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Multiplicity of solutions for a fourth order equation with power-type nonlinearity

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Abstract Let B be the unit ball in $\mathbb{R}^N$, $N \geq 3$ and n be the exterior unit normal vector on the boundary. We consider radial solutions to

$$\Delta^2 u = \lambda(1 + \mathrm{sign}(p)u)^p \quad \text{in } B, \quad u = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B$$

where $\lambda \geq 0$. For positive p we assume $5 \leq N \leq 12$ and $p > \frac{N+4}{N-4}$, or $N \geq 13$ and $\frac{N+4}{N-4} < p < p_c$, where p_c is a constant depending on N. For negative p we assume $4 \leq N \leq 12$ and $p < p_c$, or $N = 3$ and $p_c^+ < p < p_c$, where p_c^+ is a constant. We show that there is a unique $\lambda_S > 0$ such that if $\lambda = \lambda_S$ there exists a radial weakly singular solution. For $\lambda = \lambda_S$ there exist infinitely many regular radial solutions and the number of radial regular solutions goes to infinity as $\lambda \to \lambda_S$.

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1 Introduction

In their well known work, Joseph and Lundgren [30] gave a complete characterization of all positive solutions of the problem

$$- \Delta u = \lambda g(u) \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B, \tag{1}$$

where $g(u) = e^u$ or $g(u) = (1 + a\,u)^p$, $ap > 0$, B is the unit ball in $\mathbb{R}^N$, and $\lambda > 0$. In particular, they found a remarkable phenomenon for $g(u) = e^u$ and $N > 2$: either (1) has at most one solution for each λ or there is a value of λ for which infinitely many solutions exists. In the case of a power nonlinearity the same alternative is valid if $N \geq 3$ and $p \notin (1, (N+2)/(N-2)]$. The multiplicity result of Joseph and Lundgren, established for radial solutions, is based on earlier work of Barenblatt for the exponential nonlinearity, who used Emden's transformation to obtain infinitely many solutions for one λ in 3 dimensions, see [21]. We recall that all positive smooth solutions of (1) are radial by the result of Gidas et al. [23].

A general problem formulated by Lions [33, Sect. 4.2 (c)] is whether it is possible to obtain a description of the solution set of higher order semilinear equations, similar to those known for (1).

In this paper we study a semilinear equation involving the bilaplacian operator and a power type nonlinearity:

$$\Delta^2 u = \lambda\,(1 + \mathrm{sign}(p) u)^p \quad \text{in } B, \quad u = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B \tag{2}$$

where $\lambda > 0$, $p \in \mathbb{R}$, $p \neq 0$ and $\mathrm{sign}(p) = 1$ if $p > 0$, $\mathrm{sign}(p) = -1$ if $p < 0$. We also treat Navier boundary conditions. We show that (2) presents a multiplicity phenomenon of radial solutions similar to the one known for the second order equation, if p is restricted to be in a region that involves a critical number p_c, defined in (7) below, which was introduced in a recent work by Gazzola and Grunau [20]. We consider only radial solutions, since all positive smooth solutions of (2) are radial, see Berchio et al. [6].

A motivation for considering negative powers in (2) stems from a model for the steady states of a simple *micro electromechanical system* (MEMS) which has the general form (see for example [32,37])

$$\begin{cases} \alpha \Delta^2 u = \left(\beta \displaystyle\int_\Omega |\nabla u|^2 \, dx + \gamma \right) \Delta u + \lambda (1-u)^{-2} \dfrac{f(x)}{(1 + \chi \int_\Omega \frac{dx}{(1-u)^2})} & \text{in } \Omega \\ 0 < u < 1 \qquad \text{in } \Omega & \\ u = 0, \quad \alpha \dfrac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega. & \end{cases}$$

We will consider this equation posed in $\Omega = B$ with $\beta = \gamma = \chi = 0$, $\alpha = 1$ and $f(x) \equiv 1$. Note that in this model only the power $p = -2$ is relevant, but we shall work with a larger range of negative powers.

Problem (2) for positive powers has been studied in [5,14–16,31] and the case of negative powers has been treated in [9,11,16,26–28,34]. For both cases it is known that there exists $\lambda^* > 0$ such that there is a radial classical solution of (2) for $0 < \lambda < \lambda^*$ and there are no solutions for $\lambda > \lambda^*$. In fact, for $0 < \lambda < \lambda^*$ there is a pointwise minimal and regular solution u_λ. It is also known that the monotone limit $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$ belongs to at least $L^1(B)$ and is a weak solution, called the extremal solution. For the proofs see for example [14] for positive powers, where the authors also proved the stability of the minimal solution, that singular solutions are weakly singular and characterized regular versus singular solutions in terms of the behavior of an autonomous first order system of ODEs. In [7] they obtained the existence of the threshold λ^* when $p = -2$ and $N = 3$, but the argument applies to any $p < 0$ and $N \geq 1$. In [15] the authors proved that if $N \geq 5$ and $p > \frac{N+4}{N-4}$ then there is a singular solution for some value of λ, and if addition $5 \leq N \leq 12$ and or $N \geq 13$ and $p < p_c$ then the extremal solution is bounded. For boundedness of the extremal solution for negative powers in [11] the authors obtain for $p = -2$ the sharp dimensions for which u^* is regular. See also [16] where they proved that if $N \leq 4$ and $p < 0$ together with $p \leq (2 - N)/2$.

An ingredient in previous arguments, e.g. in [15], is a relation of (2) with entire solutions to

$$\Delta^2 U = \operatorname{sign}(p) U^p, \quad U > 0 \quad \text{in } \mathbb{R}^N. \tag{3}$$

In the case of positive p and $p < p_c$ the authors in [15] showed that the positive entire solutions of (3) oscillate infinitely around the explicit singular solution, see also [20] for positive powers, and [27] in the case $N = 3$ and $p = -2$. The stability of the entire solutions of (3) has been studied in [31], where the author also obtained that for $N \geq 13$ and $p \geq p_c$ the entire solutions are ordered. A similar phenomenon for some negative p is proved in [28].

Problem (2) has a resemblance with the case of an exponential nonlinearity studied in [1,2,12], and [13] where we have obtained recently multiplicity results similar to the ones in this work.

To introduce our results we define the notion of weak solution for (2). If $p > 0$ we call u a weak solution of (2) if

$$\begin{cases} u \in L^1(B),\, u \geq 0 \text{ a.e., } (1+u)^p \in L^1(B), \text{ and} \\ \displaystyle\int_B u \Delta^2 \varphi = \lambda \int_B (1+u)^p \varphi \quad \forall \varphi \in C^4(\bar{B}), \quad \varphi|_{\partial B} = \nabla \varphi|_{\partial B} = 0. \end{cases} \tag{4}$$

If $p < 0$ a weak solution u of (2) is

$$\begin{cases} u \in L^1(B),\, 0 \leq u < 1 \text{ a.e., } (1-u)^p \in L^1(B), \text{ and} \\ \displaystyle\int_B u \Delta^2 \varphi = \lambda \int_B (1-u)^p \varphi \quad \forall \varphi \in C^4(\bar{B}), \quad \varphi|_{\partial B} = \nabla \varphi|_{\partial B} = 0. \end{cases} \tag{5}$$

If $p > 0$ a weak solution u to (2) is called singular if $u \notin L^\infty(B)$ and regular otherwise. If $p < 0$ a weak solution u to (2) is called singular if $\|u\|_{L^\infty(B)} = 1$ and regular if $\|u\|_{L^\infty(B)} < 1$. By standard regularity theory regular solutions are C^∞. Radial solutions can be only singular at the origin, that is, if u is a radial solution then $u(r)$ is smooth for $r \in (0, 1)$.

For $p > 1$ a radial singular solution $u = u(r)$ of (2) is called weakly singular if

$$\lim_{r\to 0} r^\tau u(r) \quad \text{exists}$$

where

$$\tau = \frac{4}{p-1},$$

while for $p < 0$ a radial singular solution $u = u(r)$ of (2) is called weakly singular if

$$\lim_{r\to 0} r^\tau (1 - u(r)) \quad \text{exists.}$$

Ferrero and Grunau [14, Theorem 3] proved that if $N \geq 5$ and $p > \frac{N+4}{N-4}$ then any radial singular weak solution of (2) is also weakly singular. For negative powers this is also true (see Sect. 6).

Theorem 1 *Assume $N \geq 4$ and $p < -1$, or $N = 3$ and $-3 < p < -1$. Then any radial singular weak solution of (2) is also weakly singular.*

1.1 Main results for positive powers

Theorem 2 *Assume $N \geq 5$ and $p > \frac{N+4}{N-4}$. Then there exists a unique $\lambda_S > 0$ such that (2) with $\lambda = \lambda_S$ admits a radial weakly singular solution and this weakly singular solution is unique.*

Let $\mathcal{C}$ denote the solution set associated to (2), that is,

$$\mathcal{C} = \{(\lambda, u) \in (0, \infty) \times C^4(\overline{B}) \ : \ u \text{ is radial and solves (2)}\}. \tag{6}$$

Theorem 3 *Assume $N \geq 5$ and $p > \frac{N+4}{N-4}$. The set $\mathcal{C}$ is homeomorphic to $\mathbb{R}$ and the identification can be done through $(\lambda, u) \in \mathcal{C} \mapsto u(0)$.*

The inverse of the above identification can be extended as $0 \mapsto (0, 0)$ and $\infty \mapsto (\lambda_S, u_S)$ where u_S is the unique weakly singular solution of Theorem 2.

Define

$$p_c = \frac{N + 2 - \sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}}{N - 6 - \sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}} \quad \text{for} \quad N \geq 3 \tag{7}$$

with

$$H_N = (N(N-4)/4)^2.$$

The constant H_N appears as the best constant in the Hardy-Rellich inequality, see [38]. We note that if $3 \leq N \leq 12$ then $p_c < 0$ and if $N \geq 13$ then $p_c > 0$.

The main multiplicity result for positive powers is the following.

Theorem 4 *Assume*

$$5 \leq N \leq 12 \quad \textit{and} \quad \frac{N+4}{N-4} < p < +\infty, \quad \textit{or} \quad N \geq 13 \quad \textit{and} \quad \frac{N+4}{N-4} < p < p_c. \tag{8}$$

Then for $\lambda = \lambda_S$ problem (2) has infinitely many radial smooth solutions. For $\lambda \neq \lambda_S$ there are finitely many radial smooth solutions and their number goes to infinity as $\lambda \to \lambda_S$.

1.2 Main results for negative powers

Theorem 5 *Assume $N \geq 4$ and $p < -1$, or $N = 3$ and $-3 < p < -1$. Then there exists a unique $\lambda_S > 0$ such that (2) with $\lambda = \lambda_S$ admits a radial weakly singular solution and this weakly singular solution is unique.*

Define $\mathcal{C}$ as in (6).

Theorem 6 *Assume $N \geq 4$ and $p < -1$, or $N = 3$ and $-3 < p < -1$. The set $\mathcal{C}$ is homeomorphic to $(0, 1)$ and the identification can be done through $(\lambda, u) \in \mathcal{C} \mapsto u(0)$.*

The inverse of the above identification can be extended as $0 \mapsto (0, 0)$ and $1 \mapsto (\lambda_S, u_S)$ where u_S is the unique weakly singular solution of Theorem 5.

Define

$$p_c^+ = \frac{N + 2 + \sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}}{N - 6 + \sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}} \quad \text{for} \quad N \geq 3, \quad N \neq 4. \tag{9}$$

Theorem 7 *Assume*

$$N = 3 \quad \textit{and} \quad p_c^+ < p < p_c, \quad \textit{or} \quad 4 \leq N \leq 12 \quad \textit{and} \quad -\infty < p < p_c. \tag{10}$$

where p_c is given in (7). Then for $\lambda = \lambda_S$ problem (2) has infinitely many radial smooth solutions. For $\lambda \neq \lambda_S$ there are finitely many radial smooth solutions and their number goes to infinity as $\lambda \to \lambda_S$.

We note that if $3 \leq N \leq 12$ then $p_c < 0$ and if $N \geq 13$ then $p_c > 0$. When $N = 3$, the range $p_c^+ < p < p_c$ can be written as

$$-2.626\ldots = -\frac{5 + \sqrt{13 - 3\sqrt{17}}}{3 - \sqrt{13 - 3\sqrt{17}}} < p < -\frac{5 - \sqrt{13 - 3\sqrt{17}}}{3 + \sqrt{13 - 3\sqrt{17}}} = -1.108\ldots$$

and when $N = 4$ we have $p_c = -1$.

1.3 Further results and comments

Concerning Theorems 2 and 5, there is a proof of existence of weakly singular solutions for positive powers in [15], but uniqueness is not treated.

For positive powers, in [15] the authors showed, using the ideas of [12], that in the range (8) one has $\lambda_S < \lambda^*$ and u^* is regular. In the case $p = -2$, in [11] they proved that u^* is regular if $1 \le N \le 8$ and singular if $N \ge 9$. We can actually complete part of this result for negative powers:

Corollary 1 *Assume that p is in the range* (10). *Then $\lambda_S < \lambda^*$ and u^* is regular.*

This corollary follows from Theorem 7, since we also prove that under (10) there are regular radial solutions for $\lambda > \lambda_S$ close to λ_S. It follows then that $\lambda^* > \lambda_S$. If u^* is singular, then by Theorem 1 it would be weakly singular and this would contradict the uniqueness part of Theorem 5.

It is natural to ask: if p is the complementary ranges to (8) and (10), more precisely, if

$$\begin{cases} N \ge 13 \text{ and } p \ge p_c, \text{ or} \\ N = 3 \text{ and } p \in (-3, p_c^+] \cup [p_c, -1), \text{ or} \\ 5 \le N \le 12 \text{ and } p_c \le p < -1, \text{ or} \\ N \ge 13 \text{ and } p < -1, \end{cases} \tag{11}$$

is u^* is singular? We know that in some cases this is true, see [11], where they proved that if $p = -2$ then u^* is singular if and only if $N \ge 9$, which is consistent with (11). But surprisingly the answer in part of the range (11) is negative. In fact, in [16] the authors show that when $N = 3$ and $p \le -1/2$ then u^* is regular. This implies that the curve of solution must bend back at λ^* and then continues to the weakly singular solution u_S. Numerical computations shown in Fig. 1 suggest that if $N = 3$ and $p \in (-3, p_c^+] \cup [p_c, -1)$, there is no oscillation as $\lambda \to \lambda_S$ and that the number of solutions is bounded independently of λ and bigger than one in some intervals. This is notably different to what happens for the Laplacian with power-type nonlinearities, where it is known that either there is uniqueness of solutions for all λ or there is some λ with infinitely many solutions, see [30].

It remains an open problem whether u^* is singular in the range (11), with $N \ge 5$ in the case of negative powers. In Fig. 2, we observe from numerical calculations that u^* is regular when $p = -1.02$ and $N = 5$ or $N = 6$. We conjecture that for each $p \in (p_c, -1)$ there exists a critical dimension N_p such that u^* is singular for $N \ge N_p$.

The computations of Figs. 1 and 2 were done for the Navier problem (12) to obtain solutions with $u(0)$ closer to 1. We found, however, that the Dirichlet problem (2) produces qualitatively similar pictures.

We complement the previous results with the following property, that relates the regularity of u^* to uniqueness of solutions.

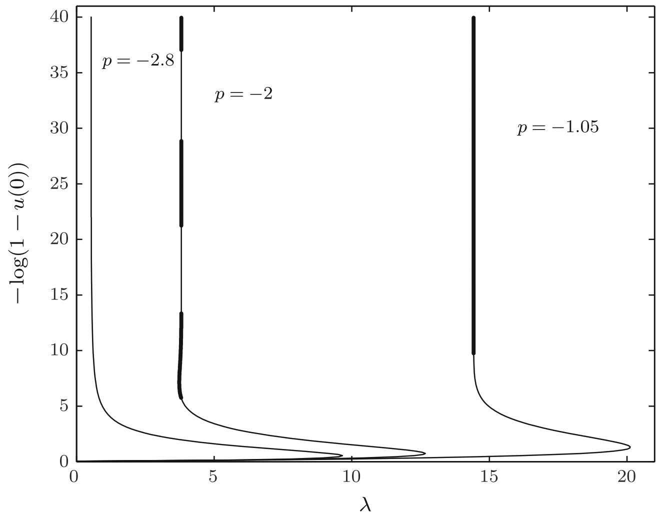

Fig. 1 Bifurcation diagram of Eq. (12) for negative powers and $N = 3$. Since oscillations have small amplitude, we draw in a *thicker line* the points to the left of λ_S in the range $-\log(1 - u(0)) > 4$. We estimate λ_S as the numerical value of λ at the highest point in each curve. For $p = -2$, the bifurcation diagram has the form described by Theorem 9

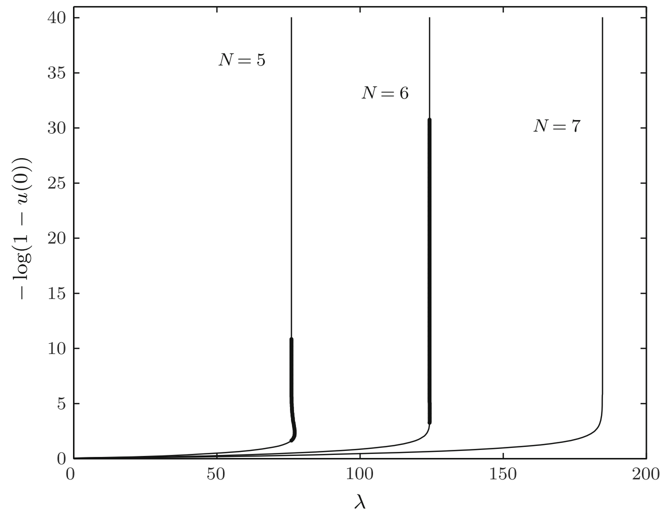

Fig. 2 Bifurcation diagram of Eq. (12) for $p = -1.02$ and several dimensions. We draw in a *thicker line* the points to the right of λ_S. We estimate λ_S as the numerical value of λ at the highest point in each curve. Here for $N = 5$ and $N = 6$ we find non-uniqueness

Proposition 2 *Assume*

(a) $N \geq 5$ *and* $p > \frac{N+4}{N-4}$, *or*
(b) $N \geq 4$ *and* $p < -1$, *or* $N = 3$ *and* $-3 < p < -1$.

Then u^* *is singular if and only if for each* $\lambda \in (0, \lambda^*)$ *Eq.* (2) *has a unique solution.*

For the problem with Navier boundary conditions

$$\begin{cases} \Delta^2 u = \lambda\,(1 + \mathrm{sign}(p)u)^p & \text{in } B \\ \quad u = \Delta u = 0 & \text{on } \partial B \end{cases} \tag{12}$$

we have similar results.

Theorem 8 *Assume*

(a) $N \geq 5$ *and* $p > \frac{N+4}{N-4}$, *or*
(b) $N \geq 4$ *and* $p < -1$, *or* $N = 3$ *and* $-3 < p < -1$.

Then there exists a unique $\lambda_S > 0$ *such that* (12) *with* $\lambda = \lambda_S$ *admits a radial weakly singular solution and this weakly singular solution is unique.*

Theorem 9 *Assume* (8) *for positive* p *or* (10) *for negative* p. *Then* (12) *with* $\lambda = \lambda_S$ *admits infinitely many regular radial solutions. For* $\lambda \neq \lambda_S$ *then* (12) *has a finite number of regular radial solutions and the number of radial regular solutions goes to infinity as* $\lambda \to \lambda_S$.

By a change of variables we transform the ODE version of (2) into a reasonable first order 4 dimensional nonlinear system, treating simultaneously positive and negative powers. The system has 2 stationary points P_1, P_2. Some properties of this or similar systems were studied in [14–16,20,27]. We review this material in Sect. 2. The existence and uniqueness of a weakly singular solution is related to the properties of the unstable manifold of P_2. This is explained in Sect. 3.

For the multiplicity results we follow the same argument as in [13] for the bilaplacian with exponential nonlinearity. This idea traces back to the work of Bamón, Flores, del Pino [3] and was subsequently applied also in [17–19]. An important step consists in finding a heteroclinic connection from P_1 to P_2. This connection was found by Gazzola, Grunau in [20] for positive powers and by Guo and Wei [27] in the case $N = 3$ and $p = -2$, based on the analysis of entire solutions. We complete this analysis for the remaining negative powers in Sect. 4. This extension is not trivial. In [20,27] the authors introduce a natural energy that decreases along trajectories that oscillate infinitely many times. However, for many negative exponents this energy does not seem useful, and we have to find an alternative argument. We find a similar difficulty in proving Theorem 1, which we do in Sect. 6.

In Sect. 5 we explain how to obtain the connection from the entire solution. This connection is then useful to establish that in the correct range of powers the unstable manifold of P_1, which gives rise to regular solutions, has a spiral structure around the unstable manifold of P_2, and this yields the multiplicity results, see Sect. 7.

The fact that the solution set is homeomorphic to $(0, \infty)$ is based on an idea of Guo and Wei [27], that asserts that the radial solutions to (2) are uniquely determined by their value at the origin. We then show that these values actually cover the whole interval $(0, \infty)$ or $(0, 1)$. This is done in Sect. 8, where we complete the proofs of Theorems 3, 6 and Proposition 2.

2 Preliminaries

2.1 Important constants

We define

$$\tau = \frac{4}{p-1}, \quad \text{and} \quad K_0 = \tau(\tau+2)(N-2-\tau)(N-4-\tau). \tag{13}$$

In the sequel we shall work in the following range of p. If p is positive:

$$N \geq 5 \quad \text{and} \quad p > \tfrac{N+4}{N-4} \tag{14}$$

and p is negative:

$$N = 3 \quad \text{and} \quad -3 < p < -1, \quad \text{or} \quad N \geq 4 \quad \text{and} \quad p < -1. \tag{15}$$

In this range we have

$$\operatorname{sign}(K_0) = \operatorname{sign}(p) = \operatorname{sign}(\tau). \tag{16}$$

Indeed, even for $p > N/(N-4)$ and $N \geq 5$, we have

$$\tau + 2 > 0, \quad p\tau > 0, \quad N-4-\tau > 0, \quad \text{and} \quad N-2-\tau > 0 \tag{17}$$

and hence $K_0 > 0$. If $p < -1$ the inequality $\tau + 2 > 0$ holds and if $N = 3$, then $N-4-\tau > 0$ for $p > -3$. Therefore for $N = 3$ and $-3 < p < -1$ or $N \geq 4$ and $p < -1$ we have (17) and hence $K_0 < 0$.

In the sequel we will write

$$\alpha = \operatorname{sign}(p).$$

and we shall use that $\alpha^2 = 1$ in some of the forthcoming computations.

For some of the arguments it will be convenient to work with the following change of variables

$$U = \left(\frac{\lambda}{\alpha K_0}\right)^{\frac{1}{p-1}} (1 + \alpha u).$$

Then (2) becomes

$$\begin{cases} \Delta^2 U = K_0 U^p & \text{in } B \\ U = \left(\dfrac{\lambda}{\alpha K_0}\right)^{\frac{1}{p-1}} \quad \text{and} \quad \dfrac{\partial U}{\partial n} = 0 & \text{on } \partial B. \end{cases} \tag{18}$$

When (16) holds the equation

$$\Delta^2 U = K_0 U^p \quad \text{in } \mathbb{R}^N$$

has an explicit radial singular solution:

$$U(r) = r^{-\tau},$$

however, this solution does not satisfy the boundary condition for the normal derivative in (18).

2.2 The Emden-Fowler transformation

With the change of variables

$$v(t) = \left(\frac{\lambda}{\alpha K_0}\right)^{\frac{1}{p-1}} e^{\tau t}\left(1 + \alpha u(r)\right), \quad r = e^t \tag{19}$$

Equation (2) is equivalent to

$$Lv(t) = K_0 v(t)^p \quad \text{for all } t < 0 \tag{20}$$

where

$$L = (\partial_t - \tau + N - 4)(\partial_t - \tau + N - 2)(\partial_t - \tau - 2)(\partial_t - \tau)$$

with the boundary conditions

$$v(0) = \left(\frac{\lambda}{\alpha K_0}\right)^{\frac{1}{p-1}}, \quad v'(0) - \tau v(0) = 0. \tag{21}$$

and the behavior at $-\infty$ of regular solutions is given by

$$\lim_{t \to -\infty} v(t)^\alpha = 0, \quad \lim_{t \to -\infty} (v'(t) - \tau v(t)) = 0.$$

The operator L can also be written in the form

$$Lv = v^{(4)} + K_3 v''' + K_2 v'' + K_1 v' + K_0 v$$

where K_0 is defined in (13) and

$$\begin{cases} K_1 = -4\tau^3 + (6N - 24)\tau^2 + (20N - 2N^2 - 40)\tau - 2N^2 + 12N - 16 \\ K_2 = 6\tau^2 + (24 - 6N)\tau - 10N + 20 + N^2 \\ K_3 = 2N - 8 - 4\tau. \end{cases} \tag{22}$$

Let

$$\begin{cases} v_1(t) = v(t)^\alpha & = C^\alpha e^{\alpha\tau t}\left(1+\alpha u(e^t)\right)^\alpha \\ v_2(t) = \alpha\left(\partial_t - \tau\right) v(t) & = Ce^{(\tau+1)t}\dfrac{du}{dr}(e^t) \\ v_3(t) = \left(\partial_t - \tau - 2 + N\right) v_2(t) & = Ce^{(\tau+2)t}\Delta u(e^t) \\ v_4(t) = \left(\partial_t - \tau - 2\right) v_3 & = Ce^{(\tau+3)t}\dfrac{d(\Delta u)}{dr}(e^t) \end{cases} \tag{23}$$

where $C = (\lambda/(\alpha K_0))^{\frac{1}{p-1}}$. Then (20) becomes

$$\begin{cases} v_1' = \alpha\tau v_1 + v_2 v_1^{1-\alpha} \\ v_2' = (\tau + 2 - N)\, v_2 + v_3 \\ v_3' = (\tau + 2)\, v_3 + v_4 \\ v_4' = \alpha K_0 v_1^{\alpha p} + (\tau - N + 4)\, v_4. \end{cases} \tag{24}$$

In the sequel we will consider this system only for solutions such that $v_1 > 0$. However, it is useful to extend it to be C^1 for all values of v_1 so that we can linearize around the origin. We do this by replacing the last equation with

$$v_4' = \alpha K_0 |v_1|^{\alpha p} + (\tau - N + 4)\, v_4.$$

Condition (21) is equivalent to

$$v_1(0) = \left(\frac{\lambda}{\alpha K_0}\right)^{\frac{1}{p-1}}, \quad v_2(0) = 0. \tag{25}$$

The only stationary points of the system (24) are

$$\begin{cases} P_1 = (0,0,0,0) \\ P_2 = (1, -\alpha\tau, -\alpha\tau\,(N-2-\tau), \alpha\,(N-2-\tau)\,\tau\,(\tau+2)) \end{cases} \tag{26}$$

The linearization of (24) around the point P_1 is given by $Z' = \bar{M} Z$ where

$$\bar{M} = \begin{bmatrix} \alpha\tau & \sigma & 0 & 0 \\ 0 & -(N-2-\tau) & 1 & 0 \\ 0 & 0 & 2+\tau & 1 \\ 0 & 0 & 0 & -(N-4-\tau) \end{bmatrix}$$

where $\sigma = 1$ if $\alpha = 1$ and $\sigma = 0$ if $\alpha = -1$. The eigenvalues of this matrix are $\alpha\tau, 2+\tau, -N+4+\tau$, and $-N+2+\tau$. Then, in the range (14) and (15), P_1 is a hyperbolic point with a 2-dimensional unstable manifold $W^u(P_1)$ and a 2-dimensional stable manifold $W^s(P_1)$.

The linearization of (24) around P_2 is given by $Z' = MZ$ where

$$M = \begin{bmatrix} \tau & 1 & 0 & 0 \\ 0 & -(N-2-\tau) & 1 & 0 \\ 0 & 0 & \tau+2 & 1 \\ pK_0 & 0 & 0 & -(N-4-\tau) \end{bmatrix} \tag{27}$$

The eigenvalues of M are given by

$$\begin{cases} \nu_1 = \tau + \frac{1}{2}\left(4 - N + \sqrt{M_1(N) + M_2(N)}\right) \\ \nu_2 = \tau + \frac{1}{2}\left(4 - N - \sqrt{M_1(N) + M_2(N)}\right) \\ \nu_3 = \tau + \frac{1}{2}\left(4 - N + \sqrt{M_1(N) - M_2(N)}\right) \\ \nu_4 = \tau + \frac{1}{2}\left(4 - N - \sqrt{M_1(N) - M_2(N)}\right) \end{cases} \tag{28}$$

where

$$M_1(N) = (N-2)^2 + 4, \qquad M_2(N) = 4\sqrt{(N-2)^2 + pK_0}$$

Note that for $N \geq 5$ and $p > (N+4)/(N-4)$ we have $0 < \tau < (N-4)/2$. If $N = 3$ and $-3 < p < -1$ then $-2 < \tau < -1$ and for $N \geq 4$ and $p < -1$, we have $-2 < \tau < 0$. Then, in all these cases

$$\nu_2 < 0 < \nu_1.$$

It can be directly checked that $M_1(N) - M_2(N) < 0$ is equivalent to $pK_0 > H_N$. The numbers p_c and p_c^+ are such that when $p = p_c$ or $p = p_c^+$ then

$$p\,K_0 = H_N.$$

See the appendix Sect. B for the explicit calculation of p_c and p_c^+. In the range (8) for positive p or (10) for negative p we have $pK_0 > H_N$, and then ν_3, ν_4 are complex conjugate with nonzero imaginary part and negative real part. More precisely in the ranges (8) or (10) we have

$$\nu_2 < Re(\nu_3) = Re(\nu_4) < 0 < \nu_1.$$

On the other hand if

$$p_c < p < +\infty \quad \text{if} \quad N \geq 13 \tag{29}$$

or

$$\begin{cases} -3 < p < p_c^+ \text{ or } p_c < p < -1 & \text{if } \; N = 3, \\ p_c < p < -1 & \text{if } \; 5 \le N \le 12, \\ p < -1 & \text{if } \; N \ge 13 \end{cases} \tag{30}$$

hold, then we have $0 < pK_0 < H_N$. Thus in this range all eigenvalues are real and ν_3, ν_4 are negative, with

$$\nu_2 < \nu_4 < \nu_3 < 0 < \nu_1.$$

In the range (8) for positive p or (10) for negative p or in the ranges (29) and (30), P_2 is a hyperbolic stationary point with a 1-dimensional unstable manifold $W^u(P_2)$ and a 3-dimensional stable manifold $W^s(P_2)$.

Concerning the eigenvectors of M we have:

Lemma 3 *The vector*

$$v^{(k)} = [1,\ \nu_k - \tau,\ (\nu_k - \tau)(\nu_k + N - 2 - \tau),\ (\nu_k - \tau)(\nu_k + N - 2 - \tau)(\nu_k - 2 - \tau)] \tag{31}$$

is eigenvector of M associated to ν_k, $k = 1, \dots, 4$. We have that $v^{(1)}$, $v^{(2)}$ are always real, and $v^{(3)}$, $v^{(4)}$ are complex conjugate if N and p are in the range (8) *or* (10). *Let us write $v^{(i)} = (v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, v_4^{(i)})$, $i = 1, \dots, 4$. If $N \ge 3$ and $-3 < p < -1$, or $N \ge 4$ and $p < -1$, or $N \ge 5$ and $p > \frac{N+4}{N-4}$ then*

$$v_1^{(1)} > 0, \quad v_2^{(1)} > 0, \quad v_3^{(1)} > 0, \quad v_4^{(1)} > 0, \tag{32}$$

and

$$v_1^{(2)} > 0, \quad v_2^{(2)} < 0, \quad v_3^{(2)} > 0, \quad v_4^{(2)} < 0. \tag{33}$$

Proof is given by the matrix M defined in (27). Let $v^{(1)} = (t_1, t_2, t_3, t_4)$ an eigenvector for M with eigenvalue ν_1. We claim that

$$\begin{aligned} &t_1 = 1 > 0, \qquad\qquad t_2 = \nu_1 - \tau > 0, \\ &t_3 = (\nu_1 + N - 2 - \tau)(\nu_1 - \tau) > 0, \\ &t_4 = (\nu_1 - 2 - \tau)(\nu_1 + N - 2 - \tau)(\nu_1 - \tau) > 0. \end{aligned}$$

In fact, since $\nu_1 > 0$, it is sufficient to prove that $\nu_1 - 2 - \tau > 0$. This holds if $\sqrt{M_1(N) + M_2(N)} > N$ and this is equivalent to $pK_0 > 0$. □

Let $V = (v_1, \dots, v_4)$. From Theorem 6 in [14] we learn that when $N \ge 5$ and $p > \frac{N+4}{N-4}$, u is a regular solution of (2) if and only if

$$\lim_{t \to -\infty} V(t) = P_1$$

while u is a weakly singular solution if and only if

$$\lim_{t \to -\infty} V(t) = P_2.$$

The same property also holds for negative powers.

Lemma 4 *Assume* $N = 3$ *and* $-3 < p < -1$*, or* $N \geq 4$ *and* $p < -1$*. Let* u *be a radial weak solution of* (2) *and* $V = (v_1, \ldots, v_4)$ *be defined as in* (23)*. Then* u *is a regular solution of* (2) *if and only if*

$$\lim_{t \to -\infty} V(t) = P_1 \tag{34}$$

while u *is a weakly singular solution if and only if*

$$\lim_{t \to -\infty} V(t) = P_2. \tag{35}$$

Proof Directly by definition we have: if u is a regular solution then (34) holds and if (35) holds then u is weakly singular. To prove the reciprocals of these statements we use Theorem 1 since we also get from its proof: either u is a regular solution and then satisfies (34) or satisfies (35) and then it is weakly singular. □

The proof that (34) implies that u is a regular solution can also be done similarly as for positive powers, see Theorem 6 in [14].

By a result of Belickiĭ, see [4] or [39, Page 25], we know that the system (24) is C^1-conjugate to its linearization around the point P_2 under the non-resonance condition:

$$Re(\nu_i) \neq Re(\nu_j) + Re(\nu_k) \quad \text{when } Re(\nu_j) < 0 < Re(\nu_k) \tag{36}$$

where $\nu_1, \ldots, \nu_4$ are the eigenvalues of M defined in (27).

Lemma 5 *If* N *and* p *are in the range* (8) *or* (10)*, then the system* (24) *is* C^1*-conjugate to its linearization around the point* P_2*.*

Proof In the range (8) or (10) we have

$$Re(\nu_2) < Re(\nu_4) = Re(\nu_3) = \tau + \frac{4-N}{2} < 0 < Re(\nu_1).$$

Thus the only relation to be verified is

$$Re(\nu_1) + Re(\nu_2) \neq Re(\nu_3) = \tau + \tfrac{4-N}{2}$$

which is equivalent to $\tau + (4-N)/2 \neq 0$, and this is true in the considered range of N and p. □

3 The unstable manifold at P_2

Let $v^{(j)}$ denote the eigenvectors of the linearization of (24) at P_2 with corresponding eigenvalue ν_j. Then $W^u(P_2)$ is one dimensional and tangent to $v^{(1)}$ at P_2. Hence, if $V = (v_1, \ldots, v_4) : (-\infty, T) \to \mathbb{R}^4$ is any trajectory in $W^u(P_2)$ there are 2 cases:

$$\langle V'(t), v^{(1)} \rangle < 0 \quad \text{for } t \text{ near } -\infty$$
$$\langle V'(t), v^{(1)} \rangle > 0 \quad \text{for } t \text{ near } -\infty.$$

The main results in this section are

Proposition 6 *Suppose that $V = (v_1, \ldots, v_4) : (-\infty, T) \to \mathbb{R}^4$ is the trajectory in $W^u(P_2)$ such that $\langle V'(t), v^{(1)} \rangle < 0$ for t near $-\infty$. Then*

(a) $v_2(t) < -\alpha\tau$ *for all $t \in (-\infty, T)$, and*
(b) $v_3(t) < -\alpha\tau(N-2-\tau)$ *for all $t \in (-\infty, T)$.*

Proposition 7 *Let $V = (v_1, \ldots, v_4) : (-\infty, T) \to \mathbb{R}^4$ be the trajectory in $W^u(P_2)$ such that $\langle V'(t), v^{(1)} \rangle > 0$ for t near $-\infty$, where T is the maximal time of existence. Then*

(a) $v_1(t) > 1$ *for all $t < T$.*
(b) *There exists a unique t_0 such that $v_2(t_0) = 0$. Moreover the trajectory of V intersects the hyperplane $\{v_2 = 0\}$ transversally.*
(c) *There exists a unique t_1 such that $v_3(t_1) = 0$. Moreover the trajectory of V intersects the hyperplane $\{v_3 = 0\}$ transversally.*

The idea of the proof of these results is similar to [15, Proposition 1].

Proof of Proposition 6 (a) The relations (32) and the hypothesis $\langle V'(t), v^{(1)} \rangle < 0$ for $t \to -\infty$ imply that for t near $-\infty$

$$\begin{cases} v_1(t) < 1, & v_2(t) < -\alpha\tau, \\ v_3(t) < -\alpha\tau(N-2-\tau), & v_4(t) < \alpha\tau(N-2-\tau)(\tau+2). \end{cases} \tag{37}$$

Assume by contradiction that $v_2(t) \geq -\alpha\tau$ for some $t < T$. Thus we may define $t_0 < T$ the smallest time such that $v_2(t) = -\alpha\tau$. Then $v_2'(t_0) \geq 0$. By (24) we have $0 \leq v_2'(t_0) = v_3(t_0) + \alpha\tau(N-2-\tau)$, that is,

$$v_3(t_0) \geq -\alpha\tau(N-2-\tau).$$

By (37) we can define $t_1 \leq t_0$ as the smallest time such that $v_3(t) = -\alpha\tau(N-2-\tau)$. Then $v_3'(t_1) \geq 0$ and (24) implies

$$v_4(t_1) \geq \alpha\tau(N-2-\tau)(\tau+2). \tag{38}$$

Again using (37), let $t_2 \leq t_1$ be the smallest time that $v_4(t) = \alpha\tau(N-2-\tau)(\tau+2)$. Then $v_4'(t_2) \geq 0$ and by (24)

$$v_1(t_2) \geq 1$$

Thanks to (37) we must have a smallest time $t_3 \le t_2$ such that $v_1(t) = 1$. But then $v_1'(t_3) \ge 0$ which by (24) implies

$$v_2(t_3) + \alpha\tau \ge 0.$$

Thus $v_2(t_3) \ge -\alpha\tau$. This cannot happen if $t_3 < t_0$ because $v_2(t) < -\alpha\tau$ for all $t < t_0$. If $t_3 = t_2 = t_1 = t_0$ then $v_1'(t_0) = v_2'(t_0) = v_3'(t_0) = v_4'(t_0)$, which means $V(t) \equiv P_2$, a contradiction. This proves that $v_2(t) < -\alpha\tau$ for all $t < T$.

(b) Let us show now that $v_3(t) < -\alpha\tau(N - 2 - \tau)$ for all $t < T$. If not, we can define $t_1 < T$ as the smallest time such that $v_3(t) = -\alpha\tau(N - 2 - \tau)$. Then $v_3'(t_1) \ge 0$ and we may repeat the same argument starting at (38) to find $t_3 \le t_1$ such that $v_2(t_3) \ge -\alpha\tau$. This is impossible and proves the result. □

Proof of Proposition 7 By (32) and the hypothesis $\langle V'(t), v^{(1)}\rangle > 0$ for $t \to -\infty$ we have

$$v_1'(t) > 0, \quad v_2'(t) > 0, \quad v_3'(t) > 0, \quad v_4'(t) > 0 \tag{39}$$

for t near $-\infty$.

Let us first prove that for $\alpha = -1$ we have

$$v_1(t) > 0 \quad \forall t < T$$

This is valid for t near $-\infty$ by the first inequality in (39). If $v_1(t) = 0$ for some t then v_1 would be constant by the equation,

$$v_1' = v_1(v_2 v_1 - \tau),$$

and this is not possible.

We claim that

$$v_3'(t) > 0 \quad \forall t < T. \tag{40}$$

To prove (40) suppose it fails. Let $s_0 < T$ be the smallest time such that $v_3'(s_0) = 0$. Using (24) we see that

$$0 = v_3'(s_0) = (2 + \tau)v_3(s_0) + v_4(s_0).$$

But $v_3(s_0) > -\alpha\tau(N - 2 - \tau)$ we deduce $v_4(s_0) < \alpha\tau(N - 2 - \tau)(2 + \tau)$. Let $s_1 \le s_0$ be the smallest time such that $v_4(t) = \alpha\tau(N - 2 - \tau)(\tau + 2)$. Then $v_4'(s_1) \le 0$ and hence

$$v_1(s_1) \le 1.$$

Let $s_2 \le s_1$ be the smallest time such that $v_1(s_2) = 1$. Then $v_1'(s_2) \le 0$ and we conclude

$$v_2(s_2) \le -\alpha\tau.$$

Let $s_3 \le s_2$ be the smallest time such that $v_2(s_3) = -\alpha\tau$. Then $v_2'(s_3) \le 0$ and we conclude

$$v_3(s_2) \le -\alpha\tau(N - 2 - \tau).$$

Now since $s_2 < s_0$, we have $v_3(s_2) > -\alpha\tau(N - 2 - \tau)$, a contradiction. This establishes our claim (40).

Since (40) holds we have then $v_3(t) > -\alpha\tau(N - 2 - \tau)$ for all $t < T$. From the second equation in (24), we have

$$v_2'' = -(N - 2 - \tau)v_2' + v_3'$$

We claim that $v_2' > 0$. By contradiction if s_0 is the smallest time such that $v_2'(s_0) = 0$ then using (40), we have that $v_2''(s_0) > 0$ so v_2 has a local minimum at s_0 which is not possible, since v_2 is increasing near $t = -\infty$. We conclude that

$$v_2'(t) > 0 \quad \forall t < T. \tag{41}$$

Similarly using

$$v_1'' = (-\tau + 2v_1v_2)\, v_1' + v_1^2 v_2', \quad \text{for} \quad \alpha = -1$$

or

$$v_1'' = \tau v_1' + v_2', \quad \text{for} \quad \alpha = 1,$$

the inequality (41), and if $\alpha = -1$, the positivity of v_1 in $(-\infty, T)$, we obtain that

$$v_1'(t) > 0 \quad \forall t < T. \tag{42}$$

and now using the fourth equation in (24), and (42), we have

$$v_4'(t) > 0 \quad \forall t < T,$$

this proves that (39) is valid for all $-\infty < t < T$.

Now using (42) the property (a) follows.

Let us prove now that

$$\sup_{t<T} v_1(t) = +\infty. \tag{43}$$

If we assume the contrary, i.e. v_1 remains bounded, then (24) implies the estimate

$$|(v_1,\ldots,v_4)'(t)| \le C|(v_1,\ldots,v_4)(t)| \quad \forall t < T,$$

for some $C > 0$ and from Gronwall's inequality we deduce that the solution is defined for all times, that is, $T = +\infty$. Since v_1 is increasing, $v_1(t) \to L < +\infty$ as $t \to \infty$, and $v_1'(t_k) \to 0$ along some sequence $t_k \to \infty$. But v_1, v_2 are increasing and $v_2(t) > -\alpha\tau$ and $v_1(t) > 1$ for all $t > -\infty$. From the equation for v_1', i.e.

$$v_1' = \tau v_1^{1-\alpha}\alpha(v_1^\alpha - 1) + v_1^{1-\alpha}(v_2 + \alpha\tau)$$

we obtain a contradiction, since $\alpha(v_1^\alpha(t) - 1) > 0$ for all $t > -\infty$. This proves (43). Now using (42) the property (a) follows.

We claim that

$$\sup_{t<T} v_2(t) > 0. \tag{44}$$

If this fails, then $v_2(t) \le 0$ for all $t < T$. Therefore by the equation for v_1' in (24)

$$0 \le v_1'(t) \le Cv_1(t).$$

Gronwall's inequality implies that v_1 cannot blow up in finite time. But $v_1(t)$ blows up as $t \to T$ and this implies that $T = +\infty$. Now let us show that $v_4(t) \to \infty$ as $t \to \infty$. Indeed, if we assume that $v_4(t) \to L < \infty$ as $t \to \infty$ then for some sequence $t_k \to \infty$, $v_4'(t_k) \to 0$. Using the equation for v_4' and (43) we obtain a contradiction. Applying the same argument and the equation for v_3', we obtain $v_3(t) \to \infty$ as $t \to \infty$, and $v_2(t) \to \infty$ as $t \to \infty$. This contradicts our assumption and proves (44).

We also have

$$\sup_{t<T} v_3(t) > 0. \tag{45}$$

In fact, using the equation for v_2' in (24):

$$v_3(t) = v_2'(t) + (N - 2 - \tau)v_2(t)$$

we see that if $v_2(t) > 0$ then $v_3(t) > 0$, because $v_2'(t) > 0$ and $N - 2 - \tau > 0$.

Finally the property b) clearly follows from (44) and that $v_2'(t) > 0$ for all $t < T$. Similarly property c) is a consequence of (45) and that $v_3'(t) > 0$ for all $t < T$. □

Proof of Theorems 2, 5 and 8 By Propositions 6 and 7 we know that $W^u(P_2) \cap \{v_2 = 0\}$ is a single point, which we call P^*. Any weakly singular radial solution of (2) gives rise, through the change of variables $v(t) = (\lambda/(\alpha K_0))^{\frac{1}{p-1}} e^{\tau t}(1 + \alpha u(e^t))$, $t \le 0$, and (23), to a solution $V : (-\infty, 0] \to \mathbb{R}^4$ of the system (24) such that the final conditions (25) hold. Since the solution is weakly singular, $\lim_{t\to-\infty} V(t) = P_2$.

Hence $V((-\infty, 0])$ is contained in $W^u(P_2)$ and therefore there are 2 possibilities: either $\langle V'(t), v^{(1)}\rangle < 0$ for t near $-\infty$ or $\langle V'(t), v^{(1)}\rangle > 0$ for t near $-\infty$. The first case is not possible, because Proposition 6 shows that V cannot satisfy the end condition $v_2(0) = 0$. Thus we are in the second case and we can apply Proposition 7 b). Therefore there exists a $t_0 > -\infty$ such that $v_2(t_0) = 0$ and by uniqueness $t_0 = 0$. Then $V(0) = P^*$, which implies that V is uniquely determined. This concludes the proof of Theorems 2 and 5. The proof of Theorem 8 is similar, using Proposition 7 c), since $v_3(0) = C\Delta u(1)$. □

4 Entire solutions for negative powers

Throughout this section we assume that p is in the range defined by:

$$p < -1 \quad \text{if } N \geq 4, \quad \text{or } -3 < p < -1 \quad \text{if } N = 3. \tag{46}$$

We consider the initial value problem

$$\begin{cases} \Delta^2 U = K_0 U^p, \quad u > 0 \quad r \in (0, R_{\max}(\beta)) \\ U(0) = 1, \quad U'(0) = 0, \quad \Delta U(0) = \beta, \quad (\Delta U)'(0) = 0 \end{cases} \tag{47}$$

where $K_0 < 0$ is given by (13). Here $[0, R_{\max}(\beta))$ is the interval of existence of the solution. The main result here is the following.

Proposition 8 *Assume $N \geq 4$ and $p < -1$, or $N = 3$ and $-3 < p < -1$. Then there is a unique $\beta^* > 0$ such that*:

(a) *If $\beta < \beta^*$ then $R_{\max}(\beta) < \infty$,*
(b) *If $\beta \geq \beta^*$ then $R_{\max}(\beta) = \infty$*
(c) *If $\beta = \beta^*$ then*

$$\lim_{r\to\infty} r^\tau U_{\beta^*}(r) = 1. \tag{48}$$

(d) *If $\beta \geq \beta^*$ then*

$$U_\beta(r) \geq U_{\beta^*}(r) + \frac{\beta - \beta^*}{2N} r^2 \quad \textit{for all } r \geq 0 \tag{49}$$

and

$$U'_\beta(r) \geq U'_{\beta^*}(r) + \frac{\beta - \beta^*}{N} r > 0 \quad \textit{for all } r \geq 0.$$

(e) *If $0 < \beta < \beta^*$ then there exists $0 < R_0 < R_{\max}(\beta)$ such that $U'_\beta(r) > 0$ for all $r \in (0, R_0)$, $U'_\beta(R_0) = 0$, $U'_\beta(r) < 0$ for $r \in (R_0, R_{\max}(\beta))$.*
(f) *If $\beta \leq 0$ then $U'_\beta(r) < 0$ for all $r \in (0, R_{\max}(\beta))$.*

McKenna and Reichel [34, Theorem 3.1] proved (a) and (b) of the above result for $N \geq 3$. In the same reference the authors showed that for any β,

$$U_\beta \leq Cr^2 \quad \text{for all } r \geq 1,$$

for some $C > 0$. Guo and Wei [27, Theorem 1.3] obtained statement c) for $p = -2$ and $N = 3$. The goal here is to extend the result to the remaining powers.

Let us introduce some notation. Given U_β the solution of the problem (47) let

$$v_\beta(t) = r^\tau U_\beta(r), \quad r = e^t, \quad -\infty < t < \log(R_{\max}(\beta)).$$

Then v_β satisfies Eq. (20) in $(-\infty, \log(R_{\max}(\beta)))$. We also define $V_\beta = (v_{\beta,1}, \ldots, v_{\beta,4})$ by (23). Then V satisfies the system (24). We then see that

$$\begin{aligned} v_{\beta,1} &= e^{\alpha\tau t} U_\beta^\alpha & v_{\beta,2} &= \alpha e^{(\tau+1)t} \frac{d}{dr} U_\beta(e^t) \\ v_{\beta,3} &= \alpha e^{(\tau+2)t} \Delta U_\beta(e^t) & v_{\beta,4} &= \alpha e^{(\tau+3)t} \frac{d}{dr} \Delta U_\beta(e^t) \end{aligned}$$

For any $\beta > 0$, from the formula

$$\frac{d\Delta U_\beta}{dr}(r) = r^{1-N} K_0 \int_0^r s^{N-1} U_\beta(s)^p ds < 0 \quad \text{for } 0 \leq r < R_{\max}(\beta) \tag{50}$$

we deduce that $\Delta U_\beta(r)$ is decreasing on $[0, R_{\max}(\beta))$ and

$$\lim_{r \to R_{\max}(\beta)} \Delta U_\beta(r) \quad \text{exists.}$$

We recall a comparison result.

Lemma 9 (McKenna and Reichel [34, Lemma 3.2]) *Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and increasing. Let $u, v \in C^4([0, R))$, $R > 0$ be such that*

$$\begin{aligned} &\forall r \in [0, R) \quad \Delta^2 u(r) - f(u(r)) \geq \Delta^2 v(r) - f(v(r)), \\ &u(0) \geq v(0), \quad u'(0) \geq v'(0), \quad \Delta u(0) \geq \Delta v(0), \quad (\Delta u)'(0) \geq (\Delta v)'(0). \end{aligned}$$

Then for all $r \in [0, R)$

$$u(r) \geq v(r), \quad u'(r) \geq v'(r), \quad \Delta u(r) \geq \Delta v(r), \quad (\Delta u)'(r) \geq (\Delta v)'(r). \tag{51}$$

Moreover:

(i) *The initial point 0 can be replaced by any initial point $\rho > 0$ if all four initial data are weakly ordered.*

(ii) *A strict inequality in one of the initial data at* $\rho \geq 0$ *or in the differential inequality on* (ρ, R) *implies a strict ordering of* u, u', Δu, $(\Delta u)'$ *and* v, v', Δv, $(\Delta v)'$ *in* (51).

Although the lemma is stated for f differentiable, the proof is also valid if $f(u) = -u^p$ $(p < 0)$ and u, v are positive.

Lemma 10 *Let* $\beta \geq \beta^*$ *so that* $R_{\max}(\beta) = \infty$. *Then* $\lim_{r\to\infty} \Delta U_\beta(r) \geq 0$ *and* $\lim_{r\to\infty} \Delta U_\beta(r) = 0$ *if and only if* $\beta = \beta^*$.

Proof If

$$\lim_{r\to\infty} \Delta U_\beta(r) < 0$$

integrating twice we deduce

$$U_\beta(r) \leq -C_1 r^2 + C_2 \quad \text{for all } r \geq 0$$

with $C_1, C_2 > 0$, which is impossible.

Assume now that

$$\lim_{r\to\infty} \Delta U_\beta(r) > 0$$

then

$$U_\beta(r) \geq cr^2 \quad \text{and} \quad U'_\beta(r) \geq cr \quad \text{for all } r \geq 0 \tag{52}$$

for some $c > 0$. Also, integrating once the Eq. (47) we see that for all $r \geq 2$:

$$(\Delta U_\beta)'(r) \geq -c \begin{cases} r^{1-N} & \text{if } N < -2p \\ r^{1-N}\log r & \text{if } N = -2p \\ r^{1+2p} & \text{if } N > -2p. \end{cases} \tag{53}$$

Let $m \in \mathbb{R}$ to be fixed and

$$v(r) = (1 + r^2)^m.$$

A computation shows that:

$$\Delta^2 v = A(2m)(1 + r^2)^{m-2} + B(2m)(1 + r^2)^{m-3} + C(2m)(1 + r^2)^{m-4}$$

where

$$\begin{aligned} A(\nu) &= \nu(\nu + N - 2)(\nu - 2)(\nu + N - 4) \\ B(\nu) &= -2\nu(\nu - 2)(\nu - 4)(\nu + N - 4) \\ C(\nu) &= \nu(\nu - 2)(\nu - 4)(\nu - 6). \end{aligned}$$

For $N \geq 3$ and $2m \in (1, 2)$ we have $A(2m) < 0$, $B(2m) < 0$ and $C(2m) < 0$. Let $b > 0$ and $w(r) = v(br)$. Then for $b > 0$ large enough

$$\Delta^2 w \leq K_0 w^p \quad \text{for all } r \geq 0,$$

(a similar calculation is done in [34, Lemma 3.5]). We choose $m \in (1/2, 1)$ close to 1 so that $2m - 3 > 1 + 2p$. From (52) and (53) there exists $r_0 > 0$ such that $U_\beta(r_0) > w(r_0)$, $U'_\beta(r_0) > w'(r_0)$, $\Delta U_\beta(r_0) > \Delta w(r_0)$ and $(\Delta U_\beta)'(r_0) > (\Delta w)'(r_0)$. By the continuous dependence of the solution to (47) there is $\beta_1 < \beta$ such that

$$U_{\beta_1}(r_0) > w(r_0), \quad U'_{\beta_1}(r_0) > w'(r_0)$$

and

$$\Delta U_{\beta_1}(r_0) > \Delta w(r_0), \quad (\Delta U_{\beta_1})'(r_0) > (\Delta w)'(r_0).$$

Using Lemma 9 we deduce that $U_{\beta_1} \geq w$ for all $r \geq r_0$. This shows that u_{β_1} is defined for all $r \geq 0$ and hence $\beta_1 \geq \beta^*$. We deduce that $\beta > \beta^*$.

Now suppose that $\beta > \beta^*$. From Lemma 9 we deduce that

$$(\Delta U_\beta)'(r) \geq (\Delta U_{\beta^*})'(r) \quad \text{for all } r \geq 0.$$

Integrating this we find

$$\lim_{r \to \infty} \Delta U_\beta(r) - \beta \geq \lim_{r \to \infty} \Delta U_{\beta^*}(r) - \beta^*$$

which implies

$$\lim_{r \to \infty} \Delta U_\beta(r) \geq \beta - \beta^* > 0.$$

□

Lemma 11 *It cannot happen than $v_{\beta^*}(t) \to +\infty$ as $t \to \infty$. If $\lim_{t \to \infty} v_{\beta^*}(t) = L$ exists, then $L = 1$.*

Proof For simplicity we write $v = v_{\beta^*}$. We have

$$(\partial_t + N - 2 - \tau)(\partial_t - \tau)v = h(t)$$

where $h(t) = e^{(\tau+2)t} \Delta U_{\beta^*}(e^t)$. Thanks to Lemma 10 $h(t) = o(e^{(\tau+2)t})$ as $t \to \infty$. Using the variation of parameters formula:

$$v(t) = Ae^{\tau t} + Be^{(\tau+2-N)t} + \frac{1}{N-2} \int_{t_0}^{t} \left[e^{\tau(t-s)} h(s) - e^{(\tau+2-N)(t-s)} h(s) \right] ds.$$

Hence

$$v(t) = o(e^{(\tau+2)t}) \quad \text{as } t \to \infty. \tag{54}$$

If $v(t) \to +\infty$ as $t \to \infty$ then (20) takes the form:

$$Lv = g(t) \quad \text{with } g(t) = o(1) \text{ as } t \to \infty.$$

The linearly independent solutions of the homogeneous equation $Lz = 0$ are $e^{\nu_i t}$ with

$$\nu_1 = \tau + 2, \quad \nu_2 = \tau + 2 - N, \quad \nu_3 = \tau + 4 - N, \quad \nu_4 = \tau$$

(except when $N = 4$, in which case they are $e^{\nu t}$ with $\nu = \tau+2, \tau+2-N, \tau$ and $te^{\tau t}$). The only positive ν_i is $\nu_1 = \tau + 2$. For simplicity we proceed assuming $N \neq 4$. The case $N = 4$ can be treated similarly. By the variation of parameters formula (Theorem 6.4 [10, Chap. 3])

$$v(t) = \sum_{i=1}^{4} c_i e^{\nu_i t} + d_1 \int_t^\infty e^{\nu_1(t-s)} g(s)\, ds + \sum_{i=2}^{4} d_i \int_0^t e^{\nu_i(t-s)} g(s)\, ds \tag{55}$$

where the integrals represent a concrete choice of a particular solution and d_i, $i = 1, \ldots, 4$ are fixed constants. Since $\nu_1 = \tau + 2 > 0$, we have

$$\int_t^\infty e^{\nu_1(t-s)} g(s)\, ds \to 0 \quad \text{as } t \to \infty.$$

For $i = 2, 3, 4$ we have $\nu_i < 0$ and hence,

$$\int_0^t e^{\nu_i(t-s)} g(s)\, ds \to 0 \quad \text{as } t \to \infty.$$

If $v(t) \to \infty$ as $t \to \infty$ we conclude that $c_1 \neq 0$ in (55), that is, $\lim_{t\to\infty} v(t)e^{-(\tau+2)t} = c$, with $c > 0$. This contradicts (54).

The rest of the proof is the same as [27, Lemma 4.3]. □

Lemma 12 *We have*

$$\limsup_{t\to\infty} v_{\beta^*}(t) > 0. \tag{56}$$

Proof We use the test-function method of Mitidieri and Pohozaev [35], see also [1,14,20]. Let us write $v = v_{\beta^*}$. Assume by contradiction that

$$\lim_{t\to\infty} v(t) = 0.$$

Let $0 < \delta < 1/2$ be fixed. Then there exists $T > 0$ such that $v(t) \leq \delta$ for all $t \geq 0$. Since the Eq. (20) is autonomous we can assume that $T = 0$.

Let $\phi \in C^4(\mathbb{R})$ be such that, $0 \leq \phi \leq 1$, $\phi(t) = 0$ for $t \leq 0$ and $t \geq 3$, $\phi(t) > 0$ for $t \in (0, 3)$, $\phi(t) = 1$ for $t \in [1, 2]$, and for $i = 1, 2, 3, 4$

$$\int_0^3 \frac{(\phi^{(i)})^2}{\phi}\, dt < +\infty.$$

Let $L > 1$ and $\phi_L(t) = \phi(t/L)$. We rewrite the Eq. (20) in the form

$$\sum_{i=1}^{4} K_i v^{(i)}(t) = K_0(v^p - v) \quad \text{for } t \in \mathbb{R} \tag{57}$$

where $K_0, \ldots, K_3$ are as before and $K_4 = 1$. Multiplying (57) by ϕ_L and integrating we find

$$\sum_{i=1}^{4} K_i(-1)^i \int_0^{3L} \phi_L^{(i)} v\, dt = K_0 \int_0^{3L} (v^p - v)\phi_L\, dt. \tag{58}$$

Since $0 < v(t) < 1$ for $t \geq 0$ and $p < 0$ we have $v(t)^p - v(t) > 0$. Thus (58) yields

$$|K_0| \int_0^{3L} (v^p - v)\phi_L\, dt \leq K \max_{i=1,\ldots,4} \int_0^{3L} v|\phi_L^{(i)}|\, dt \tag{59}$$

where $K = \sum_{i=1}^{4} |K_i|$. Let $\varepsilon > 0$ to be fixed later on. Using

$$v|\phi_L^{(i)}| \leq \varepsilon v^2 \phi_L + C_\varepsilon \frac{(\phi_L^{(i)})^2}{\phi_L}$$

we obtain from (59)

$$\int_0^{3L} \left(|K_0|(v^p - v) - \varepsilon K v^2\right) \phi_L\, dt \leq C_\varepsilon K \max_{i=1,\ldots,4} \int_0^{3L} \frac{(\phi_L^{(i)})^2}{\phi_L}\, dt. \tag{60}$$

Recall that $0 < v(t) \leq \delta$ for all $t \geq 0$. Since $0 < \delta < 1/2$,

$$\min_{v \in (0,\delta]} |K_0|(v^p - v) > 0.$$

Therefore we can fix $\varepsilon > 0$ sufficiently small so that

$$c(\delta) = \min_{v \in (0,\delta]} |K_0|(v^p - v) - \varepsilon K v^2 > 0.$$

This implies $|K_0|(v(t)^p - v(t)) - \varepsilon K v(t)^2 \geq c(\delta) > 0$ for all $t \geq 0$. It follows from this and (60) that

$$c(\delta)L \leq c(\delta)\int_0^{3L} \phi_L(t)\,dt \leq C_\varepsilon K \max_{i=1,\ldots,4} \int_0^{3L} \frac{(\phi_L^{(i)})^2}{\phi_L}\,dt.$$

But

$$\int_0^{3L} \frac{(\phi_L^{(i)})^2}{\phi_L}\,dt = L^{1-2i}\int_0^{3} \frac{(\phi^{(i)})^2}{\phi}\,dt \leq C_i L^{1-2i}.$$

Then

$$c(\delta)L \leq C_\varepsilon K \max_{i=1,\ldots,4} C_i L^{1-2i} \quad \text{for all } L > 1,$$

which is not possible. □

Lemma 13 *We have*

$$\liminf_{t\to\infty} v_{\beta^*}(t) > 0. \tag{61}$$

Proof We write $v = v_{\beta^*}$, $V = (v_1, \ldots, v_4) = V_{\beta^*}$ and $U = U_{\beta^*}$. Suppose by contradiction that $\liminf_{t\to\infty} v(t) = 0$. Then, since (56) holds, there is a sequence (t_k) such that $t_k \to \infty$, $t_{k+1} \geq t_k + 1$, $v(t_k) \to 0$, $v'(t_k) = 0$ and $v''(t_k) \geq 0$. Let $R_k = e^{t_k}$ and define

$$U_k = \frac{1}{v(t_{k+1})R_{k+1}^{-\tau}} U(R_{k+1}r).$$

Then U_k satisfies

$$\Delta^2 U_k = v(t_{k+1})^{p-1} K_0 U_k^p \quad \text{in } \mathbb{R}^N,$$
$$U_k(1) = 1, \quad U_k(R_k/R_{k+1}) = \frac{v(t_k)R_k^{-\tau}}{v(t_{k+1})R_{k+1}^{-\tau}} = \frac{U(R_k)}{U(R_{k+1})}.$$

But $\frac{d}{dr}U \geq 0$ and $R_{k+1} \geq R_k$. Therefore,

$$U_k(R_k/R_{k+1}) \leq 1.$$

We compute

$$\begin{aligned}\Delta U_k(r) &= \frac{R_{k+1}^{\tau+2}}{v(t_{k+1})}\Delta U(R_{k+1}r)\\ &= \frac{R_{k+1}^{\tau+2}}{v(t_{k+1})}e^{-(\tau+2)(t_{k+1}+t)}\big[v''(t_{k+1}+t)+(N-2-2\tau)v'(t_{k+1}+t)\\ &\quad -\tau(N-2-\tau)v(t_{k+1}+t)\big].\end{aligned}$$

Therefore

$$\Delta U_k(1)\geq 0 \quad \text{and} \quad \Delta U_k(R_k/R_{k+1})\geq 0.$$

Define now $u_k = 1-U_k$. Then u_k satisfies

$$\begin{aligned}&\Delta^2 u_k = |K_0|v(t_{k+1})^{p-1}(1-u_k)^p \quad \text{in } \mathbb{R}^N\\ &u_k(1)=0, \quad u_k(R_k/R_{k+1})\geq 0, \quad \Delta u_k(1)\leq 0, \quad \Delta u_k(R_k/R_{k+1})\leq 0.\end{aligned} \tag{62}$$

Let $D_k = B_1(0)\backslash \overline{B}_{R_k/R_{k+1}}(0)$. Let λ_k be the first eigenvalue for $-\Delta$ with Dirichlet boundary condition in the annulus D_k and $\phi_k>0$ be an associated eigenfunction, that is

$$\begin{cases}-\Delta\phi_k = \lambda_k\phi_k & \text{in } D_k\\ \quad\ \phi_k = 0 & \text{on } \partial D_k.\end{cases}$$

Then $\Delta^2\phi_k = \lambda_k^2\phi_k$. Multiplying (62) equation by ϕ_k and integrating by parts we obtain

$$\begin{aligned}|K_0|v(t_{k+1})^{p-1}\int_{D_k}(1-u_k)^p\phi_k\,dx &= \int_{D_k}\Delta^2 u_k\phi_k\,dx\\ = \int_{\partial D_k}\left[\frac{\partial\Delta u_k}{\partial n}\phi_k - \Delta u_k\frac{\partial\phi_k}{\partial n}+\frac{\partial u_k}{\partial n}\Delta\phi_k - u_k\frac{\partial\Delta\phi_k}{\partial n}\right]&\\ +\int_{D_k}u_k\Delta^2\phi_k\,dx.&\end{aligned}$$

But on ∂D_k, $\phi_k=\Delta\phi_k=0$, $\frac{\partial\phi_k}{\partial n}\leq 0$ and $\frac{\partial\Delta\phi_k}{\partial n}\geq 0$. Hence

$$\Delta u_k\frac{\partial\phi_k}{\partial n}\geq 0 \quad \text{and} \quad u_k\frac{\partial\Delta\phi_k}{\partial n}\geq 0 \quad \text{on } \partial D_k.$$

Using also the inequality $(1-u)^p\geq u$ for $0\leq u<1$ it follows that

$$|K_0|v(t_{k+1})^{p-1}\leq\lambda_k^2.$$

But since the annulus D_k has a width that does not converge to zero, λ_k remains uniformly bounded, even if $R_k/R_{k+1} \to 0$. It follows that $v(t_{k+1})$ remains bounded away from zero as $k \to \infty$, which is a contradiction. $\square$

Lemma 14 *There exists $C > 1$ such that for all $t \geq 0$*

$$\frac{1}{C} \leq v_{\beta^*,1}(t) \leq C, \quad -C \leq v_{\beta^*,2}(t) \leq 0, \quad -C \leq v_{\beta^*,3}(t) \leq 0, \quad 0 \leq v_{\beta^*,4}(t) \leq C,$$

and

$$|v_{\beta^*}^{(i)}(t)| \leq C \quad \forall i = 0, \ldots, 4.$$

Proof In this proof we omit β^* from the notation. The estimate (61) implies that $v_1(t) \leq C$ for all $t \geq 0$. Using the equation for v_4 we find that for $0 \leq t_0 \leq t$

$$v_4(t) = e^{-(N-4-\tau)t}\left[e^{(N-4-\tau)t_0} v_4(t_0) + |K_0| \int_{t_0}^{t} e^{(N-4-\tau)s} v_1(s)^{|p|}\, ds\right]$$

a formula that shows that v_4 remains bounded as $t \to \infty$. Similarly, integrating the equation for v_3 we obtain for $0 \leq t \leq t_0$

$$v_3(t) = e^{(\tau+2)t}\left[e^{-(\tau+2)t_0} v_3(t_0) + \int_{t_0}^{t} e^{-(\tau+2)s} v_4(s)\, ds\right].$$

Since v_4 is bounded and $\tau + 2 > 0$ the integral $\int_{t_0}^{\infty} e^{-(\tau+2)s} v_4(s)\, ds$ exists. Using Lemma 10 we know that $v_3(t) = o(e^{(\tau+2)t})$ as $t \to \infty$, a condition that implies

$$v_3(t_0) = -e^{(\tau+2)t_0} \int_{t_0}^{\infty} e^{-(\tau+2)s} v_4(s)\, ds.$$

The fact that v_4 is bounded and this formula imply that v_3 is bounded. Repeating the same argument that we used for v_4 we may prove that v_2 remains bounded as $t \to \infty$. Writing the equation for v_1 in the form

$$\frac{d}{dt}\left(e^{-|\tau|t} v_1(t)\right) = \left(e^{-|\tau|t} v_1(t)\right)^2 e^{|\tau|t} v_2(t)$$

and integrating over $0 \leq t_0 \leq t$ yields

$$v_1(t) = \frac{e^{|\tau|t}}{\frac{e^{|\tau|t_0}}{v_1(t_0)} - \int_{t_0}^{t} e^{|\tau|s} v_2(s)\, ds}.$$

But the fact that $\frac{d}{dr}U \geq 0$ implies that $v_2 \leq 0$, and since $|v_2|$ is bounded, we deduce from the above formula that v_1 is bounded below by a positive constant.

Finally, $v_3 \leq 0$ is due to $\Delta U \geq 0$ and $v_4 \geq 0$ is because $\frac{d}{dr}\Delta U \leq 0$. The estimates for v and its derivatives follow from the estimates for v_i and (23), (24). □

For v solving (20) for all $t \in \mathbb{R}$ and $v(t) > 0$ for all $t \in \mathbb{R}$ we define

$$E(t) = \frac{1}{2}(v''(t))^2 - \frac{K_2}{2}(v'(t))^2 - \frac{K_0}{2}v(t)^2 + K_0\frac{v(t)^{p+1}}{p+1}.$$

As we will see, this energy is useful if $K_3 > 0$ and $K_1 < 0$. We note that $K_3 > 0$ is equivalent to $\tau < (N-4)/2$, which always holds for negative exponents. However, the sign of K_1 is not constant in the range (10).

Lemma 15 *Assume $K_1 < 0$. Suppose $v > 0$ solves (20) for all $t \in \mathbb{R}$. If $t_1 < t_2$ and $v'(t_1) = 0$, $v'(t_2) = 0$ then*

$$E(t_2) \leq E(t_1)$$

with strict inequality unless v is constant in $[t_1, t_2]$.

Proof Using the equation

$$E(t_2)-E(t_1)=\int_{t_1}^{t_2} E'(t)\,dt = v'''v'\Big|_{t_1}^{t_2} + K_3v''v'\Big|_{t_1}^{t_2} - K_3\int_{t_1}^{t_2}(v''(t))^2\,dt + K_1\int_{t_1}^{t_2}(v'(t))^2\,dt. \tag{63}$$

The lemma follows once we know that $K_3 > 0$ and $K_1 < 0$. □

Lemma 16 *Assume $K_1 < 0$. Then*

$$\int_0^\infty v'_{\beta^*}(s)^2\,ds < +\infty, \quad \int_0^\infty v''_{\beta^*}(s)^2\,ds < +\infty \tag{64}$$

Proof It is a consequence of (63) and the fact that v is bounded and bounded away from zero, and that the derivatives of v remain bounded as $t \to \infty$. □

Lemma 17 *Assume $K_2K_3 - K_1 > 0$. Then (64) holds.*

Proof Using (63) in the interval $[t_0, t_1]$ with $t_0 \leq t_1$ and that v and its derivatives are uniformly bounded by Lemma 14 we obtain

$$K_3\int_{t_0}^{t_1}(v'')^2 - K_1\int_{t_0}^{t_1}(v')^2 = O(1) \tag{65}$$

with $O(1)$ bounded independently of t_0 and t_1.

Multiplying (20) by v and integrating over $[t_0, t_1]$ we find

$$\left[v'''v - v''v' + K_3 v''v - \frac{K_3}{2}(v')^2 + K_2 v'v + \frac{K_1}{2}v^2\right]_{t_0}^{t_1} + \int_{t_0}^{t_1}(v'')^2 - K_2\int_{t_0}^{t_1}(v')^2 + K_0\int_{t_0}^{t_1}v^2 - K_0\int_{t_0}^{t_1}v^{p+1} = 0.$$

Using Lemma 14 we deduce that

$$\int_{t_0}^{t_1}(v'')^2 - K_2\int_{t_0}^{t_1}(v')^2 + K_0\int_{t_0}^{t_1}v^2 - K_0\int_{t_0}^{t_1}v^{p+1} = O(1)$$

where $O(1)$ is bounded independently of t_0 and t_1. Hence by (65)

$$\frac{K_2K_3 - K_1}{K_3}\int_{t_0}^{t_1}(v')^2 + |K_0|\int_{t_0}^{t_1}(v^2 - v^{p+1}) = O(1). \tag{66}$$

Then

$$\int_{t_0}^{t_1}(v^2 - v^{p+1}) \le C \tag{67}$$

with a constant C independent of t_0, t_1. But just integrating (20) on $[t_0, t_1]$ and using the bound on v and its derivatives (c.f. Lemma 14) we find

$$\int_{t_0}^{t_1}(v - v^p) = O(1), \tag{68}$$

where $O(1)$ is bounded independently of t_0, t_1. Using the inequality $v - v^p \le v^2 - v^{p+1}$ for all $v > 0$ together with (68) yields

$$-C \le \int_{t_0}^{t_1}(v^2 - v^{p+1}).$$

We deduce from this and (67) that

$$\int_{t_0}^{t_1}(v^2 - v^{p+1}) = O(1).$$

Hence, by (66) we have $\int_0^\infty (v')^2 < \infty$. The relation (65) gives also $\int_0^\infty (v'')^2 < \infty$. $\square$

Proof of Proposition 8 (a) and (b) are proved in [34, Theorem 3.1].

(c) We write $v = v_{\beta^*}$. An explicit computation using (22) shows that for all $N \geq 3$ and all $\tau < (N-4)/2$ we have $K_1 < 0$ or $K_2 K_3 - K_1 > 0$, see the Appendix, Sect. A. If $K_1 < 0$ we may apply Lemma 16 and if $K_2 K_3 - K_1 > 0$ we apply Lemma 17 to conclude that (64) holds.

Let (t_k) be a strictly increasing sequence such that $t_k \to +\infty$, $\lim_{k\to\infty}(t_{k+1} - t_k) = 0$, and

$$v'(t_k) \to 0 \quad \text{as } k \to \infty.$$

If $t \geq s \geq 0$ we have by (64)

$$|v'(t) - v'(s)| \leq C|t - s|^{1/2}.$$

Hence for $t \in [t_k, t_{k+1}]$

$$|v'(t)| \leq |v'(t_k)| + C(t_{k+1} - t_k)^{1/2}.$$

This shows that $v'(t) \to 0$ as $t \to \infty$. Using then elliptic estimates we deduce

$$v^{(i)}(t) \to 0 \quad \text{as } t \to \infty$$

for $i = 1, 2, 3, 4$. Using the equation we also deduce that $v(t) \to 1$ as $t \to \infty$. We hence obtain that $(v_1(t), \ldots, v_4(t)) \to P_2$ as $t \to \infty$.

(d) Let $\beta > \beta^*$. Using Lemma 9 we see that $(\Delta U_\beta)' \geq (\Delta U_{\beta^*})'$ for all $r \geq 0$. Since $\Delta U_\beta(0) - \Delta U_{\beta^*}(0) = \beta - \beta^*$, integrating we deduce that

$$(r^{N-1}(U'_\beta - U'_{\beta^*}))' \geq (\beta - \beta^*) r^{N-1} \quad \text{for all } r \geq 0.$$

Integrating successively we deduce $U'_\beta(r) \geq U'_{\beta^*}(r) + (\beta - \beta^*)\frac{r}{N} > 0$ for all $r > 0$ and (49).

(e) Let $0 < \beta < \beta^*$. Then by (50), ΔU_β is decreasing. But it is also positive at 0 and it cannot be positive in $(0, R_{\max}(\beta))$, because otherwise U_β would be an entire solution. Hence ΔU_β changes sign exactly once in $(0, R_{\max}(\beta))$. Using then the formula

$$r^{N-1} U'_\beta(r) = \int_0^r s^{N-1} \Delta U_\beta(s)\, ds \quad \forall r > 0 \tag{69}$$

we see that U'_β is first increasing, then decreasing. It has to be negative at some point because otherwise U_β would be entire. Thus U'_β vanishes at exactly one point R_0, is positive on $(0, R_0)$ and negative on $(R_0, R_{\max}(\beta))$.

(f) If $\beta \leq 0$ then $\Delta U_\beta(0) \leq 0$ and then $\Delta U_\beta(r) < 0$ for all $r \in (0, R_{\max}(\beta))$. Hence by (69), $U'_\beta(r) < 0$ for all $r \in (0, R_{\max}(\beta))$. □

5 Heteroclinic connection from P_1 to P_2

Proposition 18 *In the following cases*:

(a) $N \geq 5$ *and* $p > \frac{N+4}{N-4}$
(b) $N \geq 4$ *and* $p < -1$
(c) $N = 3$ *and* $-3 < p < -1$

system (24) *has an heteroclinic orbit from* P_1 *to* P_2.

When p is positive this result is related to the properties of the initial value problem:

$$\begin{aligned} &\Delta^2 U = K_0 |U|^{p-1} U \quad r \in (0, R_{\max}(\beta)) \\ &U(0) = 1, \quad U'(0) = 0, \quad \Delta U(0) = \beta, \quad (\Delta U)'(0) = 0 \end{aligned} \tag{70}$$

where $[0, R_{\max}(\beta))$ is the maximal interval of existence, and for negative p it is related to the initial value problem (47).

In the case of negative powers we will deduce Proposition 18 from Proposition 8, and in the case of positive powers we will use the following result.

Theorem 10 (Gazzola and Grunau [20, Theorem 2]) *Assume* $N \geq 5$, $p > \frac{N+4}{N-4}$. *Then there exists a unique* $\beta^* < 0$ *such that* $R_{\max}(\beta^*) = +\infty$ *and*

$$\lim_{r \to \infty} r^\tau U_{\beta^*}(r) = 1. \tag{71}$$

Moreover:

(a) *If* $\beta < \beta^*$ *there exists* $0 < R_1 < R$ *such that* $U_\beta(R_1) = 0$ *and* $\lim_{r \to R_{\max}(\beta)} U_\beta(r) = -\infty$.
(b) *If* $0 > \beta > \beta^*$ *there exists* $0 < R_0 < R_{\max}(\beta)$ *such that* $U'_\beta(r) < 0$ *for* $r \in (0, R_0)$, $U'_\beta(R_0) = 0$, $U'_\beta(r) > 0$ *for* $r \in (R_0, R_{\max}(\beta))$, *and* $\lim_{r \to R_{\max}(\beta)} U_\beta(r) = \infty$.

The these properties we add the following:

Lemma 19 *Assume* $N \geq 5$, $p > \frac{N+4}{N-4}$ *and let* U_β *be the solution to* (70). *If* $\beta < \beta^*$ *then*

$$U'_\beta(r) \leq U'_{\beta^*}(r) + \frac{\beta - \beta^*}{N} r \quad \text{for all } r \in [0, R_{\max}(\beta))$$

and

$$U_\beta(r) \leq U_{\beta^*}(r) + \frac{\beta - \beta^*}{2N} r^2 \quad \text{for all } r \in [0, R_{\max}(\beta)). \tag{72}$$

If $\beta \geq 0$ *then* $U'_\beta(r) > 0$ *for all* $r \in (0, R_{\max}(\beta))$.

Proof of Proposition 18 We fix β^* is such that either (48) or (71) holds. Define $v(t) = r^\tau U_{\beta^*}(r)$, $r = e^t$, $t \in \mathbb{R}$. Then v satisfies Eq. (20) for all $t \in \mathbb{R}$. We also define $V = (v_1, \ldots, v_4)$ by (23). Then V satisfies the system (24) for all $t \in \mathbb{R}$. Since U_{β^*} is smooth at the origin

$$\lim_{t\to-\infty} V(t) = P_1$$

and (71) tells us that

$$\lim_{t\to\infty} v_1(t) = 1.$$

The proofs of Theorem 10 and of Proposition 8 actually yield:

$$\lim_{t\to\infty} V(t) = P_2.$$

See indeed Proposition 3 in [20] for p positive and Sect. 4 for negative p. □

Proof of Lemma 19 Let $\beta < \beta^*$ (recall that $\beta^* < 0$). Using Lemma 9 we see that $(\Delta U_\beta)' \le (\Delta U_{\beta^*})'$ for all $r \ge 0$. Since $\Delta U_\beta(0) - \Delta U_{\beta^*}(0) = \beta - \beta^*$, integrating we deduce that

$$(r^{N-1}(U'_\beta - U'_{\beta^*}))' \le (\beta - \beta^*) r^{N-1} \quad \text{for all } r \ge 0.$$

Integrating successively we deduce $U'_\beta(r) \le U'_{\beta^*}(r) + (\beta - \beta^*)\frac{r}{N} < 0$ for all $r > 0$ and (72).

If $\beta \ge 0$ then $\Delta U_\beta(0) \ge 0$. Since

$$\frac{d\Delta U_\beta}{dr}(r) = r^{1-N} K_0 \int_0^r s^{N-1} U_\beta(s)^p ds > 0$$

for $0 \le r < R_{\max}(\beta)$ we have $\Delta U_\beta(r) > 0$ for all $r \in (0, R_{\max}(\beta))$. Hence by (69), $U'_\beta(r) > 0$ for all $r \in (0, R_{\max}(\beta))$. □

Lemma 20 *Assume* $V : (T, \infty) \to \mathbb{R}^4$, $V = (v_1, v_2, v_4, v_4)$ *is a solution to* (24) *such that* $\lim_{t\to\infty} V(t) = P_2$ *and either*

$$\frac{V'(t)}{|V'(t)|} + \frac{v^{(2)}}{|v^{(2)}|} \to 0 \quad \textit{as } t \to +\infty \tag{73}$$

or

$$\frac{V'(t)}{|V'(t)|} - \frac{v^{(2)}}{|v^{(2)}|} \to 0 \quad \textit{as } t \to +\infty. \tag{74}$$

Then V *cannot be extended to a connection from* P_1 *to* P_2.

Proof The case of positive p was already treated in [15, Proposition 1].

Assume first that (73) holds. Then by (33) we have

$$v_1'(t) < 0, \quad v_2'(t) > 0, \quad v_3'(t) < 0, \quad v_4'(t) > 0$$

for all t near $+\infty$ and

$$\begin{aligned} &v_1(t) > 1, \quad v_2(t) < -\alpha\tau, \\ &v_3(t) > -\alpha\tau(N-2-\tau), \quad v_4(t) < \alpha\tau(N-2-\tau)(\tau+2) \end{aligned} \tag{75}$$

for all t near $+\infty$. We claim that

$$v_2(t) < -\alpha\tau \quad \forall t > T. \tag{76}$$

Assume by contradiction that this fails. Then from (75) we can define $t_1 > T$ to be the last time such that $v_2(t_1) = -\alpha\tau$. Then $v_2'(t_1) \leq 0$. Using the Eq. (24) we deduce that

$$v_3(t_1) \leq -\alpha\tau(N-2-\tau).$$

Then thanks to (75) we can define $t_2 \geq t_1$ to be the last time such that $v_3(t_2) = -\alpha\tau(N-2-\tau)$. This implies that $v_3'(t_2) \geq 0$ and by the system (24)

$$v_4(t_2) \geq \alpha\tau(N-2-\tau)(\tau+2).$$

Let $t_3 \geq t_2$ be the last time such that $v_4(t_3) = \alpha\tau(N-2-\tau)(\tau+2)$. Then $v_4'(t_3) \leq 0$. We deduce from (24) that

$$v_1(t_3) \leq 1.$$

Let $t_4 \geq t_3$ be the last time such that $v_1(t_4) = 1$. Then $v_1'(t_4) \geq 0$ and by (24)

$$v_2(t_4) \geq -\alpha\tau.$$

But $v_2(t) < -\alpha\tau$ for all $t \in (t_1, \infty)$, which is a contradiction. This proves claim (76) and shows that the trajectory defined by V cannot come from P_1.

Assume now that (74) holds. We claim that in this case

$$v_3(t) < -\alpha\tau(N-2-\tau) \quad \text{for all } t > T. \tag{77}$$

The proof is similar as before. Note that under the assumption (74) we have the opposite inequalities in (75). If the statement (77) fails we can defined the last time t_1 such that $v_3(t_1) = -\alpha\tau(N-2-\tau)$. Then define successively $t_2 \geq t_1$ such that $v_4(t_2) = \alpha\tau(N-2-\tau)(\tau+2)$, $v_4'(t_2) \geq 0$, $t_3 \geq t_2$ such that $v_1(t_3) = 1$, $v_1'(t_3) \leq 0$, $t_4 \geq t_3$ such that $v_2(t_4) = -\alpha\tau$ and $v_2'(t_4) \geq 0$, which leads to $v_3(t_4) \geq -\alpha\tau(N-2-\tau)$ which yields a contradiction. This shows that the trajectory cannot come from P_1. □

6 Proof of Theorem 1

We assume here $N = 3$ and $-3 < p < -1$ or $N \geq 4$ and $p < -1$. Let u be a radial singular weak solution to (2). We define $v(t)$, $t \leq 0$ by (19) and $v_1, \ldots, v_4$ by (23).

The arguments are very similar to those of Sect. 4, so we will skip some of the proofs. The first step is to prove the following estimates.

Lemma 21 *We have*

$$\limsup_{t \to -\infty} v(t) > 0.$$

The proof is the same as for Lemma 12.

Lemma 22 *We have*

$$\liminf_{t \to -\infty} v(t) > 0. \tag{78}$$

The proof is analogous to that of Lemma 13.

Lemma 23 *There is $C > 0$ such that for $i = 1, 2, 3, 4$*

$$|v_i(t)| \leq Ce^{\tau t} \quad \textit{for all } t \leq 0.$$

Proof We assume that $0 \leq u(r) < 1$ for $0 < r \leq 1$ which implies $0 < v(t) \leq e^{\tau t}$ for all $t \leq 0$. We regard (20) as an elliptic equation, or use interpolation inequalities such as in Chapter 6 of [22] to obtain: for $t \leq -1$ and $i = 1, 2, 3, 4$

$$|v^{(i)}(t)| \leq C \sup_{[t-1,t+1]} (|v| + |v^p|).$$

Since $v(t) \leq e^{\tau t}$ and v bounded away from zero by (78), we deduce that $|v^{(i)}| \leq Ce^{\tau t}$ for all $t \leq 0$. Using the formulas (23) we deduce the result for $v_1, \ldots, v_4$. □

Lemma 24 *There exists $C > 1$ such that for all $t \geq 0$*

$$\frac{1}{C} \leq v_1(t) \leq C, \quad |v_2(t)| \leq C, \quad |v_3(t)| \leq C, \quad |v_4(t)| \leq C,$$

and

$$|v^{(i)}(t)| \leq C \quad \forall i = 0, \ldots, 4.$$

Proof **Case of dimension $N \geq 5$.** By [16, Theorem 6] we know that there exists some constant $C > 0$ such that

$$\frac{1}{C} r^{-\tau} \leq 1 - u(r) \leq Cr^{-\tau} \quad \text{for all } 0 < r \leq 1.$$

This means that $\frac{1}{C} \le v(t) \le C$ for all $t \le 0$. By interpolation inequalities [22, Chapter 6] we obtain: for $t \le -1$ and $i = 1, 2, 3, 4$

$$|v^{(i)}(t)| \le C \sup_{[t-1,t+1]} (|v| + |v^p|).$$

Since v is bounded and bounded away from zero, we deduce that $v^{(i)}$ remains uniformly bounded. The estimates for $v_1, \ldots, v_4$ follow from (23).

Case $N = 3, 4$**.** We observe that (78) implies that $v_1(t) \le C$ for all $t \le 0$ and some $C > 0$. Let $t_0 \le t_1 \le 0$. Integrating the equation for v_4' we find

$$v_4(t_0) = e^{-(N-4-\tau)t_0} \left[v_4(t_1) e^{(N-4-\tau)t_1} - |K_0| \int_{t_0}^{t_1} e^{(N-4-\tau)s} v_1(s)^{|p|} \, ds \right]$$

Since $N - 4 - \tau > 0$ and v_1 is uniformly bounded above by Lemma 22 the integral above converges as $t_0 \to -\infty$. We will prove that

$$v_4(t_1) e^{(N-4-\tau)t_1} = |K_0| \int_{-\infty}^{t_1} e^{(N-4-\tau)s} v_1(s)^{|p|} \, ds \tag{79}$$

holds for all $t_1 \le 0$. Indeed, suppose this fails for some $t_1 \le 0$. Then

$$v_4 \sim e^{-(N-4-\tau)t}. \tag{80}$$

where we use the notation

$$f \sim g \quad \text{if} \quad \lim_{t \to -\infty} f(t)/g(t) \text{ exists and is no zero,}$$

for $f, g : (-\infty, 0] \to \mathbb{R}$ such that $g(t) \neq 0$. Integrating the equation for v_3' on $[t, t_1]$ with $t \le t_1$ we find

$$v_3(t) = e^{(\tau+2)t} \left[v_3(t_1) e^{-(\tau+2)t_1} - \int_t^{t_1} e^{-(\tau+2)s} v_4(s) \, ds \right]. \tag{81}$$

Since $\tau + 2 > 0$, we find from (80) that $v_3(t) \sim e^{-(N-4-\tau)t}$. Integrating the equation for v_2' in $[t, t_2]$ with $t \le t_2 \le 0$ we obtain

$$v_2(t) = e^{-(N-2-\tau)t} \left[v_2(t_2) e^{(N-2-\tau)t_2} - \int_t^{t_2} e^{(N-2-\tau)s} v_3(s) \, ds. \right]$$

But we know by Lemma 23 that $|v_2(t)| \le Ce^{\tau t}$ for all $t \le 0$. Since $\tau+2-N < \tau < 0$ we must have

$$v_2(t_2)e^{(N-2-\tau)t_2} = \int_{-\infty}^{t_2} e^{(N-2-\tau)s} v_3(s)\, ds \quad \text{for all } t_2 \le 0. \tag{82}$$

We deduce from this formula that

$$v_2(t) \sim e^{-(N-4-\tau)t}. \tag{83}$$

Writing the equation for v_1' in the form

$$\frac{d}{dt}\left(e^{-|\tau|t} v_1(t)\right) = \left(e^{-|\tau|t} v_1(t)\right)^2 e^{|\tau|t} v_2(t)$$

and integrating over $t \le t_2 \le 0$ yields

$$v_1(t) = \frac{e^{|\tau|t}}{\frac{e^{|\tau|t_2}}{v_1(t_2)} + \int_t^{t_2} e^{|\tau|s} v_2(s)\, ds}. \tag{84}$$

Assume $N = 4$. Then, since $v_2(t) \sim e^{-(N-4-\tau)t}$ we have

$$\int_t^{t_2} e^{|\tau|s} v_2(s)\, ds \sim t,$$

and hence $v_1(t) \sim \frac{e^{|\tau|t}}{|t|}$. But $v = 1/v_1$ and $1 - u(e^t) = Ce^{-\tau t} v(t)$. From this we deduce that $1 - u(e^t) \sim |t|$ as $t \to -\infty$, which is impossible because $1 - u(r) \le 1$ for all $0 < r \le 1$. This proves (79) when $N = 4$.

Assume now $N = 3$. Then by (83), $e^{|\tau|s} v_2(s) \sim e^s$ and therefore the integral $\int_t^{t_2} e^{|\tau|s} v_2(s)\, ds$ has a finite limit as $t \to -\infty$. If

$$\frac{e^{|\tau|t_2}}{v_1(t_2)} + \int_{-\infty}^{t_2} e^{|\tau|s} v_2(s)\, ds \ne 0$$

for some $t_2 \le 0$ then by (84)

$$v_1(t) \sim e^{-\tau t}.$$

This means $v(t) \sim e^{\tau t}$, and hence $1-u(e^t) = Ce^{-\tau t}v(t) \sim 1$. Thus, $\lim_{r\to 0}(1-u(r))$ exists and is not zero. Then u is a regular solution, which we assume is not. Therefore

$$\frac{e^{|\tau|t_2}}{v_1(t_2)} + \int_{-\infty}^{t_2} e^{|\tau|s} v_2(s)\, ds = 0$$

for all $t_2 \le 0$. This formula yields

$$v_1(t) = -\frac{e^{|\tau|t}}{\int_{-\infty}^{t} e^{|\tau|s} v_2(s)\, ds}$$

which implies $v_1(t) \sim e^{-(1+\tau)t}$, by (83) so that $v(t) \sim e^{(1+\tau)t}$. Then $1-u(e^t) = Ce^{-\tau t}v(t) \sim e^t$ and therefore

$$1-u(r) \sim r \quad \text{as } r \to 0. \tag{85}$$

Then $(1-u(r))^{-p} \sim r^{-p}$ and this belongs to $L^q(B)$ for $q < 3/|p|$, in 3 dimensions. By L^p regularity $u \in W^{4,q}(B)$ and in 3 dimensions this is contained in $C^{1,\alpha}(\overline{B})$ for some $\alpha > 0$ if $1/q - 1 < 0$. But q can be chosen such that $q > 1$ because $|p| < 3$. Therefore $u \in C^{1,\alpha}$ for some $\alpha > 0$, but this contradicts (85). We have then established (79) also in the case $N = 3$.

Using now (79) and the fact that v_1 is bounded we deduce that v_4 is bounded. This and (81) imply that v_3 remains bounded as $t \to -\infty$. By (82) v_2 remains bounded. Consider now formula (84). Knowing that v_2 remains bounded we see that $\int_t^{t_2} e^{|\tau|s} v_2(s)\, ds$ has a limit as $t \to -\infty$. If

$$\frac{e^{|\tau|t_2}}{v_1(t_2)} + \int_{-\infty}^{t_2} e^{|\tau|s} v_2(s)\, ds \ne 0$$

for some $t_2 \le 0$ then $v_1(t) \sim e^{|\tau|t}$ and this implies that $1-u(r) \sim 1$ as $r \to 0$. But then u is a regular solution, which we assume is not. Therefore

$$\frac{e^{|\tau|t_2}}{v_1(t_2)} + \int_{-\infty}^{t_2} e^{|\tau|s} v_2(s)\, ds = 0$$

for all $t_2 \le 0$. This formula yields

$$v_1(t) = -\frac{e^{|\tau|t}}{\int_{-\infty}^{t} e^{|\tau|s} v_2(s)\, ds}$$

and since v_2 is bounded we deduce from it that v_1 is bounded below. This concludes the proof. □

Proof of Theorem 1 The proof is similar to the one of Proposition 8 part c). We claim that

$$\int_{-\infty}^{0} v'(s)^2\,ds < +\infty, \quad \int_{-\infty}^{0} v''(s)^2\,ds < +\infty \tag{86}$$

In the case $K_1 < 0$ this can be proved multiplying the equation by v' and integrating in some interval $[t, 0]$ (similar to using E in Lemma 16). In the case $K_2K_3 - K_1 > 0$ the same proof as in Lemma 17 yields (86).

Since for all $N \ge 3$ and all $\tau < (N-4)/2$ we have $K_1 < 0$ or $K_2K_3 - K_1 > 0$, see the Appendix, Sect. A, we obtain the validity of (86) in any case. Using (86) one may prove as in Proposition 8 part c) that $v(t) \to 1$ as $t \to -\infty$ and hence u is a weakly singular solution. □

7 Proof of Theorems 4 and 7

Throughout this section we assume (8) for positive powers and (10) for negative powers. Let P_1, P_2 be the stationary points of the system (24) defined in (26). Then P_1 has a 2-dimensional unstable manifold $W^u(P_1)$ while P_2 has a 1-dimensional unstable manifold $W^u(P_2)$ and a 3-dimensional stable manifold $W^s(P_2)$.

Let $V_0 : \mathbb{R} \to \mathbb{R}^4$ be the heteroclinic connection from P_1 to P_2 of Proposition 18 and $\hat{V}_0 = V_0(-\infty, \infty)$. Then $\hat{V}_0$ is contained in both $W^u(P_1)$ and $W^s(P_2)$.

Lemma 25 *$W^u(P_1)$ and $W^s(P_2)$ intersect transversally on points of $\hat{V}_0$. More precisely for points $Q \in \hat{V}_0$ sufficiently close to P_2 there directions in the tangent plane to $W^u(P_1)$ which are almost parallel to $v^{(1)}$, the tangent vector to $W^u(P_2)$ at P_2.*

Proof Let $U_\beta(r)$ be the solution to (70) or (47) defined in the maximal interval $[0, R_{\max}(\beta))$. Let β^* denote the unique value of β such that $R_{\max}(\beta^*) = \infty$ and

$$\lim_{r\to\infty} r^\tau U_{\beta^*}(r) \quad \text{exists.}$$

In Lemma 19 (for positive p) and in Proposition 8 d) (for negative p) is shown that for $\alpha\beta < \alpha\beta^*$ the following estimate holds:

$$\alpha U'_\beta(r) \le \alpha U'_{\beta^*}(r) - \alpha\frac{\beta^* - \beta}{N} r \quad \forall r \in [0, R_{\max}(\beta)).$$

Then $\frac{\partial U_\beta}{\partial \beta}(r)|_{\beta=\beta^*}$ satisfies the linearized equation at U_{β^*} and

$$\frac{\partial U'_\beta}{\partial \beta}\Big|_{\beta=\beta^*}(r) \ge \frac{1}{N} r \quad \forall r \ge 0. \tag{87}$$

Let $v(t) = e^{\tau t} U_{\beta^*}(e^t)$, $t \in \mathbb{R}$. Let $V = (v_1, \ldots, v_4)$ be defined by (23) and let $Z = \frac{\partial V}{\partial \beta}|_{\beta=\beta^*}$. Then $Z = (z_1, \ldots, z_4)$ satisfies

$$Z' = (M + R(t))Z$$

where M is the matrix defined in (27) and

$$R(t) = \begin{bmatrix} (1-\alpha)(v_1^{-\alpha} v_2 - \tau) & v_1^{1-\alpha} - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ pK_0(v_1^{\alpha p - 1} - 1) & 0 & 0 & 0 \end{bmatrix}.$$

Recall that $V(t) \to P_2$ as $t \to \infty$. Moreover the convergence is exponential, that is there are $C, \sigma > 0$ such that $|V(t) - P_2| \leq Ce^{-\sigma t}$ for all $t \geq 0$. This follows from the Hartman-Grobman theorem (see Theorem 7.1 in [29, Chap. IX] or Theorem 1.3.1 in [25, Chap. 1]), which shows that the system (24) is C^0-conjugate to its linearization near P_2. Recall that the eigenvalues of M are $\nu_1 > 0 > \nu_2$ and ν_3, ν_4 which have negative real part and nonzero imaginary part. Let $v^{(i)} \in \mathbb{C}^4$ denote an eigenvector associated to ν_i. By Theorem 8.1 in [10, Chap. 3] there are solutions φ_k to

$$\varphi_k' = (M + R(t))\varphi_k, \quad t > 0$$

such that $\lim_{t\to\infty} \varphi_k(t) e^{-\nu_k t} = v^{(k)}$. It follows from this that $Z = \sum_{i=1}^4 c_i \varphi_i$ for some constants $c_1, \ldots, c_4 \in \mathbb{C}$. The condition (87) and the definitions in (23) imply that $|z_2(t)| \geq ce^{(2+\tau)t}$ for some $c > 0$ and all $t \geq 0$. But $\tau + 2 > 0$, so $|Z(t)| \to \infty$ as $t \to \infty$. Since $\nu_1 > 0$ and ν_2, ν_3, ν_4 have negative real part, we conclude that $c_1 \neq 0$ and

$$Z(t) = c_1 v^{(1)} e^{\nu_1 t} + o(e^{\nu_1 t}) \quad \text{as } t \to \infty.$$

Since $v^{(1)}$ is the tangent vector to $W^u(P_2)$, we have that $\frac{\partial V}{\partial \beta}$ is not tangent to $W^s(P_2)$ for t large. On the other hand $\frac{\partial V}{\partial \beta}$ is tangent to $W^u(P_1)$ by construction. This shows that $W^s(P_2)$ and $W^u(P_1)$ intersect transversally on points of $\hat{V}_0$ close to P_2. By the invertibility of the flow away from the stationary points, $W^s(P_2)$ and $W^u(P_1)$ intersect transversally on all points of $\hat{V}_0$. □

Proof of Theorems 4 and 7 We will write generic points in the phase space $\mathbb{R}^4$ as (v_1, v_2, v_3, v_4). Let $\{e_j \ : \ j = 1, \ldots, 4\}$ denote the canonical basis of $\mathbb{R}^4$.

By Propositions 6 and 7 we know that $W^u(P_2) \cap \{v_2 = 0\}$ is a single point, which we call $P^* = (P_1^*, P_2^*, P_3^*, P_4^*)$. Let $\mathcal{E} = W^u(P_1) \cap \{v_2 = 0\}$. Each regular radial solution of (2) corresponds to exactly one point $v = (v_1, \ldots, v_4) \in \mathcal{E}$ with $v_1 > 0$.

The multiplicity results are consequence of the following claims:

(a) $\mathcal{E}$ contains a spiral $\mathcal{S}$ about the point P^*,

(b) $\mathcal{S}$ is contained in a 2-dimensional C^1 surface $\Sigma \subseteq \{v_2 = 0\}$, and
(c) the plane generated by e_2, e_3, e_4 is transversal to the tangent plane to Σ at P^*.

More precisely, by (a) we mean that after a C^1 diffeomorphism of a neighborhood of P^* to a neighborhood of the origin in $\mathbb{R}^4$, which maps P^* to the origin, the curve $\mathcal{S}$ can be parametrized by a C^1 function of the form $(r(s)\cos(s), r(s)\sin(s), 0, 0)$, $s \in [0, \infty)$, such that $r(s) > 0$ for all $s \geq 0$ and $r(s) \to 0$ as $s \to \infty$. Moreover one can choose this diffeomorphism such that Σ corresponds to part of the surface $\{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : \ x_3 = x_4 = 0\}$.

Assume (a), (b) and (c) have been proved and define the hyperplane $H_\lambda = \{v_1 = (\lambda/\alpha K_0)^{\frac{1}{p-1}}\}$ where $\lambda > 0$. After the C^1 diffeomorphism described above we can assume that $\mathcal{S} = \{(r(s)\cos(s), r(s)\sin(s), 0, 0) \ : \ s \geq 0\}$ and $\Sigma \cap B_\rho = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : \ x_3 = x_4 = 0\} \cap B_\rho$ for some $\rho > 0$. The hyperplane H_λ is transformed into a C^1 hypersurface. If $\lambda = \lambda_S$ then H_λ contains the origin and the condition c) ensures that it is transversal to Σ at the origin. By transversality, $H_\lambda \cap \Sigma$ is a C^1 curve through the origin contained in $\{x_3 = x_4 = 0\}$ (in the new coordinates). Using polar coordinates in $\{x_3 = x_4 = 0\}$ we then see that H_λ intersects the spiral $\mathcal{S}$ infinitely many times, which means that (2) has infinitely many radial regular solutions. If $\lambda \neq \lambda_S$ but λ is close to λ_S, $H_\lambda \cap \mathcal{E}$ contains a large number of points, which yields a large number of radial regular solutions of (2).

In what follows we will prove (a), (b) and (c). Let X_t denote the flow generated by (24), that is, $X_t(\xi)$ is the solution to (24) at time t with initial condition $X_0(\xi) = \xi \in \mathbb{R}^4$. For fixed ξ, $X_t(\xi)$ is defined for t in a maximal open interval containing 0.

Since (24) is C^1-conjugate to its linearization around the point P_2 by Lemma 5, there is an open neighborhood N_{P_2} of P_2 and a C^1 diffeomorphism $R : N_{P_2} \to N_0$ to an open neighborhood N_0 of 0 such that $RX_tR^{-1} = L_t$ where L_t is the flow generated by M, and the formula holds in some neighborhood of the origin.

Let D be the 3 dimensional disk $D = \{v = (v_1, \ldots, v_4) \ : \ v_2 = 0, \ |v - P^*| < 1\}$, which by Proposition 7 is transversal to $W^u(P_2)$. Let $B^s \subseteq W^s(P_2) \cap N_{P_2}$ be an open neighborhood of P_2 relative to $W^s(P_2)$ diffeomorphic to a 3 dimensional disk. By choosing smaller neighborhoods if necessary, we may apply the λ-lemma of Palis [36]. Let D_t be the connected component of $X_t(D) \cap N_{P_2}$ that contains $X_t(P^*)$. Then, given $\varepsilon > 0$ there exists some $t_0 < 0$, $|t_0|$ large, such that D_{t_0} contains a 3-dimensional C^1 manifold $\mathcal{M}$ that is a εC^1-close to B^s, which means that there is a diffeomorphism $\eta : \mathcal{M} \to B^s$ such that $\|i - \eta\|_{C^1(\mathcal{M})} \leq \varepsilon$ where $i : \mathcal{M} \to \mathbb{R}^4$ is the inclusion map.

Chose some point $Q \in \hat{V}_0$ such that $Q \in N_{P_2}$. By Lemma 25 we may choose a C^1 curve contained in $W^u(P_1)$, say $\Gamma = \{\gamma(s) \ : \ |s| < \delta\}$ with $\gamma : (-\delta, \delta) \to \mathbb{R}^4$ a C^1 function with $\gamma(0) = Q$, $\gamma'(0)$ not tangent to $W^s(P_2)$ at Q. We can assume also that this curve is contained in N_{P_2}. Choosing ε small we can assume that Γ intersects $\mathcal{M}$.

We summarize in the next lemma several properties that we prove later on in this section.

Lemma 26 *For large t, $X_t(\Gamma) \cap \mathcal{M}$ is a single point that we call P_t and the following properties hold*:

1. *The collection of the points P_t for large t forms a spiral.*

2. *There exists a 2 dimensional C^1 manifold $\tilde{\Sigma}$ that contains P_t for all t large.*
3. *Let Q_{t_0} be the intersection of $\mathcal{M}$ with $W^u(P_2)$. Then the tangent plane to $\tilde{\Sigma}$ at Q_{t_0} becomes parallel to the one generated by $Re(v^{(3)})$, $Im(v^{(3)})$ (the eigenvector corresponding to ν_3, ν_4) as $\varepsilon \to 0$.*
4. *Moreover, for $s > 0$ suitably small the time t such that $X_t(\gamma(s)) \in \mathcal{M}$ satisfies*

$$s = ce^{-\nu_1 t} + o(e^{-\nu_1 t}) \quad as\ t \to \infty \tag{88}$$

where $c > 0$.

Let $\tilde{\mathcal{S}}$ denote the collection $\{P_t : t \geq t_1\}$ where t_1 is suitably large. Define $\mathcal{S} = X_{-t_0}(\tilde{\mathcal{S}})$ and $\Sigma = X_{-t_0}(\tilde{\Sigma})$. Since X_{-t_0} is a smooth diffeomorphism from $\mathcal{M}$ to a neighborhood of P^* inside the hyperplane $\{v_2 = 0\}$ we see that $\mathcal{S}$ is a spiral contained in a C^1 surface Σ. The points of $\mathcal{S}$ belong to $W^u(P_1)$ because they were obtained though the flow from points in $X_t(\Gamma)$. This proves parts (a) and (b).

We now prove statement c). It is sufficient to show that inside the space $\{v_2 = 0\}$ the plane generated by e_3, e_4 is transversal to the tangent space to Σ at P^*. Let $V = (v_1, \ldots, v_4) : (-\infty, 0] \to \mathbb{R}^4$ denote the trajectory corresponding to the weakly singular solution, that is, $\lim_{t\to-\infty} V(t) = P_2$, $v_2(0) = 0$. To prove our claim we need to transport the plane generated by e_3 and e_4 back along V to P_2 and this is accomplished by solving the linearized equation around V. More precisely, let $Z, \tilde{Z} : (-\infty, 0] \to \mathbb{R}^4$ be solutions to the linearization of (24) around V, that is, $Z = (z_1, z_2, z_3, z_4)$ satisfies for $t < 0$

$$\begin{cases} z_1' = (\alpha\tau + (1-\alpha)v_1^{-\alpha} v_2)z_1 + v_1^{1-\alpha} z_2 \\ z_2' = -(N-2-\tau)z_2 + z_3 \\ z_3' = (2+\tau)z_3 + z_4 \\ z_4' = -(N-4-\tau)z_4 + pK_0 v_1^{\alpha p-1} z_1 \end{cases} \tag{89}$$

and similarly for $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4)$. As final conditions we take $Z(0) = e_3$, $\tilde{Z}(0) = e_4$.

By Theorem 8.1 in [10, Chap. 3] there are solutions $\varphi_k : (-\infty, 0] \to \mathbb{C}^4$ to (89) such that

$$\lim_{t\to-\infty} \varphi_k(t)e^{-\nu_k t} = v^{(k)} \tag{90}$$

where $v^{(1)}, \ldots, v^{(4)}$ are the eigenvectors of M. Recall that $v^{(1)}, v^{(2)}$ are real, and $v^{(3)}, v^{(4)}$ are complex conjugate. Thus one can assume that φ_1, φ_2 are real φ_3, φ_4 are complex conjugate. Then

$$Z(t) = \sum_{i=1}^{4} c_i\varphi_i(t), \quad \text{and} \quad \tilde{Z}(t) = \sum_{i=1}^{4} \tilde{c}_i\varphi_i(t)$$

for some constants $c_1, \ldots, c_4, \tilde{c}_1, \ldots, \tilde{c}_4 \in \mathbb{C}$. We note that $c_1, c_2, \tilde{c}_1, \tilde{c}_2$ are real and $c_3\varphi_3(t) + c_4\varphi_4(t) \in \mathbb{R}$, $\tilde{c}_3\varphi_3(t) + \tilde{c}_4\varphi_4(t) \in \mathbb{R}$ for all $t \leq 0$.

We claim that

$$c_2 \neq 0 \quad \text{or} \quad \tilde{c}_2 \neq 0. \tag{91}$$

Assume, by contradiction, that $c_2 = 0$ and $\tilde{c}_2 = 0$. Define

$$f(t) = e^{(N-4-2\tau)t}\left(\frac{z_4(t)\tilde{z}_1(t)}{v_1^{1-\alpha}} - z_3(t)\tilde{z}_2(t) + z_2(t)\tilde{z}_3(t) - \frac{z_1(t)\tilde{z}_4(t)}{v_1^{1-\alpha}}\right), \quad \forall t \leq 0.$$

A calculation using (89) shows that f is constant. Using the final conditions for Z and $\tilde{Z}$ we see that $f(0) = 0$ and hence

$$f(t) = 0 \quad \forall t \leq 0.$$

Using (90), (31) and the assumption $c_2 = 0$, $\tilde{c}_2 = 0$ we can compute

$$\lim_{t \to -\infty} f(t) = (c_3\tilde{c}_4 - \tilde{c}_3 c_4)B$$

where

$$\begin{aligned} B &= (\nu_3 - \tau)(\nu_3 + N - 2 - \tau)(\nu_3 - 2 - \tau) - (\nu_3 - \tau)(\nu_3 + N - 2 - \tau)(\nu_4 - \tau) \\ &\quad + (\nu_4 - \tau)(\nu_4 + N - 2 - \tau)(\nu_3 - \tau) - (\nu_4 - \tau)(\nu_4 + N - 2 - \tau)(\nu_4 - 2 - \tau) \\ &= -\frac{1}{2}M_2(N)\sqrt{M_1(N) - M_2(N)} \end{aligned}$$

Thus $B \in i\mathbb{R}$, $B \neq 0$ and we conclude that $(c_3\tilde{c}_4 - \tilde{c}_3 c_4) = 0$. This means that there exists a $\lambda \in \mathbb{C}$ such that $\tilde{c}_k = \lambda c_k$, $k = 3, 4$. Since $c_3\varphi_3(t) + c_4\varphi_4(t) \in \mathbb{R}$, $\tilde{c}_3\varphi_3(t) + \tilde{c}_4\varphi_4(t) \in \mathbb{R}$ for all $t \leq 0$, $\nu_1 > 0$ and we assume that $c_2 = \tilde{c}_2 = 0$, we must have $\lambda \in \mathbb{R}$. Using $Z(0) = e_3$ and $\tilde{Z}(0) = e_4$ we see that

$$(\tilde{c}_1 - \lambda c_1)\varphi_1(0) = e_4 - \lambda e_3.$$

But $\varphi_1 = cV'$, for some constant $c \in \mathbb{R}$, since both solve (89) and both tend to 0 as $t \to -\infty$. We know that $v_2'(0) > 0$ by Proposition 7 and this implies $\tilde{c}_1 - \lambda c_1 = 0$, a contradiction.

Finally, the condition (91) implies the assertion c). Indeed, let us recall that $\Sigma = X_{-t_0}(\tilde{\Sigma})$ where $\tilde{\Sigma}$ is defined in Lemma 26 and $t_0 < 0$, with $|t_0|$ large. Let Q_{t_0} be the intersection of $\tilde{\Sigma}$ with $W^u(P_2)$. By 3. of Lemma 26 the tangent plane to $\tilde{\Sigma}$ at Q_{t_0} is almost parallel to the plane generated by $Re(v^{(3)})$, $Im(v^{(3)})$ (the eigenvector corresponding to ν_3). The condition (91) shows that for $|t_0|$ large at least one of the vectors $Z(t_0)$ or $\tilde{Z}_0$ is transversal to the tangent plane to $\tilde{\Sigma}$ at Q_{t_0}, since one of these vectors contains a component almost in the direction of $v^{(2)}$.

To finish the proof of Theorem 4 we still need to verify one assertion: for $\lambda \neq \lambda_S$ (2) has at most a finite number of solutions. We will do this in Proposition 31 of Sect. 8. □

Proof of Lemma 26 Let us recall that by Lemma 5 there is a C^1 diffeomorphism $R : N_{P_2} \to N_0$ from an open neighborhood N_{P_2} of P_2 to an open neighborhood N_0 of 0 with $R(P_2) = 0$, $\det(R'(P_2)) > 0$, such that $RX_tR^{-1} = L_t$ where $L_t = e^{Mt}$ is the flow generated by M, and the formula holds in some neighborhood of the origin.

Thus we may assume that P_2 is at the origin, and after a further linear change of variables, that $W^s(P_2)$ in a neighborhood of the origin is $\{(y_1, \ldots, y_4) : y_1 = 0\}$ and $B^s = \{(y_1, \ldots, y_4) : y_1 = 0,\ |y| < \delta\}$ for some $\delta > 0$. We can also assume that the heteroclinic orbit V_0 near the origin in the new variables is given by

$$V_0(t) = (0, c_2e^{\nu_2 t},\ c_3Re(e^{\nu_3 t}),\ c_4Im(e^{\nu_3 t})), \quad t \geq 0 \tag{92}$$

for some constants c_2, c_3, c_4. By Lemma 20 the curve V_0 cannot have a direction that becomes parallel to $e_2 = (0, 1, 0, 0)$ as $t \to \infty$. Since $|\nu_2| > |Re(\nu_3)|$ by (28), $c_3 \neq 0$ or $c_4 \neq 0$. By choosing ε small, we can assume that the normal vector to $\mathcal{M}$ near P^* is almost parallel to $e_1 = (1, 0, 0, 0)$. Thus by passing to a subset of $\mathcal{M}$ we may assume that $\mathcal{M}$ is a C^1 graph over the variables (y_2, y_3, y_4), that is, there exists a C^1 function $\psi : \{y' = (y_2, y_3, y_4) \in \mathbb{R}^3,\ |y'| < \delta\} \to \mathbb{R}$ with $\psi(0) > 0$ such that

$$\mathcal{M} = \{(\psi(y'), y') : y' \in \mathbb{R}^3,\ |y'| < \delta\}.$$

By Lemma 25 the tangent plane to $W^u(P_1)$ at points close to the new origin (i.e. P_2) contains vectors almost parallel to $e_1 = (1, 0, 0, 0)$ and hence $\gamma_1'(0) \neq 0$. Using the implicit function theorem we see that for large t the intersection of $\mathcal{M}$ and $L_t(\Gamma)$ occurs at points of the form

$$P_t = (\gamma_1(s)e^{\nu_1 t}, \gamma_2(s)e^{\nu_2 t}, \gamma_3(s)Re(e^{\nu_3 t}), \gamma_4(s)Im(e^{\nu_3 t}))$$

where $s = ce^{-\nu_1 t} + o(e^{-\nu_1 t})$ as $t \to \infty$ for some $c > 0$. Since $c_3 \neq 0$ or $c_4 \neq 0$ in (92) we can define a surface

$$\tilde{\Sigma} = \{y = (y_1, y_2, y_3, y_4) : |y| < \delta,\ y_1 = \psi(y_2, y_3, y_4),\ y_2 = g(y_3, y_4)\}$$

that contains the points P_t, where g is smooth away from the origin and has the property

$$g(y_3, y_4) = O(|(y_3, y_4)|^\beta)$$

with $\beta = \nu_2/Re(\nu_3)$. Thanks to (28) we see that $\beta > 1$. Therefore g is C^1 and $\tilde{\Sigma}$ is a C^1 surface. □

Proof of Theorem 9 By Propositions 6 and 7 we know that $W^u(P_2) \cap \{v_3 = 0\}$ is a single point, which we call $\bar{P}^* = (\bar{P}_1^*, \bar{P}_2^*, \bar{P}_3^*, \bar{P}_4^*)$.

As in Theorem 4, the multiplicity results asserted in Theorem 9 are consequence of the following claims:

(a) $\mathcal{E} := W^u(P_1) \cap \{v_3 = 0\}$ contains a spiral $\mathcal{S}$ about the point $\bar{P}^*$,
(b) $\mathcal{S}$ is contained in a 2-dimensional C^1 surface $\Sigma \subseteq \{v_3 = 0\}$, and

(c) the plane through $\bar{P}^*$ parallel to e_2, e_3, e_4 is transversal to the tangent plane to Σ at $\bar{P}^*$.

The proofs are similar to the Dirichlet case, now changing $v_2 = 0$ for $v_3 = 0$. So to prove c) it will be sufficient now to show that inside the space $\{v_3 = 0\}$ the plane generated by e_2, e_4 is transversal to the tangent space to Σ at $\bar{P}^*$. We define now Z satisfying (89) with the final condition $Z(0) = e_2$, and $\tilde{Z}$ remains unchanged. In the same form we claim that (91) holds. Indeed using the same argument as before with $Z(0) = e_2$ and $\tilde{Z}(0) = e_4$, we find

$$(\tilde{c}_1 - \lambda c_1)\varphi_1(0) = e_4 - \lambda e_2.$$

But we know by Proposition 7 that $v_3'(0) > 0$ and this implies $\tilde{c}_1 - \lambda c_1 = 0$, a contradiction. The rest of the proof is the same. □

8 Structure of the solution set

The initial value problems (70) for positive p and (47) for negative p yield solutions to problem (2). Indeed, let U_β be the solution of (70) or (47) defined in the maximal interval of existence $[0, R_{\max}(\beta))$. Set

$$I = \begin{cases} (\beta^*(p), 0) & \text{for } N \geq 5,\ p > \frac{N+4}{N-4} \\ (0, \beta^*(p)) & \text{for } N = 3 \text{ and } -3 < p < -1, \text{ or } N \geq 4 \text{ and } p < -1, \end{cases}$$

where $\beta^*(p)$ is the critical value obtained in Proposition 8 for negative p and in Theorem 10 for positive p. Thanks to these results we know that if $\beta \in I$ then U_β' vanishes exactly at $R_0(\beta)$, and for β outside I, U_β' does not vanish. It is not difficult to verify that $R_0(\beta)$ defines a C^1 function of $\beta \in I$. Let us introduce, for $\beta \in I$ the function

$$u_\beta(r) = \operatorname{sign}(p)\left[\frac{U_\beta(R_0(\beta)r)}{U_\beta(R_0(\beta))} - 1\right], \quad 0 \leq r \leq 1. \tag{93}$$

Then u_β is a solution of (2) for the value of $\lambda_\beta = |K_0| U_\beta(R_0(\beta))^{p-1} R_0(\beta)^4$.

As in Sect. 7, we let $\mathcal{E} = W^u(P_1) \cap \{v_2 = 0\}$ and recall that each regular radial solution of (2) corresponds to exactly one point $v = (v_1, \ldots, v_4) \in \mathcal{E}$ with $v_1 > 0$. Define $\mathcal{E}_0 = W^u(P_1) \cap \{v_2 = 0, v_1 > 0\}$. For $\beta \in I$ we let $V_\beta = (v_{\beta,1}, \ldots, v_{\beta,4}) : (-\infty, T(\beta)) \to \mathbb{R}^4$ be the function obtained from $v_\beta(t) = U_\beta(e^t)$ for $t < T(\beta)$ through the transformations (23), where $T(\beta) = \log(R_{\max}(\beta))$. Define also $T_0(\beta) = \log(R_0(\beta))$ for $\beta \in I$. Then V_β satisfies (24) and $v_{2,\beta}(T_0(\beta)) = 0$. Since $V_\beta(-\infty, T(\beta))$ lies in $W^u(P_1)$ we have $V_\beta(T_0(\beta)) \in \mathcal{E}$. Let us define $\phi : I \to \mathbb{R}^4$ by

$$\phi(\beta) = V_\beta(T_0(\beta)) \quad \text{for all } \beta \in I.$$

Then by construction $\phi(\beta) \in \mathcal{E}_0$ for all $\beta \in I$.

Lemma 27 *Let P^* be the intersection of $W^u(P_2)$ with $\{v_2 = 0\}$. Then*

$$\lim_{\beta\to\beta^*} \phi(\beta) = P^*, \qquad \lim_{\beta\to\beta^*} T_0(\beta) = +\infty$$

and

$$\lim_{\beta\to\beta^*} u_\beta(0) = \begin{cases} \infty & \text{for positive } p \\ 1 & \text{for negative } p. \end{cases}$$

Proof Let $\mathcal{M}$, Q and $\Gamma = \{\gamma(s) \ : \ |s| < \delta\}$ with $\gamma : (-\delta, \delta) \to \mathbb{R}^4$ be as in the proof of Theorem 4. Let $\Gamma_0 = \{\gamma(s) \ : \ 0 < s < \delta\}$. We note that for $\beta \in I$ and β close to β^* there is some time $t_1(\beta)$ such that $V_\beta(t_1(\beta)) \in \Gamma_0$. As $\beta \to \beta^*$, $V_\beta(t_1(\beta)) \to Q$ Fixing δ sufficiently small, we may define the function $\tau : \Gamma_0 \to \mathbb{R}_+$ where $\tau(p)$ is such that $X_{\tau(p)}(p) \in \mathcal{M}$. Then τ is continuous, and by (88)

$$\frac{1}{C}\log(1/s) \le \tau(\gamma(s)) \le C\log(1/s) \quad \text{for } 0 < s < \delta$$

where $C > 0$ is some constant. This shows that $\tau(p) \to +\infty$ as $p \to Q$ and then $T_0(\beta) \to \infty$ as $\beta \to \beta^*$. As in the proof of Lemma 26 one can also show that as $p \to Q$, $p \in \Gamma_0$ the point $X_{\tau(p)}(p)$ approaches the intersection of $\mathcal{M}$ with $W^u(P_2)$. This shows that $\phi(\beta) \to P^*$ as $\beta \to \beta^*$. Finally, since $T_0(\beta) \to \infty$ as $\beta \to \beta^*$ we see from formula (93) that $u_\beta(0) \to \infty$ as $\beta \to \beta^*$ if p is positive and $u_\beta(0) \to 1$ as $\beta \to \beta^*$ if p is negative. □

Lemma 28 *We have*

$$\lim_{\beta\to 0} \phi(\beta) = 0 \quad \textit{and} \quad \lim_{\beta\to 0} u_\beta(0) = 0.$$

Proof Using the implicit function theorem there is $\delta > 0$ such that for $\lambda > 0$ small there is a unique small solution $\underline{u}_\lambda$ of (2). The map $\lambda \mapsto \underline{u}_\lambda$ is C^1 into $C^4(\overline{B})$. Set

$$\tilde{U}_\lambda(r) = \frac{1 + \alpha\, \underline{u}_\lambda(A_\lambda r)}{1 + \alpha\, \underline{u}_\lambda(0)}$$

where

$$A_\lambda = \left(\frac{|K_0|(1 + \alpha\, \underline{u}_\lambda(0))^{1-p}}{\lambda}\right)^{1/4}.$$

Then $\tilde{U}_\lambda$ is the solution of (70) with $\beta = \beta(\lambda)$ where $\beta(\lambda) := \alpha A_\lambda^2 \Delta \underline{u}_\lambda(0)/(1 + \alpha\, \underline{u}_\lambda(0))$ by uniqueness of that initial value problem. In particular $u_\beta = \underline{u}_\lambda$ if $\beta = \beta(\lambda)$.

Since $\underline{u}_\lambda \to 0$ as $\lambda \to 0$, using standard elliptic estimates one can prove that

$$\frac{\underline{u}_\lambda}{\lambda} \to \frac{1}{8N(N+2)}(1 - r^2)^2 \quad \text{as } \lambda \to 0$$

in $C^4(\overline{B})$. It follows that $\beta(\lambda) = O(\lambda^{1/2})$ as $\lambda \to 0$. Thus for small $\beta \in I$ the solution of the shooting problem (70) or (47) is $\tilde{U}_\lambda$ with $\lambda > 0$ such that $\beta(\lambda) = \beta$, and this $\lambda > 0$ is uniquely determined. Then as $\beta \to 0$, $\lambda \to 0$ and $U_\beta(0) = u_\lambda(0) \to 0$. Also $R_0(\beta) = 1/A_\lambda \to 0$ and $\phi(\beta) \to 0$ as $\beta \to 0$ (since $\phi(\beta)$ is expressed in terms of derivatives of u_λ). □

Analogously to [27, Lemma 5.1] we have:

Lemma 29 *Let $p \in \mathbb{R}$. Suppose that u_1, u_2 are smooth radial solutions of (2) associated to parameters $\lambda_1 > 0$, $\lambda_2 > 0$ such that $u_1(0) = u_2(0)$. Then $\lambda_1 = \lambda_2$ and $u_1 \equiv u_2$.*

Proof First we consider the case $p \neq 0$. Suppose we have smooth radial solutions u_1, u_2 of (2) associated to parameters $\lambda_1 > \lambda_2$ such that $u_1(0) = u_2(0) = \kappa$.

Let $\alpha = \text{sign}(p)$. For $j = 1, 2$ define

$$v_j(r) = \frac{1 + \alpha\, u_j(\lambda_j^{-1/4} r)}{1 + \alpha\,\kappa}, \quad \text{for } r \in [0, \lambda_j^{1/4}]$$

The v_j satisfies

$$\begin{aligned} \Delta^2 v_j &= f(v_j) \quad \text{for } r \in [0, \lambda_j^{1/4}] \\ v_j(0) &= 1, \quad v_j'(0) = 0, \quad (\Delta v_j)'(0) = 0 \\ v_j(\lambda_j^{1/4}) &= \frac{1}{1 + \alpha\kappa}, \quad v_j'(\lambda_j^{1/4}) = 0 \end{aligned}$$

where $f(t) = \alpha(1+\alpha\kappa)^{p-1} t^p$. Note that f is increasing. Since u_1 and u_2 are decreasing functions on $(0, 1)$, we have that

$$\alpha v_j'(r) < 0 \quad \text{for all } r \in (0, \lambda_j^{1/4}). \tag{94}$$

Assume that $\alpha \Delta v_1(0) < \alpha \Delta v_2(0)$. Then by Lemma 9 $\alpha\, v_1(r) < \alpha\, v_2(r)$ for all $r \in [0, \lambda_2^{1/4}]$. In particular $\alpha\, v_1(\lambda_2^{1/4}) < \alpha\, v_2(\lambda_2^{1/4}) = \alpha/(1 + \alpha\kappa)$ which is impossible because (94) implies that $\alpha\, v_1(r) > \alpha/(1 + \alpha\kappa)$ for all $r \in [0, \lambda_1^{1/4})$.

Assume now that $\alpha \Delta v_1(0) > \alpha \Delta v_2(0)$. Then by Lemma 9 $\alpha v_1(r) > \alpha v_2(r)$, $\alpha v_1'(r) > \alpha v_2'(r)$, $\alpha \Delta v_1(r) > \alpha \Delta v_2(r)$, $\alpha(\Delta v_1)'(r) > \alpha(\Delta v_2)'(r)$ for all $r \in [0, \lambda_2^{1/4}]$. Since v_1 is defined up to $\lambda_1^{1/4}$, v_2 can be extended to $[0, \lambda_1^{1/4}]$ and the previous inequalities are valid in this interval. Evaluating at $\lambda_1^{1/4}$ we deduce that

$$0 = \alpha v_1'(\lambda_1^{1/4}) > \alpha v_2'(\lambda_1^{1/4}). \tag{95}$$

Since $w = \alpha \Delta v_2$ satisfies $\Delta w = \alpha f(v_j) > 0$ it is subharmonic and hence $w(r_1) \le w(r_2)$ for all $0 \le r_1 \le r_2 \le \lambda_1^{1/4}$. But the Green function for the bilaplacian in the ball of radius $R > 0$ with Dirichlet boundary conditions $G(x, y)$ satisfies $G(x, y) \ge$

$c(R - |x|)^2(R - |y|)^2$ for some $c > 0$, see [24]. This implies that $\alpha \Delta v_2(\lambda_2^{1/4}) > 0$ and therefore $w(r) > 0$ for all $r \in [\lambda_2^{1/4}, \lambda_1^{1/4}]$. Thus

$$r^{N-1}\alpha v_2'(r) = \int_{\lambda_2^{1/4}}^{r} t^{N-1}\alpha \Delta v_2(t)\,dt > 0 \quad \text{for all } r \in (\lambda_2^{1/4}, \lambda_1^{1/4}].$$

In particular $\alpha v_2'(\lambda_1^{1/4}) > 0$ which contradicts (95).

It follows that $\Delta v_1(0) = \Delta v_2(0)$ and hence $v_1 \equiv v_2$. This implies that $\lambda_1 = \lambda_2$ and that $u_1 \equiv u_2$.

The remaining case $p = 0$, can be solved explicitly and we find that $\lambda = 8(N + 2)N\,u(0)$. □

Proof of Theorems 3 and 6 To prove this we will show that

$$\mathcal{E}_0 = \{\phi(\beta) \,:\, \beta \in I\} \tag{96}$$

Indeed, by construction $\phi(\beta) \in \mathcal{E}_0$ for each $\beta \in I$. To prove $\mathcal{E}_0 \subseteq \{\phi(\beta) \,:\, \beta \in I\}$ we need to show that given any radial regular solution u of (2) there exists $\beta \in I$ such that $u = u_\beta$. Note that if p is negative and u is a radial regular solution for some $\lambda > 0$ then $0 < u(0) < 1$. Using Lemma 29 it is sufficient to find $\beta \in I$ such that $u(0) = u_\beta(0)$. We have by Lemma 27 that $u_\beta(0) \to +\infty$ as $\beta \to \beta^*$ if p is positive and $u_\beta(0) \to 1$ if p is negative. By Lemma 28 we know that $u_\beta(0) \to 0$ as $\beta \to 0$. Since $u_\beta(0)$ varies continuously with β there is $\beta \in I$ such that $u(0) = U_\beta(0)$. □

Lemma 30 *The map $\phi : I \to \mathbb{R}^4$ is a real analytic.*

Proof Let $\beta_0 \in I$. Then there is $r_0 > 0$ and $\delta > 0$ such that $U_\beta(r_0)$ is well defined for all $\beta \in (\beta_0 - \delta, \beta_0 + \delta)$ and is analytic. Fix $R_0(\beta_0) < R_1 < R_{\max}(\beta_0)$. Then, by standard theory of ODE, by taking $\delta > 0$ smaller if necessary we find that $U_\beta(r)$ is well defined for all $r \in [0, R_1]$. Moreover $U_\beta(r)$ is analytic with respect to $\beta \in (\beta_0 - \delta, \beta_0 + \delta)$ and $r \in (R_0(\beta) - \delta, R_0(\beta) + \delta)$ (see [8]). Since $\frac{\partial}{\partial\beta}U_\beta(R_0(\beta_0))|_{\beta=\beta_0} \neq 0$, by the implicit function theorem the map $\beta \mapsto R_0(\beta)$ is analytic in a neighborhood of β_0. It follows that $\phi(\beta) = V_\beta(T_0(\beta))$ is analytic in a neighborhood of β_0. □

Proposition 31 *Assume p is in the range (8) or (10). If $\lambda \neq \lambda_S$, then there is at most a finite number of regular radial solutions of (2).*

Proof By (96) and Lemmas 27 and 28 we can consider P_1 and P^* as the endpoints of $\mathcal{E}_0$. If $\lambda = 0$ then $u = 0$ is the only solution of (2). Let $\lambda \neq 0$, $\lambda \neq \lambda^*$. By analyticity $\mathcal{E}_0 \cap \{v_1 = \lambda\}$ can only accumulate at either P_1 or P^*. Since P^* is not included in $\{v_1 = \lambda\}$ accumulation in P^* is not possible. Similarly, since $P_1 \notin \{v_1 = \lambda\}$ the set $\mathcal{E}_0 \cap \{v_1 = \lambda\}$ cannot accumulate at P_1. Thus $\mathcal{E}_0 \cap \{v_1 = \lambda\}$ consists of a finite number of points, which correspond to regular radial solutions of (2). □

Proof of Proposition 2 By [7] and [14] there exists λ^* such that if $0 \leq \lambda < \lambda^*$ then (2) has a minimal smooth solution $\underline{u}_\lambda$ and if $\lambda > \lambda^*$ then (20) has no weak solution.

Although in [7] the authors deal with (2) when $p = -2$ and $N = 3$, the proof applies to any $p < 0$ and $N \geq 1$. The limit $u^* = \lim_{\lambda \nearrow \lambda^*} \underline{u}_\lambda$ exists pointwise, belongs to $H^2(B)$ and is a weak solution to (20) in the sense (4) or (5). The functions $\underline{u}_\lambda$, $0 \leq \lambda < \lambda^*$ and u^* are radially symmetric and radially decreasing.

Assume u^* is singular. Fix $\bar{\lambda} \in (0, \lambda^*)$ and let v be a smooth radial solution to (2) with parameter $\bar{\lambda}$. Since $\lambda \in (0, \lambda^*) \to \underline{u}_\lambda(0)$ depends continuously on λ, and since $\lim_{\lambda \to \lambda^*} u_\lambda(0) \to \infty$ we see that there exists some $\lambda \in (0, \lambda^*)$ such that $v(0) = \underline{u}_\lambda(0)$. By Lemma 29 we conclude that $\bar{\lambda} = \lambda$ and $v = \underline{u}_\lambda$.

Now assume that for all $0 < \lambda < \lambda^*$ there is a unique solution. Then this solution has to be the minimal $\underline{u}_\lambda$ and is therefore regular. This shows that $\lambda_S \geq \lambda^*$. Since we always have the opposite inequality we deduce $\lambda_S = \lambda^*$. We claim that $u^* = u_S$. If u^* is not regular then it has to be weakly singular and by uniqueness $u^* = u_S$. So, suppose that u^* is regular. Since $\underline{u}_\lambda \leq u^*$ and u^* is regular there would be a constant C such that $(1 + \text{sign}(p)\underline{u}_\lambda)^p \leq C$ for all $0 < \lambda < \lambda^*$ and all $0 \leq r \leq 1$. Recall the family of solution u_β constructed in (93). By Lemma 27

$$\lim_{\beta \to \beta^*} (1 + \text{sign}(p) u_\beta(0))^p = \infty.$$

But each u_β corresponds to some u_λ by uniqueness, which gives a contradiction. $\square$

Appendix A: Sign of some constants

We see that K_1 is a cubic polynomial in τ and that $\frac{N-4}{2}$, which corresponds to the critical exponent $p^* = \frac{N+4}{N-4}$, is a root. Then

$$K_1 = -4\left(\tau - \frac{N-4}{2}\right)(\tau - \tau^-)(\tau - \tau^+)$$

where

$$\tau^+ = \frac{N-4}{2} + \frac{\sqrt{(N-2)^2+4}}{2}, \quad \tau^- = \frac{N-4}{2} - \frac{\sqrt{(N-2)^2+4}}{2}.$$

Since $\tau -_< (N-4)/2 < \tau^+$ for all $N \geq 3$ we see that $K_1 < 0$ on $(\tau^-, (N-4)/2)$ and (τ^+, ∞).

Using formulas (22) we see that $K_2 K_3 - K_1$ is a cubic polynomial in τ which can be written as

$$K_2 K_3 - K_1 = -20\left(\tau - \frac{N-4}{2}\right)(\tau - \tau_a)(\tau - \tau_b)$$

where

$$\tau_a = \frac{N-4}{2} - \frac{\sqrt{(N-2)^2+4}}{2\sqrt{5}}, \quad \tau_b = \frac{N-4}{2} + \frac{\sqrt{(N-2)^2+4}}{2\sqrt{5}}.$$

Since $\tau_a < (N-4)/2 < \tau_b$ for all $N \geq 3$ we see that $K_2K_3 - K_1 > 0$ on $(-\infty, \tau_a)$ and $((N-4)/2, \tau_b)$. It follows that for all $\tau < (N-4)/2$ we have $K_1 < 0$ or $K_2K_3 - K_1 > 0$.

Appendix B: Calculation of p_c and p_c^+

The relation $pK_0 = H_N$ can be written as a fourth order polynomial in τ, that is

$$(\tau+4)(\tau+2)(N-2-\tau)(N-4-\tau) = H_N.$$

This polynomial has four real roots given by

$$\tau_1^{\pm} = \frac{N-6}{2} \pm \frac{1}{2}\sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}$$

and

$$\tau_2^{\pm} = \frac{N-6}{2} \pm \frac{1}{2}\sqrt{4 + N^2 + 4\sqrt{N^2 + H_N}}.$$

We can check that $\tau_2^{\pm}$ are always outside the range (14) or (15), because $\tau_2^+ \geq N-2$ (the corresponding value of p satisfies $1 < p \leq (N+2)/(N-2)$), and $\tau_2^- \leq -4$, $(0 \leq p < 1)$ and so we write

$$p_c = \frac{4+\tau_1^-}{\tau_1^-} \quad \text{and} \quad p_c^+ = \frac{4+\tau_1^+}{\tau_1^+}.$$

Note that p_c^+ only appears as a critical power when $N = 3$, since for $N = 4$, $\tau_1^+ = 0$ and for $N > 4$, we have $\tau_1^+ > (N-4)/2$, $(1 < p < (N+4)/(N-4))$.

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