VARIANTS OF KATO'S INEQUALITY AND REMOVABLE SINGULARITIES

By

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Abstract. An inequality reminiscent of Kato's inequality is presented. Motivated by this, we discuss some criteria to decide whether a singularity of the equation $\Delta u = g$ in $\Omega \setminus K$ comes from a Radon measure or not. As an application, we extend a lemma of H. Brezis and P. L. Lions on isolated singularities to the case where the singularity lies on a compact manifold.

1 Introduction and main results

The original motivation for this work is the following remark, which is related to Kato's inequality (see Kato [K]). First, let us recall one of its many versions. Consider $\Omega \subset \mathbb{R}^N$ an open set, and let $v \in L^1(\Omega)$ be such that $\Delta v \in L^1(\Omega)$. Then

$$\Delta|v| \geq \operatorname{sign}(v)\Delta v \qquad \text{in } \mathcal{D}'(\Omega), \tag{1}$$

where $\operatorname{sign}(s) = 1$ if $s > 0$, -1 if $s < 0$ and zero at $s = 0$. If we assume in addition that v is continuous in Ω, it is easy to verify that

$$\Delta|v| = \operatorname{sign}(v)\Delta v \qquad \text{in } \mathcal{D}'([v \neq 0]). \tag{2}$$

Comparison between (1) and (2) suggests that the inequality in (1) should be a consequence of the fact that $|v|$ achieves its minimum on the set $[v = 0]$, where one has $\Delta|v| \geq 0$ in a suitable sense.

Motivated by this fact, Y. Li posed the following question: suppose $u \in L^1(\Omega)$ is such that $u \geq 0$ a.e. in Ω and $u \equiv 0$ on a compact set K in some reasonable sense. Set $g = \Delta u$ in $\mathcal{D}'(\Omega \setminus K)$ and assume that $g \in L^1(\Omega \setminus K)$ (no conditions on Δu are prescribed on the "zero set" K). Let $\tilde{g}$ be the extension of g to Ω such that $\tilde{g} \equiv 0$ on K. Define

$$\mu := \Delta u - \tilde{g} \qquad \text{in } \mathcal{D}'(\Omega),$$

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so that $\operatorname{supp}(\mu) \subset K$. Is it true that μ is a nonnegative distribution? In this case it has to be a Radon measure supported in K (see Schwartz [S]).

We have given a positive answer to this question in the following theorem, which includes the case where $u \in C(\Omega)$ and $K \subset [u = 0]$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset, and $u \in L^1(\Omega)$ such that $u \geq 0$ a.e. in Ω. Let $K \subset \Omega$ be compact. Set*

$$g := \Delta u \quad \text{in } \mathcal{D}'(\Omega \setminus K).$$

Assume that $g \in L^1(\Omega \setminus K)$ and that

$$\limsup_{r \downarrow 0} \sup_{x \in K} \fint_{B_r(x)} u = 0. \tag{3}$$

Let $\tilde{g}$ be the extension of g to Ω such that $\tilde{g} \equiv 0$ on K. Then $\Delta u \geq \tilde{g}$ in $\mathcal{D}'(\Omega)$; in other words,

$$\int_\Omega u \Delta \varphi \geq \int_\Omega \tilde{g} \varphi, \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0. \tag{4}$$

As we have pointed out before, the theorem above implies the following

Corollary 2. *Let $\Omega \subset \mathbb{R}^N$ be open, bounded, and $u \in C(\Omega)$ be a nonnegative function. Let $K \subset \Omega$ be a compact subset such that $u \equiv 0$ on K. Set*

$$g := \Delta u \quad \text{in } \mathcal{D}'(\Omega \setminus K),$$

and assume that $g \in L^1(\Omega \setminus K)$. Let $\tilde{g}$ be the extension of g to Ω such that $\tilde{g} \equiv 0$ on K. Then $\Delta u \geq \tilde{g}$ in $\mathcal{D}'(\Omega)$; in other words,

$$\int_\Omega u \Delta \varphi \geq \int_\Omega \tilde{g} \varphi, \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0. \tag{5}$$

Remark 1. We shall see later that if the set K is sufficiently small and a certain growth condition for u near K is prescribed, then one really has the equality $\Delta u = \tilde{g}$ in $\mathcal{D}'(\Omega)$ (see Corollary 7). This is not the general case, though, as one can see by very simple examples. For instance, if $u(x) := \frac{1}{2}|x_N|$ for $x \in \mathbb{R}^N$, then $\Delta u = dx'$ in $\mathcal{D}'(\mathbb{R}^N)$, where dx' denotes $(N-1)$-dimensional Lebesgue measure on $[x_N = 0]$.

Remark 2. A consequence of this theorem is that $\mu = \Delta u - \tilde{g}$ is a nonnegative distribution, and hence a Radon measure. This implies that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for any $1 \leq p < N/(N-1)$ (see Bénilan–Brezis–Crandall [BeBrC]).

Remark 3. The same theorem holds under the weaker hypothesis that Ω is just open, $K \subset \Omega$ is relatively closed, and

$$\lim_{r \downarrow 0} \sup_{x \in A} \fint_{B_r(x)} u = 0 \qquad \text{for all } A \subset K \text{ compact.}$$

In fact, for any $\delta > 0$ set

$$\widehat{\Omega}_\delta := \left\{ x \in \Omega : d(x, \partial\Omega) > \delta \text{ and } |x| < 1/\delta \right\}.$$

Now fix $\delta > 0$ and let $\psi \in C_0^\infty(\widehat{\Omega}_{2\delta})$, $0 \le \psi \le 1$ and $\psi \equiv 1$ in $\widehat{\Omega}_{3\delta}$. We can then apply Theorem 1 in $\widehat{\Omega}_\delta$ to $\hat{u} := u\psi$, $\hat{K} := K \cap \overline{\widehat{\Omega}_{2\delta}}$, and conclude that (4) holds for all $\varphi \in C_0^\infty(\widehat{\Omega}_{3\delta})$.

Remark 4. A simple application of the Besicovitch Covering Lemma implies that condition (3) is equivalent to

$$\lim_{r \downarrow 0} \frac{1}{r^N} \int_{N_r(K)} u = 0, \tag{6}$$

where $N_r(K)$ denotes the r-neighborhood of K, i.e.,

$$N_r(K) = \{ x \in \mathbb{R}^N : \operatorname{dist}(x, K) < r \}.$$

The assumption required in (3) (or equivalently (6)) is probably too strong but we do not know how to weaken it in this general setting. In the case where $K \subset \Omega$ is a smooth manifold of codimension 1, we have been able to relax the hypothesis (3) by assuming that

$$\lim_{r \downarrow 0} \fint_{\Xi_r(K)} u = 0,$$

where $\Xi_r = \Xi_r(K)$ is the tubular neighborhood of K with radius r. In other words, for such singular sets, one can replace the factor $1/r^N$ in (6) by $1/r$, and still get the same conclusion of Theorem 1. More precisely,

Theorem 3. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $M^{N-1} \subset \Omega$ be a compact, smooth manifold, without boundary, of codimension 1. Let $u \in L^1_{\mathrm{loc}}(\Omega)$, and assume that there exists $g \in L^1_{\mathrm{loc}}(\Omega)$ such that*

$$\Delta u = g \quad \text{in } \mathcal{D}'(\Omega \setminus M).$$

If

$$\lim_{r \downarrow 0} \frac{1}{r} \int_{\Xi_r} |u| = 0, \tag{7}$$

then, for each $\varphi \in C_0^\infty(\Omega)$, $\frac{1}{r^2}\int_{\Xi_r} u\varphi$ *converges as* $r \downarrow 0$, *and*

$$\lim_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r} u\varphi = \frac{1}{2}\int_\Omega u\Delta\varphi - g\varphi, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{8}$$

In particular, if we suppose in addition that $u \geq 0$ *a.e. in* Ω, *then*

$$\Delta u \geq g \quad \textit{in } \mathcal{D}'(\Omega). \tag{9}$$

Remark 5. As mentioned in Remark 2, *a posteriori* we conclude from (9) that $u \in W^{1,p}_{\text{loc}}(\Omega)$ for $1 \leq p < N/(N-1)$, in which case condition (7) is equivalent to $u = 0$ in M in the sense of the trace.

Next, we study the case where the singular set M is a compact manifold of codimension $k \geq 2$. It turns out that, in this case, the condition $u \geq 0$ a.e. in Ω already suffices to conclude that $-\Delta u$ is a (nonnegative) measure on M. More precisely, we have

Theorem 4. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set and let* $M \subset \Omega$ *be a compact, smooth manifold without boundary of codimension* $k \geq 2$. *Let* $u \in L^1_{\text{loc}}(\Omega)$, $u \geq 0$ *a.e. in* Ω, *and assume there exists* $g \in L^1_{\text{loc}}(\Omega)$ *such that*

$$\Delta u = g \quad \textit{in } \mathcal{D}'(\Omega \setminus M).$$

Set

$$\mu := \Delta u - g \quad \textit{in } \mathcal{D}'(\Omega), \tag{10}$$

which is a distribution supported on M.

Then

$$\mu \textit{ is a nonpositive measure on } M \tag{11}$$

and, for any $\varphi \in C_0^\infty(\Omega)$, *we have*

$$\langle \mu, \varphi\rangle = \begin{cases} -2(k-2)\displaystyle\lim_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r} u\varphi & \textit{if } k \geq 3, \\ -2\displaystyle\lim_{r\downarrow 0}\frac{1}{r^2|\log r|}\int_{\Xi_r} u\varphi & \textit{if } k = 2. \end{cases} \tag{12}$$

We should mention that the conclusion (11) holds true in a much more general setting. In fact, a classical result in potential theory states that if in the statement above one replaces M by a compact set of zero H^1-capacity K (this includes the case of a smooth manifold of codimension $k \geq 2$), then μ, defined by (10), is a nonpositive measure on K (see L. L. Helms [H], Theorem 7.7). We present in Section 5 a completely independent proof of this result in our special case in order to deduce (12), which is used to prove Theorems 5 and 6 below.

Even if we do not assume any conditions on the sign of u, we can still characterize the case when μ is a measure in terms of the growth of $|u|$ near M. More precisely, we have proved the following

Theorem 5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $M \subset \Omega$ be a compact, smooth manifold without boundary of codimension $k \geq 3$. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ (here we* **do not** *assume that $u \geq 0$ a.e. in Ω) and assume there exists $g \in L^1_{\mathrm{loc}}(\Omega)$ such that*

$$\Delta u = g \quad \text{in } \mathcal{D}'(\Omega \setminus M).$$

Set

$$\mu := \Delta u - g \quad \text{in } \mathcal{D}'(\Omega),$$

which is a distribution supported in M. Then μ is a Radon measure if and only if

$$\frac{1}{r^2}\int_{\Xi_r} |u| \quad \text{remains bounded as } r \downarrow 0. \tag{13}$$

In this case, for all $\varphi \in C_0^\infty(\Omega)$ we have

$$\lim_{r\downarrow 0} \frac{1}{r^2}\int_{\Xi_r} u\varphi = -\frac{1}{2(k-2)}\langle \mu, \varphi\rangle. \tag{14}$$

Moreover,

$$\lim_{r\downarrow 0} \frac{1}{r^2}\int_{\Xi_r} |u| = \frac{1}{2(k-2)}\|\mu\|, \tag{15}$$

where $\|\mu\| := \sup\left\{ \int_M w\, d\mu \ ;\ w \in C(M),\ \|w\|_\infty \leq 1\right\}$ denotes the usual norm of Radon measures on M.

Remark 6. Using a formula deduced in Section 3, we show (see Remark 9) that (14) still holds if one replaces (13) by

$$\lim_{r\downarrow 0} \frac{1}{r}\int_{\Xi_r} |u| = 0. \tag{16}$$

On the other hand, if one takes, for instance, the function $u(x) = x_1/|x|^3$ in $\mathbb{R}^3\setminus\{0\}$, then $\Delta u = cD_{x_1}\delta_0$ for some constant $c \neq 0$. In the notation of Theorem 5, let $M := \{0\}$, $g \equiv 0$ and $\mu := cD_{x_1}\delta_0$, so that μ is a distribution of order 1 and

$$\lim_{r\downarrow 0} \frac{1}{r}\int_{B_r} |u| > 0. \tag{17}$$

The example above suggests the following

Open problem. Let $M \subset \Omega$ be a compact, smooth manifold without boundary of codimension $k \geq 3$. Let u and g be as in the statement of Theorem 5, and set $\mu := \Delta u - g$ in $\mathcal{D}'(\Omega)$. If (16) holds, is μ a measure?

There is also a result analogous to Theorem 5 in the case of codimension $k = 2$:

Theorem 6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $M \subset \Omega$ be a compact, smooth manifold without boundary of codimension $k = 2$. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ (here we* **do not** *assume that $u \geq 0$ a.e. in Ω) and assume there exists $g \in L^1_{\mathrm{loc}}(\Omega)$ such that*

$$\Delta u = g \quad \text{in } \mathcal{D}'(\Omega \setminus M).$$

Set

$$\mu := \Delta u - g \quad \text{in } \mathcal{D}'(\Omega),$$

which is a distribution supported in M. Then μ is a Radon measure if and only if

$$\frac{1}{r^2|\log r|} \int_{\Xi_r} |u| \quad \text{remains bounded as } r \downarrow 0. \tag{18}$$

In this case, for all $\varphi \in C_0^\infty(\Omega)$ we have

$$\lim_{r \downarrow 0} \frac{1}{r^2|\log r|} \int_{\Xi_r} u\varphi = -\frac{1}{2}\langle \mu, \varphi \rangle. \tag{19}$$

Moreover,

$$\lim_{r \downarrow 0} \frac{1}{r^2|\log r|} \int_{\Xi_r} |u| = \frac{1}{2}\|\mu\|. \tag{20}$$

As a consequence of Theorems 5 and 6 we have the following removable singularity statement:

Corollary 7 ***(Removable singularity).*** *Under the assumptions of Theorems 5 and 6 above, $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$ if and only if*

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} |u| = 0, \quad \text{for } k \geq 3, \tag{21}$$

$$\lim_{r \downarrow 0} \frac{1}{r^2|\log r|} \int_{\Xi_r} |u| = 0, \quad \text{for } k = 2. \tag{22}$$

Next, we give an application of Theorem 4, by extending an earlier result of Brezis–Lions [BrL] originally concerning the study of isolated singularities.

Theorem 8. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $M^{N-k} \subset \Omega$ be a compact manifold without boundary of codimension $k \geq 2$. Let $u \in L^1_{\rm loc}(\Omega \setminus M)$ be such that*

$$\begin{aligned} &\Delta u \in L^1_{\rm loc}(\Omega \setminus M) \quad \text{in the sense of distributions on } \Omega \setminus M,\\ &u \geq 0 \quad \text{a.e. in } \Omega,\\ &\Delta u \leq au + f \quad \text{a.e. in } \Omega \setminus M, \end{aligned}$$

where a is a nonnegative constant and $f \in L^1_{\rm loc}(\Omega)$.

Then $u \in L^1_{\rm loc}(\Omega)$, and there exist $h \in L^1_{\rm loc}(\Omega)$ and a nonnegative Radon measure μ supported on M such that

$$-\Delta u = h + \mu \quad \text{in } \mathcal{D}'(\Omega). \tag{23}$$

Since a compact manifold M of codimension $k \geq 2$ is a set of zero H^1-capacity, and also because of the linear nature of Theorem 8, the classical result we mention just after Theorem 4 leads us to state the following

Open problem. Suppose that in the statement of Theorem 8 one replaces the smooth manifold M by a compact set K of zero H^1-capacity. Can one still conclude that $u \in L^1_{\rm loc}(\Omega)$, and that there exists $h \in L^1_{\rm loc}(\Omega)$ such that (23) holds for some μ supported on K? (Note that potential theory would tell us that μ is necessarily a nonnegative Radon measure.)

If the answer to the open problem above is affirmative, it gives a sort of *linear* version of a general result of P. Baras and M. Pierre (see [BaPi]).

An immediate consequence of Theorem 8 is the following

Corollary 9. *Let $M \subset \Omega$ be as above. Assume $f : \mathbb{R}_+ \to \mathbb{R}$ is continuous and*

$$\liminf_{t \to \infty} \frac{f(t)}{t} > -\infty.$$

Suppose $u, f(u) \in L^1_{\rm loc}(\Omega \setminus M)$, $u \geq 0$ a.e. in Ω, and

$$-\Delta u = f(u) \quad \text{in } \mathcal{D}'(\Omega \setminus M).$$

Then $u, f(u) \in L^1_{\rm loc}(\Omega)$ and

$$-\Delta u = f(u) + \mu \quad \text{in } \mathcal{D}'(\Omega) \tag{24}$$

for some nonnegative Radon measure μ supported on M.

A simple application of Corollaries 7 and 9 allows us to recapture the following consequence of a removable singularity result which was originally proved by L. Véron for the case $k > 2$ (see [V1]).

Corollary 10. *Under the hypotheses of Corollary 9, if*

$$\begin{aligned} &\liminf_{t\to\infty} t^{-k/(k-2)} f(t) > 0, \quad \text{for } k > 2, \\ (25) \qquad &\liminf_{t\to\infty} e^{-at} f(t) > 0, \quad \text{for } k = 2, \text{for all } a > 0, \end{aligned}$$

then $\mu = 0$, *i.e.,*

$$-\Delta u = f(u) \quad \text{in } \mathcal{D}'(\Omega).$$

Warning. The result of Corollary 10 may seem misleading at first. For instance, assume $k \geq 3$ and $f(t) = t^{k/(k-2)}$. Although it implies that $-\Delta u = u^{k/(k-2)}$ in $\mathcal{D}'(\Omega)$, one **cannot** conclude solely from this equation that u is smooth. What Corollary 10 tells us is that the eventual singularities of u are not detectable at the distribution level. In fact, a result of Mazzeo–Pacard [MPa] says that, given some compact manifolds in Ω (not necessarily with the same codimension), and for certain values of $p > 1$, depending on their codimension, one can construct nonnegative solutions of the equation $-\Delta u = u^{k/(k-2)}$ in $\mathcal{D}'(\Omega)$, whose singularities lie precisely on the prescribed manifolds. See Véron [V2] for details.

2 Proof of Theorem 1

In this section, we use the following notation.

Notation. For an open set $U \subset \mathbb{R}^N$ and $\delta > 0$, we write

$$U_\delta = \{x \in U \mid d(x, \partial U) > \delta\},$$

and for any set $A \subset \mathbb{R}^N$ and $\delta > 0$, we let

$$N_\delta(A) = \{x \in \mathbb{R}^N : d(x, A) < \delta\}.$$

We also use the standard notation for averages:

$$\fint_E v \, d\mu = \frac{\int_E v \, d\mu}{\int_E 1 \, d\mu}.$$

Proof of Theorem 1. Take $\rho \in C_0^\infty(B_1)$ such that $\rho \geq 0$ in $\mathbb{R}^N$ and $\int_{\mathbf{R}^N} \rho = 1$. For any $\varepsilon > 0$, define $\rho_\varepsilon(x) := \varepsilon^{-N} \rho(x/\varepsilon)$ on $\mathbb{R}^N$, $u_\varepsilon := \rho_\varepsilon * u$ and $g_\varepsilon := \rho_\varepsilon * \tilde{g}$ on Ω_ε. Using this notation, one can easily check that

$$\Delta u_\varepsilon = g_\varepsilon \quad \text{on } \Omega_{2\varepsilon} \setminus \overline{N_{2\varepsilon}(K)}. \tag{26}$$

For $\varepsilon > 0$, let

$$\eta_\varepsilon := \max_{N_{2\varepsilon}(K)} u_\varepsilon.$$

Step 1. Condition (3) implies that

$$\lim_{\varepsilon \downarrow 0} \eta_\varepsilon = 0.$$

In particular, $u_\varepsilon \to 0$ uniformly on K as $\varepsilon \downarrow 0$.

Proof. For $z \in \overline{N_{2\varepsilon}(K)}$, let $x \in K$ be such that $|x - z| \le 2\varepsilon$. Since $B_\varepsilon(z) \subset B_{3\varepsilon}(x)$ for all $\varepsilon > 0$, we have

$$\begin{aligned} u_\varepsilon(z) &= \frac{1}{\varepsilon^N} \int_{B_\varepsilon(z)} \rho\left(\frac{z-y}{\varepsilon}\right) u(y)dy \\ &\le \frac{C}{\varepsilon^N} \int_{B_\varepsilon(z)} u \le \frac{C}{\varepsilon^N} \int_{B_{3\varepsilon}(x)} u \\ &= \frac{3^N C}{(3\varepsilon)^N} \int_{B_{3\varepsilon}(x)} u \to 0 \end{aligned}$$

uniformly in $z \in \overline{N_{2\varepsilon}(K)}$ as $\varepsilon \to 0$, by (3). This concludes the proof of the claim.

Step 2. There exists a measurable set $L(u) \subset \Omega$ such that

$$[\liminf u_\varepsilon > 0] \subset L(u) \subset \Omega \setminus K$$

and

$$\int_\Omega u \Delta\varphi \ge \int_{L(u)} \tilde{g}\varphi, \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \ge 0. \tag{27}$$

Proof. It follows from Kato's inequality (with $|\cdot|$ replaced by sign^+ in (1)) that

$$\begin{aligned} \Delta(u_\varepsilon - \eta_\varepsilon)^+ &\ge \operatorname{sign}^+(u_\varepsilon - \eta_\varepsilon)\, \Delta(u_\varepsilon - \eta_\varepsilon) \\ &= \chi_{[u_\varepsilon > \eta_\varepsilon]} \Delta u_\varepsilon \quad \text{in } \mathcal{D}'(\Omega_{2\varepsilon}), \end{aligned} \tag{28}$$

where $\chi_{[u_\varepsilon > \eta_\varepsilon]}$ is the characteristic function of the set $[u_\varepsilon > \eta_\varepsilon]$.

Since $u_\varepsilon \le \eta_\varepsilon$ on $\overline{N_{2\varepsilon}(K)}$, it follows from (26) and (28) that

$$\Delta(u_\varepsilon - \eta_\varepsilon)^+ \ge \chi_{[u_\varepsilon > \eta_\varepsilon]} g_\varepsilon \quad \text{in } \mathcal{D}'(\Omega_{2\varepsilon}). \tag{29}$$

Now, given $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \ge 0$, if $\varepsilon > 0$ is so small that $\operatorname{supp}\varphi \subset\subset \Omega_{2\varepsilon}$, (29) implies that

$$\int_\Omega (u_\varepsilon - \eta_\varepsilon)^+ \Delta\varphi \ge \int_\Omega \chi_{[u_\varepsilon > \eta_\varepsilon]} g_\varepsilon \varphi. \tag{30}$$

Since $u_\varepsilon \to u$ in $L^1_{\text{loc}}(\Omega)$ and $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$, we conclude that

$$\int_\Omega (u_\varepsilon - \eta_\varepsilon)^+ \Delta\varphi \to \int_\Omega u \Delta\varphi \quad \text{as } \varepsilon \to 0. \tag{31}$$

On the other hand, take a sequence $\varepsilon_n \downarrow 0$. Up to a subsequence of $(\varepsilon_n)_{n\geq 1}$, we have

$$\begin{aligned} &u_{\varepsilon_n} \to u && \text{a.e. in } \Omega,\\ &g_{\varepsilon_n} \to \tilde{g} && \text{a.e. in } \Omega,\\ &|g_{\varepsilon_n}| \leq h && \text{a.e. in } \Omega,\ \forall n \geq 1,\ \text{for some } h \in L^1(\Omega). \end{aligned}$$

Set

$$L(u) := \liminf_{n\to\infty}\,[u_{\varepsilon_n} > \eta_{\varepsilon_n}] = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}[u_{\varepsilon_n} > \eta_{\varepsilon_n}].$$

Note that, by our choice of η_ε, we have $K \subset \Omega \setminus L(u)$. By Fatou's Lemma, which may be applied here since $g_{\varepsilon_n} \geq -h$ a.e. in Ω, we have

$$\int_\Omega \chi_{L(u)}\tilde{g}\varphi = \int_\Omega \liminf_{n\to\infty} \chi_{[u_{\varepsilon_n}>\eta_{\varepsilon_n}]} g_{\varepsilon_n}\varphi \leq \liminf_{n\to\infty} \int_\Omega \chi_{[u_{\varepsilon_n}>\eta_{\varepsilon_n}]} g_{\varepsilon_n}\varphi. \tag{32}$$

It then follows from (30), (31) and (32) that

$$\int_\Omega u\Delta\varphi \geq \int_\Omega \chi_{L(u)}\tilde{g}\varphi, \quad \forall\varphi \in C_0^\infty(\Omega),\ \varphi \geq 0.$$

Step 3. Proof of Theorem 1 completed. Given $\lambda > 0$, let $h_\lambda \in C^\infty(\mathbb{R}^N)$ be such that $h_\lambda \equiv 0$ on $\overline{N_\lambda(K)}$ and $h_\lambda > 0$ outside $\overline{N_\lambda(K)}$.

If we apply (27) in Step 2 with the function u replaced by $u + h_\lambda$ (note that condition (3) is still satisfied if we replace u by $u + h_\lambda$), we get

$$\int_\Omega (u + h_\lambda)\Delta\varphi \geq \int_\Omega \chi_{L(u+h_\lambda)}(\tilde{g} + \Delta h_\lambda)\varphi, \quad \forall\varphi \in C_0^\infty(\Omega),\ \varphi \geq 0. \tag{33}$$

Now, for a.e. $x \in \Omega \setminus \overline{N_\lambda(K)}$, we have $(u + h_\lambda)_{\varepsilon_n}(x) \to u(x) + h_\lambda(x) > 0$ as $n \to \infty$. By the definition of the set $L(u + h_\lambda)$, we conclude that $x \in L(u + h_\lambda)$ for a.e. $x \in \Omega \setminus \overline{N_\lambda(K)}$; in other words,

$$\chi_{L(u+h_\lambda)} = 1 \quad \text{a.e. in } \Omega \setminus \overline{N_\lambda(K)}.$$

In view of (33) and the relation above, for any $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \int_\Omega u\Delta\varphi &\geq \int_\Omega \chi_{L(u+h_\lambda)}(\tilde{g} + \Delta h_\lambda)\varphi - \int_\Omega \Delta h_\lambda\varphi \\ &= \int_{\Omega\setminus\overline{N_\lambda(K)}} (\tilde{g} + \Delta h_\lambda)\varphi + \int_{\overline{N_\lambda(K)}} \chi_{L(u+h_\lambda)}\tilde{g}\varphi - \int_{\Omega\setminus\overline{N_\lambda(K)}} \Delta h_\lambda\varphi \\ &= \int_{\Omega\setminus\overline{N_\lambda(K)}} \tilde{g}\varphi + \int_{\overline{N_\lambda(K)}} \chi_{L(u+h_\lambda)}\tilde{g}\varphi \\ &= \int_\Omega \tilde{g}\varphi + o(1), \end{aligned}$$

where $o(1)$ is a quantity which converges to 0 as $\lambda \downarrow 0$. In the expression above, let $\lambda \downarrow 0$ to finally conclude that

$$\int_\Omega u\Delta\varphi \geq \int_\Omega \tilde{g}\varphi, \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0. \qquad \square$$

Remark 7. It is noteworthy that the proof of Theorem 1 is somewhat simpler if one assumes that u is continuous at each point of K. In fact, in this case, Step 1 is unnecessary and one can apply the other steps of the proof directly to the function u instead of to its convolution.

3 Some useful formulas

Let us recall some standard results.

Given a compact smooth manifold M^{N-k} (with or without boundary) embedded in $\mathbb{R}^N$ with codimension $k \geq 1$, we define its distance function $d : \mathbb{R}^N \to \mathbb{R}_+$ by $d(x) := \operatorname{dist}(x, M)$. The case $k = N$ is included, i.e., M may be a finite collection of points. It is a well-known fact that for $\delta > 0$ small enough, the set $\overline{N_\delta(M)}$ is a smooth manifold with boundary, also called the δ-tubular neighborhood of M, which from now on we denote by $\Xi_\delta(M)$, and when no confusion arises, simply by Ξ_δ. The distance function d is Lipschitz in $\mathbb{R}^N$, is smooth in $\Xi_\delta \setminus M$ and satisfies (for the second property see Véron [V2])

$$|\nabla d| = 1 \quad \text{a.e. in } \mathbb{R}^N; \tag{34}$$

$$\Delta d = \frac{k-1}{d} + a_0 \quad \text{in } \Xi_\delta \setminus M, \tag{35}$$

where a_0 is a bounded function in $\Xi_\delta \setminus M$.

For each $x \in \Xi_\delta$, there exists a unique element $\pi(x) \in M$ for which the distance function is realized, i.e., such that $|x - \pi(x)| = d(x)$. The projection $\pi : \Xi_\delta \to M$ thus defined is also smooth.

For simplicity, from now on we assume that Ξ_2 is a smooth tubular neighborhood of M.

Finally, let us recall that if $v \in L^1(\mathbb{R}^N)$, we have by the coarea formula (see Evans and Gariepy [EG])

$$\int_{\Xi_\delta} v = \int_0^\delta \int_{\partial \Xi_r} v \, d\sigma \, dr.$$

Lemma 11. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $M \subset \Omega$ be a compact, smooth manifold without boundary of codimension $k \geq 1$.*

Let $u \in L^1_{\mathrm{loc}}(\Omega)$, and assume there exists $g \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\Delta u = g \quad \text{in } \mathcal{D}'(\Omega \setminus M).$$

Set

$$\mu := \Delta u - g \quad \text{in } \mathcal{D}'(\Omega),$$

which is a distribution supported on M.

For $k \geq 1$ and $t, r > 0$, define

$$G_k(r,t) = \begin{cases} \dfrac{1}{k} & \text{if } 0 < t < r, \\ -\dfrac{k-1}{k}\dfrac{r^k}{t^k} & \text{if } 0 < r < t. \end{cases} \tag{36}$$

Then for any $R \in (0,1)$ fixed and $\varphi \in C_0^\infty(\Omega)$, all the limits below exist and

(a) *if $k \geq 3$, then*

$$\begin{aligned} \frac{1}{2(k-2)}\langle \mu, \varphi\rangle = \lim_{r \downarrow 0} \Bigg\{ & -\frac{1}{r^2}\int_{\Xi_r} u\varphi + \\ & + \frac{1}{r^2}\int_0^R G_k(r,t)\left(\int_{\Xi_t} 2u\nabla\varphi\cdot\nabla d + u\varphi a_0\right) dt \Bigg\}; \end{aligned} \tag{37}$$

(b) *if $k = 2$, then*

$$\begin{aligned} \frac{1}{2}\langle \mu, \varphi\rangle = \lim_{r \downarrow 0} \Bigg\{ & -\frac{1}{r^2|\log r|}\int_{\Xi_r} u\varphi + \\ & + \frac{1}{r^2|\log r|}\int_0^R G_2(r,t)\left(\int_{\Xi_t} 2u\nabla\varphi\cdot\nabla d + u\varphi a_0\right) dt \Bigg\}; \end{aligned} \tag{38}$$

(c) *if $k = 1$, then*

$$\begin{aligned} -\frac{1}{2}\langle \mu, \varphi\rangle = \lim_{r \downarrow 0} \Bigg\{ & -\frac{1}{r^2}\int_{\Xi_r} u\varphi + \frac{1}{r}\left(\lim_{t \downarrow 0}\int_{\partial\Xi_t} u\varphi\right) \\ & + \frac{1}{r^2}\int_0^r \left(\int_{\Xi_t} 2u\nabla\varphi\cdot\nabla d + u\varphi a_0\right) dt \Bigg\}. \end{aligned} \tag{39}$$

Proof. The idea of the proof is first to derive the following

Claim. For any $\varphi \in C_0^\infty(\Omega)$, the function $s \to \int_{\partial\Xi_s} u\varphi$ is C^1 on $(0,1)$ and

$$\begin{aligned} \langle \Delta u, \varphi\rangle = \int_{\Xi_s} u\Delta\varphi - 2\int_{\partial\Xi_s} u\nabla\varphi\cdot\nabla d + s^{k-1}\frac{d}{ds}\left(\frac{1}{s^{k-1}}\int_{\partial\Xi_s} u\varphi\right) - \\ - \int_{\partial\Xi_s} u\varphi a_0. \end{aligned} \tag{40}$$

Proof of (40). We first assume that u is smooth.

Fix a smooth, nonincreasing function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(t) = 0$ for $t \geq 1$ and $\Phi(t) = 1$ for $t \leq 0$. For $\varepsilon > 0$, set

$$\Phi_\varepsilon(t) = \Phi\Big(\frac{t-1}{\varepsilon}\Big).$$

Now let $\varphi \in C_0^\infty(\Omega)$ and for ε, $s > 0$, define

$$\varphi_{s,\varepsilon}(x) = \begin{cases} \varphi(x) & \text{if } x \in \Xi_s, \\ \varphi(x)\Phi_\varepsilon(d(x)/s) & \text{if } x \in \Xi_{s(1+\varepsilon)} \setminus \Xi_s, \\ 0 & \text{if } x \in \Omega \setminus \Xi_{s(1+\varepsilon)}. \end{cases}$$

Observe that $\varphi_{s,\varepsilon} \equiv \varphi$ in Ξ_s and $\varphi_{s,\varepsilon} \equiv 0$ in $\Omega \setminus \Xi_{s(1+\varepsilon)}$. We now compute in $\Xi_{s(1+\varepsilon)} \setminus \Xi_s$, using (35),

$$\begin{aligned} \Delta\varphi_{s,\varepsilon} &= \Delta\varphi\,\Phi_\varepsilon(d/s) + \frac{2}{s}\nabla\varphi\cdot\nabla d\,\Phi_\varepsilon'(d/s) \\ &\quad + \frac{1}{s^2}\varphi\Big\{\Phi_\varepsilon''(d/s) + \frac{s}{d}\Phi_\varepsilon'(d/s)(k-1+a_0 d)\Big\}. \end{aligned}$$

Since $\varphi_{s,\varepsilon}$ is an admissible test function, we obtain

$$\langle\Delta u, \varphi_{s,\varepsilon}\rangle = \int_\Omega u\Delta\varphi\Phi_\varepsilon(d/s) + I_1 + I_2 + I_3 + I_4, \tag{41}$$

where

$$\begin{aligned} I_1 &= \frac{2}{s}\int_\Omega u\nabla\varphi\cdot\nabla d\,\Phi_\varepsilon'(d/s), \\ I_2 &= \frac{1}{s^2}\int_\Omega u\varphi\,\Phi_\varepsilon''(d/s), \\ I_3 &= \frac{k-1}{s}\int_\Omega u\frac{\varphi}{d}\,\Phi_\varepsilon'(d/s), \\ I_4 &= \frac{1}{s}\int_\Omega u\varphi a_0\,\Phi_\varepsilon'(d/s). \end{aligned}$$

Next we find the limit as $\varepsilon \downarrow 0$ of the four previous integrals. For this purpose, we compute

$$I_1 = \frac{2}{\varepsilon s}\int_{\Xi_{s(1+\varepsilon)}\setminus\Xi_s} \Phi'\Big(\frac{d/s-1}{\varepsilon}\Big)u\nabla\varphi\cdot\nabla d$$

and by the coarea formula

$$\begin{aligned} &= \frac{2}{\varepsilon s}\int_s^{s(1+\varepsilon)}\Big\{\Phi'\Big(\frac{r/s-1}{s}\Big)\int_{\partial\Xi_r} u\nabla\varphi\cdot\nabla d\Big\}\,dr \\ &= 2\int_0^1\Big\{\Phi'(t)\int_{\partial\Xi_{(1+\varepsilon t)s}} u\nabla\varphi\cdot\nabla d\Big\}\,dt. \end{aligned}$$

We now let $\varepsilon \downarrow 0$:

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} I_1 &= 2\int_0^1 \Phi'(t)\Big\{\int_{\partial\Xi_s} u\nabla\varphi\cdot\nabla d\Big\}\,dt \\ &= -2\int_{\partial\Xi_s} u\nabla\varphi\cdot\nabla d. \end{aligned} \tag{42}$$

We now proceed with I_2:

$$\begin{aligned} I_2 &= \frac{1}{s^2}\int_\Omega u\varphi\,\Phi_\varepsilon''(d/s) \\ &= \frac{1}{\varepsilon^2 s^2}\int_{\Xi_{s(1+\varepsilon)}\setminus\Xi_s} u\varphi\,\Phi''\Big(\frac{d/s-1}{\varepsilon}\Big) \\ &= \frac{1}{\varepsilon s}\int_0^1\Big\{\Phi''(t)\int_{\partial\Xi_{s(1+\varepsilon t)}} u\varphi\Big\}\,dt. \end{aligned}$$

Integrating by parts, we have

$$I_2 = \Big[\frac{1}{\varepsilon s}\Phi'(t)\int_{\partial\Xi_{s(1+\varepsilon t)}} u\varphi\Big]_{t=0}^{t=1} - \int_0^1\bigg(\Phi'(t)\Big[\frac{d}{d\lambda}\int_{\partial\Xi_\lambda} u\varphi\Big]_{\lambda=s(1+\varepsilon t)}\bigg)\,dt.$$

Letting $\varepsilon \downarrow 0$, we arrive at

$$\lim_{\varepsilon\downarrow 0} I_2 = \frac{d}{ds}\int_{\partial\Xi_s} u\varphi. \tag{43}$$

The computations for I_3, I_4 are similar; they yield

$$\lim_{\varepsilon\downarrow 0} I_3 = -\frac{k-1}{s}\int_{\partial\Xi_s} u\varphi, \tag{44}$$

$$\lim_{\varepsilon\downarrow 0} I_4 = -\int_{\partial\Xi_s} u\varphi a_0. \tag{45}$$

Thus, passing to the limit as $\varepsilon \downarrow 0$ in (41) and using (42)–(45), we get

$$\begin{aligned} \langle\Delta u,\varphi\rangle = \int_{\Xi_s} u\Delta\varphi - 2\int_{\partial\Xi_s} u\nabla\varphi\cdot\nabla d + \frac{d}{ds}\int_{\partial\Xi_s} u\varphi - \frac{k-1}{s}\int_{\partial\Xi_s} u\varphi - \\ - \int_{\partial\Xi_s} u\varphi a_0. \end{aligned} \tag{46}$$

But

$$\frac{d}{ds}\int_{\partial\Xi_s} u\varphi - \frac{k-1}{s}\int_{\partial\Xi_s} u\varphi = s^{k-1}\frac{d}{ds}\Big(\frac{1}{s^{k-1}}\int_{\partial\Xi_s} u\varphi\Big); \tag{47}$$

therefore, combining (46) with (47) we find (40).

We now consider u as in the statement of the lemma, i.e., $u \in L^1_{\text{loc}}(\Omega)$ so that $\mu := \Delta u - g$ is a distribution with support in M, where $g \in L^1_{\text{loc}}(\Omega)$. Using a density argument and the fact that $u \in W^{1,1}_{\text{loc}}(\Omega \setminus M)$, we deduce that the function $s \mapsto \int_{\partial \Xi_s} u\varphi$ is C^1 on $(0,1)$ and that

$$\begin{aligned}\langle \mu, \varphi\rangle = \int_{\Xi_s} (u\Delta\varphi - g\varphi) - 2\int_{\partial\Xi_s} u\nabla\varphi\cdot\nabla d + s^{k-1}\frac{d}{ds}\Big(\frac{1}{s^{k-1}}\int_{\partial\Xi_s} u\varphi\Big) - \\ - \int_{\partial\Xi_s} u\varphi a_0.\end{aligned} \tag{48}$$

At this point, we distinguish the three cases: (a) $k \geq 3$, (b) $k = 2$, and (c) $k = 1$.

(a) Case $k \geq 3$. Fix $R \in (0,1)$ and let $0 < t < R$. Dividing (48) by s^{k-1} and integrating over $s \in (t, R)$, we get

$$\begin{aligned}\frac{1}{(k-2)t^{k-2}}\langle\mu,\varphi\rangle = o(1)\frac{1}{t^{k-2}} - \frac{1}{t^{k-1}}\int_{\partial\Xi_t} u\varphi - \\ - \int_t^R \Big\{\frac{1}{s^{k-1}}\int_{\partial\Xi_s} u\varphi a_0 + 2u\nabla\varphi\cdot\nabla d\Big\}\, ds,\end{aligned} \tag{49}$$

where $o(1)$ denotes a quantity that goes to zero as $t \to 0$. Multiplying (49) by t^{k-1} and integrating over $t \in (0, r)$ with $0 < r < R$, we obtain

$$\begin{aligned}\frac{1}{2(k-2)}\langle\mu,\varphi\rangle = o(1) - \frac{1}{r^2}\int_{\Xi_r} u\varphi - \\ - \frac{1}{r^2}\int_0^r t^{k-1}\int_t^R \frac{1}{s^{k-1}}\int_{\partial\Xi_s} v\, ds\, dt,\end{aligned} \tag{50}$$

where

$$v = 2u\nabla\varphi\cdot\nabla d + u\varphi a_0. \tag{51}$$

We now integrate by parts in the last term on the right hand side of (50):

$$\begin{aligned}\int_t^R \frac{1}{s^{k-1}}\Big(\frac{d}{ds}\int_{\Xi_s} v\Big)\, ds &= \Big[\frac{1}{s^{k-1}}\int_{\Xi_s} v\Big]_{s=t}^{s=R} - \int_t^R (1-k)\frac{1}{s^k}\int_{\Xi_s} v\, ds \\ &= \frac{1}{R^{k-1}}\int_{\Xi_R} v - \frac{1}{t^{k-1}}\int_{\Xi_t} v + \\ &\quad + (k-1)\int_t^R \frac{1}{s^k}\int_{\Xi_s} v\, ds.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{r^2}\int_0^r t^{k-1}\int_t^R \frac{1}{s^{k-1}}\int_{\partial\Xi_s} v\, ds\, dt = \frac{r^{k-2}}{kR^{k-1}}\int_{\Xi_R} v - \frac{1}{r^2}\int_0^r\int_{\Xi_t} v\, dt + \\ + \frac{k-1}{r^2}\int_0^r t^{k-1}\int_t^R \frac{1}{s^k}\int_{\Xi_s} v\, ds\, dt,\end{aligned} \tag{52}$$

and changing the order of integration in the last term of (52) gives

$$\int_0^r t^{k-1}\int_t^R \frac{1}{s^k}\int_{\Xi_s} v\,ds\,dt = \frac{1}{k}\int_0^r \int_{\Xi_s} v\,ds + \frac{r^k}{k}\int_r^R \frac{1}{s^k}\int_{\Xi_s} v\,ds. \tag{53}$$

Then, (52) in combination with (53) yields

$$\begin{aligned}\frac{1}{r^2}\int_0^r t^{k-1}\int_t^R \frac{1}{s^{k-1}}\int_{\partial\Xi_s} v\,ds\,dt &= \frac{r^{k-2}}{kR^{k-1}}\int_{\Xi_R} v - \frac{1}{kr^2}\int_0^r\int_{\Xi_t} v\,dt +\\ &\quad + \frac{k-1}{k}r^{k-2}\int_r^R \frac{1}{s^k}\int_{\Xi_s} v\,ds.\end{aligned} \tag{54}$$

Hence, using (54) in (50), we conclude that

$$\begin{aligned}\frac{1}{2(k-2)}\langle\mu,\varphi\rangle &= o(1) - \frac{1}{r^2}\int_{\Xi_r} u\varphi + \frac{1}{kr^2}\int_0^r\int_{\Xi_t} v\,dt -\\ &\quad - \frac{k-1}{k}r^{k-2}\int_r^R \frac{1}{s^k}\int_{\Xi_s} v\,ds\\ &= o(1) - \frac{1}{r^2}\int_{\Xi_r} u\varphi + \frac{1}{r^2}\int_0^R \Big(G_k(r,t)\int_{\Xi_t} v\Big)dt,\end{aligned}$$

where G_k is given by (36).

This establishes (37).

We now deal with

(b) Case $k = 2$. Note that (48) is still valid; and, since $k = 2$, it takes the form

$$\langle\mu,\varphi\rangle = o(1) - \int_{\partial\Xi_s} v + s\frac{d}{ds}\Big(\frac{1}{s}\int_{\partial\Xi_s} u\varphi\Big),$$

where v is given by (51). Dividing the last equation by s and integrating over $s \in (t, R)$, we get

$$(\log R - \log t)\langle\mu,\varphi\rangle = o(1)|\log t| - \int_t^R \frac{1}{s}\int_{\partial\Xi_s} v\,ds - \frac{1}{t}\int_{\partial\Xi_t} u\varphi.$$

Multiplying by t and integrating over $t \in (0, r)$, we obtain

$$\frac{1}{2}\langle\mu,\varphi\rangle = o(1) - \frac{1}{r^2|\log r|}\int_0^r t\int_t^R \frac{1}{s}\int_{\partial\Xi_s} v\,ds\,dt - \frac{1}{r^2|\log r|}\int_{\Xi_r} u\varphi. \tag{55}$$

Integrating by parts yields

$$\begin{aligned}\int_t^R \frac{1}{s}\int_{\partial\Xi_s} v\,ds &= \Big[\frac{1}{s}\int_{\Xi_s} v\Big]_{s=t}^{s=R} + \int_t^R \frac{1}{s^2}\int_{\Xi_s} v\,ds\\ &= \frac{1}{R}\int_{\Xi_R} v - \frac{1}{t}\int_{\Xi_t} v + \int_t^R \frac{1}{s^2}\int_{\Xi_s} v\,ds.\end{aligned}$$

Hence, using Fubini, we get

$$\int_0^r t\int_t^R \frac{1}{s}\int_{\partial\Xi_s} v\,ds = \frac{r^2}{2R}\int_{\Xi_R} v - \int_0^r\int_{\Xi_t} v\,dt + \int_0^r t\int_t^R \frac{1}{s^2}\int_{\Xi_s} v\,ds\,dt$$

$$= \frac{r^2}{2R}\int_{\Xi_R} v - \frac{1}{2}\int_0^r\int_{\Xi_t} v\,dt + \frac{r^2}{2}\int_r^R \frac{1}{s^2}\int_{\Xi_s} v\,ds. \tag{56}$$

So, from (55) and (56), we infer that

$$\begin{aligned}\frac{1}{2}\langle\mu,\varphi\rangle &= o(1) - \frac{1}{r^2|\log r|}\int_{\Xi_r} u\varphi + \frac{1}{2r^2|\log r|}\int_0^r\int_{\Xi_t} v\,dt- \\ &\quad - \frac{1}{2|\log r|}\int_r^R \frac{1}{t^2}\int_{\Xi_t} v\,dt \\ &= o(1) - \frac{1}{r^2|\log r|}\int_{\Xi_r} u\varphi + \frac{1}{r^2|\log r|}\int_0^R \Big(G_2(r,t)\int_{\Xi_t} v\Big)\,dt,\end{aligned}$$

where G_2 is given by (36) with $k = 2$.

This proves (38).

Finally

(c) Case $k = 1$. This time (48) becomes

$$\langle\mu,\varphi\rangle = \int_\Omega \Delta u\varphi - g\varphi - \int_{\partial\Xi_s} v + \frac{d}{ds}\int_{\partial\Xi_s} u\varphi.$$

Integrate the previous relation over $s \in (t,\lambda)$:

$$(\lambda - t)\langle\mu,\varphi\rangle = o(1) - \int_{\Xi_\lambda\setminus\Xi_t} v + \int_{\partial\Xi_\lambda} u\varphi - \int_{\partial\Xi_t} u\varphi, \tag{57}$$

where $o(1) \to 0$ as $\lambda \to 0$. Since $v = 2u\nabla\varphi\cdot\nabla d + u\varphi a_0 \in L^1_{\mathrm{loc}}(\Omega)$, letting $t \downarrow 0$ in (57), we see that $\lim_{t\downarrow 0}\int_{\partial\Xi_t} u\varphi$ exists and

$$\lambda\langle\mu,\varphi\rangle = o(1) - \int_{\Xi_\lambda} v + \int_{\partial\Xi_\lambda} u\varphi - \Big(\lim_{t\downarrow 0}\int_{\partial\Xi_t} u\varphi\Big). \tag{58}$$

We now integrate (58) over $\lambda \in (0,r)$ and divide by r^2 to find

$$\frac{1}{2}\langle\mu,\varphi\rangle = o(1) + \frac{1}{r^2}\int_{\Xi_r} u\varphi - \frac{1}{r}\Big(\lim_{t\downarrow 0}\int_{\partial\Xi_t} u\varphi\Big) - \frac{1}{r^2}\int_0^r\int_{\Xi_t} v\,dt,$$

which concludes the proof of the lemma. □

4 Proof of Theorem 3

Set $\mu = \Delta u - g$. Suppose (7) holds. Then, since $\lim_{r\downarrow 0}\int_{\partial\Xi_r} u\varphi$ exists by Lemma 11, we conclude that

$$\lim_{r\downarrow 0}\int_{\partial\Xi_r} u\varphi = 0, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{59}$$

On the other hand, given $\varepsilon > 0$, (7) implies that there exists $\delta > 0$ such that

$$\int_{\partial\Xi_r} |u| \leq \varepsilon, \quad \forall r \in (0, \delta).$$

Therefore, we have

$$\begin{aligned}\left| \int_0^r \left(\int_{\Xi_t} 2u\nabla\varphi \cdot \nabla d + u\varphi a_0 \right) dt \right| &\leq C \int_0^r \left(\int_{\Xi_t} |u| \right) dt \\ &\leq C \int_0^r \varepsilon t \, dt = \varepsilon \frac{r^2}{2}, \quad \forall r \in (0, \delta).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we deduce that

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_0^r \left(\int_{\Xi_t} 2u\nabla\varphi \cdot \nabla d + u\varphi a_0 \right) dt = 0. \tag{60}$$

Inserting (59) and (60) into (39), we get

$$\frac{1}{2}\langle \mu, \varphi \rangle = \lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} u\varphi, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Now (8) follows since, by definition, $\mu = \Delta u - g$. This completes the proof of the theorem. □

5 Proof of Theorem 4

We prove Theorem 4 only for the case of codimension $k \geq 3$, the case $k = 2$ being entirely analogous.

Let G_k be the function defined by (36). Then, using the fact that $u \geq 0$ a.e. in Ω, we have

$$\begin{aligned}&\left| \int_0^R G_k(r,t) \left(\int_{\Xi_t} 2u\nabla\varphi \cdot \nabla d + u\varphi a_0 \right) dt \right| \leq \\ &\qquad \leq C \int_0^r \left(\int_{\Xi_t} u \right) dt + C \int_r^R \frac{r^k}{t^k} \left(\int_{\Xi_t} u \right) dt \\ &\qquad \leq Cr \int_{\Xi_r} u + C \int_r^R \frac{r^k}{t^k} \left(\int_{\Xi_t} u \right) dt, \quad \forall r \in (0, R).\end{aligned} \tag{61}$$

Choose $R_1 \in (0, R)$ so small that $CR_1 < \frac{1}{2}$.

Applying (37) with $R := R_1$ and $\varphi \in C_0^\infty(\Omega)$, $\varphi \equiv 1$ on Ξ_{R_1}, we have by (61) and our choice of R_1

$$\frac{1}{2} \int_{\Xi_r} u - C \int_r^R \frac{r^k}{t^k} \left(\int_{\Xi_t} u \right) dt \leq Cr^2, \quad \forall r \in (0, R_1). \tag{62}$$

We shall use (62) and a bootstrap argument to conclude that

$$\int_{\Xi_r} u \le Cr^2, \quad \forall r \in (0, R_1). \tag{63}$$

In fact, since $\int_{\Xi_t} u$ is uniformly bounded for $t \in (0, R_1)$,

$$\int_r^R \frac{r^k}{t^k} \left(\int_{\Xi_t} u \right) dt \le Cr, \quad \forall r \in (0, R_1). \tag{64}$$

In particular, (62) and (64) imply that

$$\frac{1}{2} \int_{\Xi_r} u \le Cr, \quad \forall r \in (0, R_1),$$

so that

$$\int_r^R \frac{r^k}{t^k} \left(\int_{\Xi_t} u \right) dt \le Cr^2, \quad \forall r \in (0, R_1). \tag{65}$$

Therefore, by (62) and (65), we conclude that estimate (63) holds.

It then follows from (63) and (61), with R replaced by R_1, that the right-hand side in (61) is bounded by Cr^3, for all $r \in (0, R_1)$. In particular,

$$\lim_{r \downarrow 0} \frac{1}{r^2} \left\{ \int_0^{R_1} G_k(r,t) \left(\int_{\Xi_t} 2u \nabla\varphi \cdot \nabla d + u\varphi a_0 \right) dt \right\} = 0. \tag{66}$$

By (37) and (66), we have

$$-\frac{1}{2(k-2)} \langle \mu, \varphi \rangle = \lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} u\varphi, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{67}$$

If we now apply (67) with estimate (63), we conclude that μ is a measure. Since $u \ge 0$ a.e. in Ω, (67) implies that μ is nonpositive. □

6 Proof of Theorems 5 and 6

Proof of Theorem 5. We split the proof of the theorem into 3 steps.
Step 1. If

$$\frac{1}{r^2} \int_{\Xi_r} |u| \quad \text{remains bounded as } r \downarrow 0, \tag{68}$$

then μ is a measure and

$$\langle \mu, \varphi \rangle = -2(k-2) \lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} u\varphi, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{69}$$

Proof. It is easy to see that condition (68) implies that

$$\lim_{r\downarrow 0}\frac{1}{r^2}\left\{\int_0^{R_1} G_k(r,t)\left(\int_{\Xi_t} 2u\nabla\varphi\cdot\nabla d + u\varphi a_0\right)dt\right\} = 0,$$

where G_k is the function defined by (36). From the limit above and (37), we deduce that (69) holds. In particular, it follows from (68) and (69) that μ is a measure and

(70) $$\|\mu\| \leq 2(k-2)\liminf_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r}|u|.$$

Step 2. If μ is a measure, then

(71) $$\frac{1}{r^2}\int_{\Xi_r}|u| \quad \text{remains bounded as } r\downarrow 0,$$

and

(72) $$\|\mu\| \geq 2(k-2)\limsup_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r}|u|.$$

Proof. In this step, we use an estimate given in the proof of Theorem 4 and the representation of the solutions of $\Delta v = \nu$ in $\mathbb{R}^N$ when ν is a measure in terms of the fundamental solution. More precisely, let $E(x) = c_N/|x|^{N-2}$ be the fundamental solution of $-\Delta$ in $\mathbb{R}^N$, $N \geq 3$, where the constant c_N is chosen so that $-\Delta E = \delta_0$. If ν is a Radon measure, then $v := E * \nu$ satisfies $-\Delta v = \nu$ in $\mathcal{D}'(\mathbb{R}^N)$.

Now let $\nu := g + \mu$ in Ω. Next, we decompose $\nu = \nu^+ - \nu^-$ into its positive and negative parts, where $\nu^\pm = g^\pm + \mu^\pm$. Let $v^\pm := E * \nu^\pm$. As observed above, we have

$$-\Delta v^\pm = \nu^\pm = g^\pm + \mu^\pm \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Moreover, $v^\pm \geq 0$ a.e. in $\mathbb{R}^N$. In particular, the functions $v^\pm$ satisfy the assumptions of Theorem 4, so that (11) holds with u and μ replaced by $v^\pm$ and $-\mu^\pm$, respectively. In other words, we have

(73) $$\int_{\Xi_r} v^\pm \leq Cr^2, \quad \forall r\in(0,1),$$

and

(74) $$\frac{1}{2(k-2)}\langle\mu^\pm,\varphi\rangle = \lim_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r} v^\pm\varphi, \quad \forall\varphi\in C_0^\infty(\Omega).$$

On the other hand, it is easy to see that $u = v^- - v^+ + w$ a.e. in Ω for some harmonic function w. Since w is bounded in some neighborhood of M, we have

(75) $$\lim_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r}|w| = 0.$$

In particular, (71) follows from (73) and (75). Moreover, if we apply (74) with a test function φ such that $\varphi \equiv 1$ in some neighborhood of M, then we have

$$\begin{aligned}\frac{1}{2(k-2)}\|\mu\| &= \frac{1}{2(k-2)}\left(\langle\mu^+,1\rangle + \langle\mu^-,1\rangle + 0\right)\\ &= \lim_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r}(v^+ + v^- + |w|) \geq \limsup_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r}|u|.\end{aligned}$$

This concludes the proof of Step 2.

Step 3. Proof of Theorem 5 completed. By Steps 1 and 2, we know that μ is a measure if and only if

$$\frac{1}{r^2}\int_{\Xi_r}|u| \quad \text{remains bounded as } r\downarrow 0,$$

in which case formula (14) holds. Moreover, applying (70) and (72), we get

$$\|\mu\| \leq 2(k-2)\,\liminf_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r}|u| \leq 2(k-2)\,\limsup_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r}|u| \leq \|\mu\|,$$

so that all the inequalities are reduced to equalities in the estimate above and (15) holds. □

Proof of Theorem 6. The proof of Theorem 6 follows the same lines as those in the previous one and is omitted. □

Remark 8. Although we derived (14) in Theorem 5 through a somewhat lengthy computation, there is a more natural approach if one assumes that the limits involved exist. Indeed, take $\varphi \in C_0^\infty(\Omega)$. Then, using l'Hôpital's rule, we obtain

$$\lim_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r}u\varphi = \lim_{r\downarrow 0}\frac{1}{2r}\int_{\partial\Xi_r}u\varphi. \tag{76}$$

But, using formula (106) of the Appendix (with $\lambda = 1$), we have

$$\begin{aligned}\lim_{r\downarrow 0}\frac{1}{r}\int_{\partial\Xi_r}u\varphi &= \lim_{r\downarrow 0}\frac{1}{r}\left\{r^{k-1}\int_{\partial\Xi_1}(u\varphi)\circ\pi_r\,\Theta(\xi,r)\,d\sigma(\xi)\right\}\\ &= \lim_{r\downarrow 0}\left\{r^{k-1}\int_{\partial\Xi_1}\frac{\partial(u\varphi)}{\partial\nu}\circ\pi_r\,\Theta + r^{k-1}\int_{\partial\Xi_1}(u\varphi)\circ\pi_r\,\frac{\partial\Theta}{\partial r}\right.\\ &\qquad\left. + (k-1)r^{k-2}\int_{\partial\Xi_1}(u\varphi)\circ\pi_r\,\Theta\right\}\\ &= \lim_{r\downarrow 0}\left\{\int_{\partial\Xi_r}\frac{\partial(u\varphi)}{\partial\nu} + \int_{\partial\Xi_r}u\varphi\left(\frac{\partial\Theta}{\partial r}\frac{1}{\Theta}\right)\circ\pi_1\right.\\ &\qquad\left. + \frac{k-1}{r}\int_{\partial\Xi_r}u\varphi\right\}.\end{aligned}$$

We can solve from the previous equations for $\lim_{r\downarrow 0}\frac{1}{r}\int_{\partial\Xi_r} u\varphi$:

$$\lim_{r\downarrow 0}\frac{1}{r}\int_{\partial\Xi_r} u\varphi = -\frac{1}{(k-2)}\lim_{r\downarrow 0}\left\{\int_{\partial\Xi_r}\frac{\partial(u\varphi)}{\partial\nu} + \int_{\partial\Xi_r} u\varphi\Big(\frac{\partial\Theta}{\partial r}\frac{1}{\Theta}\Big)\circ\pi_1\right\}.$$

Integrating by parts and using estimates in the Appendix, we find

$$\lim_{r\downarrow 0}\frac{1}{r}\int_{\partial\Xi_r} u\varphi = -\frac{1}{(k-2)}\langle\mu,\varphi\rangle. \tag{77}$$

Thus, (76) and (77) combined yield

$$\lim_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r} u\varphi = -\frac{1}{2(k-2)}\langle\mu,\varphi\rangle.$$

Remark 9. Formula (14) in Theorem 5 holds under weaker conditions than that stated in the theorem, namely that $\frac{1}{r^2}\int_{\Xi_r}|u|$ remains bounded as $r\downarrow 0$, or equivalently, that $\Delta u = \mu + g$ with $g\in L^1_{\mathrm{loc}}(\Omega)$ and μ a Radon measure supported in M. For example, it is easy to check that if

$$\frac{1}{r}\int_{\Xi_r}|u|\to 0 \qquad \text{as } r\downarrow 0, \tag{78}$$

then (14) holds, i.e. (in codimension $k\geq 3$),

$$\lim_{r\downarrow 0}\frac{1}{r^2}\int_{\Xi_r} u\varphi = -\frac{1}{2(k-2)}\langle\mu,\varphi\rangle \qquad \forall\varphi\in C_0^\infty(\Omega).$$

This suggests the following

Open problem. Let $\Omega\subset\mathbb{R}^N$ be a bounded open set and let $M\subset\Omega$ be a compact, smooth manifold without boundary of codimension $k\geq 3$. Let $u\in L^1_{\mathrm{loc}}(\Omega)$ and assume there exists $g\in L^1_{\mathrm{loc}}(\Omega)$ such that $\Delta u = g$ in $\mathcal{D}'(\Omega\setminus M)$. Set $\mu := \Delta u - g$ in $\mathcal{D}'(\Omega)$. Assume that

$$\lim_{r\downarrow 0}\frac{1}{r}\int_{\Xi_r}|u| = 0.$$

Is μ a measure?

The requirement (78) cannot be further relaxed, for instance, by asking instead that

$$\frac{1}{r}\int_{\Xi_r}|u| \text{ remains bounded as } r\downarrow 0. \tag{79}$$

For example, if $u(x) = x_1/|x|^3$ in $\mathbb{R}^3$, then $\Delta u = cD_{x_1}\delta_0$ for some constant $c\neq 0$, and $\frac{1}{r}\int_{\Xi_r}|u|$ remains bounded away from 0 as $r\downarrow 0$. In any case, if (79) holds, then from the formulas in Lemma 11 we see that μ has to be a distribution of order 1.

7 Proof of Theorem 8

As in [BrL], the proof consists of the following two steps.

Step 1. $u \in L^1_{\mathrm{loc}}(\Omega)$ and there exists a constant $C > 0$ such that

$$\int_{\Xi_r} u \le C \begin{cases} r^2 & \text{if } k \ge 3, \\ r^2 \log \frac{1}{r} & \text{if } k = 2, \end{cases} \tag{80}$$

for all $r > 0$ sufficiently small.

Step 2. Set $h := -\Delta u$ a.e. in $\Omega \setminus M$. Then $h \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega h\varphi \le -\int_\Omega u\Delta\varphi, \quad \forall \varphi \in \mathcal{F}, \tag{81}$$

where the class $\mathcal{F}$ of admissible test functions is defined by

$$\mathcal{F} := \left\{ \varphi \in C_0^\infty(\Omega) \;\middle|\; \begin{array}{l} \varphi \ge 0 \text{ in } \Omega, \\ \exists \lambda > 0 \text{ such that } \nabla\varphi \cdot \nabla d = 0 \text{ on } \Xi_\lambda \end{array} \right\}. \tag{82}$$

By Steps 1 and 2, we conclude that $u \in L^1_{\mathrm{loc}}(\Omega)$, and we can write

$$-\Delta u = h + \mu \quad \text{in } \mathcal{D}'(\Omega)$$

for some function $h \in L^1_{\mathrm{loc}}(\Omega)$ and some distribution μ supported on M. Since $u \ge 0$ a.e. in Ω, we deduce from Theorem 4 that μ is a positive measure (note the change of sign in the definition of μ). In other words, in order to show that Theorem 8 holds, it suffices to prove Steps 1 and 2.

Proof of Step 1. Consider the function

$$\bar{u}(r) = \frac{1}{r^{k-1}} \int_{\partial\Xi_r} u \, d\sigma = \int_{\partial\Xi_1} u \circ \pi_r \, \Theta(\sigma, r) \, d\sigma, \quad 0 < r < 1,$$

where Θ is a smooth function defined on $\partial\Xi_1 \times [0,1]$ which arises from the change of variables (see (107) and Lemma 12 in the Appendix), and π_r is defined by

$$\pi_r(x) = \pi(x) + r\frac{x - \pi(x)}{d(x)}, \quad x \in \Xi_2 \setminus M.$$

We use the function $\bar{u}$ to prove (80) in a similar way as in Brezis–Lions [BrL]. In order to get some of its properties, suppose for a moment that $u \in C^\infty(\Omega \setminus M)$; then

$$\frac{d\bar{u}}{dr}(r) = \int_{\partial\Xi_1} \frac{\partial u}{\partial \nu} \circ \pi_r \, \Theta(\xi, r) \, d\sigma(\xi) + \int_{\partial\Xi_1} u \circ \pi_r \frac{\partial\Theta}{\partial r}(\xi, r) \, d\sigma(\xi).$$

Hence, by Corollary 13,

$$r^{k-1}\frac{d\bar{u}}{dr}(r) = \int_{\partial\Xi_r} \frac{\partial u}{\partial \nu}\, d\sigma + r^{k-1}\int_{\partial\Xi_1} u\circ\pi_r \frac{\partial\Theta}{\partial r}(\xi, r)\, d\sigma(\xi);$$

and, integrating by parts, we have

$$= -\int_{\Xi_{r_0}\setminus\Xi_r} \Delta u + \int_{\partial\Xi_{r_0}} \frac{\partial u}{\partial \nu}\, d\sigma + r^{k-1}\int_{\partial\Xi_1} u\circ\pi_r \frac{\partial\Theta}{\partial r}(\xi, r)\, d\sigma(\xi),$$

for any $r_0 > 0$ small enough. Throughout this step, we denote by ν the unit normal vector to Ξ_r, pointing out of Ξ_r (which explains the minus sign in front of $\int_{\Xi_{r_0}\setminus\Xi_r} \Delta u$ in the expression above).

For a general $u \in L^1_{\rm loc}(\Omega\setminus M)$ with $\Delta u \in L^1_{\rm loc}(\Omega\setminus M)$, by using Fubini's Theorem and the fact that $u \in W^{1,1}_{\rm loc}(\Omega\setminus M)$, it follows by density that $\bar{u} \in C^1(0,1)$, $d\bar{u}/dr$ is absolutely continuous on $(0,1)$, and

$$r^{k-1}\frac{d\bar{u}}{dr}(r) = -\int_{\Xi_{r_0}\setminus\Xi_r} \Delta u + \int_{\partial\Xi_{r_0}} \frac{\partial u}{\partial \nu}\, d\sigma + r^{k-1}\int_{\partial\Xi_1} u\circ\pi_r \frac{\partial\Theta}{\partial r}(\xi, r)\, d\sigma(\xi)$$

is still true for a.e. $r_0 > 0$ small (which will be fixed later).

We now proceed with the main computation. The next formulas hold for a.e. $r \in (0,1)$. We have

$$\tag{83}\begin{aligned}\frac{1}{r^{k-1}}\frac{d}{dr}\Big(r^{k-1}\frac{d\bar{u}}{dr}\Big) &= \frac{1}{r^{k-1}}\int_{\partial\Xi_r} \Delta u \\ &\quad + \frac{1}{r^{k-1}}\frac{d}{dr}\Big\{r^{k-1}\int_{\partial\Xi_1} u\circ\pi_r \frac{\partial\Theta}{\partial r}(\xi, r)\, d\sigma(\xi)\Big\} \\ &= \frac{1}{r^{k-1}}\int_{\partial\Xi_r} \Delta u + \frac{k-1}{r} I_1 + I_2,\end{aligned}$$

where

$$I_1 = \int_{\partial\Xi_1} u\circ\pi_r \frac{\partial\Theta}{\partial r}(\xi, r)\, d\sigma(\xi) = \frac{1}{r^{k-1}}\int_{\partial\Xi_r} u\Big(\frac{1}{\Theta}\frac{\partial\Theta}{\partial r}\Big)\circ\pi_1$$

and

$$\begin{aligned}I_2 &= r^{k-1}\frac{d}{dr}\int_{\partial\Xi_1} u\circ\pi_r \frac{\partial\Theta}{\partial r}(\xi, r)\, d\sigma(\xi) \\ &= r^{k-1}\int_{\partial\Xi_1} \frac{\partial u}{\partial\nu}\circ\pi_r \frac{\partial\Theta}{\partial r} + r^{k-1}\int_{\partial\Xi_1} u\circ\pi_r \frac{\partial^2\Theta}{\partial r^2} \\ &= \int_{\partial\Xi_r} \frac{\partial u}{\partial\nu}\Big(\frac{1}{\Theta}\frac{\partial\Theta}{\partial r}\Big)\circ\pi_1 + \int_{\partial\Xi_r} u\Big(\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial r^2}\Big)\circ\pi_1.\end{aligned}$$

At this point it is convenient to set

$$\vartheta^{(i)}(x) = \frac{1}{\Theta(\pi_1(x), d(x))} \frac{\partial^i \Theta}{\partial r^i}(\pi_1(x), d(x)), \qquad i = 1, 2.$$

Then we can rewrite I_1 and I_2 as

$$I_1 = \frac{1}{r^{k-1}} \int_{\partial \Xi_r} u\vartheta^{(1)}, \tag{84}$$
$$I_2 = \int_{\partial \Xi_r} \frac{\partial u}{\partial \nu}\vartheta^{(1)} + \int_{\partial \Xi_r} u\vartheta^{(2)}.$$

Integrating the expression for I_2 by parts, we get

$$\begin{aligned} I_2 = & - \int_{\partial(\Xi_{r_0} \setminus \Xi_r)} u \frac{\partial \vartheta^{(1)}}{\partial \nu} + \int_{\Xi_{r_0} \setminus \Xi_r} u\Delta\vartheta^{(1)} - \int_{\Xi_{r_0} \setminus \Xi_r} \Delta u \vartheta^{(1)} \\ & + \int_{\partial \Xi_{r_0}} \frac{\partial u}{\partial \nu}\vartheta^{(1)} + \int_{\partial \Xi_r} u\vartheta^{(2)}. \end{aligned} \tag{85}$$

From Corollary 13 in the Appendix (in combination with the lower bound for Θ of Lemma 9), we obtain the following estimates for $\vartheta^{(i)}$:

$$|D^j \vartheta^{(1)}| \le C d^{k-j-1}, \quad j = 0, 1, 2; \tag{86}$$
$$|\vartheta^{(2)}| \le C d^{k-2}. \tag{87}$$

Therefore,

$$I_1 = \frac{1}{r^{k-1}} \int_{\partial \Xi_r} u\vartheta^{(1)} \le C \int_{\partial \Xi_r} u \tag{88}$$

and

$$I_2 \le C \int_{\partial \Xi_r} u + C \int_{\Xi_{r_0} \setminus \Xi_r} u d^{k-3} - \int_{\Xi_{r_0} \setminus \Xi_r} \Delta u \vartheta^{(1)} + C. \tag{89}$$

Combining (83), (88) and (89), we find

$$\begin{aligned} \frac{1}{r^{k-1}} \frac{d}{dr}\left(r^{k-1} \frac{d\bar{u}}{dr}\right) \le & \frac{1}{r^{k-1}} \int_{\partial \Xi_r} \Delta u + \frac{C}{r} \int_{\partial \Xi_r} u + C \int_{\partial \Xi_r} u + \\ & + C \int_{\Xi_{r_0} \setminus \Xi_r} u d^{k-3} - \int_{\Xi_{r_0} \setminus \Xi_r} \Delta u \vartheta^{(1)} + C. \end{aligned}$$

Then, multiplying the last inequality by r^{k-1} and integrating with respect to r

yields

$$
\begin{aligned}
(90)\qquad & r_0^{k-1}\frac{d\bar{u}}{dr}(r_0) - r^{k-1}\frac{d\bar{u}}{dr}(r) \le \\
&\quad \le C\int_r^{r_0} s^{k-2}\int_{\partial\Xi_s} u\,d\sigma\,ds + C\int_r^{r_0} s^{k-1}\int_{\partial\Xi_s} u\,d\sigma\,ds \\
&\quad + C\int_r^{r_0} s^{k-1}\int_{\Xi_{r_0}\setminus\Xi_s} ud^{k-3}\,d\sigma\,ds \\
&\quad + \int_r^{r_0}\int_{\partial\Xi_s} \Delta u\,d\sigma\,ds - \int_r^{r_0} s^{k-1}\int_{\Xi_{r_0}\setminus\Xi_s} \Delta u\,\vartheta^{(1)}\,dx\,ds \\
&\quad + Cr_0^k.
\end{aligned}
$$

We now estimate each term on the right-hand side of (90). We start with

$$
(91)\qquad \int_r^{r_0} s^{k-2}\int_{\partial\Xi_s} u\,d\sigma\,ds \le r_0^{k-2}\int_r^{r_0}\int_{\partial\Xi_s} u\,d\sigma\,ds \le \int_r^{r_0} s^{k-1}\bar{u}(s)\,ds.
$$

Similarly,

$$
(92)\qquad \int_r^{r_0} s^{k-1}\int_{\partial\Xi_s} u\,d\sigma\,ds \le \int_r^{r_0} s^{k-1}\bar{u}(s)\,ds.
$$

The third term on the right-hand side of (90) is, by Fubini,

$$
\begin{aligned}
(93)\qquad \int_r^{r_0} s^{k-1}\int_{\Xi_{r_0}\setminus\Xi_s} ud^{k-3}\,d\sigma\,ds &= \int_r^{r_0}\int_s^{r_0}\int_{\partial\Xi_\lambda} s^{k-1}u\lambda^{k-3}\,d\sigma\,d\lambda\,ds \\
&= \int_r^{r_0}\int_{\partial\Xi_\lambda} u\lambda^{k-3}\int_r^{\lambda} s^{k-1}\,ds\,d\sigma\,d\lambda \\
&\le C\int_r^{r_0}\int_{\partial\Xi_\lambda} u\lambda^{2k-3}d\sigma\,d\lambda \\
&\le C\int_r^{r_0} s^{k-1}\bar{u}(s)\,ds,
\end{aligned}
$$

since $\lambda^{2k-3} \le 1$, for $0 < \lambda \le r_0 \le 1$. We now estimate the fifth term in (90), again using Fubini:

$$
\begin{aligned}
\int_r^{r_0} s^{k-1}\int_{\Xi_{r_0}\setminus\Xi_s} \Delta u\vartheta^{(1)}\,d\sigma\,ds &= \int_r^{r_0}\int_s^{r_0}\int_{\partial\Xi_\lambda} s^{k-1}\Delta u\,\vartheta^{(1)}\lambda^{k-1}\,d\sigma\,d\lambda\,ds, \\
&= \int_r^{r_0}\int_{\partial\Xi_\lambda} \Delta u\,\vartheta^{(1)}\,\lambda^{k-1}\int_r^{\lambda} s^{k-1}\,ds\,d\sigma\,d\lambda \\
&= \int_r^{r_0}\int_{\partial\Xi_\lambda} \Delta u\,O(r_0^{2k-1})\,d\sigma\,d\lambda,
\end{aligned}
$$

where $O(r_0^{2k-1})$ denotes a function bounded by Cr_0^{2k-1}. Hence the fourth and fifth terms of (90) combined yield

$$\int_r^{r_0}\int_{\partial\Xi_s}\Delta u\,d\sigma\,ds-\int_r^{r_0}s^{k-1}\int_{\Xi_{r_0}\setminus\Xi_s}\Delta u\,\vartheta^{(1)}\,d\sigma\,ds=$$
$$=\int_r^{r_0}\int_{\partial\Xi_\lambda}\Delta u\left(1+\;O(r_0^{2k-1})\;\right)d\sigma\,d\lambda.$$

We now **fix** $r_0>0$ so small that $1/2\le 1+O(r_0^{2k-1})\le 3/2$. Since $\Delta u\le au+f$ a.e. in $\Omega\setminus M$,

$$\begin{aligned}\int_r^{r_0}\int_{\partial\Xi_s}\Delta u\,d\sigma\,ds-\int_r^{r_0}s^{k-1}\int_{\Xi_{r_0}\setminus\Xi_s}\Delta u\,\vartheta^{(1)}\,d\sigma\,ds&\le\\ \le C\int_r^{r_0}\int_{\partial\Xi_\lambda}au+f\,d\sigma\,d\lambda&\\ \le C\int_r^{r_0}s^{k-1}\bar u(s)\,ds+\int_{\Xi_\lambda}f&\end{aligned}\tag{94}$$

(in the hypotheses of the theorem, after replacing f with f^+ we may assume that $f\ge 0$ a.e. in Ω). Hence, from (90)–(94), we get

$$-r^{k-1}\frac{d\bar u}{dr}(r)\le C\int_r^{r_0}s^{k-1}\bar u(s)\,ds+C.$$

We now proceed exactly as in [BrL]. Take $0<R<r_0$ to be chosen later and define

$$\psi_R(r):=\int_r^R s^{k-1}\bar u(s)\,ds,\qquad 0<r<R.$$

With this notation, we have

$$-r^{k-1}\frac{d\bar u}{dr}(r)\le C\,\psi_R(r)+C_R,$$

where C_R is a constant that depends on R, but C is independent of R. After integration, we find

$$\bar u(r)-\bar u(R)\le C\int_r^R\frac{\psi(s)}{s^{k-1}}\,ds+C_R\int_r^R\frac{ds}{s^{k-1}},$$

and therefore

$$\bar u(r)\le C\int_r^R\frac{1}{s^{k-1}}\psi_R(s)\,ds+C_R\Big(1+\frac{1}{r^{k-2}}\Big)\tag{95}$$

if $k\ge 3$. In the case $k=2$, we have to replace $1/r^{k-2}$ by $|\log r|$ in the second term on the right-hand side. Since ψ_R is nonincreasing, we thus obtain

$$r^{k-1}\bar u(r)\le CR\psi_R(r)+C_R.$$

Integrating once more, we get

$$\begin{aligned}\psi_R(r) = \int_r^R s^{k-1}\bar{u}(s)\,ds &\le CR\int_r^R \psi_R(s)\,ds + C_R \\ &\le CR^2\psi_R(r) + C_R.\end{aligned} \tag{96}$$

Now choose $0 < R < r_0$ such that $(1 - CR^2) \ge 1/2$; then from (96) we see that

$$\psi_R(r) \le C,$$

with C independent of $r \in (0, R)$. By letting $r \to 0$, we conclude that $u \in L^1_{\mathrm{loc}}(\Omega)$. Moreover, from (95), we see that

$$\bar{u}(r) \le C \begin{cases} \dfrac{1}{r^{k-2}} & \text{if } k \ge 3, \\ |\log r| & \text{if } k = 2, \end{cases}$$

which implies the estimate

$$\int_{\Xi_r} u = \int_0^r s^{k-1}\bar{u}(s)\,ds \le C \begin{cases} r^2 & \text{if } k \ge 3, \\ r^2|\log r| & \text{if } k = 2. \end{cases}$$

This concludes the proof of the first step.

Proof of Step 2. First, note that to prove the whole statement of Step 2, it is enough to show that (81) holds. In fact, suppose that (81) has already been established. By the assumptions of the theorem, we know that $h \ge -au - f$ a.e. in Ω, and $au + f \in L^1_{\mathrm{loc}}(\Omega)$ by Step 1. If we take an admissible test function $\varphi \in \mathcal{F}$ such that $\varphi \equiv 1$ in some small neighborhood of M, then we have

$$0 \le \int_\Omega (h + au + f)\varphi \le -\int_\Omega u\Delta\varphi + \int_\Omega (au + f)\varphi < \infty,$$

which implies that $h \in L^1_{\mathrm{loc}}(\Omega)$.

We now proceed with the proof of (81).

Let $\varphi \in \mathcal{F}$. Since (81) is trivially satisfied if $\varphi \equiv 0$ near M (in fact, we have equality in (81) in this case), there is no loss of generality if we assume that $\operatorname{supp}\varphi \subset \Xi_1$ and $\varphi \not\equiv 0$ near M. Next, fix $\lambda > 0$ such that $\nabla\varphi \cdot \nabla d = 0$ in Ξ_λ.

Let $\Phi \in C^3(\mathbb{R})$ be a convex function such that $\Phi(t) = 0$ for $t \ge 1$, and $\Phi(0) = 1$, to be given explicitly below.

For $0 < \varepsilon < 1$, and if $k \ge 3$, define

$$\varphi_\varepsilon(x) := \begin{cases} \varphi(x)\,\Phi\left(\dfrac{\varepsilon^{k-2}}{d(x)^{k-2}}\right) & \text{if } x \in \Xi_1 \setminus \Xi_\varepsilon, \\ 0 & \text{otherwise;} \end{cases}$$

if $k = 2$, we let

$$\varphi_\varepsilon(x) := \begin{cases} \varphi(x)\,\Phi\left(\dfrac{\log \frac{1}{d(x)}}{\log \frac{1}{\varepsilon}}\right) & \text{if } x \in \Xi_1 \setminus \Xi_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, we have $\varphi_\varepsilon \in C_0^3(\Omega)$ and $\varphi_\varepsilon \equiv 0$ on Ξ_ε. In particular,

$$\int_\Omega h\varphi_\varepsilon = -\int_\Omega \Delta u \varphi_\varepsilon = -\int_\Omega u\Delta\varphi_\varepsilon. \tag{97}$$

In the argument that follows, we assume $k \geq 3$, the proof of (81) when $k = 2$ being entirely analogous.

If we compute $\Delta\varphi_\varepsilon$ explicitly on $\Xi_1 \setminus \Xi_\varepsilon$ and use (34) and (35), we get (recall that $\varphi_\varepsilon \equiv 0$ outside this set)

$$\begin{aligned} \Delta\varphi_\varepsilon = \Delta\varphi\,\Phi\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right) - 2(k-2)\nabla\varphi\cdot\nabla d\,\Phi'\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right)\frac{\varepsilon^{k-2}}{d^{k-1}} + \\ + \varphi\frac{\varepsilon^{k-2}}{d^{k-1}}\left\{(k-2)^2\Phi''\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right)\frac{\varepsilon^{k-2}}{d^{k-1}} + \Phi'\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right)O(1)\right\}, \end{aligned} \tag{98}$$

where $O(1)$ is a quantity which remains bounded as $\varepsilon \downarrow 0$.

Note that

$$\Phi\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right) \to \Phi(0) = 1$$

and

$$\Phi'\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right)\frac{\varepsilon^{k-2}}{d^{k-1}} \to 0,$$

both limits being uniform in any compact subset of $\Omega \setminus M$ as $\varepsilon \downarrow 0$.

Since $\Delta\varphi\,\Phi\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right)$ is uniformly bounded and $\nabla\varphi\cdot\nabla d = 0$ on Ξ_λ, we conclude that

$$\begin{aligned} \int_\Omega \left[\Delta\varphi\,\Phi\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right) - 2(k-2)\nabla\varphi\cdot\nabla d\,\Phi'\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right)\frac{\varepsilon^{k-2}}{d^{k-1}}\right] u \longrightarrow \\ \longrightarrow \int_\Omega u\Delta\varphi \quad \text{as } \varepsilon \downarrow 0. \end{aligned} \tag{99}$$

Next, we analyze the behavior of the term between brackets in (98). Before that, let us make a special choice of the function Φ.

Let $a \geq 3$ be a number sufficiently large to be chosen below. Define $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(t) := \begin{cases} (1-t)^{a+1} & \text{if } t \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\Phi \in C^3(\mathbb{R})$, $\Phi(t) = 0$ for $t \geq 1$, and $\Phi(0) = 1$.

For $x \in \Xi_1 \setminus \Xi_\varepsilon$, we have

$$\begin{aligned}
&\frac{\varepsilon^{k-2}}{d^{k-1}}\left\{(k-2)^2\Phi''\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right)\frac{\varepsilon^{k-2}}{d^{k-1}} + \Phi'\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right)O(1)\right\} = \\
&\quad = (a+1)\frac{\varepsilon^{k-2}}{d^{k-1}}\left(1 - \frac{\varepsilon^{k-2}}{d^{k-2}}\right)^{a-1}\left\{a(k-2)^2\frac{\varepsilon^{k-2}}{d^{k-1}} - \left(1 - \frac{\varepsilon^{k-2}}{d^{k-2}}\right)O(1)\right\} \\
&\quad = (a+1)\frac{\varepsilon^{k-2}}{d^{k-1}}\left(1 - \frac{\varepsilon^{k-2}}{d^{k-2}}\right)^{a-1}\left\{\frac{\varepsilon^{k-2}}{d^{k-1}}\Big(a(k-2)^2 + O(1)\,d\Big) - O(1)\right\}.
\end{aligned}$$

Now choose $K > 0$ and then $a \geq 3$ both so large that

$$\frac{a(k-2)^2}{2} \geq K \geq |O(1)|, \quad \text{for } 0 < \varepsilon < 1.$$

Then we get

$$\begin{aligned}
&\frac{\varepsilon^{k-2}}{d^{k-1}}\left\{(k-2)^2\Phi''\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right)\frac{\varepsilon^{k-2}}{d^{k-1}} + \Phi'\left(\frac{\varepsilon^{k-2}}{d^{k-2}}\right)O(1)\right\} = \\
&\qquad \geq (a+1)\frac{\varepsilon^{k-2}}{d^{k-1}}\left(1 - \frac{\varepsilon^{k-2}}{d^{k-2}}\right)^{a-1}\left\{\frac{a(k-2)^2}{2}\frac{\varepsilon^{k-2}}{d^{k-1}} - O(1)\right\} =: H.
\end{aligned}$$

Next we split the estimate for a lower bound of H into two cases, depending on how near the point x is with respect to the singular set M:

Case 1. $\dfrac{a(k-2)^2}{2}\dfrac{\varepsilon^{k-2}}{d^{k-1}} \geq K.$

In this case, by our very choice of K, the expression defining H must be nonnegative, i.e., $H \geq 0$.

Case 2. $\dfrac{a(k-2)^2}{2}\dfrac{\varepsilon^{k-2}}{d^{k-1}} < K.$

If the inequality above holds, we have

$$\begin{aligned}
H &\geq -(a+1)\frac{\varepsilon^{k-2}}{d^{k-1}}\left(1 - \frac{\varepsilon^{k-2}}{d^{k-2}}\right)^{a-1} K \\
&\geq -(a+1)\frac{2K}{a(k-2)^2}K > -\frac{8}{3}K^2 =: -C.
\end{aligned}$$

In both cases, we have

$$H \geq -C, \tag{100}$$

for some constant $C > 0$ independent of ε and φ.

It now follows from (98)–(100) and Fatou's Lemma (recall that $h \geq -au - f \in L^1_{\mathrm{loc}}(\Omega)$) that, if we let $\varepsilon \downarrow 0$ in (97), we get

$$\int_\Omega h\varphi \leq -\int_\Omega \Delta u\varphi + C\int_\Omega u\varphi, \quad \forall \varphi \in \mathcal{F}, \tag{101}$$

which is "almost" the inequality we want to prove. In any case, the argument presented at the beginning of this step, applied to (101), already gives $h \in L^1_{\mathrm{loc}}(\Omega)$. Next, we show how the constant $C > 0$ above can be removed.

Given any small $\delta > 0$, let $\eta_\delta \in C_0^\infty(\Xi_\delta)$ be such that $0 \leq \eta_\delta \leq 1$ and $\eta_\delta \equiv 1$ on $\Xi_{\delta/2}$. Note that $\varphi\eta_\delta$ still belongs to $\mathcal{F}$ so that, after replacing φ in (101) by $\varphi\eta_\delta$, we get

$$\int_\Omega h\varphi\eta_\delta \leq -\int_\Omega u\Delta(\varphi\eta_\delta) + C\int_\Omega u\varphi\eta_\delta.$$

On the other hand, since $\varphi(1-\eta_\delta) \in C_0^\infty(\Omega \setminus M)$,

$$\int_\Omega h\varphi(1-\eta_\delta) = -\int_\Omega u\Delta(\varphi(1-\eta_\delta)).$$

Now adding both relations, we obtain

$$\int_\Omega h\varphi \leq -\int_\Omega u\Delta\varphi + C\int_\Omega u\varphi\eta_\delta.$$

If we let $\delta \downarrow 0$ in the inequality above, we get (81), as claimed. This concludes the proof of Step 2.

8 Proof of Corollary 10

Let $u \in L^1_{\mathrm{loc}}(\Omega)$, $u \geq 0$ a.e. in Ω, be as in Corollary 9. Since $f(u) \in L^1_{\mathrm{loc}}(\Omega)$, (25) implies that

$$u \in L^{k/(k-2)}_{\mathrm{loc}}(\Omega) \quad \text{if } k > 2, \tag{102}$$

$$e^{au} \in L^1_{\mathrm{loc}}(\Omega) \quad \text{if } k = 2, \text{ for all } a > 0. \tag{103}$$

If $k > 2$, we apply Hölder's inequality to conclude from (102) (using the fact that $|\Xi_r| \sim r^k$ as $r \downarrow 0$) that

$$\lim_{r\downarrow 0} \frac{1}{r^2}\int_{\Xi_r} u = 0.$$

By Corollary 7, we must have $\mu = 0$ in (24), which proves the result in the case $k \geq 3$.

Let us now suppose $k = 2$. For $a > 0$ fixed, we have by Jensen's inequality and (103) that

$$e^{\frac{1}{|\Xi_r|}\int_{\Xi_r} au} \leq \frac{1}{|\Xi_r|}\int_{\Xi_r} e^{au} \leq \frac{C_a}{|\Xi_r|}, \quad \forall r > 0 \text{ small},$$

where $C_a > 0$ is a constant depending on a. We conclude that

$$\frac{1}{|\Xi_r|}\int_{\Xi_r} au \leq \log \frac{C_a}{|\Xi_r|}. \tag{104}$$

Let $0 < \alpha_1 \leq \alpha_2$ be such that $\alpha_1 r^2 \leq |\Xi_r| \leq \alpha_2 r^2$ for all $r > 0$ small. From (104), we get

$$\frac{1}{\alpha_2 r^2 \log 1/r}\int_{\Xi_r} au \leq \frac{\log(C_a/\alpha_1 r^2)}{\log 1/r} = 2 + \frac{\log(C_a/\alpha_1)}{\log 1/r}.$$

By letting $r \downarrow 0$, we deduce that

$$\limsup_{r\downarrow 0} \frac{1}{r^2|\log r|}\int_{\Xi_r} u \leq \frac{2\alpha_2}{a}, \qquad \forall a > 0.$$

If we take $a \uparrow \infty$, then we have

$$\lim_{r\downarrow 0} \frac{1}{r^2|\log r|}\int_{\Xi_r} u = 0.$$

We now invoke Corollary 7 to get the result in the case $k = 2$. □

Appendix

In the sequel, we assume that Ξ_r is a tubular neighborhood of M^{N-k} of radius r, where M^{N-k} is a compact manifold without boundary in $\mathbb{R}^N$ of codimension $k \geq 1$. We use here the same notation as in Section 3. Before stating the lemma below, let us recall the definition of the projection $\pi_r : \Xi_2 \setminus M \to \partial\Xi_r$:

$$\pi_r(x) := \pi(x) + r\,\frac{x - \pi(x)}{d(x)}.$$

Note that, if $0 < r, \lambda \leq 2$, then $\pi_r|_{\partial\Xi_\lambda} : \partial\Xi_\lambda \to \partial\Xi_r$ is a smooth diffeomorphism between the manifolds $\partial\Xi_\lambda$ and $\partial\Xi_r$.

Throughout the Appendix, we will use the notation

$$\Theta(x, r) := \frac{1}{r^{k-1}} J(\pi_r|_{\partial\Xi_\lambda}), \quad x \in \partial\Xi_\lambda, \quad r, \lambda \in (0, 2], \tag{105}$$

where $J(\pi_r|_{\partial\Xi_\lambda})$ denotes the Jacobian of the map $\pi_r|_{\partial\Xi_\lambda}$, so that

$$\int_{\partial\Xi_r} v = \int_{\partial\Xi_\lambda} v \circ \pi_r(\xi)\,\Theta(\xi, r) r^{k-1}\, d\sigma_\lambda(\xi), \quad \forall v \in L^1(\partial\Xi_r), \tag{106}$$

or equivalently, by the coarea formula,

$$\int_{\Xi_r} v = \int_0^r \int_{\partial\Xi_\lambda} v \circ \pi_s(\xi)\, \Theta(\xi, s) s^{k-1}\, d\sigma_\lambda(\xi)\, ds, \quad \forall v \in L^1(\Xi_r). \tag{107}$$

We should remark at this point that the choice of the normalization factor $1/r^{k-1}$ comes from the degeneracy rate of $J(\pi_r|_{\partial\Xi_\lambda})$ as $r \downarrow 0$, as we shall see in Lemma 12.

In the next lemma, we present some properties of this function, which were used in some of the main results in this paper. We handle only the case of codimension $k \geq 2$. Since we are mostly interested in the limiting behavior of $\Theta(\cdot, r)$ as $r \downarrow 0$, we consider Θ as a function defined on $\partial\Xi_1 \times (0, 2]$, i.e., we take $\lambda = 1$ in equations (106) and (107).

Lemma 12. *Suppose $M \subset \mathbb{R}^N$ is a compact manifold without boundary of codimension $k \geq 2$. Then $\Theta \in C^\infty(\partial\Xi_1 \times [0, 2])$ and satisfies:*

(i) *there exists $a > 0$ such that $\Theta \geq a > 0$ on $\partial\Xi_1 \times [0, 2]$;*

(ii) *there exist smooth functions α, β defined on $\partial\Xi_1$ such that*

$$\Theta(\xi, r) = \alpha(\xi) + r^k \beta(\xi), \quad \forall(\xi, r) \in \partial\Xi_1 \times [0, 2]. \tag{108}$$

Proof. Instead of computing $J(\pi_s|_{\partial\Xi_1})$ directly in (105) to get the desired properties of Θ, we try to find another representation for the function Θ. We proceed as follows.

Given a small geodesic neighborhood $U \subset M$, let $h : U \times B_2^k \to \pi^{-1}(U) \times \operatorname{int} \Xi_2$ be a diffeomorphism such that $h(z_1, 0) = z_1$, $\pi(h(z_1, \cdot)) = z_1$, and $h(z_1, \cdot)$ is an affine linear isometry for each $z_1 \in U$.

Using the parametrization of Ξ_r induced by h and the coarea formula, we have

$$\begin{aligned}
\int_{\Xi_r \cap \pi^{-1}(U)} v &= \int_U \int_{B_r^k} v \circ h\, Jh \\
&= \int_0^r \int_U \int_{\partial B_s^k} v \circ h\, Jh\, d\sigma_s dz_1 ds \\
&= \int_0^r \int_U \int_{S^{k-1}} v(h(z_1, s\zeta))\, Jh(z_1, s\zeta) s^{k-1}\, d\sigma(\zeta) dz_1\, ds \\
&= \int_0^r \int_U \int_{S^{k-1}} v \circ h \circ j_s(z_1, \zeta)\, Jh \circ j_s(z_1, \zeta) s^{k-1}\, d\sigma(\zeta) dz_1 ds,
\end{aligned}$$

where $j_s(z_1, \zeta) := (z_1, s\zeta)$.

Therefore, we get the following expression for the integral of v on $\Xi_r \cap \pi^{-1}(U)$:

$$\begin{aligned}
&\int_{\Xi_r \cap \pi^{-1}(U)} v \\
&\quad = \int_0^r \int_{\partial\Xi_1 \cap \pi^{-1}(U)} v \circ \pi_s \left[Jh \circ j_s \circ h^{-1}\, J(h^{-1}|_{\partial\Xi_1}) \right] s^{k-1}\, d\sigma ds,
\end{aligned} \tag{109}$$

where we have used the fact that, by our very choice of h, we must have $\pi_s = h \circ j_s \circ h^{-1}$ on $\partial\Xi_1$. If we compare the identities (107) and (109), we then conclude that

$$\Theta = Jh \circ j_r \circ h^{-1}\, J(h^{-1}|_{\partial\Xi_1}) \quad \text{on } \big(\partial\Xi_1 \cap \pi^{-1}(U)\big) \times (0,2]. \tag{110}$$

Since U was an arbitrary small geodesic neighborhood of M and h was a diffeomorphism, (110) immediately implies that $\Theta \in C^\infty(\partial\Xi_1 \times [0,2])$ and $\Theta > 0$ on $\partial\Xi_1 \times [0,2]$, so that (i) must hold.

In order to prove (ii), we first rewrite (110) as

$$\begin{aligned}\Theta(h(z),r) = Jh(z_1, rz_2)\, J(h^{-1}|_{\partial\Xi_1})(h(z)),&\\ \forall (z,r) \in \big(U \times S^{k-1}\big) \times [0,2].&\end{aligned} \tag{111}$$

By choosing a smaller open subset of U if necessary, we may assume we have a parametrization $p : \mathbb{R}^{N-k} \to U$. Next, define $\tilde{h} : \mathbb{R}^{N-k} \times B_2^k \to \Xi_2$ by $\tilde{h}(y_1,y_2) := h(p(y_1),y_2)$, so that

$$J\tilde{h}(y_1,y_2) = Jh(p(y_1),y_2)\, Jp(y_1). \tag{112}$$

In view of (111) and (112), in order to show that Θ may be written as (108), it suffices to prove the following decomposition for $J\tilde{h}$:

$$J\tilde{h}(y) = \tilde{\alpha}(y_1) + \tilde{\beta}(y), \quad \forall y = (y_1,y_2) \in \mathbb{R}^{N-k} \times B_2^k, \tag{113}$$

where $\tilde{\alpha}, \tilde{\beta}$ are smooth and $\tilde{\beta}(y_1,y_2)$ is a homogeneous polynomial of order k with respect to the y_2-variable, for each $y_1 \in \mathbb{R}^{N-k}$.

From the properties of h, we may write it more explicitly as

$$h(z_1,z_2) = z_1 + T(z_1)z_2, \quad \forall (z_1,z_2) \in U \times B_2^k,$$

for some linear isometry $T(z_1) : \mathbb{R}^k \to \mathbb{R}^{N-k}$, $z_1 \in U$, so that

$$\tilde{h}(y_1,y_2) = p(y_1) + T(p(y_1))y_2 =: p(y_1) + \tilde{T}(y_1)y_2,$$

which implies

$$\begin{aligned} J\tilde{h}(y_1,y_2) &= \det\Big(Dp(y_1) + D\tilde{T}(y_1)y_2, \tilde{T}(y_1)\Big)\\ &= \det\Big(Dp(y_1), \tilde{T}(y_1)\Big) + \det\Big(D\tilde{T}(y_1)y_2, \tilde{T}(y_1)\Big). \end{aligned}$$

Now (113) follows if we take

$$\begin{aligned} \tilde{\alpha}(y_1) &:= \det\Big(Dp(y_1), \tilde{T}(y_1)\Big),\\ \tilde{\beta}(y_1,y_2) &:= \det\Big(D\tilde{T}(y_1)y_2, \tilde{T}(y_1)\Big). \end{aligned}$$

In particular, note that $\tilde{\beta}(y_1, \cdot)$ is a homogeneous polynomial of order k. As we have already remarked, (111), (112) and (113) imply (ii). This concludes the proof of the lemma. □

The following corollary gives some estimates we needed in the proof of Theorem 8.

Corollary 13. *For any $j \geq 0$ and $x \in \Xi_1 \setminus M$,*

$$D_x^j \left[\frac{\partial^i \Theta}{\partial r^i}(\pi_1(x), d(x)) \right] = \begin{cases} O(d^{k-i-j}) & \text{if } 1 \leq i \leq k, \\ 0 & \text{if } i > k. \end{cases} \tag{114}$$

In particular, estimates (86) *and* (87) *hold.*

Proof. First, we see from (108) that we only need to prove (114) for $1 \leq i \leq k$. If we differentiate (108) with respect to r and evaluate the resulting expression at the point $(\xi, r) = (\pi_1(x), d(x))$, for some $x \in \Xi_2 \setminus M$, we get

$$\frac{\partial^i \Theta}{\partial r^i}(\pi_1(x), d(x)) = \frac{k!}{(k-i)!} d(x)^{k-i} \beta(\pi_1(x)). \tag{115}$$

In particular, (114) with $j = 0$ (and any $i \leq k$) follows from the expression above.

Next, assume $j \geq 1$. Instead of differentiating (115) directly with respect to x, we write it in terms of conveniently chosen local coordinates, as we did in the proof of Lemma 12.

For a sufficiently small geodesic neighborhood $U \subset M$, we can find a parametrization $p : \mathbb{R}^{N-k} \to U$ and a diffeomorphism $h : U \times B_2^k \to \pi^{-1}(U) \cap \operatorname{int} \Xi_2$ such that $h(z_1, 0) = z_1$, $\pi(h(z_1, \cdot)) = z_1$, and $h(z_1, \cdot)$ is an affine linear isometry for each $z_1 \in U$.

Define $\tilde{h}(y) := h(p(y_1), y_2)$, $y \in B_{10}^{N-k} \times B_2^k$, so that $\tilde{h}$ is a diffeomorphism between $B_{10}^{N-k} \times B_2^k$ and $\pi^{-1}(p(B_{10}^{N-k})) \cap \operatorname{int} \Xi_2 =: \mathcal{V}$; moreover, the derivatives of $\tilde{h}$ and $\tilde{h}^{-1}$ are bounded (which explains why we defined $\tilde{h}$ using B_{10}^{N-k}, instead of $\mathbb{R}^{N-k}$).

Given $x \in \mathcal{V} \setminus M$, let $y \in B_{10}^{N-k} \times B_2^k \setminus \{0\}$ be such that $\tilde{h}(y) = x$. Using the properties of $\tilde{h}$ (or rather of h), we may write (115) as

$$\begin{aligned} \frac{\partial^i \Theta}{\partial r^i}(\pi_1(x), d(x)) &= \frac{k!}{(k-i)!} |y_2|^{k-i} \beta(\pi_1(\tilde{h}(y))) \\ &= \frac{k!}{(k-i)!} |y_2|^{k-i} \beta(\tilde{h}(y_1, \frac{y_2}{|y_2|})) \\ &=: \frac{k!}{(k-i)!} |y_2|^{k-i} \tilde{\beta}(y_1, \frac{y_2}{|y_2|}) =: F_i(y). \end{aligned} \tag{116}$$

One can now check that the derivatives of F satisfy

$$|D^j F_i(y)| \leq C_{ij}|y_2|^{k-i-j}, \quad \forall y \in B_{10}^{N-k} \times B_2^k \setminus \{0\}, \quad \forall j \geq 1.$$

If we now apply the chain rule to (116), then the estimates above and the boundedness of the derivatives of $\tilde{h}^{-1}$ imply that (114) holds for $j \geq 1$.

Finally, estimates are readily checked using (114) and the fact that $\Theta \geq a > 0$ on $\partial\Xi_1 \times [0, 1]$. □

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