POSITIVE VERSUS FREE BOUNDARY SOLUTIONS TO A SINGULAR ELLIPTIC EQUATION

By

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Abstract. The equation

$$-\Delta u = \chi_{\{u>0\}}\left(-\frac{1}{u^\beta} + \lambda f(x,u)\right)$$

in Ω with Dirichlet boundary condition on $\partial\Omega$ has a maximal solution $u_\lambda \geq 0$ for every $\lambda > 0$. For λ less than a constant λ^*, the solution vanishes inside the domain; and for $\lambda > \lambda^*$, the solution is positive. We obtain optimal regularity of u_λ even in the presence of the free boundary.

1 Introduction

The elliptic problem

$$\left\{\begin{array}{rcll} -\Delta u &=& g_\lambda(x,u) & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{array}\right. \tag{1}$$

with a singular nonlinearity g_λ arises as limit of some equations modelling catalytic and enzymatic reactions; see [1] and [9] for an account. The domain $\Omega \subset \mathbb{R}^n$ is smooth and bounded. We are interested in studying (1) with

$$g_\lambda(x,u) = \chi_{\{u>0\}}\Big(-\frac{1}{u^\beta} + \lambda f(x,u)\Big), \tag{2}$$

by convention $g_\lambda(x,0) = 0$, where $\beta > 0$ is a constant and the function $f : \Omega \times [0,\infty) \to [0,\infty)$ is measurable in x, $f \not\equiv 0$, and is nondecreasing, concave and sublinear in the second variable u; the latter requirement is equivalent to

$$\lim_{u\to\infty} \frac{f(x,u)}{u} = 0 \quad \text{uniformly for } x \in \Omega.$$

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In addition, we assume that $f_u(x,\cdot)$ is continuous on $(0,\infty)$ for a.e. $x \in \Omega$. We say $u \geq 0$ is a weak solution of (1) if $u \in L^1(\Omega)$,

$$\chi_{\{u>0\}}\Big(-\frac{1}{u^\beta} + \lambda f(x,u)\Big)\delta \in L^1(\Omega),$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$, and for all $\varphi \in C^2(\overline{\Omega})$ with $\varphi = 0$ on $\partial\Omega$, we have

$$\int_\Omega u(-\Delta\varphi) = \int_{\{u>0\}} \Big(-\frac{1}{u^\beta} + \lambda f(x,u)\Big)\varphi.$$

By a positive classical solution we mean a function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ which is positive in Ω and satisfies (1) in the usual sense.

The distance function to the boundary $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ plays an important role in our arguments because of the interplay of the singular term $1/u^\beta$ and the zero Dirichlet boundary condition. If we drop the function f from the expression (2), in dimension $n = 1$ the equation (1) reduces to the ordinary differential equation $u'' = 1/u^\beta$ and the function $u(t) = t^{2/(1+\beta)}$ is a positive solution defined in $(0,\infty)$ with $u(0) = 0$. Intuitively, this suggests that in higher dimensions there may exist positive solutions u of (1) whose behavior near the boundary is like $\delta^{2/(1+\beta)}$. Therefore, looking at the principal part of the linearized operator of (1) at such a solution, one finds

$$-\Delta - c\frac{1}{\delta^2}$$

for some constant $c > 0$. In fact, the singular potential $c\delta^{-2}$ plays a crucial role in the stability of such a solution and is related to the Hardy inequality

$$\int_\Omega \frac{\varphi^2}{\delta^2} \leq C\int_\Omega |\nabla\varphi|^2,$$

which is valid for all $\varphi \in C_c^\infty(\Omega)$; see [4]. This is similar to what happens to the minimal solution in some semilinear elliptic equations with a convex nonlinearity $g_\lambda(x,u) = \lambda f(u)$. Standard examples are $f(u) = e^u$ and $f(u) = (1+u)^p$ for $p > 1$; see [12, 11, 15, 7, 3, 5, 14]. One of the main features is that classical solutions exist for $0 < \lambda < \tilde{\lambda}$. A result in [14] states that when $\lambda = \tilde{\lambda}$, there is a unique weak solution $\tilde{u}$, which may be singular and which is called the extremal solution. For $\lambda > \tilde{\lambda}$, there is no weak solution; see [3]. If, for instance, $f(u) = e^u$ and Ω is the unit ball $B_1(0)$ in $\mathbb{R}^n$ with $n \geq 10$, then the extremal solution is known explicitly by the formula $\tilde{u}(x) = -2\log(|x|)$; see [13]. So the linearized operator at $\tilde{u}$ is

$$-\Delta - c\frac{1}{|x|^2}.$$

The linearized stability of this solution is then related to another Hardy inequality, namely

$$\frac{(n-2)^2}{4}\int_\Omega \frac{\varphi^2}{|x|^2} \le \int_\Omega |\nabla\varphi|^2$$

for all $\varphi \in C_c^\infty(\Omega)$; see [5].

In the present work, we determine a constant $\lambda^* > 0$ such that problem (1) admits a positive solution for $\lambda > \lambda^*$ and no positive weak solution exists if $0 < \lambda < \lambda^*$ (see Theorem 2.1 below). But, in contrast to the above mentioned convex nonlinearity, if $0 < \lambda < \lambda^*$, problem (1) has solutions which vanish on a set of positive measure, exhibiting a free boundary.

Let us mention some already known results for particular cases of equation (1). In [10] and [6], the authors studied the problem where f is bounded and depends only on x. They proved some results on existence, uniqueness and stability of solutions. A variational approach was carried out in [16] in order to obtain optimal regularity $C^{1,(1-\beta)/(1+\beta)}_{\text{loc}}(\Omega)$ for minimizers of the energy $\int \frac{1}{2}|\nabla u|^2 + (u^+)^{1-\beta}$ in the convex set $\{u \in H^1(\Omega) : u = 1 \text{ on } \partial\Omega\}$. One of the ideas behind the proof is that minimizers are preserved under certain scaling. This is not exactly the case for our problem (1); see Theorem 2.1. In [17], they studied the equation with $g_\lambda(x,u) = -K(x)/u^\beta + \lambda u^p$ with $0 < p < 1$, but only considered positive solutions. The weight K could change sign; but when $\inf_\Omega K > 0$, they found results similar to ours. Problems involving singular functions with different behavior from g_λ were addressed in [8] and [15].

2 Main results

We are in position to state our main results. By means of an approximation procedure, we construct the maximal solution u_λ of (1). The difficulty in obtaining its regularity is the presence of a free boundary for $0 < \lambda < \lambda^*$.

Theorem 2.1. *Assume $0 < \beta < 1$. Then there is a unique maximal weak solution u_λ to (1) for any $\lambda > 0$. Moreover, there exists $\lambda^* \in (0,\infty)$ such that for $\lambda > \lambda^*$, the maximal solution u_λ is positive in Ω and belongs to $C(\overline{\Omega}) \cap C^{1,\mu}_{\text{loc}}(\Omega)$ for all $0 < \mu < 1$. Moreover, $a\delta \le u_\lambda \le b\delta$ in Ω, where a, b are positive constants depending only on Ω, $\lambda > 0$ and f. If $f \in C^1(\overline{\Omega} \times [0,\infty)$, then actually u_λ is a classical solution.*

For $0 < \lambda \le \lambda^*$, the maximal solution u_λ has optimal regularity $C(\overline{\Omega}) \cap C^{1,\gamma}_{\text{loc}}(\Omega)$ with $\gamma = (1-\beta)/(1+\beta)$; and for $0 < \lambda < \lambda^*$, the set $\{u_\lambda = 0\}$ has positive measure.

The maximal solution u_λ is obtained as the (decreasing) limit of the maximal solutions $u_{\lambda,\varepsilon}$ to

$$(3)\qquad \begin{cases} -\Delta u + \dfrac{u}{(u+\varepsilon)^{1+\beta}} & = \lambda f(x,u) \quad \text{in } \Omega \\ \qquad\qquad\quad u & = 0 \qquad\qquad \text{on } \partial\Omega \end{cases}$$

as $\varepsilon \to 0$. This approach is inspired by the work in [9]. We also prove that $u_{\lambda,\varepsilon} \to u_\lambda$ uniformly as $\varepsilon \to 0$; see Proposition 3.9. First we show that u_ε converges pointwisely to the maximal subsolution u of the problem

$$(4)\qquad \begin{cases} -\Delta u + \chi_{\{u>0\}}\dfrac{1}{u^\beta} & = \lambda f(x,u) \quad \text{in } \Omega, \\ \qquad\qquad\quad u & = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases}$$

We progressively regularize this maximal subsolution, obtaining precise estimates of its derivatives, similarly to [16]. These estimates allow us to verify that the subsolution u satisfies (1). Our problem (1) can be viewed as a perturbation by λf of the minimization studied in [16], because for parameter values $0 < \lambda < \lambda^*$ the maximal solution u_λ possesses a free boundary, so it vanishes on a set of positive measure. But $u_\lambda > 0$ for $\lambda > \lambda^*$; see the proof of Lemma 3.11. The borderline is for $\lambda = \lambda^*$, where $u_{\lambda^*} > 0$ a.e., but it could be positive (not only a.e.) according to Theorem 2.4; see also Remark 2.2 (C). Therefore, every solution of (1) vanishes on a set for $\lambda \leq \lambda^*$. Nothing prevents us from having a solution of (1) different from u_λ for $\lambda > \lambda^*$ and vanishing somewhere, but the maximal one is positive.

Remark 2.2. (A) The weak solutions of (1) belong to $H_0^1(\Omega)$; see [10] and [6] for alternative proofs.

(B) Any weak solution u satisfies $u^{-\beta}\chi_{\{u>0\}} \in L^1(\Omega)$ (and not just $u^{-\beta}\chi_{\{u>0\}}\delta \in L^1(\Omega)$).

(C) We define the extremal solution as $u^* = u_{\lambda^*}$. As we shall see, u^* is positive a.e. in Ω, although it can vanish at some points in Ω (this makes sense because it is continuous). We make this more precise in terms of β in Theorem 2.4 and show that in some circumstances $u^* > 0$ in Ω. The optimality of this situation is discussed in Example 2.5.

(D) The extremal solution u^* is unique in the class of weak solutions which are positive a.e. in Ω. A similar result in [14] deals with convex nonlinearities.

(E) For $\beta \geq 1$ and any $\lambda \geq 0$, there is no weak solution of (1) which is positive a.e. in Ω. This statement was already proved in less generality in [6].

For the sake of completeness, we give a proof of the above statements in Section 4.

The question of stability of the maximal solution u_λ for $\lambda \geq \lambda^*$ leads us to define, for a function $u \in L^1_{\text{loc}}(\Omega)$, $u > 0$ a.e. in Ω, the expression

$$\Lambda(u) = \inf_{\varphi \in C_c^\infty(\Omega)} \frac{\displaystyle\int_\Omega |\nabla\varphi|^2 - \frac{\partial g_\lambda}{\partial u}(x,u)\varphi^2}{\displaystyle\int_\Omega \varphi^2}. \tag{5}$$

Note that $\frac{\partial g_\lambda}{\partial u}(x,u)$ contains the term $u^{-1-\beta}$; thus for a general $u > 0$ a.e. $\Lambda(u)$ makes sense, but can be $-\infty$. This is the first eigenvalue of the linearization of problem (1).

Theorem 2.3. *Assume* $0 < \beta < 1$. *For* $\lambda > \lambda^*$, *the maximal solution* u_λ *of (1) is stable, that is,* $\Lambda(u_\lambda) > 0$. *For* $\lambda = \lambda^*$, *the extremal solution* u^* *is weakly stable, in the sense that* $\Lambda(u^*) \geq 0$. *Conversely, if* u *is a weak solution of (1) for some* $\lambda \geq \lambda^*$ *such that* u *is positive a.e. and* $\Lambda(u) \geq 0$, *then* u *coincides with the maximal solution (i.e.,* $u = u_\lambda$*).*

The stability property allow us to obtain the positivity for the extremal solution u^* under some restrictions on β.

Theorem 2.4. *Let* $\beta \in (0,1)$. *There exists* $c > 0$ *such that* $u^* \geq c\delta^{2/(1+\beta)}$ *if one assumes*

$$\frac{3\beta + 1 + 2\sqrt{\beta^2+\beta}}{\beta+1} > \frac{n}{2}. \tag{6}$$

In particular, u^* *is positive in* Ω *(and not only a.e.).*

Our result appears to be close to optimal regarding the behavior of u^* near the boundary in view of the example that follows.

Example 2.5. There exists a function $f = f(x)$ such that problem

$$\begin{cases} -\Delta u + \frac{1}{u^\beta} &= f(x) & \text{in } A = \{r : R < r < 1\} \\ \qquad\quad u &= 0 & \text{on } \partial B_1 \\ \qquad\quad u &= c(1-R)^\alpha & \text{on } \partial B_R \end{cases}$$

has a solution $u \sim \delta^{2/(1+\beta)}$ near ∂B_1. More details are given in Section 6.

Theorem 2.4 also sheds some light on the problem we explain in the sequel. It is natural to ask whether or not there is a characterization for the maximal solution u_λ in terms of stability similar to Theorem 2.3 when $0 < \lambda < \lambda^*$. The situation in the range $0 < \lambda < \lambda^*$ is more delicate, because the maximal solution

u_λ vanishes in parts of the domain, and therefore any solution to (1) vanishes on a set of positive measure. In the same spirit, whenever $\lambda \geq \lambda^*$ one may ask whether the characterization of the maximal solution given in Theorem 2.3 is valid for any solution (not known a priori to be positive a.e.). One possible approach would be to say that a solution $u \in C(\Omega)$ to (1) is weakly stable if

$$\int_\omega \frac{\partial g_\lambda}{\partial u}(x,u)\varphi^2 \leq \int_\omega |\nabla\varphi|^2, \quad \varphi \in C_c^\infty(\omega), \tag{7}$$

where ω is the open set

$$\omega = \{x \in \Omega : u(x) > 0\}.$$

Assume now that $u \in C(\Omega)$ is a weakly stable solution of (1) in the sense of (7). Is it true that it has to be the maximal solution? It turns out that the answer is negative in general.

Example 2.6. Let Ω be the interval $(-2,2)$. There exists a smooth function $f = f(x)$ and a continuous solution u to (1) in Ω with $\lambda = 1$, such that $u > 0$ in $(-2,0) \cup (0,2)$, but $u(0) = 0$. Moreover u satisfies the condition (7), but u is *not* the maximal solution. Indeed, first note that $\lambda^* \leq 1$ because $u > 0$ a.e. If $\lambda^* = 1$, then by Remark 2.2 (D) (uniqueness of u^*) we would infer that $u^* = u$, which is not possible by Theorem 2.4. Hence $\lambda^* < 1$; and then u cannot be the maximal solution u_λ, because $u(0) = 0$ and $u_\lambda \geq a\delta$ (with $a > 0$). See the explicit computations in Section 6.

Remark 2.7. The stability of the maximal solution for $\lambda \geq \lambda^*$ implies that the map $\lambda \mapsto u_\lambda$ is continuous for $\lambda \geq \lambda^*$, considered as a map from $(0,\infty) \subset \mathbb{R}$ to $L^1(\Omega)$.

It is natural then to ask whether $\lambda \mapsto u_\lambda$ is continuous for *all* $\lambda > 0$. We can easily show that u_λ is continuous from the right. This conclusion follows from the characterization of u_λ as the unique maximal subsolution to (4); see Corollary 3.8. On the other hand, if $\lambda_k \nearrow \lambda$ with $\lambda_k < \lambda$, the increasing limit $u = \lim_{\lambda_k \nearrow \lambda} u_{\lambda_k}$ exists and is a subsolution of (4). But is it the maximal one? The answer is negative in general, and examples can be easily constructed by applying the next proposition. For instance, take Ω to be the interval $(0,1)$ and $f(u) \equiv 1$. From the proposition below, one concludes that $u_\lambda \equiv 0$ for all $0 < \lambda < \lambda^*$; but Theorem 2.4 says that $u^* > 0$ in Ω. Hence the branch $\lambda \mapsto u_\lambda$ of maximal solutions has a discontinuity at λ^*. In addition, it is easy to deduce from the iterative scheme of Lemma 3.1 that the branch $\lambda \mapsto u_\lambda$ is nondecreasing.

Proposition 2.8. *Assume Ω is an interval in $\mathbb{R}$ and that f depends only on u. Then, for any $\lambda > 0$, the maximal solution is either identically zero or positive in Ω.*

A similar statement can be found in [10], where they claim that if Ω is an interval in $\mathbb{R}$ and $f \equiv 1$, any minimizer of the corresponding energy is either zero or positive in Ω.

3 The maximal solution and its regularity

The proof of Theorem 2.1 is divided into a series of lemmas. We start by constructing a maximal subsolution of problem (4). It turns out that this maximal subsolution is indeed the maximal solution u_λ of (1).

Throughout this section, $\beta \in (0,1)$ and $\alpha = 2/(1+\beta)$.

Lemma 3.1. *There exists a unique maximal subsolution of* (4).

Proof. The perturbed problem (3) has a unique maximal solution u_ε (we omit the dependence on λ). First observe that there exists a fixed maximal supersolution $\overline{U}$ of (3) (independent of ε); just take $\overline{U} = kY$ with k sufficiently large constant, where Y denotes the solution to

$$\begin{cases} -\Delta Y &= 1 \quad \text{in } \Omega, \\ \quad Y &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{8}$$

The existence of a solution to (3) is clear; beginning with $u_0 = \overline{U}$, the sequence u_n of solutions of

$$\begin{cases} -\Delta u_n + \frac{u_n}{(u_{n-1}+\varepsilon)^{1+\beta}} &= \lambda f(x, u_{n-1}) \quad \text{in } \Omega \\ \qquad\qquad u_n &= 0 \qquad\qquad\quad \text{on } \partial\Omega \end{cases}$$

converges monotonically to a solution u_ε of (3). Moreover, u_ε is the maximal solution to (3) in $[0, \overline{U}]$. Choosing k larger if necessary, any subsolution of $\underline{u}$ to the equation (3) satisfies $\underline{u} \le \overline{U}$; and hence u_ε is the maximal solution to (3).

Observe that if $0 < \varepsilon_1 < \varepsilon_2$ we have $u_{\varepsilon_1} \le u_{\varepsilon_2}$. Therefore, the pointwise limit

$$u = \lim_{\varepsilon \to 0} u_\varepsilon$$

exists. We claim that u is a maximal subsolution to (4). Indeed, take $\varphi \in C^2(\overline{\Omega})$, $\varphi \ge 0$, $\varphi = 0$ on $\partial\Omega$. Then

$$\int_\Omega u_\varepsilon(-\Delta\varphi) + \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon + \varepsilon)^{1+\beta}}\varphi = \lambda \int_\Omega f(x, u_\varepsilon)\varphi. \tag{9}$$

Observe that

$$\liminf_{\varepsilon \to 0} \frac{u_\varepsilon}{(u_\varepsilon + \varepsilon)^{1+\beta}} \ge \frac{1}{u^\beta}\chi_{\{u>0\}}.$$

Using Fatou's lemma on the left hand side and dominated convergence on the right hand side of (9), we obtain

$$\int_\Omega u(-\Delta\varphi) + \int_{\{u>0\}} \frac{1}{u^\beta}\varphi \le \lambda \int_\Omega f(x,u)\varphi.$$

It is now easy to conclude that u is the maximal subsolution of (4). Observe that any subsolution v of (4) is also a subsolution of (3) and therefore $v \le u_\varepsilon$ for all $\varepsilon > 0$, implying $v \le u$. □

The maximal subsolution u of (4) is regular. First, we prove that it is continuous in $\overline{\Omega}$. This fact can be derived from the following local estimates of u.

Lemma 3.2. *Let u be the maximal subsolution of* (4)*. For every ball $B_r(p) \subset \Omega$, there exist constants $c_0, \tau > 0$ depending only on n and β such that*

$$u \ge \tau \fint_{\partial B_r(p)} u \qquad \text{a.e. in } B_{r/2}(p)$$

whenever

$$\fint_{\partial B_r(p)} u \ge c_0 r^\alpha.$$

We state as a separate result the first step for proving Lemma 3.2; see [16] for a proof.

Lemma 3.3. *Let $B = B_1(0)$ be the unit ball in $\mathbb{R}^n$ and let $\tilde{u} \in H^{1/2}(\partial B)$. There exist positive constants c_0 and $c_1 > 0$ (both depending only on n and β) such that if*

$$\fint_{\partial B} \tilde{u} \ge c_0,$$

then there exists a solution $w \in H^1(B)$ of

$$\begin{cases} -\Delta w + \frac{1}{w^\beta} &= 0 & \text{in } B, \\ \qquad\quad w &\ge c_1(1-|x|)\fint_{\partial B_r} \tilde{u} & \text{in } B, \\ \qquad\quad w &= \tilde{u} & \text{on } \partial B. \end{cases} \tag{10}$$

Proof of Lemma 3.2. We assume that $p = 0$ and rescale u by defining

$$\tilde{u}(y) = r^{-\alpha} u(ry).$$

Hence $\tilde{u}$ satisfies the inequality

$$-\Delta\tilde{u} + \chi_{\{\tilde{u}>0\}} \frac{1}{\tilde{u}^\beta} \le \lambda r^{2-\alpha} f(ry, r^\alpha \tilde{u}(y)) \quad \text{in } B_1.$$

Observe that

$$\fint_{\partial B_1} \tilde{u} = r^{-\alpha} \fint_{\partial B_r} u.$$

By Lemma 3.3, if

$$\fint_{\partial B_1} \tilde{u} \geq c_0,$$

there exists $w \in H^1(B)$ satisfying (10). Therefore, the problem

$$\begin{cases} -\Delta \tilde{v} + \dfrac{1}{\tilde{v}^\beta} & = g(y, \tilde{v}) \quad \text{in } B_1 \\ \qquad \tilde{v} & = \tilde{u} \qquad\quad\ \text{on } \partial B_1 \end{cases}$$

has a maximal solution $\tilde{v} \geq w$; here we have used the notation

$$g(y, v) = \lambda r^{2-\alpha} f(ry, r^\alpha v).$$

We rescale back $\tilde{v}$, i.e., define

$$v(x) = r^\alpha \tilde{v}(x/r)$$

and set

$$z(x) = \begin{cases} u(x) & \text{if } x \in \Omega \setminus B_r, \\ v(x) & \text{if } x \in B_r. \end{cases}$$

We claim that z is a subsolution of (4). Indeed, let $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. Then

$$\begin{aligned} \int_\Omega \nabla z \nabla \varphi &= \int_\Omega \nabla u \nabla \varphi + \int_{B_r} \nabla (v - u) \nabla \varphi \\ &= \int_\Omega \Big(-\chi_{\{u>0\}} \frac{1}{u^\beta} + \lambda f(x, u) \Big) \varphi + \int_{\partial B_r} \frac{\partial (v-u)}{\partial \nu} \varphi \\ &\quad + \int_{B_r} \Big(-\frac{1}{v^\beta} + \lambda f(x, v) + \chi_{\{u>0\}} \frac{1}{u^\beta} - \lambda f(x, u) \Big) \varphi \\ &\leq \int_\Omega \Big(-\chi_{\{z>0\}} \frac{1}{z^\beta} + \lambda f(x, z) \Big) \varphi. \end{aligned}$$

In the previous computation, we have proceeded formally, since at this point it is not known whether $v - u$ has a normal derivative on ∂B_r. But the calculation is justified by the lemma below (with $w = v - u$). Since u is the maximal subsolution, we have $u \geq v$ on B_r, and therefore

$$u \geq \frac{1}{2} c_1 \fint_{\partial B_r} u \quad \text{in } B_{r/2}.$$

□

The next result completes the proof of Lemma 3.2.

Lemma 3.4. *Let $w \in W^{1,1}(B_1)$ be such that $w \geq 0$ a.e. on B_1 and $w = 0$ on ∂B_1. Suppose that $\Delta w \in L^1(B_1)$. Then $\partial w/\partial \nu \leq 0$ in the sense that*

$$\int_{B_1} \varphi \Delta w + \nabla w \nabla \varphi \leq 0 \quad \forall \varphi \in C^1(\overline{B}_1), \varphi \geq 0. \tag{11}$$

Proof. Assume for a moment that w is smooth in B_1, and let $\varphi \in C^1(\overline{B}_1)$. Then for $0 < r < 1$,

$$\begin{aligned}\frac{d}{dr} \fint_{\partial B_r} w\varphi &= \fint_{\partial B_r} \frac{\partial (w\varphi)}{\partial \nu} \\ &= \frac{c_n}{r^{n-1}} \int_{B_r} \Delta(w\varphi) \\ &= \frac{c_n}{r^{n-1}} \left\{ \int_{B_r} \varphi \Delta w + \nabla w \nabla \varphi + \int_{\partial B_r} w \frac{\partial \varphi}{\partial \nu} \right\}.\end{aligned}$$

Integrating over $r \in (R, 1)$ with $0 < R < 1$ yields

$$\fint_{\partial B_1} w\varphi - \fint_{\partial B_R} w\varphi = \int_R^1 \frac{c_n}{r^{n-1}} \left\{ \int_{B_r} \varphi \Delta w + \nabla w \nabla \varphi + \int_{\partial B_r} w \frac{\partial \varphi}{\partial \nu} \right\} dr. \tag{12}$$

By approximation, this relation holds also for a function w as in the statement of the lemma. If we assume now that $\varphi \geq 0$, since $w \geq 0$, $w = 0$ on ∂B_1 (12) implies that

$$\int_R^1 \frac{c_n}{r^{n-1}} \left\{ \int_{B_r} \varphi \Delta w + \nabla w \nabla \varphi + \int_{\partial B_r} w \frac{\partial \varphi}{\partial \nu} \right\} dr \leq 0. \tag{13}$$

Observe that by the hypotheses on w, the quantity in brackets is continuous in r. So, dividing (13) by $1 - R$ and letting $R \nearrow 1$, we conclude that (11) holds. $\square$

The continuity of u can now be achieved.

Corollary 3.5. *The maximal subsolution u of* (4) *is continuous in $\overline{\Omega}$ (up to redefinition on a set of measure 0). Moreover, u belongs to $C^{1,\mu}$ for any $0 < \mu < 1$ restricted to the open set $\omega = \{u > 0\}$ and satisfies*

$$-\Delta u + \frac{1}{u^\beta} = \lambda f(x, u) \quad \text{in } \omega.$$

Proof. We work with the following precise representative of u:

$$U(x) = \lim_{r \to 0} \fint_{\partial B_r(x)} u.$$

Since $u \in H_0^1(\Omega)$, the limit exists a.e. and $U = u$ a.e. in Ω. Note that U is upper semicontinuous. Indeed, one can write $u = v + w$, where w solves

$$\begin{cases} -\Delta w &= \lambda f(x, u) \quad \text{in } \Omega \\ \quad w &= 0 \quad\quad\quad\quad \text{on } \partial\Omega \end{cases} \tag{14}$$

and therefore

$$\begin{cases} -\Delta v &= -\chi_{\{u>0\}}\dfrac{1}{u^\beta} - \chi_{\{u=0\}}\lambda f(x,u) \quad \text{in } \Omega, \\ \quad v &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{15}$$

Since u is bounded, $w \in C^{1,\mu}(\overline{\Omega})$ for every $0 < \mu < 1$. Therefore,

$$U(x) = w(x) + \lim_{r\to 0} \fint_{\partial B_r(x)} v.$$

But v is subharmonic, and thus the previous limit is actually an infimum. Since for fixed r the function

$$x \to \fint_{\partial B_r(x)} v$$

is continuous, we see that U is upper semicontinuous.

Let $q \in \Omega$ be such that $U(q) > 0$. Then for some $r > 0$ small enough

$$\fint_{\partial B_r(q)} u \geq c_0 r^\alpha,$$

and by Lemma 3.2

$$u(x) \geq \tau \fint_{\partial B_r(q)} u > 0 \quad \text{in } B_{r/2}(q).$$

This shows that the set $\omega = \{U > 0\}$ is open. Furthermore, by (14) and (15), u satisfies

$$-\Delta u + \frac{1}{u^\beta} = \lambda f(x,u) \quad \text{in } B_{r/2}(q).$$

Therefore, it is (up to a representative) $C^{1,\mu}$ in $B_{r/2}(q)$ for all $0 < \mu < 1$.

Finally, we show that U is continuous in $\overline{\Omega}$. We start by showing that U is continuous at $p \in \partial\omega \cap \Omega$.

Before proving this, observe that if a function $h \in C^{1,\mu}(B_1)$ for some $0 < \mu < 1$, then

$$\left|\fint_{\partial B_r} h - h(0)\right| \leq |Dh|_{\mu,B_1} r^{1+\mu}, \tag{16}$$

where $|.|_{\mu,B_1}$ is the Hölder semi-norm defined by

$$|Dh|_{\mu,B_1} := \sup_{x,y\in B_1} \frac{|Dh(x) - Dh(y)|}{|x-y|^\mu}.$$

This follows easily from the fact that for $z \in \partial B_r$ one has $h(z) - h(0) = Dh(\xi)z$ for some ξ in the segment $\overline{0z}$. Take now $\fint_{\partial B_r}$ in the expression

$$\begin{aligned} h(z) - h(0) - Dh(0)z &\leq \left|(Dh(\xi) - Dh(0))z\right| \\ &\leq |Dh|_{\mu,B_1} r^{1+\mu} \end{aligned}$$

to get (16).

Recall the decomposition $u = v + w$, where v is subharmonic and $w \in C^{1,\mu}(\overline{\Omega})$ for $\mu \in (0,1)$. Fix $\mu \in (0,1)$, let $x \in \Omega$ with $|x-p| < \operatorname{dist}(p, \partial\Omega)/2$, and set $R = |x-p|$. Then, since v is subharmonic, using (16) we get, for $0 < r < R$,

$$\begin{aligned} u(x) &= v(x) + w(x) \\ &\leq \fint_{\partial B_r(x)} (v+w) + Cr^{1+\mu} \\ &= \fint_{\partial B_r(x)} u + Cr^{1+\mu}. \end{aligned}$$

Multiplying the previous relation by r^{n-1} and integrating over $r \in (0,R)$ yields

$$\begin{aligned} u(x)|B_R| &\leq \int_{B_R(x)} u + CR^{n+1+\mu} \\ &\leq \int_{B_{2R}(p)} u + CR^{n+1+\mu}. \end{aligned} \tag{17}$$

But, if

$$\fint_{\partial B_r(p)} u \geq c_0 r^\alpha$$

for some $r \in (0, 2R)$, then $U(p) > 0$ by Lemma 3.2, which is impossible. Hence, integrating, we obtain

$$\int_{B_{2R}(p)} u \leq CR^{\alpha+n};$$

and combining with (17), we get

$$u(x) \leq CR^\alpha + CR^{1+\mu} \leq C|x-p|^\alpha.$$

If $p \in \partial\Omega$, then, since $u \leq C\delta$, we have

$$\sup_{B_r(p)\cap\Omega} u \leq C \sup_{B_r(p)\cap\Omega} \delta \to 0$$

as $r \to 0$. □

We need better local estimates on the derivatives of u in order to obtain the precise regularity.

Lemma 3.6. *Let u be the maximal subsolution of* (4). *Then for all $\Omega' \subset\subset \Omega$, we have*

$$|Du| \leq Cu^{(1-\beta)/2} \quad \text{in } \Omega',$$

where C depends only on $\operatorname{dist}(\Omega', \partial\Omega)$, *$n$, β,* $\sup_\Omega u$ *and* $\sup_\Omega f(x, u(x))$.

Moreover, if $f \in C^1(\overline{\Omega} \times [0, \infty))$ *then*

$$|D^2 u| \le C u^{-\beta} \quad \text{in } \Omega'$$

and C *depends also on* f.

Proof. We write $u = v + w$, where w is the solution of (14) and v is subharmonic, as in Corollary 3.5.

Let $p \in \Omega'$ and assume $p = 0$, $u(p) > 0$. For $s > 0$, define

$$u_s(y) = s^{-\alpha} u(sy)$$

and define w_s, v_s accordingly. Note that

$$Du_s(y) = s^{1-\alpha} Du(sy).$$

Let us fix $s > 0$ so that

$$u_s(0) = s^{-\alpha} u(0) = 2c_0,$$

where c_0 is the constant from Lemma 3.2. Note that $s \le s_0$, where $s_0 = \left(u(0)/2c_0\right)^{1/\alpha}$ depends only on c_0 and $\|u\|_{L^\infty(\Omega')}$.

Recall that we have a bound $\|w\|_{C^{1,\mu}(\overline{\Omega})} \le C$ for any $0 < \mu < 1$. We fix here some $\mu \in (\alpha - 1, 1)$. In particular (see analogous notation (16)),

$$|Dw|_{\mu,\Omega} \le C.$$

For w_s we then have

$$|Dw_s|_{\mu,\Omega/s} \le s^{1-\alpha+\mu} C$$

(note that the domain of w_s is Ω/s).

Let

$$r_0 = \min(\operatorname{dist}(\Omega', \partial\Omega), 1)$$

and consider some $0 < r \le r_0$, which will be fixed later, depending only on $\operatorname{dist}(\Omega', \partial\Omega)$, n, β, $\sup_\Omega u$, $\sup_\Omega f(x, u(x))$. Then, using the subharmonicity of v and (16),

$$\begin{aligned}
\fint_{\partial B_r} u_s &= \fint_{\partial B_r} v_s + \fint_{\partial B_r} w_s \\
&\ge v_s(0) + w_s(0) - |Dw_s|_{\mu,\Omega/s} r^{1+\mu} \\
&= u_s(0) - |Dw|_{\mu,\Omega}\, s^{1-\alpha+\mu}\, r^{1+\mu} \qquad (18)\\
&\ge 2c_0 - |Dw|_{\mu,\Omega}\, s_0^{1-\alpha+\mu}\, r^{1+\mu} \\
&\ge c_0 r^\alpha \Big(2 - \frac{|Dw|_{\mu,\Omega}}{c_0} (s_0 r)^{1-\alpha+\mu}\Big).
\end{aligned}$$

Now fix r so small that

$$2 - \frac{|Dw|_{\mu,\Omega}}{c_0}(s_0 r)^{1-\alpha+\mu} \geq 1;$$

we see that this choice depends only on the quantities as claimed. Then by Lemma 3.2, we have

$$u_s \geq r \fint_{\partial B_r} u_s \quad \text{in } B_{r/2}. \tag{19}$$

Define h by

$$\begin{cases} \Delta h &= 0 \quad \text{in } B_{r/2}, \\ h &= u_s \quad \text{on } \partial B_{r/2}. \end{cases}$$

We claim that

$$\fint_{\partial B_{r/2}} h = \fint_{\partial B_{r/2}} u_s \leq C.$$

Indeed, similarly to (18), we have

$$\begin{aligned} \fint_{\partial B_{r/2}} u_s &\geq v_s(0) + w_s(0) - |Dw|_{\mu,\Omega}\, s^{1+\mu-\alpha}(r/2)^{1+\mu} \\ &\geq u_s(0) - |Dw|_{\mu,\Omega}\, s_0^{1+\mu-\alpha}(r/2)^{1+\mu} \\ &\geq c_0(r/2)^\alpha\Big(2 - \frac{1}{c_0}|Dw|_{\mu,\Omega}\,\big(\tfrac{1}{2}s_0 r)\big)^{1-\alpha+\mu}\Big) \\ &\geq c_0(r/2)^\alpha, \end{aligned}$$

by our choice of r. Hence, by Lemma 3.2,

$$2c_0 = u_s(0) \geq r \fint_{\partial B_{r/2}} u_s.$$

Since r has been fixed, we obtain the bound

$$\fint_{\partial B_{r/2}} h = \fint_{\partial B_{r/2}} u_s \leq C.$$

Then, by standard properties of harmonic functions,

$$|Dh|, |D^2 h| \leq C \quad \text{in } B_{r/4}. \tag{20}$$

Let now z be the solution of

$$\begin{cases} -\Delta z &= -\dfrac{1}{u_s^\beta} + \lambda s^{2-\alpha} f(sy, s^\alpha u_s(y)) \quad \text{in } B_{r/2}, \\ z &= 0 \quad \text{on } \partial B_{r/2}. \end{cases} \tag{21}$$

By (18) and (19), the right hand side of (21) is bounded in L^∞ by a constant. Hence, by Schauder estimates,

$$|Dz| \le C \quad \text{in } B_{r/2}.$$

If, moreover, $f \in C^1(\overline{\Omega} \times [0,\infty))$, we have a bound for the first derivatives of the right hand side of (21), i.e.,

$$|D(s^{2-\alpha} f(sy, s^\alpha u_s(y)))| \le C; \tag{22}$$

and then

$$|D^2 z| \le C \quad \text{in } B_{r/2}.$$

Combining (20) and (22), we see that

$$|Du_s| \le C \quad \text{in } B_{r/4}; \tag{23}$$

and if $f \in C^1(\overline{\Omega} \times \mathbb{R}^+)$, then

$$|D^2 u_s| \le C \quad \text{in } B_{r/4}.$$

In particular, $|Du_s(0)| \le C$ and if $f \in C^1$ also $|D^2 u_s(0)| \le C$. By the definition of u_s, this yields the result. □

Putting the previous estimates together, we obtain the best regularity.

Lemma 3.7. *The maximal subsolution u of* (4) *belongs to $C^{1,\gamma}_{\text{loc}}(\Omega)$ where $\gamma = (1-\beta)/(1+\beta)$.*

Proof. Let $\Omega' \subset\subset \Omega$. We use notation similar to that of the previous lemma, i.e., fix $\mu \in (\alpha - 1, 1)$, set

$$r_0 = \min\left(\operatorname{dist}(\Omega', \partial\Omega)/8, 1\right),$$
$$s_0 = \left(\frac{\max_\Omega u}{2c_0}\right)^{1/\alpha},$$

and fix $0 < r < r_0$ so that

$$2 - \frac{1}{c_0} |Dw|_{\mu,\Omega}\, (s_0 r)^{1+\mu-\alpha} \ge 1.$$

Pick $x, y \in \Omega'$, $x \ne y$, and let us show that

$$|Du(x) - Du(y)| \le C|x-y|^\gamma.$$

There are two cases to consider.

Case 1. For either x or y, and to fix the notation say for x, we have

$$u(x) \geq 2c_0\Big(\frac{8}{r}|x-y|\Big)^{\alpha}. \tag{24}$$

By translating, we can assume that $x = 0$. Let $s = (u(0)/2c_0)^{1/\alpha}$ and define u_s as in Lemma 3.6. In that lemma we showed (23), but Schauder estimates also imply

$$|Du_s|_{\mu,B_{r/4}} \leq C, \tag{25}$$

where C is a constant only depending on n, β, $\mathrm{dist}(\Omega', \partial\Omega)$, $\max_\Omega u$ and $\max_{x\in\Omega} f(x, u(x))$.

Observe that by (24) we have

$$s = \Big(\frac{u(0)}{2c_0}\Big)^{1/\alpha} \geq \frac{8}{r}|x-y|;$$

and since $x = 0$ by our translation, we have

$$|y/s| \leq r/8. \tag{26}$$

We can thus apply (25) to the points $0 = x/s$ and y/s to conclude that

$$|Du_s(0) - Du_s(y/s)| \leq C|y/s|^{\mu}$$

and hence

$$|Du(0) - Du(y)| \leq C|y|^{\alpha-1}(|y|/s)^{1-\alpha+\mu}.$$

By using (26) in the previous estimate, and observing that $\alpha-1=\gamma$ and $1-\alpha+\mu > 0$, we arrive at

$$|Du(0) - Du(y)| \leq C|y|^{\gamma}.$$

Case 2. We have

$$u(x), u(y) \leq 2c_0\Big(\frac{8}{r}|x-y|\Big)^{\alpha}.$$

Then by Lemma 3.6,

$$|Du(x)| \leq Cu(x)^{(1-\beta)/2} \leq C|x-y|^{\gamma},$$

and for $Du(y)$ in a similar manner. □

The maximal subsolution of (4) is indeed a solution of (1).

Corollary 3.8. *The maximal subsolution u of* (4) *satisfies (1).*

Proof. We know that $u \in H_0^1(\Omega) \cap C^{1,\gamma}_{\mathrm{loc}}(\Omega)$ with $\gamma = (1-\beta)/(1+\beta)$ and also that u satisfies the equation (4) in the open set $\omega = \{u > 0\}$. If ω had a smooth boundary, then the statement of this lemma would be equivalent to saying that $\partial u/\partial \nu = 0$ on $\partial\omega \cap \Omega$, with ν the unit outward normal vector to $\partial\omega$. This is indeed the case by continuity of Du. Since $\partial\omega$ may not be smooth, we argue by approximation. Our aim is to show that

$$\int_\Omega \nabla u \nabla \varphi = \int_\omega \Big(-\frac{1}{u^\beta} + \lambda f(x,u)\Big)\varphi$$

for all $\varphi \in C_c^\infty(\Omega)$. Since $\nabla u = 0$ a.e. on the set $\Omega \setminus \omega$, this is equivalent to showing that

$$\int_\omega \nabla u \nabla \varphi = \int_\omega \Big(-\frac{1}{u^\beta} + \lambda f(x,u)\Big)\varphi.$$

Take $\varphi \in C_c^\infty(\Omega)$ and $\eta \in C_c^\infty(\omega)$. Then, using the equation in ω, we have

$$\int_\omega \nabla u \nabla \varphi \eta + \int_\omega \nabla u \nabla \eta \varphi = \int_\omega \Big(-\frac{1}{u^\beta} + \lambda f(x,u)\Big)\varphi\eta.$$

This is valid for all $\eta \in C_c^\infty(\omega)$; by approximation, it is also valid for $\eta_\theta = u^\theta$. Let us verify that

$$\int_\omega \nabla u \nabla \eta_\theta \varphi \to 0 \quad \text{as } \theta \to 0.$$

Indeed, we have $\nabla \eta^\theta = \theta u^{\theta-1}\nabla u$. Therefore,

$$\Big|\int_\omega \nabla u \nabla \eta_\theta \varphi\Big| \le \theta \|\varphi\|_\infty \int_{\omega\cap S} u^{\theta-1}|\nabla u|^2,$$

where $S = \mathrm{supp}(\varphi) \subset \Omega$. Using Lemma 3.6, we obtain

$$|\nabla u| \le C u^{(1-\beta)/2}$$

on S with C independent of θ. Hence

$$\Big|\int_\omega \nabla u \nabla \eta_\theta \varphi\Big| \le C\theta \|\varphi\|_\infty \int_{\omega\cap S} u^{\theta-\beta}.$$

Letting $\theta \to 0$ and recalling that $\chi_{\{u>0\}}u^{-\beta} \in L^1_{\mathrm{loc}}(\Omega)$, we obtain the result. □

A consequence of the above estimates is that we have nice convergence of the sequence u_ε defined as the maximal solution of (3) to u, the maximal solution of (1).

Proposition 3.9. *Let u_ε denote the maximal solution of* (3) *and u the maximal solution of (1). Then $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$.*

Proof. By the construction of u in Lemma 3.1, we know that $u_\varepsilon \to u$ pointwise; moreover, $u = \inf_{\varepsilon>0} u_\varepsilon$. Furthermore,

$$u_\varepsilon \to u \quad \text{in } H_0^1(\Omega). \tag{27}$$

Indeed, $\|u_\varepsilon\|_{H_0^1} \le C$ for some constant independent of $\varepsilon > 0$. Hence, for a sequence, we have $u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$. To conclude (27), we only need to verify that $\|u_\varepsilon\|_{H_0^1} \to \|u\|_{H_0^1}$. Multiplying (3) by u_ε and integrating on Ω, we find

$$\|u_\varepsilon\|_{H_0^1}^2 = \int_\Omega \lambda f(x,u_\varepsilon)u_\varepsilon - \frac{u_\varepsilon^2}{(u_\varepsilon+\varepsilon)^{1+\beta}}.$$

Recall that the sequence u_ε is uniformly bounded (by a fixed supersolution) and that

$$\frac{u_\varepsilon^2}{(u_\varepsilon+\varepsilon)^{1+\beta}} \le u_\varepsilon^{1-\beta}.$$

Thus, by dominated convergence,

$$\|u_\varepsilon\|_{H_0^1}^2 \to \int_\Omega \lambda f(x,u)u - u^{1-\beta} = \|u\|_{H_0^1}^2.$$

We write $u_\varepsilon = v_\varepsilon + w_\varepsilon$, where w_ε is the solution to

$$\begin{cases} -\Delta w_\varepsilon &= \lambda f(x,u_\varepsilon) \quad \text{in } \Omega, \\ w_\varepsilon &= 0 \qquad\qquad\ \ \text{on } \partial\Omega, \end{cases}$$

so that we have a uniform bound $\|w_\varepsilon\|_{C^{1,\mu}(\overline{\Omega})} \le C$, and v_ε is subharmonic.

Let $r > 0$ and $K = \{x \in \Omega \mid \operatorname{dist}(x,\partial\Omega) \ge 2r\}$. Then

$$\begin{aligned} \sup_K |u_\varepsilon - u| &= \sup_K u_\varepsilon - u \\ &= \sup_{x\in K} v_\varepsilon(x) + w_\varepsilon(x) - u(x) \\ &\le \sup_{x\in K} \fint_{\partial B_r(x)} (u_\varepsilon - u) + Cr^{1+\gamma}, \end{aligned}$$

where we have used that v is subharmonic, formula (16) for w_ε and u, the bound $\|w_\varepsilon\|_{C^{1,\mu}(\overline{\Omega})} \le C$ and $u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$ with $\gamma = (1-\beta)/(1+\beta)$. Letting $\varepsilon \to 0$ and using (27), we see that

$$\limsup_{\varepsilon\to 0} \sup_K |u_\varepsilon - u| \le Cr^{1+\gamma}.$$

On the other hand, recall that there exists $C > 0$ such that $u_\varepsilon \le C\delta$ in Ω. Therefore,

$$\sup_{\Omega\setminus K} u_\varepsilon - u \le \sup_{\Omega\setminus K} C\delta \le Cr;$$

and hence

$$\limsup_{\varepsilon\to 0}\sup_{\Omega}|u_\varepsilon - u| \le Cr^{1+\gamma} + Cr.$$

Since r was arbitrary, we conclude that

$$\limsup_{\varepsilon\to 0}\sup_{\Omega}|u_\varepsilon - u| = 0.$$

□

The next lemmas complete the proof of Theorem 2.1.

Lemma 3.10. *Problem (1) has no positive weak solution for $\lambda > 0$ small.*

Proof. Let φ_1 be the first eigenfunction of the Laplacian with zero Dirichlet boundary data. Multiplying (1) by φ_1 and integrating, we find

$$\begin{aligned}\lambda\int_\Omega f(x,u)\varphi_1 &= \int_\Omega \lambda_1 u\varphi_1 + \frac{\varphi_1}{u^\beta}\\ &\ge c\int_\Omega \varphi_1,\end{aligned}\tag{28}$$

where $c > 0$ is a constant such that

$$\lambda_1 u + \frac{1}{u^\beta} \ge c \quad \text{for all } u > 0.$$

But all solutions u remain bounded as $\lambda \to 0$; therefore, if they exist for all $\lambda > 0$, we get a contradiction from (28). □

We establish next the existence of a positive maximal solution which must coincide with the maximal solution u_λ found in Corollary 3.8.

Lemma 3.11. *Set*

$$\lambda^* = \inf\{\lambda > 0 : (1) \textit{ has a positive a.e. solution}\}.$$

Then $\lambda^* < \infty$; and for all $\lambda \ge \lambda^*$, (1) has a positive a.e. weak solution.

Proof. The method of sub- and supersolutions in the $L^1(\Omega)$ setting (see [3]) is well-suited to obtain the conclusions. We shall establish two claims:

(a) for $\lambda > 0$ large enough, there is a positive subsolution $\underline{U}$; and

(b) for any $\lambda > 0$, there is a supersolution $\overline{U}$ (with $\overline{U} \ge \underline{U}$).

To establish (a), note that $f(x, \delta^\alpha(x)) \ge 0$ and is not identically zero in Ω. Solve

$$\begin{cases} -\Delta\zeta_1 = f(x,\delta^\alpha) & \text{in } \Omega,\\ \zeta_1 = 0 & \text{on } \partial\Omega.\end{cases}$$

Then, for some constant $c_0 > 0$, we have

$$\zeta_1 \geq 2c_0 Y,$$

where Y is the solution to (8). Take

$$\zeta = \zeta_1 - c_0 Y \quad \text{and} \quad \underline{U} = k\zeta,$$

where k is to be chosen below. Then

$$-\Delta \underline{U} + \frac{1}{\underline{U}^\beta} = k\alpha\zeta^{\alpha-1}(f(x,\delta^\alpha) - c_0) + \zeta^{\alpha-2}\left(k^{-\beta} - k\alpha(\alpha-1)|\nabla\zeta|^2\right).$$

Using that $|\nabla\zeta|^2$ is bounded from below near $\partial\Omega$ and choosing k large enough, we have

$$k^{-\beta} - k\alpha(\alpha-1)|\nabla\zeta|^2 \leq 0$$

near $\partial\Omega$. In the interior of the domain, for k large enough, one obtains

$$-kc_0\alpha\zeta^{\alpha-1} + k^{-\beta}\zeta^{\alpha-2} \leq 0.$$

We only need to achieve

$$k\alpha\zeta^{\alpha-1}f(x,\delta^\alpha) \leq \lambda f(x,\underline{U}) = \lambda f(x,k\zeta^\alpha). \tag{29}$$

Using that $\delta \leq C\zeta$ for some $C > 0$, once k has been fixed we can choose λ large to obtain (29).

To prove (b), it suffices to find a supersolution $\overline{U}$ of (1) with $\delta \leq C\overline{U}$ for some large C. Consider $\overline{U} = MY$, where Y is the solution to (8). Then $\overline{U}$ is a supersolution of (1) provided

$$\lambda f(x, MY) \leq M. \tag{30}$$

But f is sublinear (uniformly in x), so (30) holds for sufficiently large M. The rest of the proof follows by an iterative scheme similar to that in the proof of Lemma 3.1. □

Lemma 3.12. *For all $\lambda > \lambda^*$, there are constants a, $b > 0$ (depending on λ) such that*

$$a\delta \leq u_\lambda \leq b\delta. \tag{31}$$

Proof. To prove the first inequality in (31), consider $\lambda' \in (\lambda^*, \lambda)$. Let $u_{\lambda'}$ denote the maximal solution of (1) with parameter λ'. We claim that for $\varepsilon > 0$

small enough, $w = u_{\lambda'} + \varepsilon\zeta$ is a subsolution of (1) with parameter λ, where ζ is the solution of

$$\begin{cases} -\Delta\zeta &= f(x, u_{\lambda'}) \quad \text{in } \Omega, \\ \zeta &= 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{32}$$

Note that $f(x, u_{\lambda'}) \geq 0$ and $f(x, u_{\lambda'}) \not\equiv 0$ (otherwise, $u_{\lambda'} \leq 0$ by the maximum principle). Hence $\zeta \geq c\delta$ for some $c > 0$.

We compute

$$\begin{aligned} -\Delta w + \frac{1}{w^\beta} &= -\frac{1}{u_{\lambda'}^\beta} + \frac{1}{(u_{\lambda'} + \varepsilon\zeta)^\beta} + \lambda' f(x, u_{\lambda'}) + \varepsilon f(x, u_{\lambda'}) \\ &\leq \lambda' f(x, u_{\lambda'}) + \varepsilon f(x, u_{\lambda'}) \\ &= \lambda f(x, w) + (\lambda' - \lambda + \varepsilon) f(x, u_{\lambda'}) + \lambda(f(x, u_{\lambda'}) - f(x, w)) \\ &\leq \lambda f(x, w) \end{aligned}$$

for $\varepsilon > 0$ small enough.

Let us establish the other estimate. Since f is sublinear in u, we can find $C > 0$ so that

$$\lambda f(u) \leq \frac{\lambda_1}{2} u + C \quad \text{for all } u \geq 0,$$

where λ_1 is the first eigenvalue of the Laplacian with zero Dirichlet boundary condition. Therefore, any solution u of (1) also satisfies

$$\begin{cases} -\Delta u - \frac{\lambda_1}{2} u &\leq C \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Take Z to be the solution of

$$\begin{cases} -\Delta Z - \frac{\lambda_1}{2} Z &= C \quad \text{in } \Omega, \\ Z &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Then $u \leq Z$ by the maximum principle, and $Z \leq C\delta$ by elliptic regularity. □

4 Proofs of the statements in Remark 2.2

(A) *A weak solution of (1) belongs to* $H_0^1(\Omega)$.

Proof. Let u be a weak solution of (1), and for $j > 0$ let

$$F_j(x) = \max\Big(-j, \chi_{\{u>0\}}\big(-\frac{1}{u^\beta} + \lambda f(x, u)\big)\Big),$$

Let u_j denote the solution of

$$\begin{cases} -\Delta u_j &= F_j \quad \text{in } \Omega, \\ u_j &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Note that $F_j \in L^\infty(\Omega)$ (because $u \le C\delta$), and so $u_j \in C^{1,\alpha}(\overline{\Omega})$. Observe also that $u_j \ge u \ge 0$. Multiplying the previous equation by u_j, we get

$$\begin{aligned}\int_\Omega |\nabla u_j|^2 &= \int_\Omega F_j u_j \\ &\le \lambda \int_\Omega f(u) u_j \\ &\le \lambda \int_\Omega f(u_j) u_j. \end{aligned} \tag{33}$$

But, using the sublinearity of f, we have

$$f(u)u \le \varepsilon u^2 + C \quad \text{for all } u > 0. \tag{34}$$

Therefore, combining (33) with (34) we obtain

$$\int_\Omega |\nabla u_j|^2 \le C.$$

Up to a subsequence u_j converges weakly in $H_0^1(\Omega)$, and the limit must be u. □

(B) *For any weak solution u, we have*

$$\|u^{-\beta}\chi_{\{u>0\}}\|_{L^1(\Omega)} \le C\lambda \|f(x, u(x))\|_{L^\infty(\Omega)}, \tag{35}$$

where C depends only on Ω.

Proof. We take as test function in the definition of weak solution $(\varphi_1+\varepsilon)^\gamma - \varepsilon^\gamma$, where φ_1 is the first eigenfunction for $-\Delta$ with zero Dirichlet boundary condition and $0 < \gamma, \varepsilon < 1$. Observe that this function belongs to $C^2(\overline{\Omega})$, vanishes on $\partial\Omega$, and satisfies

$$-\Delta\big((\varphi_1+\varepsilon)^\gamma - \varepsilon^\gamma\big) = \gamma\lambda_1(\varphi_1+\varepsilon)^{\gamma-1}\varphi_1 - \gamma(\gamma-1)(\varphi_1+\varepsilon)^{\gamma-2}|\nabla\varphi_1|^2 \ge 0.$$

Therefore,

$$\int_{\{u>0\}} \big((\varphi_1+\varepsilon)^\gamma - \varepsilon^\gamma\big)\frac{1}{u^\beta} \le \lambda \int_\Omega f(x,u)\big((\varphi_1+\varepsilon)^\gamma - \varepsilon^\gamma\big) \le C\lambda \|f(x,u(x))\|_{L^\infty(\Omega)}.$$

Taking $\varepsilon \to 0$ and then $\gamma \to 0$, and using Fatou's lemma, we obtain (35). □

(C) *The extremal solution* u^* *is positive a.e. in* Ω.

Proof. For $\lambda > \lambda^*$, we have $u_\lambda > 0$ in Ω by Lemma 3.12. Thus, by item (B),

$$\int_\Omega \frac{1}{u_\lambda^\beta} \le C$$

with C independent of λ (for λ near λ^*). Letting $\lambda \searrow \lambda^*$, we see that

$$\int_\Omega \frac{1}{u_{\lambda^*}^\beta} < \infty;$$

and therefore $u^* = u_{\lambda^*} > 0$ a.e. in Ω. □

(D) *Uniqueness of u^* in the class of solutions which are positive a.e.*

Proof. We just sketch the main points. The argument is an adaptation of the analogous result proved in [14] for convex nonlinearities.

(i) If there are two different solutions u_1 and u_2 of (1) for $\lambda = \lambda^*$ which are positive a.e., then a convex combination of them is a strict subsolution which is positive a.e.

(ii) Assume that u is a strict subsolution of (1) which is positive a.e. Let v be the solution of

$$\begin{cases} -\Delta v &= -\dfrac{1}{u^\beta} + \lambda^* f(x,u) \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega \end{cases}$$

and take the solution Y of (8). Then for $\varepsilon > 0$ small enough, we have $v - \varepsilon Y \geq \varepsilon\delta$ and $v - \varepsilon Y \geq u$ and

$$\begin{cases} -\Delta(v - \varepsilon Y) &\leq -\dfrac{1}{(v-\varepsilon Y)^\beta} + \lambda^* f(v - \varepsilon Y) - \varepsilon \quad \text{in } \Omega, \\ v - \varepsilon Y &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

(iii) By item (ii), there exists a positive weak solution w to

$$\begin{cases} -\Delta w &= -\dfrac{1}{w^\beta} + \lambda^* f(w) - \varepsilon \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

(iv) We now consider $\lambda' < \lambda^*$ but very close to λ^*, and the function

$$W = \frac{\lambda'}{\lambda^*} w + \varepsilon' Y.$$

We first choose $0 < \varepsilon' < \varepsilon$ and then $\lambda' \in (0, \lambda^*)$ so close to λ^* as to have $\varepsilon' < \varepsilon\lambda'/\lambda^*$ and

$$w \leq \frac{\lambda'}{\lambda^*} w + \varepsilon' Y \quad \text{and} \quad -\frac{\lambda'}{\lambda^*} w^{-\beta} \leq -\Big(\frac{\lambda'}{\lambda^*} w + \varepsilon' Y\Big)^{-\beta}.$$

This is possible because $w \leq C\delta$ for some constant $C > 0$. Then W satisfies

$$\begin{cases} -\Delta W &\leq -\dfrac{1}{W^\beta} + \lambda' f(W) \quad \text{in } \Omega, \\ W &= 0 \quad \text{on } \partial\Omega; \end{cases}$$

and so (1) has a positive solution for a parameter $\lambda' < \lambda^*$, which contradicts the minimality of λ^*. □

(E) *We show that for $\beta \geq 1$ there is no weak solution of (1) which is positive a.e., for any $\lambda \geq 0$.*

Proof. Suppose there is a weak solution u, positive a.e. Then by Lemma 3.12, $u \leq C\delta$. Therefore, by (B) we conclude that

$$\frac{1}{C}\int_\Omega \frac{1}{\delta^\beta} \leq \int_\Omega \frac{1}{u^\beta} < \infty,$$

which is impossible for $\beta \geq 1$. □

5 Stability and the extremal solution

Proof of Theorem 2.3. Let $\lambda > \lambda^*$ and let u_λ denote the maximal solution.

Step 1. For $\lambda > \lambda^*$ the maximal solution $u = u_\lambda$ (we drop the dependence on λ) is weakly stable, that is

$$\Lambda(u) \geq 0, \tag{36}$$

where $\Lambda(u)$ was defined in (5).

We prove (36) by using the perturbation (4) used in Lemma 3.1. Omitting the dependence on λ, recall that $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$. Note that since u_ε is the maximal solution to (4), it satisfies a corresponding stability inequality

$$\int_\Omega \left(\frac{\beta u_\varepsilon - \varepsilon}{(u_\varepsilon + \varepsilon)^{2+\beta}} + \lambda f_u(x, u_\varepsilon)\right)\varphi^2 \leq \int_\Omega |\nabla\varphi|^2, \quad \varphi \in C_c^\infty(\Omega). \tag{37}$$

By Lemma 3.12, $u_\varepsilon \geq u \geq a\delta$ for some $a > 0$; hence one can let $\varepsilon \to 0$ in (37) and obtain (36).

Step 2. Let us show that

$$\Lambda(u) > 0. \tag{38}$$

We consider a variation of problem (1). Recall that $\lambda > \lambda^*$ is fixed. We introduce a new parameter θ and consider the singular elliptic equation

$$\begin{cases} -\Delta u &= \chi_{\{u>0\}}\left(-\frac{1}{u^\beta} + \lambda f(x,u) + \theta\right) & \text{in } \Omega, \\ u &\geq 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \tag{39}$$

We recover (1) when $\theta = 0$.

An analysis similar to that in the proof of Theorem 2.1 and in Step 1 above can be done for this equation; we summarize the properties as follows:

(i) (39) has a supersolution $\overline{U} = KY$, where K is a large constant and Y is the solution to (8);

(ii) if (39) has bounded positive subsolution, then it has a maximal positive solution $\overline{u}_\theta$;

(iii) the same proof as in Step 1 above applies and shows that

$$\Lambda(\overline{u}_\theta) = \inf_{\varphi \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 - \big(\beta(\overline{u}_\theta)^{-\beta-1} + \lambda f_u(x, \overline{u}_\theta)\big)\varphi^2}{\int_\Omega \varphi^2} \geq 0.$$

The main observations needed to conclude (38) are

(40) there exists $\theta_0 < 0$ such that (39) has a positive subsolution for $\theta > \theta_0$

and

(41) $\Lambda(\overline{u}_\theta)$ is strictly increasing with θ.

Indeed, assuming (40) and (41), we have a maximal solution $\overline{u}_\theta$ for some $\theta < 0$. But then

$$0 \leq \Lambda(\overline{u}_\theta) < \Lambda(\overline{u}_0),$$

and $\overline{u}_0$ is just the maximal solution of (1).

Proof of (40). This proof is similar to that of Lemma 3.12. Fix $\lambda' \in (\lambda^*, \lambda)$ and let $u_{\lambda'}$ denote the maximal solution of (1) with parameter λ' and Y denote the solution to (8). Let ζ be the solution to

$$\begin{cases} -\Delta\zeta = f(x, u_{\lambda'}) & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

Now choose $\varepsilon \in (0, \lambda - \lambda')$ and then $\theta_0 < 0$ with $|\theta_0|$ small enough so that

$$|\theta_0| Y \leq \tfrac{1}{2}\varepsilon\zeta.$$

For $\theta > \theta_0$, set

$$w = u_{\lambda'} + \varepsilon\zeta + \theta Y.$$

Observe that

$$w \geq \varepsilon\zeta + \theta Y > 0.$$

We compute

$$\begin{aligned} -\Delta w + \frac{1}{w^\beta} &= -\frac{1}{u_{\lambda'}^\beta} + \frac{1}{(u_{\lambda'} + \varepsilon\zeta - \theta Y)^\beta} + \lambda' f(x, u_{\lambda'}) + \varepsilon f(x, u_{\lambda'}) + \theta \\ &\leq \lambda f(x, w) + (\lambda' - \lambda + \varepsilon) f(x, u_{\lambda'}) + \theta \\ &\leq \lambda f(x, w) + \theta. \end{aligned}$$

This proves the claim.

Proof of (41). Let $\theta_0 < \theta_1 < \theta_2$ and let $\overline{u}_{\theta_1}$, $\overline{u}_{\theta_2}$ denote the maximal solution of (39) with parameters θ_1 and θ_2. Note that

$$\overline{u}_{\theta_1} < \overline{u}_{\theta_2}. \tag{42}$$

Let ψ_1 and ψ_2 denote the first eigenfunctions, i.e.,

$$\begin{cases} -\Delta\psi_i - \beta(\overline{u}_{\theta_i})^{-\beta-1}\psi_i - \lambda f_u(x,\overline{u}_{\theta_i})\psi_i &= \Lambda(\overline{u}_{\theta_i})\psi_i & \text{in } \Omega, \\ \psi_i &= 0 & \text{on } \partial\Omega, \end{cases}$$

$i = 1, 2$, normalized so that $\|\psi_i\|_{L^2} = 1$ (note that $(\overline{u}_{\theta_i})^{-\beta-1} \le C\delta^{-\beta-1}$ and therefore the existence of these eigenfunctions in $H_0^1(\Omega)$ can be obtained using, e.g., Hardy's inequality). Then

$$\begin{aligned} \Lambda(\overline{u}_{\theta_1}) &= \int_\Omega |\nabla\psi_1|^2 - \big(\beta(\overline{u}_{\theta_1})^{-\beta-1} + \lambda f_u(x,\overline{u}_{\theta_1})\big)\psi_1^2 \\ &\le \int_\Omega |\nabla\psi_2|^2 - \big(\beta(\overline{u}_{\theta_1})^{-\beta-1} + \lambda f_u(x,\overline{u}_{\theta_1})\big)\psi_2^2 \\ &< \int_\Omega |\nabla\psi_2|^2 - \big(\beta(\overline{u}_{\theta_2})^{-\beta-1} + \lambda f_u(x,\overline{u}_{\theta_2})\big)\psi_2^2 \\ &= \Lambda(\overline{u}_{\theta_2}), \end{aligned}$$

where the last inequality is strict because $\psi_2 > 0$ and (42).

Step 3. Let us prove the converse, that is: assume u is a positive weak solution of (1) and that $\Lambda(u) \ge 0$; then $u = u_\lambda$.

Proof. Since $\Lambda(u) \ge 0$, we have

$$\int_\Omega \frac{\partial g_\lambda}{\partial u}(x,u)\varphi^2 \le \int_\Omega |\nabla\varphi|^2 \quad \forall\varphi \in H_0^1(\Omega). \tag{43}$$

Subtracting the equations for u, u_λ, multiplying by φ and integrating by parts yields

$$\int_\Omega \nabla(u_\lambda - u)\nabla\varphi \le \int_\Omega (g_\lambda(x,u_\lambda) - g_\lambda(x,u))\varphi, \quad \varphi \in H_0^1(\Omega).$$

Taking $\varphi = (u_\lambda - u)^+$ in the previous relation, we obtain

$$\int_\Omega |\nabla(u_\lambda - u)^+|^2 \le \int_\Omega (g_\lambda(x,u_\lambda) - g_\lambda(x,u))(u_\lambda - u)^+; \tag{44}$$

and using (43) with the same φ yields

$$\int_\Omega \frac{\partial g_\lambda}{\partial u}(x,u)(u_\lambda - u)^{+2} \le \int_\Omega |\nabla(u_\lambda - u)^+|^2. \tag{45}$$

Combining (44) and (45), we obtain

$$\int_\Omega \frac{\partial g_\lambda}{\partial u}(x,u)(u_\lambda - u)^{+2} \le \int_\Omega |\nabla(u_\lambda - u)^+|^2 \le \int_\Omega (g_\lambda(x,u_\lambda) - g_\lambda(x,u))(u_\lambda - u)^+;$$

and therefore

$$\int_{\{u_\lambda > u\}} (u_\lambda - u)\left(g_\lambda(x,u) + \frac{\partial g_\lambda}{\partial u}(x,u)(u_\lambda - u) - g_\lambda(x,u_\lambda)\right) \le 0.$$

But g_λ is strictly concave on $(0,\infty)$; and therefore $|\{u_\lambda > u\}| = 0$, which proves the claim. $\square$

Proof of Theorem 2.4. We work always with $\lambda > \lambda^*$, and obtain estimates that are independent of λ. Then we let $\lambda \to \lambda^*$. Denote by $u = u_\lambda$ the maximal solution of (1) (dropping the parameter λ for convenience).

Step 1. We first prove that for any

$$1 \le p < 3\beta + 1 + 2\sqrt{\beta^2 + \beta} \tag{46}$$

and any ball $B_R(x)$ such that $B_{2R}(x) \subset \Omega$, we have

$$\left(\frac{1}{|B_R|}\int_{B_R} u^{-p}\right)^{1/p} \le CR^{-2/(1+\beta)}, \tag{47}$$

where C is independent of λ and R.

Proof. We multiply the equation (1) by $\eta^2 u^{-2j-1}$, where $j > 0$ and $\eta \in C_c^\infty(B_{2R})$ and is such that

$$0 \le \eta \le 1 \quad \text{and} \quad \eta \equiv 1 \text{ on } B.$$

Using that $\lambda f(x,u) \ge 0$, we obtain

$$-(2j+1)\int_\Omega u^{-2j-2}|\nabla u|^2\eta^2 + 2\int_\Omega u^{-2j-1}\eta\nabla u\nabla\eta + \int_\Omega u^{-\beta-2j-1}\eta^2 \ge 0. \tag{48}$$

We rewrite (48) as

$$\begin{aligned} 2\int_\Omega u^{-2j-1}\eta\nabla u\nabla\eta + \int_\Omega u^{-\beta-2j-1}\eta^2 \; &\ge (2j+1-\varepsilon)\int_\Omega u^{-2j-2}|\nabla u|^2\eta^2 \\ &\quad + \varepsilon\int_\Omega u^{-2j-2}|\nabla u|^2\eta^2. \end{aligned} \tag{49}$$

But

$$|\nabla(\eta u^{-j})|^2 = j^2 u^{-2j-2}|\nabla u|^2\eta^2 - 2ju^{-2j-1}\eta\nabla u\nabla\eta + u^{-2j}|\nabla\eta|^2,$$

that is,

(50)
$$u^{-2j-2}|\nabla u|^2\eta^2 = \frac{1}{j^2}|\nabla(\eta u^{-j})|^2 j^2 u^{-2j-2}|\nabla u|^2\eta^2 + \frac{2}{j}u^{-2j-1}\eta\nabla u\nabla\eta - \frac{1}{j^2}u^{-2j}|\nabla\eta|^2.$$

Combining (49) and (50), we find

$$\frac{2j+1-\varepsilon}{j^2}\int_\Omega |\nabla(\eta u^{-j})|^2 \le C\int_\Omega u^{-2j}|\nabla\eta|^2 + \int_\Omega u^{-\beta-2j-1}\eta^2, \tag{51}$$

where the constant C depends on ε and j (we are not interested in taking j very large, so we omit the explicit dependence here).

We now use the weak stability of the maximal solution u, that is, $\Lambda(u) \ge 0$, where $\Lambda(u)$ is defined in (5). We take $\varphi = \eta u^{-j}$ to obtain

$$\int_\Omega |\nabla(\eta u^{-j})|^2 \ge \beta\int_\Omega \eta^2 u^{-\beta-2j-1}.$$

The last inequality and (51) yield

$$\Big(\beta\frac{2j+1-\varepsilon}{j^2} - 1\Big)\int_{B_R} u^{-\beta-2j-1} \le C\int_\Omega u^{-2j}|\nabla\eta|^2. \tag{52}$$

If

$$\beta\frac{2j+1-\varepsilon}{j^2} > 1, \tag{53}$$

we deduce from (52) that

$$\int_B u^{-\beta-2j-1} \le C,$$

which is (47) with $p = \beta + 2j + 1$ (where C is independent of λ.) Finally note that (53) can be satisfied for some $\varepsilon > 0$ if

$$j < \beta + \sqrt{\beta^2+\beta}.$$

In terms of $p = \beta + 2j + 1$, this is exactly the same as (46).

To get the dependence on R as stated in (47), we use a scaling argument. Assume that the ball B_R is centered at the origin and define $\tilde{u}(y) = R^{-2/(1+\beta)}u(Ry)$. Then $\tilde{u}$ satisfies

$$-\Delta\tilde{u} + \frac{1}{\tilde{u}^\beta} \ge 0 \qquad \text{in } \frac{1}{R}\Omega \tag{54}$$

and is still weakly stable. Then apply the preceding computation to $\tilde{u}$.

Step 2. There exists $c > 0$ independent of λ such that if the assumption (6) holds, then

$$u_\lambda \ge c\delta^{2/(1+\beta)}, \quad \lambda > \lambda^*.$$

Proof. We still work with $\tilde{u}(y) = R^{-2/(1+\beta)}u(Ry)$. Set $v = \tilde{u}^{-1}$; then v satisfies

$$\begin{cases} -\Delta v & \leq v^{2+\beta} \quad \text{in } B_1, \\ v & > 0 \quad \text{in } B_1, \end{cases} \tag{55}$$

and from (47)

$$\int_{B_1} v^p \leq C$$

for p satisfying (46). Using (55) and an iteration argument, it is easy to show that if

$$p > (\beta+1)\frac{n}{2}, \tag{56}$$

then

$$v \leq C \quad \text{on } B_{1/2}.$$

But the inequalities (46) and (56) are compatible only when

$$\frac{3\beta+1+2\sqrt{\beta^2+\beta}}{\beta+1} > \frac{n}{2},$$

which is the same as (6). □

6 Examples

Proof of Example 2.5. Consider the annulus $A = \{r : R < r < 1\} \subset \mathbb{R}^n$, $r = |x|$, $0 < R < 1$. Given $0 < \beta < 1$, let $\alpha = 2/(1+\beta)$ and choose $c > 0$ such that $c^{-\beta-1} = \alpha(\alpha-1)$. The function $u = c(1-r)^\alpha$ is a solution of the equation

$$\begin{cases} -\Delta u + 1/u^\beta & = f \quad & \text{in } A, \\ u & = 0 \quad & \text{on } \partial B_1, \\ u & = c(1-R)^\alpha \quad & \text{on } \partial B_R, \end{cases} \tag{57}$$

where

$$f(x) = f(r) = c\alpha(1-r)^{\alpha-1} > 0.$$

We claim that the first eigenvalue of the linearized operator is positive, that is,

$$\inf_{\varphi \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 - \beta u^{-\beta-1}\varphi^2}{\int_\Omega \varphi^2} > 0. \tag{58}$$

Indeed,

$$\begin{aligned}\beta\int_\Omega u^{-1-\beta}\varphi^2 &= \beta\int_\Omega (c(1-r)^\alpha)^{-1-\beta}\varphi^2\\ &= \beta c^{-1-\beta}\int_\Omega (1-r)^{-\alpha(-1-\beta)}\varphi^2\\ &= \beta\alpha(\alpha-1)\int_\Omega (1-r)^{-2}\varphi^2.\end{aligned}$$

Observe that $\beta\alpha(\alpha-1)\le\frac{1}{4}$ (with equality only if $\alpha=3/2$, i.e., $\beta=1/3$). But the Hardy inequality states that

$$\frac{1}{4}\int_{B_1}\frac{\varphi^2}{\delta^2}\le\int_{B_1}|\nabla\varphi|^2\quad\text{for all }\varphi\in C_c^\infty(B_1),$$

where the distance function to the boundary for B_1 is $\delta(r)=1-r$. Moreover, it is known from [4] that

$$\inf_{\varphi\in C_c^\infty(\Omega)}\frac{\int_\Omega|\nabla\varphi|^2-\frac{1}{4}\int_\Omega\varphi^2/\delta^2}{\int_\Omega\varphi^2}>0,\tag{59}$$

when $\Omega=B_1$; therefore (58) follows.

It is worth mentioning that the methods applied for (1) can be used for (57). This indicates that the extremal function u^* cannot satisfy an estimate of the form

$$u^*\ge c\delta^\gamma$$

for an exponent γ smaller than $2/(1+\beta)$. In this sense, the conclusion of Theorem 2.4 is optimal. □

Proof of Example 2.6. We start by constructing a one-dimensional variation of the previous example.

Let $w_\varepsilon:\mathbb{R}\to\mathbb{R}^+$ be a family of smooth convex functions such that

$$\begin{aligned}&w_\varepsilon(x)=|x| &&\text{for } |x|>\varepsilon,\\ &0<w_\varepsilon(x)\le\varepsilon &&\text{for } |x|\le\varepsilon,\\ &|w_\varepsilon'(x)|\le 1 &&\text{for all } x\in\mathbb{R}.\end{aligned}$$

Let

$$u_\varepsilon(x)=c(1-w_\varepsilon)^\alpha,\quad x\in(-1,1),$$

where (as before) $\alpha=2/(1+\beta)$ and $c>0$ is defined by $c^{-\beta-1}=\alpha(\alpha-1)$. A computation similar to the one in the previous example shows that

$$-u_\varepsilon''+\frac{1}{u_\varepsilon^\beta}=f_\varepsilon\quad\text{in }(-1,1),$$

where

$$f_\varepsilon = c^{-\beta}(1-w_\varepsilon)^{-\alpha\beta}\left(1-c^{\beta+1}\alpha(\alpha-1)(w_\varepsilon')^2\right)+c\alpha(1-w_\varepsilon)^{\alpha-1}w_\varepsilon''. \tag{60}$$

We claim that for $\varepsilon > 0$ sufficiently small u_ε is weakly stable in $(-1,1)$ in the sense of relation (7), i.e.,

$$\beta\int_{-1}^{1} u_\varepsilon^{-1-\beta}\varphi^2 \le \int_{-1}^{1}\varphi'^2 \quad \forall\varphi\in C_c^\infty(-1,1). \tag{61}$$

Indeed,

$$\beta u_\varepsilon^{-1-\beta} = \beta c^{-\beta-1}(1-w_\varepsilon)^{-2} = \beta\alpha(\alpha-1)(1-w_\varepsilon)^{-2} \le \frac{1}{4}(1-w_\varepsilon)^{-2}.$$

Therefore,

$$\begin{aligned}
\beta\int_{-1}^{1} u_\varepsilon^{-1-\beta}\varphi^2 &\le \frac{1}{4}\int_{-1}^{1}\frac{\varphi^2}{(1-w_\varepsilon)^2}\\
&= \frac{1}{4}\int_{-1}^{1}\frac{\varphi^2}{(1-|x|)^2}+\frac{1}{4}\int_{-\varepsilon}^{\varepsilon}\left((1-w_\varepsilon)^{-2}-(1-|x|)^{-2}\right)\varphi^2\\
&= \frac{1}{4}\int_{-1}^{1}\frac{\varphi^2}{(1-|x|)^2}+\frac{1}{4}\int_{-\varepsilon}^{\varepsilon}\frac{(1-w_\varepsilon)(1-|x|)}{(1-w_\varepsilon)^2(1-|x|)^2}(w_\varepsilon-|x|)\varphi^2\\
&\le \frac{1}{4}\int_{-1}^{1}\frac{\varphi^2}{(1-|x|)^2}+C\varepsilon\int_{-1}^{1}\varphi^2.
\end{aligned}$$

Using the inequality of [4] given by (59), one concludes that for ε small enough (61) holds. From now on, we *fix* this $\varepsilon > 0$.

We remark that by Theorems 2.1 and 2.3, for the problem

$$\begin{cases} -u_\varepsilon'' &= \chi_{\{u_\varepsilon>0\}}\Big(-\dfrac{1}{u_\varepsilon^\beta}+\lambda f_\varepsilon(x)\Big) & \text{in } (-1,1),\\ u_\varepsilon &= 0 & \text{at } x=-1,1,\end{cases}$$

we have $\lambda^* = 1$ and $u^* = u_\varepsilon$.

Define

$$u(x) = \begin{cases} u_\varepsilon(x-1) & \text{for } x\in(0,2),\\ u_\varepsilon(x+1) & \text{for } x\in(-2,0),\end{cases}$$

and

$$f(x) = \begin{cases} f_\varepsilon(x-1) & \text{for } x\in(0,2),\\ f_\varepsilon(x+1) & \text{for } x\in(-2,0).\end{cases}$$

Then u is continuous (even $C^{1,\gamma}(\overline{(-2,2)})$, $\gamma=(1-\beta)/(1+\beta)$) and is a solution of

$$\begin{cases} -u'' &= \chi_{\{u>0\}}\Big(-\dfrac{1}{u^\beta}+\lambda f(x)\Big) & \text{in } (-2,2),\\ u &= 0 & \text{at } x=-2,2,\end{cases}$$

and satisfies the condition (7). □

Proof of Proposition 2.8. Recall that we assume that Ω is an interval in $\mathbb{R}$, say $\Omega = (A, B)$. Let u_λ be the maximal solution of (1), and suppose that $u_\lambda \not\equiv 0$ but that u_λ vanishes at a point in Ω. Let I denote a connected component of the open set $\{u_\lambda > 0\}$. Then $I \neq \Omega$.

We want to consider u_λ restricted to I and also translated. For this purpose, we introduce some notation; for $\tau \in \mathbb{R}$ such that $I + \tau \subset \Omega$, we define

$$v_\tau(x) = \begin{cases} u_\lambda(x - \tau) & \text{if } x \in I + \tau, \\ 0 & \text{if } x \in \Omega \setminus (I + \tau). \end{cases}$$

We claim that for every τ such that $I + \tau \subset \Omega$, v_τ is a subsolution of (1).

In fact, suppose $I = (a, b)$. If a, $b \in \Omega$, then actually $u_\lambda'(a) = u_\lambda'(b) = 0$ (the derivatives exist), and hence v_τ is not just a subsolution, but also a solution of (1). Suppose that $a \notin \Omega$, i.e., $I = (a, b) \subset \Omega = (a, B)$, $b < B$. Let $\varphi \in C^2(\Omega)$ with $\varphi(a) = \varphi(B) = 0$ and $\varphi \geq 0$. The following formal computation can be justified using Lemma 3.4:

$$\begin{aligned} \int_\Omega \varphi' v_\tau' &= \int_a^b u' \varphi'(\cdot + \tau) \\ &= -\int_a^b u'' \varphi(\cdot + \tau) - \varphi(a + \tau) u_\lambda'(a) \\ &\leq -\int_a^b -\frac{1}{u_\lambda^\beta} + \lambda f(u) \\ &= \int_\Omega -\frac{1}{v_\tau^\beta} + \lambda f(v_\tau). \end{aligned}$$

If v_1, v_2 are two subsolutions of (1), then $\max(v_1, v_2)$ is also a subsolution. This statement is standard, and we omit its proof.

It is possible to find $\tau_1, \ldots, \tau_m$ such that $I + \tau_i \subset \Omega$ and $\bigcup_{i=1}^m (I + \tau_i) = \Omega$. Then

$$v = \max_{i=1,\ldots,m} v_{\tau_i}$$

is a subsolution of (1) which is positive in Ω. Since u_λ is the maximal subsolution of (1), it follows that u_λ is positive in Ω, in contradiction with our assumptions. □

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