u+\delta)u \leq q g_\delta(u+\delta) \quad \text{for all } u \in \mathbb{R}. \tag{3.4}$$

Indeed, if $u \geq 0$ then $g_\delta(u+\delta) = (u+\delta)^{(p-1)\frac{N}{2}}$ and

$$g_\delta'(u+\delta) = (p-1)\frac{N}{2}(u+\delta)^{(p-1)\frac{N}{2}-1}.$$

Therefore

$$g_\delta'(u+\delta)u \le g_\delta'(u+\delta)(u+\delta) = (p-1)\frac{N}{2}(u+\delta)^{(p-1)\frac{N}{2}} \le q(u+\delta)^{(p-1)\frac{N}{2}}$$

and (3.4) holds. If $-\delta \le u \le 0$, then

$$g_\delta'(u+\delta)u \le 0 \le q g_\delta(u+\delta).$$

Finally, if $u \le -\delta$ then both sides in (3.4) are zero.

Using (3.4) we deduce that

$$\int_{\Omega_1} \frac{g_\delta'(u+\delta)u}{(|x|^2+\delta)^{\frac{N}{2}}} \le q I_\delta(u) \tag{3.5}$$

and therefore

$$\eta_\theta(I_\delta(u_n)) - \frac{1}{p+1}\eta_\theta'(I_\delta(u_n)) \int_{\Omega_1} \frac{g_\delta'(u_n+\delta)u_n}{(|x|^2+\delta)^{\frac{N}{2}}} \ge \frac{1}{\theta}\left(1-\frac{q}{p+1}\right) I_\delta(u) \ge 0.$$

Under the assumption $I_\delta(u_n) \ge 2\theta$ we find

$$\|u_n\|_{H_0^1} \le C \quad \text{for all } n \tag{3.6}$$

with $C$ independent of $n$. If $I_\delta(u_n) \le \theta$ we obtain the same conclusion because

$$\eta_\theta(I_\delta(u_n)) - \frac{1}{p+1}\eta_\theta'(I_\delta(u_n)) \int_{\Omega_1} \frac{g_\delta'(u_n+\delta)u_n}{(|x|^2+\delta)^{\frac{N}{2}}} = 0.$$

If $\theta \le I_\delta(u_n) \le 2\theta$ then using (3.5)

$$\eta_\theta(I_\delta(u_n)) - \frac{1}{p+1}\eta_\theta'(I_\delta(u_n)) \int_{\Omega_1} \frac{g_\delta'(u_n+\delta)u_n}{(|x|^2+\delta)^{\frac{N}{2}}} \ge -\frac{C}{\theta} I_\delta(u_n) \ge -C.$$

This proves (3.6) also in this case. By taking a subsequence we can assume that $u_n$ converges weakly to $u$ in $H_0^1(\Omega_\epsilon)$. Then $u_n^{p+1}$ and $u_n^{(p-1)N/2}$ converge strongly in $L^1(\Omega_\epsilon)$. From

$$o(1) = E_{\delta,\epsilon,\theta}'(u_n)(u_n-u) = \int_{\Omega_\epsilon} \nabla u_n(\nabla u_n - \nabla u) - \int_{\Omega_\epsilon} \frac{(u_n^+)^p(u_n-u)}{|x|^2+\delta}$$
$$+\eta_\theta'(I_\delta(u_n)) \int_{\Omega_1} \frac{g_\delta'(u_n)(u_n-u)}{(|x|^2+\delta)^{N/2}}$$

we then deduce that $\int_{\Omega_\epsilon} \nabla u_n(\nabla u_n - \nabla u) \to 0$ which implies the strong convergence $u_n \to u$ in $H_0^1(\Omega_\epsilon)$. □

The *mountain pass* geometry is a consequence of the following observation. If $\|u\|_{H_0^1(\Omega_\epsilon)}$ is sufficiently small then $E_{\delta,\epsilon,\theta}(u) = \tilde{E}(u)$ where

$$\tilde{E}(u) = \frac{1}{2}\int_{\Omega_\epsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} \frac{(u^+)^{p+1}}{|x|^2+\delta}$$

is the functional without penalization.

Let $\phi_2 > 0$ be the principal eigenfunction of $-\Delta$ in $\Omega_2$ with $\phi|_{\partial\Omega_2} = 0$, normalized such that $\|\phi_2\|_{L^2(\Omega_2)} = 1$ and fix $A > 0$ large such that

$$E_{0,\epsilon,\theta} = \frac{1}{2}A^2 \int_{\Omega_2} |\nabla\phi_2|^2 - \frac{1}{p+1}A^{p+1} \int_{\Omega_2} \frac{\phi_2^{p+1}}{|x|^2} < 0.$$

Define

$$c_{\delta,\epsilon,\theta} = \inf_{\gamma} \max_{t\in[0,1]} E(\gamma(t))$$

where the infimum ranges over all continuous paths $\gamma : [0,1] \to H_0^1(\Omega_\epsilon)$ such that $\gamma(0) = 0$ and $\gamma(1) = A\phi_2$.

**Lemma 3.2** *There exists a critical point $u$ of $E_{\delta,\epsilon,\theta}$ with critical value $c_{\delta,\epsilon,\theta}$. Moreover there exists a constant $C$ independent of $\epsilon, \theta, \delta$ such that*

$$c_{\delta,\epsilon,\theta} \le C.$$

*Proof* The functional $E_{\delta,\epsilon,\theta}$ satisfies the assumptions of the Mountain Pass Lemma and then has a *mountain pass* critical point. Since the weight $\dfrac{1}{|x|^2+\delta}$ is bounded we see that there exist $\rho > 0$ such that

$$\inf\{\tilde{E}(u) : \|u\|_{H_0^1(\Omega_\epsilon)} = \rho\} > 0$$

and then also

$$\inf\{E_{\delta,\epsilon,\theta}(u) : \|u\|_{H_0^1(\Omega_\epsilon)} = \rho\} > 0.$$

The upper bound for $c_{\delta,\epsilon,\theta}$ is proved by taking $\gamma(t) = tA\phi_2$, since $\max_{t\in[0,1]} E_{\delta,\epsilon,\theta}(tA\phi_2)$ is bounded uniformly in $\delta, \epsilon, \theta$. □

Moreover we will see that the *mountain pass* level $c_{\delta,\epsilon,\theta}$ admits a uniform bound from below away from zero. This is important to show later that the solutions we find are not trivial.

**Lemma 3.3** *There exist $\theta_0 > 0$ and $c_0 > 0$ independent of $\epsilon, \theta, \delta$ such that*

$$c_{\delta,\epsilon,\theta} \ge c_0$$

*for* $0 < \theta \le \theta_0$.

*Proof* Since in $\Omega_\epsilon \setminus \Omega_1$ the weight $\frac{1}{|x|^2+\delta}$ is bounded uniformly in $\epsilon, \delta$, we can fix $\rho > 0$ independently of $\epsilon, \delta$ such that

$$\frac{1}{2}\int_{\Omega_\epsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon\setminus\Omega_1} \frac{(u^+)^{p+1}}{|x|^2+\delta} \ge \frac{1}{4}\|u\|_{H_0^1}^2 \quad \text{for all } \|u\|_{H_0^1} \le \rho.$$

By Hölder's inequality

$$\int_{\Omega_1} \frac{(u^+)^{p+1}}{|x|^2+\delta} \le \left(\int_{\Omega_1} (u^+)^{2^*}\right)^{\frac{N-2}{N}} \left(\int_{\Omega_1} \frac{(u^+)^{(p-1)\frac{N}{2}}}{(|x|^2+\delta)^{\frac{N}{2}}}\right)^{\frac{2}{N}}$$

(this last inequality motivates the definition of $I_\delta$ for the penalization). Using Sobolev's inequality we obtain

$$\int_{\Omega_1} \frac{(u^+)^{p+1}}{|x|^2+\delta} \le C\|u\|^2_{H^1_0} I_\delta(u)^{\frac{2}{N}}.$$

It follows that

$$E(u) \ge \frac{1}{4}\|u\|^2_{H^1_0} - C\rho^2 I_\delta(u)^{\frac{2}{N}} + \eta_\theta(I_\delta(u)) \quad \text{for all } \|u\|_{H^1_0} \le \rho. \tag{3.7}$$

Let $h(t) = \eta_\theta(t) - C\rho^2 t^{\frac{2}{N}}$. If $t \ge 2\theta$ then

$$h(t) = t^{\frac{2}{N}} \left( \frac{t^{\frac{N-2}{N}}}{\theta} - C\rho^2 \right) \ge t^{\frac{2}{N}} \left( 2^{\frac{N-2}{N}} \theta^{-\frac{2}{N}} - C\rho^2 \right).$$

Having fixed $\rho > 0$ we take $\theta_0 > 0$ small so that

$$2^{\frac{N-2}{N}} \theta^{-\frac{2}{N}} - C\rho^2 \ge 1 \quad \text{for } 0 < \theta \le \theta_0. \tag{3.8}$$

Then

$$h(t) \ge t^{\frac{2}{N}} \quad \text{for } t \ge 2\theta. \tag{3.9}$$

For $0 \le t \le 2\theta$:

$$h(t) \ge -C\rho^2 t^{\frac{2}{N}} \ge -C\rho^2 (2\theta)^{\frac{2}{N}} \tag{3.10}$$

Let $\gamma : [0,1] \to H^1_0(\Omega_\epsilon)$ be continuous such that $\gamma(0) = 0$ and $\gamma(1) = A\phi_2$. We take $\rho > 0$ small so that $A\|\phi_2\|_{H^1_0} > \rho$. Let $t^*$ be defined by

$$t^* = \min\{\, t \in [0,1] : \|\gamma(t)\|_{H^1_0} \ge \rho \text{ or } I_\delta(\gamma(t)) \ge 1 \,\}.$$

Then $t^*$ is well defined and we have the properties $\|\gamma(t)\|_{H^1_0} \le \rho$, $I_\delta(\gamma(t)) \le 1$ for $0 \le t \le t^*$ and one of the following cases: either $\|\gamma(t^*)\|_{H^1_0} = \rho$ or $I_\delta(\gamma(t^*)) = 1$.

Assume first that $\|\gamma(t^*)\|_{H^1_0} = \rho$. Then using (3.7), (3.9) and (3.10)

$$E(\gamma(t^*)) \ge \frac{1}{4}\|\gamma(t^*)\|^2_{H^1_0} - C\rho^N (2\theta)^{\frac{2}{N}} = \frac{1}{4}\rho^2 - C\rho^2(2\theta)^{\frac{2}{N}}.$$

Choosing $\theta_0$ smaller we can achieve

$$\frac{1}{4}\rho^2 - C\rho^2(2\theta)^{\frac{2}{N}} \ge \frac{1}{8}\rho^2 \quad \text{for } 0 < \theta \le \theta_0. \tag{3.11}$$

Then we obtain

$$E(\gamma(t^*)) \ge \frac{1}{8}\rho^2.$$

Assume now that that $I_\delta(\gamma(t^*)) = 1$. We may also assume that $\theta_0 \le \frac{1}{2}$. Then by (3.7) and (3.9)

$$E(\gamma(t^*)) \ge \|\gamma(t^*)\|^2_{H^1_0} + I_\delta(\gamma(t^*))^{\frac{2}{N}} \ge 1.$$

It follows that the mountain pass level $c_{\delta,\epsilon,\theta}$ satisfies

$$c_{\delta,\epsilon,\theta} \geq \min\left(\frac{\rho^2}{8}, 1\right)$$

provided $0 < \theta \leq \theta_0$, and $0 < \theta_0 \leq \frac{1}{2}$ is such that (3.8) and (3.11) hold. □

### 3.3 Uniform estimates for the *mountain pass* critical points

We claim that there exists $C$ independent of $\delta, \theta, \epsilon$ such that for all $\delta > 0, \theta > 0$ and $\epsilon > 0$, if $u \in H_0^1(\Omega_\epsilon)$ is a mountain pass of the functional $E_{\delta,\epsilon,\theta}(u)$ then

$$\|u\|_{H_0^1(\Omega_\epsilon)} \leq C \tag{3.12}$$

and

$$I_\delta(u) \leq C\theta. \tag{3.13}$$

The argument is the same as in the proof of Lemma 3.1. Indeed, since $E_{\delta,\epsilon,\theta}(u) \leq C$ we have

$$\begin{aligned} C &\geq E_{\delta,\epsilon,\theta}(u) - \frac{1}{p+1} E'_{\delta,\epsilon,\theta}(u)u \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \|u\|^2_{H_0^1} + \eta_\theta(I_\delta(u)) - \frac{q}{p+1}\eta'_\theta(I_\delta(u))I_\delta(u). \end{aligned}$$

If $I_\delta(u) \geq 2\theta$ then

$$\eta_\theta(I_\delta(u)) - \frac{q}{p+1}\eta'_\theta(I_\delta(u))I_\delta(u) = \frac{1}{\theta}\left(1 - \frac{q}{p+1}\right) I_\delta(u)$$

and we deduce (3.12) and (3.13) with $C$ independent of $\delta$, $\theta$, $\epsilon$. If $I_\delta(u) \leq \theta$ we obtain the same conclusion because

$$\eta_\theta(I_\delta(u)) - \frac{q}{p+1}\eta'_\theta(I_\delta(u))I_\delta(u) = 0.$$

If $\theta \leq I_\delta(u) \leq 2\theta$ then

$$\begin{aligned} \eta_\theta(I_\delta(u)) - \frac{q}{p+1}\eta'_\theta(I_\delta(u))I_\delta(u) &\geq -\frac{q}{p+1}\eta'_\theta(I_\delta(u))I_\delta(u) \\ &\geq -\frac{C}{\theta} I_\delta(u) \geq -C \end{aligned}$$

and this proves (3.12) in this case.

### 3.4 A local supersolution

In order to control the mountain pass solutions close to the singularity we will use the supersolution to (1.4) mentioned in the introduction, which we construct next.

Fix $r_0 > 0$ small and define

$$D = \{x \in \Omega_\epsilon \ : |x| < r_0\},\ \Gamma_1 = \partial\Omega_\epsilon \cap \{|x| < r_0\} \text{ and } \Gamma_2 = \Omega_\epsilon \cap \{|x| = r_0\}.$$

Since we assume that the curve that joins $\Omega_1$ and $\Omega_2$ along which runs $C_\epsilon$ is fixed, and $0 \in \partial\Omega_1 \cap \partial\Omega_\epsilon$, if we take $r_0 > 0$ small then $D$ is independent of $\epsilon$.

Let us write $d_{\Gamma_1}(x) = dist(x, \Gamma_1)$. Let $\zeta$ be defined as the solution to

$$\begin{cases} -\Delta\zeta = \dfrac{d_{\Gamma_1}^p}{|x|^2} & \text{in } D \\ \zeta = 0 & \text{on } \Gamma_1, \ \zeta = d_{\Gamma_1} \text{ on } \Gamma_2. \end{cases}$$

The function $d_{\Gamma_1}^p/|x|^2$ belongs to $L^q(D)$ for any $1 \le q < \frac{N}{2-p}$ if $p < 2$ and for any $q \ge 1$ if $p \ge 2$. Therefore, in any case, there exists a $q > N$ such that $d_{\Gamma_1}^p/|x|^2 \in L^q(D)$. By elliptic $L^p$ estimates and the Morrey embedding $\zeta \in \mathcal{C}^{1,\beta}(\overline{D})$, for some $\beta > 0$. Hence there is some constant $C > 0$ such that $\zeta \le C d_{\Gamma_1}$. Setting $\lambda_0 = C^{-\frac{p}{p-1}} > 0$ and $w = \lambda\zeta$, the function $w$ satisfies

$$\begin{cases} -\Delta w \ge \dfrac{w^p}{|x|^2} & w > 0 \text{ in } D \\ w = 0 & \text{on } \Gamma_1 \ w \ge \lambda d_{\Gamma_1} \quad \text{on } \Gamma_2 \end{cases} \tag{3.14}$$

for any $0 \le \lambda \le \lambda_0$ and furthermore $w(x) \le C d_{\Gamma_1}(x)$ for some constant $C$. In the sequel we fix $\lambda = \lambda_0$ and $w = \lambda_0\zeta$.

### 3.5 Comparison

We claim that there is $\theta_1 > 0$ such that if $0 < \theta \le \theta_1$ then any mountain pass critical point $u$ of $E_{\delta,\epsilon,\theta}$ satisfies

$$u \le w \text{ in } D.$$

Indeed, as $I(u) \le C\theta$, c.f. (3.13), by the energy estimate (3.12) the classical $L^\infty$ and $\mathcal{C}^\beta$-estimates gives us that for any $K$ compact, $K \subset (\Omega_1 \cup \Gamma_1) \setminus \{0\}$,

$$\|u\|_{L^\infty(K)} \to 0 \quad \text{as } \theta \to 0$$

uniformly in $\epsilon, \delta$.

Then by bootstrapping $\|u\|_{\mathcal{C}^{1,\beta}(K)} \le C$ uniformly in $\epsilon, \delta$. Thus there is $\theta_1 > 0$ independent of $\epsilon, \delta$, such that for $0 < \theta \le \theta_1$ we have

$$u \le \lambda d_{\Gamma_1} \quad \text{on } \Gamma_2.$$

From (3.3) we have

$$-\Delta u \le \frac{u^p}{|x|^2 + \delta} \quad \text{in } D,$$

and therefore

$$-\Delta(u - w) \le \frac{u^p - w^p}{|x|^2 + \delta} \quad \text{in } D.$$

Multiplying by $(u-w)^+$ and integrating on $D$ we find

$$\int_D |\nabla(u-w)^+|^2 \le \int_D \frac{u^p - w^p}{|x|^2+\delta}(u-w)^+ \le C\left(\int_D |(u-w)^+|^{2^*}\right)^{\frac{N-2}{N}} \left(\int_D \frac{|u|^{(p-1)\frac{N}{2}}}{(|x|^2+\delta)^{\frac{N}{2}}}\right)^{\frac{2}{N}}$$
$$\le \left(\int_D |(u-w)^+|^{2^*}\right)^{\frac{N-2}{N}} I_\delta(u)^{\frac{2}{N}}.$$

Hence, using (3.13)

$$\int_D |\nabla(u-w)^+|^2 \le C\theta^{\frac{2}{N}} \int_D |\nabla(u-w)^+|^2.$$

Taking $\theta_1 > 0$ smaller if necessary we conclude that $(u-w)^+ \equiv 0$ in $D$, that is, $u \le w$ in $D$.

### 3.6 Control of the penalization

From this point on we fix a value of $\theta > 0$ such that $\theta \le \min(\theta_0, \theta_1)$. Let $u_{\epsilon,\delta}$ denote the mountain pass solution found in Lemma 3.2. We need to prove the following

**Lemma 3.4** *There exists* $\epsilon_0 > 0$ *such that*

$$I_\delta(u_{\epsilon,\delta}) < \theta \quad \textit{for all } 0 < \epsilon \le \epsilon_0 \textit{ and all } 0 < \delta \le \epsilon_0.$$

If we prove the above claim, then we have that

$$E_{\delta,\epsilon,\theta}(u_{\epsilon,\delta}) = \frac{1}{2}\int_{\Omega_\epsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} \frac{(u^+)^{p+1}}{|x|^2+\delta}.$$

*Proof* By contradiction. Assume that there are sequences of positive numbers $\epsilon_n \to 0, \delta_n \to 0$ such that $I_{\delta_n}(u_{\epsilon_n,\delta_n}) \ge \theta$. Let us write $u_n = u_{\epsilon_n,\delta_n}$. By (3.12) for some subsequence $u_n \rightharpoonup u$ in $H^1(\Omega_1)$ weakly and in $L^{p+1}(\Omega_1)$ strongly. Moreover $u \le w$ in $D$.

Let us show that

$$I_{\delta_n}(u_n) \to I_0(u) \quad \text{as } n \to \infty \tag{3.15}$$

where

$$I_0 = \int_{\Omega_1} \frac{(u^+)^{(p-1)\frac{N}{2}}}{|x|^N}.$$

Indeed,

$$I_{\delta_n}(u_n) = \int_{\Omega_1} \frac{g_{\delta_n}(u_n+\delta_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}}$$

and $\dfrac{g_{\delta_n}(u_n+\delta_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}} \to \dfrac{(u^+)^{(p-1)\frac{N}{2}}}{(|x|^2+\delta)^{\frac{N}{2}}}$ pointwise in $\Omega_1$. We also have that $\dfrac{g_{\delta_n}(u_n+\delta_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}}$ is bounded uniformly in $\Omega_1 \setminus D$. In $D$ the following inequalities hold

$$\frac{g_{\delta_n}(u_n+\delta_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}} \le \frac{g_{\delta_n}(w+\delta_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}} \le C\frac{(w+\delta_n)^{(p-1)\frac{N}{2}}}{(|x|^2+\delta_n)^{\frac{N}{2}}}.$$

But $w \le C|x|$, and therefore

$$\frac{g_{\delta_n}(u_n+\delta_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}} \le C\frac{(|x|+\delta_n)^{(p-1)\frac{N}{2}}}{(|x|^2+\delta_n)^{\frac{N}{2}}} \le C(|x|+\delta_n^{1/2})^{(p-1)\frac{N}{2}-N}.$$

If $(p-1)\frac{N}{2}-N \ge 0$ then this quantity is uniformly bounded and if $(p-1)\frac{N}{2}-N < 0$ then

$$\frac{g_{\delta_n}(u_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}} \le C|x|^{(p-1)\frac{N}{2}-N}$$

which is integrable. By the dominated convergence theorem we deduce the validity of (3.15). A consequence of (3.15) and (3.13) is that

$$I_0(u) \le C\theta. \tag{3.16}$$

We claim that $u$ satisfies

$$\begin{cases} -\Delta u + (p-1)\dfrac{N}{2}\eta_\theta'(I_0(u))\dfrac{\chi_{[u>0]}}{|x|^N}u^{(p-1)\frac{N}{2}-1} \le \dfrac{u^p}{|x|^2}, & \text{in } \Omega_1 \\ u=0 & \text{on } \partial\Omega_1 \end{cases} \tag{3.17}$$

To prove this let $\varphi \in \mathcal{C}^1(\overline{\Omega}_1)$, $\varphi \ge 0$ with $\varphi = 0$ on $\partial\Omega_1$. Multiplying (3.3) by $\varphi$ and integrating in $\Omega_1$ yields

$$\int_{\Omega_1} \nabla u_n \nabla\varphi + \eta_\theta'(I_{\delta_n}(u_n)) \int_{\Omega_1} \frac{g_{\delta_n}'(u_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}}\varphi = \int_{\Omega_1} \frac{(u_n^+)^p}{|x|^2+\delta_n}\varphi.$$

As before

$$\int_{\Omega_1} \frac{(u_n^+)^p}{|x|^2+\delta_n}\varphi \to \int_{\Omega_1} \frac{(u^+)^p}{|x|^2}\varphi \quad \text{as } n \to \infty.$$

Using Fatou's lemma

$$\int_{\Omega_1} \frac{\chi_{[u>0]}u^{(p-1)\frac{N}{2}-1}}{(|x|^2)^{\frac{N}{2}}}\varphi \le \int_{\Omega_1} \frac{g_{\delta_n}'(u_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}}\varphi$$

and this proves (3.17). □

Now, multiplying (3.17) by $u^+$ and integrating in $\Omega_1$ yields

$$\int_{\Omega_1} |\nabla u^+|^2 \le \int_{\Omega_1} \frac{(u^+)^{p+1}}{|x|^2} \le C\left(\int_{\Omega_1} (u^+)^{2^*}\right)^{\frac{N-2}{N}} \left(\int_{\Omega_1} \frac{(u^+)^{(p-1)\frac{N}{2}}}{|x|^N}\right)^{\frac{2}{N}}$$
$$\le C\int_{\Omega_1} |\nabla u^+|^2 I_0(u)^{\frac{2}{N}},$$

so that by (3.16)

$$\int_{\Omega_1} |\nabla u^+|^2 \le C\theta^{\frac{2}{N}} \int_{\Omega_1} |\nabla u^+|^2.$$

Since $\theta > 0$ is small, we conclude that $u^+ \equiv 0$ in $\Omega_1$ and this implies that $I_0(u) = 0$. But $I_0(u) = \lim_{n\to\infty} I_{\delta_n}(u_n) \ge \theta$, which is a contradiction.

3.7 Passing to the limit and end of the proof

If $0 < \epsilon \le \epsilon_0$ and $0 < \delta \le \epsilon_0$ the mountain pass solution $u_{\epsilon,\delta}$ of Lemma 3.2 satisfies

$$\begin{cases} -\Delta u = \dfrac{(u^+)^p}{|x|^2+\delta} & \text{in } \Omega_\epsilon \\ u = 0 & \text{on } \partial\Omega_\epsilon \end{cases} \tag{3.18}$$

Then $u_{\epsilon,\delta} \ge 0$. Recall that by Lemma 3.3

$$E_{\delta,\epsilon,\theta}(u_{\epsilon,\delta}) \ge c_0 > 0.$$

Thus $u_{\epsilon,\delta} \not\equiv 0$ and then $u_{\epsilon,\delta} > 0$ in $\Omega_\epsilon$ by the strong maximum principle.

Now, for fixed $0 < \epsilon \le \epsilon_0$ we let $\delta \to 0$. By (3.12) there is a sequence $\delta_n \to 0$ such that $u_{\epsilon,\delta_n}$ converges weakly in $H_0^1(\Omega_\epsilon)$, and in the $\mathcal{C}^1$ norm on compact sets of $\overline{\Omega}_\epsilon \setminus \{0\}$. Since we have $u_{\epsilon,\delta_n} \le w$ we can use dominated convergence to show that

$$\int_{\Omega_\epsilon} \frac{u^p_{\epsilon,\delta_n}}{|x|^2+\delta_n}\varphi \to \int_{\Omega_\epsilon} \frac{u^p_\epsilon}{|x|^2}\varphi \quad \text{as } \delta_n \to 0 \quad \text{for any } \varphi \in \mathcal{C}^1(\overline{\Omega}_\epsilon).$$

Thus $u_\epsilon$ satisfies

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2}, & u \ge 0 \text{ in } \Omega_\epsilon \\ u = 0 & \text{on } \partial\Omega_\epsilon \end{cases}$$

Multiplying (3.18) by $u_{\epsilon,\delta_n}$ we find that

$$E_{\delta_n,\epsilon,\theta}(u_{\epsilon,\delta_n}) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\epsilon} \frac{u^p_{\epsilon,\delta_n}}{|x|^2+\delta_n}$$

and by dominated convergence, using $u_{\epsilon,\delta_n} \le w$ in $D$, we see that

$$E_{\delta_n,\epsilon,\theta}(u_{\epsilon,\delta_n}) \to \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\epsilon} \frac{u^p_\epsilon}{|x|^2} \quad \text{as } n \to \infty.$$

Since $E_{\delta_n,\epsilon,\theta}(u_{\epsilon,\delta_n}) \ge c_0 > 0$ by Lemma 3.3 we deduce that $u_\epsilon > 0$. This concludes the proof of Theorem 1.3.

## 4 Some remarks on other powers

Consider

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^\alpha}, \quad u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$ is a bounded domain with smooth boundary, $0 < p < \frac{N+2}{N-2}$ and $0 \leq \alpha < \frac{N}{2} + 1$. For $\alpha \in \mathbb{R}$ let

$$p^*(\alpha) = \frac{2(N-\alpha)}{N-2}.$$

Hölder's inequality and the Sobolev embedding yield the following result.

**Lemma 4.1** *If* $0 \leq \alpha < \frac{N}{2} + 1$ *and* $1 \leq p < p^*(\alpha)$ *there exists* $C > 0$ *such that*

$$\left( \int_\Omega \frac{|u|^p}{|x|^\alpha} \right)^{1/p} \leq C \left( \int_\Omega |\nabla u|^2 \right)^{1/2} \quad \textit{for all } u \in H_0^1(\Omega). \tag{4.2}$$

If $0 \leq \alpha \leq 2$ we can take $p = p^*(\alpha)$ in (4.2) by the Caffarelli–Kohn–Nirenberg inequality [7].

As we will see, the number $p^*(\alpha) - 1 = \frac{N+2(1-\alpha)}{N-2}$ is a critical exponent for (4.1) if $0 \in \overline{\Omega}$. Note that $0 < p^*(\alpha) - 1 < 1$ for $2 < \alpha < \frac{N}{2} + 1$ and $1 < p^*(\alpha) - 1 < \frac{N+2}{N-2}$ for $0 < \alpha < 2$. Notice also that if $\alpha \geq \frac{N}{2} + 1$ then $p^*(\alpha) \leq 1$.

Thanks to inequality (4.2) we have the following result.

**Proposition 4.2** *Let* $0 \leq \alpha < \frac{N}{2} + 1$ *and* $0 < p < p^*(\alpha) - 1$, $p \neq 1$. *Then problem* (4.1) *has a solution* $u \in H_0^1(\Omega)$.

*Proof* Consider the functional

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega \frac{(u^+)^{p+1}}{|x|^\alpha}, \quad u \in H_0^1(\Omega).$$

Under the hypotheses on $p$ and $\alpha$, it is standard to verify that $J$ is $C^1$. If $0 < p < 1$ and $p < p^*(\alpha) - 1$, we can prove existence by minimization. Indeed, write $p^* = p^*(\alpha)$. By Hölder's inequality and (4.2)

$$\int_\Omega \frac{(u^+)^{p+1}}{|x|^\alpha} \leq \left( \int_\Omega \frac{(u^+)^{p^*}}{|x|^\alpha} \right)^{\frac{p+1}{p^*}} \left( \int_\Omega \frac{1}{|x|^\alpha} \right)^{\frac{p^*-p-1}{p^*}} \leq C \left( \int_\Omega |\nabla u|^2 \right)^{\frac{p+1}{2}} \tag{4.3}$$

since $1/|x|^\alpha$ is integrable. This shows that $J$ is coercive. It is also lower semi-continuous for the weak topology: if $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, for any $\delta > 0$

$$\begin{aligned}\int_\Omega \frac{|(u_n^+)^{p+1} - (u^+)^{p+1}|}{|x|^\alpha} &= \int_{B_\delta(0)} \cdots + \int_{\Omega\setminus B_\delta(0)} \cdots \\ &\le C\delta^{(N-\alpha)\frac{p^*-p-1}{p^*}}\left(\|u_n\|_{H_0^1}^{p+1} + \|u\|_{H_0^1}^{p+1}\right) \\ &\quad + \delta^{-\alpha}\int_{\Omega\setminus B_\delta(0)} |(u_n^+)^{p+1} - (u^+)^{p+1}|.\end{aligned}$$

This implies that for $\delta > 0$

$$\limsup_{n\to\infty}\int_\Omega \frac{|(u_n^+)^{p+1} - (u^+)^{p+1}|}{|x|^\alpha} \le C\delta^{(N-\alpha)\frac{p^*-p-1}{p^*}}$$

and hence

$$\lim_{n\to\infty}\int_\Omega \frac{(u_n^+)^{p+1}}{|x|^\alpha} = \int_\Omega \frac{(u^+)^{p+1}}{|x|^\alpha}. \tag{4.4}$$

Therefore $J$ attains a minimum at a function $u \in H_0^1(\Omega)$ and one can check that it must be non-zero. This yields a solution $u \in H_0^1(\Omega)$ of (4.1).

If $p > 1$ and $p < p^*(\alpha) - 1$ there is still a solution by the mountain pass theorem. Indeed, $J$ verifies the geometric conditions of the theorem by (4.3). It also verifies the Palais-Smale condition. In fact, if $(u_n)$ is a sequence in $H_0^1(\Omega)$ such that $J(u_n)$ remains bounded and $J'(u_n) \to 0$ in $H^{-1}(\Omega)$, the usual argument shows that $u_n$ is bounded, because the nonlinearity satisfies the Ambrosetti–Rabinowitz condition. To prove that it has a strongly convergent subsequence it is sufficient to show that for some subsequence (4.4) holds, which was essentially done above. □

Using the Pohozaev multiplier as in Sect. 2 and the strong maximum principle combined with the Picone identity, similarly as in the introduction, we can prove the following non existence result. Since the argument is essentially the same as before, we omit the proof.

**Proposition 4.3** *(1) Assume* $0 \in \overline{\Omega}$ *and* $\Omega$ *starshaped with respect to the origin. If* $p \ge p^*(\alpha) - 1$ *then there is not solution* $u \in H_0^1(\Omega)$ *to* (4.1).
*(2) Assume* $0 \in \Omega$*. If* $p \ge 1$ *and* $\alpha > 2$ *there is no weak solution of* (4.1).

Part (2) in this Proposition reflects a local obstruction to existence. When $0 \in \partial\Omega$ and $0 < \alpha < p + 1$ this local obstruction does not exist and one has the same result as in Theorem 1.3:

**Theorem 4.4** *Suppose,* $1 < p < \frac{N+2}{N-2}$ *and* $0 < \alpha < p + 1$*. Assume that* $\Omega_\epsilon$ *is a dumbbell domain such that* $0 \in \partial\Omega_1 \cap \partial\Omega_\epsilon$*. Then there exist* $\epsilon_0 > 0$ *such that if* $0 < \epsilon < \epsilon_0$*, problem* (4.1) *has a solution.*

*Proof* The proof of the above theorem is basically the same as of Theorem 1.3, and therefore we only sketch the necessary modifications. First we redefine the functionals

$$E_{\delta,\epsilon,\theta}(u) = \frac{1}{2}\int_{\Omega_\epsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} \frac{(u^+)^{p+1}}{|x|^\alpha + \delta} + \eta_\theta(I_\delta(u)),$$

and

$$I_\delta(u) = \int_{\Omega_1} \frac{g_\delta(u+\delta)}{(|x|^\alpha+\delta)^{\frac{N}{2}}},$$

where $\epsilon$, $\delta$, $\theta > 0$ and $\eta_\theta$, $g_\delta$ are as in (3.1) and (3.2). Then Sects. 3.1, 3.2 and 3.3 remain the same.

The construction of a local supersolution is very similar. With the notation of Sect. 3.4 we take

$$\begin{cases} -\Delta\zeta = \dfrac{d_{\Gamma_1}^p}{|x|^\alpha} & \text{in } D \\ \zeta = 0 & \text{on } \Gamma_1,\ \zeta = d_{\Gamma_1} \text{ on } \Gamma_2. \end{cases}$$

The function $d_{\Gamma_1}^p/|x|^\alpha$ belongs to $L^q(D)$ for any $1 \le q < \frac{N}{\alpha-p}$ if $p < \alpha$ and for any $q \ge 1$ if $p \ge \alpha$. Since $\alpha < p+1$, there exists a $q > N$ such that $d_{\Gamma_1}^p/|x|^\alpha \in L^q(D)$. By elliptic $L^p$ estimates and the Morrey embedding $\zeta \in C^{1,\beta}(\overline{D})$, for some $\beta > 0$. Hence there is some constant $C > 0$ such that $\zeta \le C d_{\Gamma_1}$. This implies that $w = \lambda\zeta$ satisfies

$$\begin{cases} -\Delta w \ge \dfrac{w^p}{|x|^\alpha} & w > 0 \text{ in } D \\ w = 0 & \text{on } \Gamma_1\ w \ge \lambda d_{\Gamma_1} \text{ on } \Gamma_2 \end{cases}$$

for $\lambda > 0$ suitably small. Moreover, $w(x) \le C d_{\Gamma_1}(x)$ for some constant $C$. The rest of the argument is essentially unchanged. □

### 4.1 Some open problems

(1) Assume $0 \in \partial\Omega$. Do the 2 types of energy solutions discussed in the introduction represent all possible behavior of energy solutions? More precisely, is it true that an energy solution is either continuous and a posteriori $C^{1,\beta}(\overline{\Omega})$, or it is discontinuous and of the form $u(x) = \varphi(x/|x|) + o(1)$ as $x \to 0$, where $\varphi$ is a positive solution of (1.3)?

(2) Assume $0 \in \partial\Omega$. Are there domains such that (1.1) admits discontinuous solutions, such as the ones of the form $\varphi(x/|x|)$ near 0?

(3) Part (2) in Proposition 4.3 is due to a local obstruction to existence. Suppose still $0 \in \Omega$, $\alpha > 2$ but $p < 1$. In this case $p^*(\alpha) - 1 < 1$. Is there a local obstruction to solutions if $\frac{N-\alpha}{N-2} \le p < 1$? The reason for the lower limit on $p$ is that when $0 < p < \frac{N-\alpha}{N-2}$ there is an explicit radial solution

$$u = Ar^{-\beta}$$

where $\beta = \frac{\alpha-2}{1-p} > 0$.

Are there domains such that for $p^*(\alpha) < p < \frac{N-\alpha}{N-2}$ problem (4.1) has solutions?

(4) Assume $0 \in \partial\Omega$, $p > 1$ and $\alpha \ge p+1$. Is there a local obstruction for the existence of solutions?

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