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Hölder Estimates for Solutions to a Singular Nonlinear Neumann Problem

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Abstract. We consider the elliptic equation $-\Delta u + u = 0$ in a bounded, smooth domain Ω in $\mathbb{R}^n$ subject to the nonlinear singular Neumann condition $\frac{\partial u}{\partial \nu} = -u^{-\beta} + f(x,u)$. Here $0 < \beta < 1$ and $f \geq 0$ is C^1. We prove estimates for solutions to the same equation with $\frac{\partial u_\varepsilon}{\partial \nu} = -\frac{u_\varepsilon}{(u_\varepsilon+\varepsilon)^{1+\beta}} + f(x,u_\varepsilon)$ on the boundary, uniformly in ε.

1. Introduction

This note is intended as a complement of previous work by the authors [2]. We study the regularity of solutions of the following nonlinear boundary value problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = -u^{-\beta} + f(x,u) & \text{on } \partial\Omega \cap \{u > 0\}, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with smooth boundary, $0 < \beta < 1$ and ν is the exterior unit normal vector to $\partial\Omega$. We assume that

$$f : \partial\Omega \times \mathbb{R} \to \mathbb{R} \text{ is } C^1 \text{ and } f \geq 0. \tag{2}$$

By a solution of (1) we mean a function $u \in H^1(\Omega) \cap C(\overline{\Omega})$ satisfying

$$\int_\Omega \nabla u \cdot \nabla \varphi + u\varphi = \int_{\partial\Omega \cap \{u>0\}} (-u^{-\beta} + f(x,u))\varphi, \quad \forall \varphi \in C_0^1\big(\Omega \cup (\partial\Omega \cap \{u > 0\})\big). \tag{3}$$

One natural approach to prove existence of solutions of (1) is the following: take $\varepsilon > 0$ and consider

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = -\dfrac{u}{(u+\varepsilon)^{1+\beta}} + f(x,u) & \text{on } \partial\Omega. \end{cases} \tag{4}$$

It is not difficult to show that under the additional assumption

$$\lim_{u\to\infty} \frac{f(x,u)}{u} = 0 \quad \text{uniformly for } x \in \Omega \tag{5}$$

(4) has a maximal solution $\overline{u}^\varepsilon$. In [2] we proved that this maximal solution satisfies an estimate of the form

$$|\nabla \overline{u}^\varepsilon| \le C(\overline{u}^\varepsilon)^{-\beta} \quad \text{in } \Omega,$$

with C independent of ε. This was an essential step in proving that the limit $\lim_{\varepsilon\to 0} u^\varepsilon$ exists and is a solution of (1). Nevertheless there could exist other solutions of (4). For instance assuming (2) and (5) problem (4) admits also a minimal nonnegative solution $\underline{u}^\varepsilon$ (it could be zero but assuming $f(\cdot,0) \not\equiv 0$ guarantees $\underline{u}^\varepsilon \not\equiv 0$). Assuming some growth conditions on f, any critical point of Φ_ε is also a solution with

$$\Phi_\varepsilon(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2) - \int_{\partial\Omega} G^\varepsilon(x,u), \tag{6}$$

where

$$G^\varepsilon(x,u) = \int_0^u g^\varepsilon(x,t)\,dt, \qquad \text{and } g^\varepsilon(x,u) = -\frac{u}{(u+\varepsilon)^{1+\beta}} + f(x,u).$$

In this note we prove the following result concerning any kind of solution to (4).

Theorem 1.1. *Suppose f satisfies* (2). *Then for any bounded solution u of* (4) *we have*

$$|\nabla u| \le C u^{-\beta} \quad \text{in } \Omega,$$

where C is independent of ε, and depends on Ω, n, β, f and $\|u\|_{L^\infty(\Omega)}$.

A consequence of the previous gradient estimate is the following convergence result (the proof is exactly as in [2]).

Corollary 1.2. *Assume* (2) *and let $\varepsilon_k \to 0$ and u^{ε_k} be a sequence of solutions of* (4) *with*

$$\|u^{\varepsilon_k}\|_{L^\infty(\Omega)} \le C,$$

where C is independent of k. Then up to a subsequence $u^{\varepsilon_k} \to u$ in $C^\mu(\overline{\Omega})$ for any $0 < \mu < \frac{1}{1+\beta}$ and u is a solution of (1).

This result enables us to consider other type of nonlinearities than in [2]. For example

Theorem 1.3. *Assume that $n \ge 3$ and $1 < p < \frac{n}{n-2}$. Then there exists a nontrivial solution to*

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ u \ge 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = -u^{-\beta} + u^p & \text{on } \partial\Omega \cap \{u > 0\}. \end{cases} \tag{7}$$

By Theorem 1.1 *this solution is $C^{\frac{1}{1+\beta}}(\overline{\Omega})$.*

Previous work with a singular Neumann condition include [3] where the authors study the evolution equation $u_t = u_{xx}$ in $(0,1)$ with Neumann conditions $u_x(0,t) = 0$, $u_x(1,t) = -u(1,t)^{-\beta}$. The initial condition is $u(x,0) = u_0(x) > 0$ and sufficiently smooth. They prove that the solution exists up to a quenching time $0 < T < \infty$ with $\lim_{t\nearrow T} u(1,t) = 0$ and they provide estimates of the type $C_1 \le (1-x)^{\frac{1}{\beta+1}} u(x,T) \le C_2$.

In higher dimensions a similar evolution problem was addressed in [6] with a positive unbounded nonlinearity such as $1/(1-u)$, but the authors only work with a time interval $[0,T)$ where $0 \le u(t) < 1$.

As mentioned earlier this work is a continuation of previous work of the authors. For this reason not all proofs are supplied here and we refer to [2].

2. Preliminaries

There are two important key points in the proof of Theorem 1.1. First there is a construction of a local subsolution. The second ingredient is a Hardy type inequality, which roughly speaking asserts that a solution that stays above the local subsolution is locally a minimum of the related energy. To make this more precise we rescale the problem to a small ball. It is convenient at this point to introduce some notation. Let $\tau_0 > 0$ be small enough to be fixed in Proposition 2.1 below. For $0 < \tau < \tau_0$ and $x_0 \in \partial\Omega$ let us write $\partial(B_\tau(x_0) \cap \Omega) = \Gamma^e \cup \Gamma^i$ where

$$\Gamma^i = \partial B_\tau(x_0) \cap \Omega, \quad \Gamma^e = B_\tau(x_0) \cap \partial\Omega$$

are the internal and external boundaries. We also decompose $\Gamma^e = \Gamma^1 \cup \Gamma^2$ with

$$\Gamma^1 = \varphi^{-1}(B_{\tau/2}(0)) \cap \partial\Omega, \quad \Gamma^2 = \Gamma^e \setminus \Gamma^1, \tag{8}$$

where φ is a smooth diffeomorphism which flattens the boundary of Ω near x_0. This means that $\varphi : W \subset \mathbb{R}^n \to B_{\tau_0}(0)$ is smooth with W an open set containing the ball $B_{\tau_0}(x_0)$ and $\varphi(W\cap\Omega) = B_{\tau_0}(0)\cap H$, $\varphi(W\cap\partial\Omega) = B_{\tau_0}(0)\cap\partial H$, $\varphi(W\setminus\overline{\Omega}) = B_{\tau_0}(0) \setminus \overline{H}$, where

$$H = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}.$$

Let us introduce the rescaled domains which allow us to work in balls of unit size:

$$\begin{aligned} B_\tau^+ &= \frac{1}{\tau}(B_\tau(x_0) \cap \Omega - x_0) = B_1(0) \cap \frac{1}{\tau}(\Omega - x_0), \quad \Omega_\tau = \frac{1}{\tau}(\Omega - x_0) \\ \Gamma_\tau^i &= \frac{1}{\tau}(\Gamma^i - x_0), \quad \Gamma_\tau^e = \frac{1}{\tau}(\Gamma^e - x_0), \quad \Gamma_\tau^k = \frac{1}{\tau}(\Gamma^k - x_0), \quad k = 1,2. \end{aligned} \tag{9}$$

Given $x_0 \in \partial\Omega$ and $0 < \tau < \tau_0$ we let v_τ be the solution of the linear equation

$$\begin{cases} -\Delta v_\tau + \tau^2 v_\tau = 0 & \text{in } B_\tau^+, \\ \dfrac{\partial v_\tau}{\partial \nu}(y) = -\operatorname{dist}(y, \Gamma_\tau^2)^{-\frac{\beta}{1+\beta}} & y \in \Gamma_\tau^1, \\ v_\tau(y) = 0 & y \in \Gamma_\tau^2, \\ v_\tau(y) = s \operatorname{dist}(y, \partial\Omega_\tau) & y \in \Gamma_\tau^i. \end{cases} \tag{10}$$

For large s its solution will be called a local subsolution because of the next lemma.

Proposition 2.1. *There exist* $\tau_0 > 0$ *and* $s_0 > 0$ *such that if* $0 < \tau < \tau_0$ *and* $s \geq s_0$ *the solution of* (10) *is positive in* B_τ^+ *and satisfies*

$$v_\tau(y) \geq cs\,\mathrm{dist}(y, \Gamma_\tau^2)^{\frac{1}{1+\beta}}, \quad \forall y \in \Gamma_\tau^1, \tag{11}$$

where $c > 0$ *is independent of* x_0, τ *and* s *(*c *depends only on* Ω, n, β*). In particular, choosing* s_0 *larger if necessary*

$$\frac{\partial v_\tau}{\partial \nu} \leq -v_\tau^{-\beta} \quad \text{on } \Gamma_\tau^1. \tag{12}$$

We will not include the proof of the statements in this section. They can be found in [2].

Next we state a Hardy type inequality.

Proposition 2.2. *There exists a constant* C_h *such that*

$$\int_{\Gamma_\tau^1} \frac{\varphi^2}{\mathrm{dist}(y, \Gamma_\tau^2)} \leq C_h \int_{B_\tau^+} |\nabla \varphi|^2, \quad \forall \varphi \in C_0^\infty(B_\tau^+ \cup \Gamma_\tau^1). \tag{13}$$

The constant C_h *can be taken independent of* τ *and* $x_0 \in \partial\Omega$ *if* $0 < \tau < \tau_0$.

Finally we mention some lemmas on linear equations with a Neumann boundary condition. Again, the proofs can be found in [2].

This is a sort of Harnack inequality.

Lemma 2.3. *Let* $a \in L^\infty(\Omega_\tau \cap B_3)$, $a \geq 0$ *and suppose that* $u \in H^1(\Omega_\tau \cap B_3)$, $u \geq 0$ *satisfies*

$$\begin{cases} -\Delta u + a(y)u = 0 & \text{in } \Omega_\tau \cap B_3 \\ \dfrac{\partial u}{\partial \nu} \leq N & \text{on } \Gamma_\tau^e, \end{cases}$$

where N *is a constant. Then there is a constant* $c_k > 0$ *such that*

$$u(y) \geq c_k\,\mathrm{dist}(y, \Gamma_\tau^e)(c_k u(y_1) - N), \quad \forall y \in B_\tau^+ \text{ and } \forall y_1 \in B_\tau^+ \cap B_{1/2}.$$

The constant c_k *can be chosen independent of* $x_0 \in \partial\Omega$ *and of* $0 < \tau < \tau_0$.

These last two estimates are standard in the theory of L^p regularity theory, see for instance [9].

Lemma 2.4. *Let* $a \in L^\infty(B_\tau^+)$. *Suppose* $u \in H^1(B_\tau^+)$ *satisfies*

$$\begin{cases} -\Delta u + a(x)u = 0 & \text{in } B_\tau^+ \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \Gamma_\tau^e, \end{cases}$$

where $g \in L^p(\Gamma_\tau^e)$ *and* $p \geq 1$. *Let* $1 \leq r < \frac{np}{n-1}$. *Then there exists* C *independent of* g *and* u *such that*

$$\|u\|_{W^{1,r}(\Omega_\tau \cap B_{3/4})} \leq C\Big(\|g\|_{L^p(\Gamma_\tau^e)} + \|u\|_{L^1(B_\tau^+)}\Big).$$

Lemma 2.5. *Let $a \in L^\infty(B_\tau^+)$ and suppose that $u \in H^1(B_\tau^+)$, $u \geq 0$ satisfies*

$$\begin{cases} -\Delta u + a(x)u \geq 0 & \text{in } B_\tau^+ \\ \dfrac{\partial u}{\partial \nu} \geq -N & \text{on } \Gamma_\tau^e, \end{cases}$$

where N is a constant. Then there is a constant $C > 0$ independent of u, N such that

$$\int_{B_{3/4}\cap B_\tau^+} u \leq C(u(x) + N) \quad \forall x \in B_{1/2} \cap B_\tau^+.$$

3. Proof of Theorem 1.1

Let u be a bounded nontrivial solution of equation (4) and write

$$M = \max\left(\sup_{x\in\partial\Omega} f(x, u(x)), \max_{\overline{\Omega}} u\right).$$

Let τ_0 and s_0 be the constants in Proposition 2.1 and fix $\widetilde{C} > 0$ such that

$$s_0 < \frac{1}{2} c_k^2 \widetilde{C}, \tag{14}$$

$$M^{1+\beta} < \tau_0 \widetilde{C}^{1+\beta}, \tag{15}$$

$$M^{1+\beta} < \frac{1}{2} c_k \widetilde{C}^{1+\beta}. \tag{16}$$

Next we fix C_0 large enough such that

$$\left(\frac{C_0}{\widetilde{C}}\right)^{1+\beta} \geq 6. \tag{17}$$

Let x_1 be a point in Ω. We distinguish two cases.

Case 1. Assume $u(x_1) \leq C_0 \operatorname{dist}(x_1, \partial\Omega)^{\frac{1}{1+\beta}}$. Consider the scaling about the point x_1 given by $\tilde{u}(y) = \tau^{-\frac{1}{1+\beta}} u(\tau y + x_1)$, with $\tau = \frac{1}{2}\operatorname{dist}(x_1, \partial\Omega)$. Then $-\Delta\tilde{u} + \tau^2\tilde{u} = 0$ in $B_1(0)$, $\tilde{u} \geq 0$ in $B_1(0)$ and $\tilde{u}(0) \leq 2^{\frac{1}{1+\beta}} C_0$. Since $\tilde{u} \geq 0$, by elliptic estimates we have $|\nabla\tilde{u}(0)| \leq C(n, \beta)C_0$, where $C(n, \beta)$ depends only on n, β. This implies $|\nabla u(x_1)| \leq C(n, \beta)C_0\tau^{-\frac{\beta}{1+\beta}} \leq C(n, \beta)C_0^{1-\beta}u(x_1)^{-\beta}$. Thus

$$|\nabla u(x_1)| \leq C(n, \beta)C_0^{1-\beta}u(x_1)^{-\beta}. \tag{18}$$

We keep the explicit dependence on C_0 for future reference.

Case 2. Assume

$$u(x_1) > C_0 \operatorname{dist}(x_1, \partial\Omega)^{\frac{1}{1+\beta}}. \tag{19}$$

Let

$$x_0 \in \partial\Omega, \quad \operatorname{dist}(x_1, \partial\Omega) = |x_0 - x_1|. \tag{20}$$

Our first task is to show that u satisfies an inequality such as (19) on all points on the line segment

$$[x_0, x_1] = \left\{ x_0 + t\frac{x_1 - x_0}{|x_1 - x_0|} : 0 \le t \le \bar{t} \right\},$$

where $\bar{t} = |x_1 - x_0|$.

Lemma 3.1. *Choosing C_0 larger if necessary (only depending on n, β and $\widetilde{C}$ as in* (17)*) we have*

$$u(x) \ge C_0 \operatorname{dist}(x, \partial\Omega)^{\frac{1}{1+\beta}} \quad \forall x \in [x_0, x_1]. \tag{21}$$

Proof. For the sake of notation we write

$$x_t = x_0 + t\frac{x_1 - x_0}{|x_1 - x_0|} \quad 0 \le t \le \bar{t},$$

and observe that $\operatorname{dist}(x_t, \partial\Omega) = |x_t - x_0| = t$. Suppose that (21) fails. Then

$$t_0 = \sup\{t \in [0, \bar{t}\,] : u(x_t) \le C_0 t^{\frac{1}{1+\beta}}\}$$

is well defined, $t_0 > 0$ and by (19) we have $t_0 < \bar{t}$. Define $g(t) = u(x_t)$. Using the same argument as in case 1, see (18), we have that

$$g'(t) \le C(n,\beta) C_0^{1-\beta} g(t)^{-\beta} \quad \text{whenever } g(t) \le C_0 t^{\frac{1}{1+\beta}}. \tag{22}$$

Let $h(t) = C_0 t^{\frac{1}{1+\beta}}$, so that $h'(t) = \frac{C_0^{1+\beta}}{1+\beta} h(t)^{-\beta}$. Then we have $g(t_0) = h(t_0)$ and by (22)

$$g'(t_0) \le C(n,\beta) C_0^{1-\beta} g(t_0)^{-\beta} = C(n,\beta) \frac{1+\beta}{C_0^{2\beta}} h'(t_0).$$

Choose C_0 larger so that $C(n,\beta)\frac{1+\beta}{C_0^{2\beta}} < \frac{1}{2}$. Then $g(t) > h(t)$ for $t \in (t_0 - \sigma, t_0)$ for some $\sigma > 0$. This is impossible. □

Define τ_1 by

$$\tau_1 = \left(\frac{u(x_1)}{\widetilde{C}} \right)^{1+\beta} \tag{23}$$

and observe that by (15) we have

$$\tau_1 < \tau_0.$$

We look now at the rescaled function u around the point $x_0 \in \partial\Omega$ given by (20): for $0 < \tau < \tau_0$ and $x_0 \in \partial\Omega$ define

$$u_\tau(y) = \tau^{-\frac{1}{1+\beta}} u(\tau y + x_0), \qquad y \in \Omega_\tau = \frac{1}{\tau}(\Omega - x_0). \tag{24}$$

At this point it is convenient to replace f with a C^1 function $\bar{f} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ with $\bar{f} \ge 0$ and $f, \frac{\partial f}{\partial u}$ bounded, and such that $f(x,u) = \bar{f}(x,u)$ for all $x \in \partial\Omega$ and

$0 \le u \le M$. Then u solves (4) with f replaced by $\bar f$ and therefore u_τ is a solution of

$$\begin{cases} -\Delta u_\tau + \tau^2 u_\tau = 0 & \text{in } \Omega_\tau, \\ \dfrac{\partial u_\tau}{\partial \nu} = g_\tau^\varepsilon(y, u_\tau) & \text{on } \partial\Omega_\tau. \end{cases} \tag{25}$$

where g_τ^ε is given by

$$g_\tau^\varepsilon(y, w) = \tau^{\frac{\beta}{1+\beta}} g^\varepsilon(\tau y + x_0, \tau^{\frac{1}{1+\beta}} w), \tag{26}$$

and

$$g^\varepsilon(x, u) = -\frac{u}{(u+\varepsilon)^{1+\beta}} + \bar f(x, u). \tag{27}$$

Observe that we have changed the definition of g^ε and g_τ^ε from the one given in the introduction replacing f by $\bar f$.

We will see that as a consequence of (21) u_τ has to be suitably large on the internal boundary Γ_τ^i.

Lemma 3.2. *For $0 < \tau \le \tau_1$ we have*

$$u_\tau(y) \ge s_0 \operatorname{dist}(y, \partial\Omega_\tau) \quad \forall y \in \Gamma_\tau^i.$$

Proof. Let $z_\tau = \frac{1}{2}\frac{x_1 - x_0}{|x_1 - x_0|} \in B_\tau^+ \cap B_{1/2}$. By (21) and the definition of u_τ we have

$$u_\tau(z_\tau) = \tau^{-\frac{1}{1+\beta}} u(\tau z_\tau + x_0) \ge \frac{C_0}{2} \ge \widetilde C, \tag{28}$$

where the last inequality is a consequence of (17). Using Harnack's Lemma 2.3 and (28) we obtain

$$u_\tau(y) \ge c_k \operatorname{dist}(y, \partial\Omega_\tau)\Big(c_k \widetilde C - \sup_{\Gamma_\tau^e} \frac{\partial u_\tau}{\partial \nu}\Big), \quad \forall y \in B_\tau^+. \tag{29}$$

From the boundary condition in (25) and the definition of M

$$\sup_{\Gamma_\tau^e} \frac{\partial u_\tau}{\partial \nu} \le \tau^{\frac{\beta}{1+\beta}} M.$$

Notice that from (16) we deduce $u(x_1)^\beta \le \frac{c_k \widetilde C^{1+\beta}}{2M}$ which is the same as

$$M\Big(\frac{u(x_1)}{\widetilde C}\Big)^\beta \le \frac{1}{2} c_k \widetilde C.$$

Thus

$$\tau^{\frac{\beta}{1+\beta}} M \le \tau_1^{\frac{\beta}{1+\beta}} M = \Big(\frac{u(x_1)}{\widetilde C}\Big)^\beta M \le \frac{1}{2} c_k \widetilde C.$$

Inserting this in (29) and recalling (14) we find

$$u_\tau(y) \ge \frac{1}{2} c_k^2 \widetilde C \operatorname{dist}(y, \partial\Omega_\tau) \ge s_0 \operatorname{dist}(y, \partial\Omega_\tau) \quad \forall y \in \Gamma_\tau^i. \qquad \square$$

The main step that we shall prove in the sequel is the following:

Proposition 3.3. *For all* $0 < \tau \le \tau_1$ *we have*

$$u_\tau \ge v_\tau \quad \text{in } B_\tau^+. \tag{30}$$

For the proof of Proposition 3.3 we consider the nonlinear problem

$$\begin{cases} -\Delta w + \tau^2 w = 0 & \text{in } B_\tau^+ \\ w = u_\tau & \text{on } \Gamma_\tau^i \cup \Gamma_\tau^2 \\ \dfrac{\partial w}{\partial \nu} = g_\tau^\varepsilon(x, w) & \text{on } \Gamma_\tau^1 \end{cases} \tag{31}$$

where we regard u_τ as data and w as the unknown. Observe that u_τ is a solution of (31).

The solutions of (31) are the critical points of the functional

$$\psi_\tau(w) = \frac{1}{2}\int_{B_\tau^+} (|\nabla w|^2 + \tau^2 w^2) - \int_{\Gamma_\tau^1} G_\tau^\varepsilon(x, w)$$

on the set

$$E_\tau = \{w \in H^1(B_\tau^+) \mid w = u_\tau \text{ on } \Gamma_\tau^i \cup \Gamma_\tau^2\},$$

where

$$G_\tau^\varepsilon(y, w) = \int_0^w g_\tau^\varepsilon(y, r)\, dr,$$

and g_τ^ε defined in (26).

We remark that any nontrivial solution u of the regularized problem (4) is positive by the strong maximum principle, the fact that $f \ge 0$ and Hopf's lemma. This implies that $u_\tau \to \infty$ in B_τ^+ as $\tau \to 0$, more precisely $u_\tau \sim \tau^{-\frac{1}{1+\beta}} u(x_0)$ in B_τ^+. As a consequence, for fixed $\varepsilon > 0$ as $\tau \to 0$ problem (31) is less singular and we have

Lemma 3.4. *For* $\tau > 0$ *small enough problem* (31) *has a unique solution.*

How small τ has to be may depend on ε.

Proof. Suppose that there exists a sequence $\tau_j \to 0$ and solutions $w_j^1, w_j^2 \in H^1(\Omega_\tau)$ to equation (31) with $w_j^1 \ne w_j^2$.

Since $w_j^1 = w_j^2 = u_{\tau_j}$ on $\Gamma_\tau^i \cup \Gamma_\tau^2$ we have $w_j^i \le \tau_j^{-\frac{1}{1+\beta}} M$ on $\Gamma_\tau^i \cup \Gamma_\tau^2$, $i = 1, 2$. Also, $\frac{\partial w_j^i}{\partial \nu} \le \bar f_{\tau_j}(y, w_j^i)$ on Γ_τ^1 where

$$\bar f_{\tau_j}(y, w) = \tau_j^{\frac{\beta}{1+\beta}} \bar f(\tau_j y + x_0, \tau_j^{\frac{1}{1+\beta}} w) \le C \tau_j^{\frac{\beta}{1+\beta}},$$

since $\bar f$ is bounded. By the maximum principle we have

$$w_j^i \le C \tau_j^{-\frac{1}{1+\beta}} \quad \text{on } B_{\tau_j}^+. \tag{32}$$

with C independent of j.

Let $w_j = w_j^1 - w_j^2$. Then w_j satisfies

$$\begin{cases} -\Delta w_j + \tau_j^2 w_j = 0 & \text{in } B_{\tau_j}^+ \\ w_j = 0 & \text{on } \Gamma_{\tau_j}^i \cup \Gamma_{\tau_j}^2 \\ \dfrac{\partial w_j}{\partial \nu} = b_j(x) w_j & \text{on } \Gamma_{\tau_j}^1, \end{cases} \tag{33}$$

where

$$b_j(x) = \frac{\partial g_{\tau_j}^\varepsilon}{\partial w}(x, \xi(x))$$

for some $\xi(x) \in [w_j^1(x), w_j^2(x)]$ (we use the notation $[a,b] = [\min(a,b), \max(a,b)]$). Now we estimate

$$b_j(x) = \frac{\partial g_{\tau_j}^\varepsilon}{\partial w}(x, \xi(x)) = \tau_j^{\frac{2}{1+\beta}} \frac{\partial g^\varepsilon}{\partial w}(\tau_j x + x_0, \tau_j^{\frac{1}{1+\beta}} \xi(x)),$$

where g^ε is defined in (27). By (32) we see that $\tau_j^{\frac{1}{1+\beta}} \xi(x) \leq C$ and since g^ε is C^1 we thus conclude that

$$b_j \to 0 \quad \text{uniformly on } \Gamma_{\tau_j}^1.$$

Thus, for j large enough the operator in (33) becomes coercive and hence $w_j = 0$ if j is large. Indeed, multiplying (33) by w_j and integrating we find

$$\int_{B_{\tau_j}^+} |\nabla w_j|^2 + \tau_j^2 \int_{B_{\tau_j}^+} w_j^2 = \int_{\Gamma_{\tau_j}^1} b_j w_j^2$$

Since $w_j = 0$ in $\Gamma_{\tau_j}^2 \cup \Gamma_{\tau_j}^i$ we have by the Sobolev trace inequality

$$\int_{B_{\tau_j}^+} |\nabla w_j|^2 + \tau_j^2 \int_{B_{\tau_j}^+} w_j^2 \leq C \|b_j\|_{L^\infty(\Gamma_{\tau_j}^1)} \int_{B_{\tau_j}^+} |\nabla w_j|^2,$$

which shows that $w_j \equiv 0$ for j large enough. □

Lemma 3.5. *Fix $s = s_0$ in Proposition (2.1) and let v_τ be the solution of (10). Assume $w, v \in E_\tau$ are subsolutions of (31) such that*

$$v \geq v_\tau \quad \text{on} \quad \Gamma_\tau^1, \qquad \text{and} \qquad v \leq w \quad \text{on} \quad \Gamma_\tau^i \cup \Gamma_\tau^2.$$

Then

$$\psi_\tau(\max(w,v)) \leq \psi_\tau(w) + \left(\frac{C}{s_0^{1+\beta}} + C\tau - \frac{1}{2} \right) \int_{B_\tau^+ \cap \{v > w\}} |\nabla(v - w)|^2,$$

where C is independent of ε, s_0, τ, v and w.

Proof. We derive first some estimates for the nonlinear terms. The functions $G^\varepsilon(x,u)$, $G_\tau^\varepsilon(x,w)$ are given by

$$G^\varepsilon(x,u) = \int_0^u g^\varepsilon(x,s)\, ds = \frac{(u+\varepsilon)^{-\beta}(\varepsilon + \beta u) - \varepsilon^{1-\beta}}{\beta\,(-1+\beta)} + \overline{F}(x,u),$$

where $\overline{F}(x,u) = \int_0^u \bar{f}(x,s)\,ds$, and

$$G_\tau^\varepsilon(x,w) = \tau^{\frac{-1+\beta}{1+\beta}} G^\varepsilon(\tau x + x_0, \tau^{\frac{1}{1+\beta}} w).$$

Note that

$$-u^{-\beta} + \bar{f}(x,u) \le g^\varepsilon(x,u) \le \bar{f}(x,u)$$

and hence we have the estimates

$$-\frac{u^{1-\beta}}{1-\beta} + \overline{F}(x,u) \le G^\varepsilon(x,u) \le \overline{F}(x,u)$$

and

$$-\frac{w^{1-\beta}}{1-\beta} + \tau^{\frac{-1+\beta}{1+\beta}} \overline{F}(\tau x + x_0, \tau^{\frac{1}{1+\beta}} w) \le G_\tau^\varepsilon(x,w) \le \tau^{\frac{-1+\beta}{1+\beta}} \overline{F}(\tau x + x_0, \tau^{\frac{1}{1+\beta}} w).$$

Let $W = \max(w,v)$. Then W satisfies

$$\begin{cases} -\Delta W + \tau^2 W \le 0 & \text{in } B_\tau^+, \\ W \le u_\tau & \text{on } \Gamma_\tau^i \cup \Gamma_\tau^2 \\ \dfrac{\partial W}{\partial \nu} \le g_\tau^\varepsilon(x,W) & \text{on } \Gamma_\tau^1. \end{cases} \tag{34}$$

We have the equality

$$\begin{aligned} \psi_\tau(W) - \psi_\tau(w) = &-\frac{1}{2}\int_{B_\tau^+} \big(|\nabla(W-w)|^2 + \tau^2 (W-w)^2\big) \\ &+ \int_{B_\tau^+} \big(\nabla W \cdot \nabla(W-w) + \tau^2 W(W-w)\big) \\ &- \int_{\Gamma_\tau^1} \big(G_\tau^\varepsilon(x,W) - G_\tau^\varepsilon(x,w)\big). \end{aligned} \tag{35}$$

Next we multiply (34) by $W - w \ge 0$ and integrate by parts. Note that $W - w = 0$ on $\Gamma_\tau^i \cup \Gamma_\tau^2$ so that

$$\begin{aligned} \int_{B_\tau^+} \nabla W \cdot \nabla(W-w) + \tau^2 W(W-w) &\le \int_{\Gamma_\tau^1} \frac{\partial W}{\partial \nu}(W-w) \\ &\le \int_{\Gamma_\tau^1} g_\tau^\varepsilon(x,W)(W-w). \end{aligned} \tag{36}$$

Combining (35) and (36) we obtain

$$\begin{aligned} \psi_\tau(W) - \psi_\tau(w) \le &-\frac{1}{2}\int_{B_\tau^+} |\nabla(W-w)|^2 \\ &- \int_{\Gamma_\tau^1} \big(G_\tau^\varepsilon(x,W) - G_\tau^\varepsilon(x,w) - g_\tau^\varepsilon(x,W)(W-w)\big). \end{aligned} \tag{37}$$

We claim that

$$-\left[G_\tau^\varepsilon(x,W) - G_\tau^\varepsilon(x,w) - g_\tau^\varepsilon(x,W)(W-w)\right] \le C(\tau + W^{-1-\beta})(W-w)^2, \tag{38}$$

where C is a constant independent of ε.

To verify (38) we consider first the case $W \le 2w$. By Taylor's theorem

$$-\left[G_\tau^\varepsilon(x,W) - G_\tau^\varepsilon(x,w) - g_\tau^\varepsilon(x,W)(W-w)\right] = \frac{1}{2}\frac{\partial g_\tau^\varepsilon}{\partial w}(x,\xi)(W-w)^2,$$

for some $w < \xi < W$. A computation shows that

$$\frac{\partial g_\tau^\varepsilon}{\partial w}(x,w) = \tau \frac{\beta \tau^{\frac{1}{1+\beta}} w - \varepsilon}{\left(\tau^{\frac{1}{1+\beta}} w + \varepsilon\right)^{2+\beta}} + \tau \bar{f}_u(\tau x + x_0, \tau^{\frac{1}{1+\beta}} w)$$

and therefore

$$\frac{\partial g_\tau^\varepsilon}{\partial w}(x,w) \le \tau\beta\left(\tau^{\frac{1}{1+\beta}} w + \varepsilon\right)^{-1-\beta} + K\tau \le \beta w^{-1-\beta} + K\tau, \tag{39}$$

where $K = \sup_{x,u} |\bar{f}_u(x,u(x))| < \infty$. Hence

$$-\left[G_\tau^\varepsilon(x,W) - G_\tau^\varepsilon(x,w) - g_\tau^\varepsilon(x,W)(W-w)\right] \le (\beta\xi^{-1-\beta} + K\tau)(W-w)^2.$$

But $\xi^{-\beta} \le w^{-\beta} \le (W/2)^{-\beta}$ and we obtain

$$-\left[G_\tau^\varepsilon(x,W) - G_\tau^\varepsilon(x,w) - g_\tau^\varepsilon(x,W)(W-w)\right] \le C(\tau + W^{-1-\beta})(W-w)^2.$$

For the case $W > 2w$ observe that

$$\begin{aligned}
-\Big[G_\tau^\varepsilon(x,W) - &G_\tau^\varepsilon(x,w) - g_\tau^\varepsilon(x,W)(W-w)\Big] \\
&= -G_\tau^\varepsilon(x,W) + G_\tau^\varepsilon(x,w) + g_\tau^\varepsilon(x,W)(W-w) \\
&\le \frac{W^{1-\beta}}{1-\beta} + \tau^{\frac{-1+\beta}{1+\beta}} \Big[\overline{F}(\tau x + x_0, \tau^{\frac{1}{1+\beta}} W) \\
&\qquad - \overline{F}(\tau x + x_0, \tau^{\frac{1}{1+\beta}} w) \\
&\qquad + \tau^{\frac{1}{1+\beta}} \bar{f}(\tau x + -x_0, \tau^{\frac{1}{1+\beta}} W)(W-w)\Big].
\end{aligned}$$

But for $W > 2w$ we have

$$\frac{W^{1-\beta}}{1-\beta} = \frac{1}{1-\beta} W^{-1-\beta} W^2 \le \frac{4}{1-\beta} W^{-1-\beta}(W-w)^2$$

and

$$\begin{aligned}
\Big|\overline{F}(\tau x + x_0, \tau^{\frac{1}{1+\beta}} W) - \overline{F}(\tau x + x_0, \tau^{\frac{1}{1+\beta}} w) + \tau^{\frac{1}{1+\beta}} \bar{f}(\tau x + x_0, \tau^{\frac{1}{1+\beta}} W)(W-w)\Big| \\
= \frac{1}{2}\tau^{\frac{2}{1+\beta}} |\bar{f}_u(\tau x + x_0, \tau^{\frac{1}{1+\beta}} \xi)|(W-w)^2,
\end{aligned}$$

for some ξ. Thus

$$-\left[G_\tau^\varepsilon(x,W) - G_\tau^\varepsilon(x,w) - g_\tau^\varepsilon(x,W)(W-w)\right] \le (CW^{-1-\beta} + K\tau)(W-w)^2.$$

Using estimate (38) in (37) we find

$$\psi_\tau(W) - \psi_\tau(w) \le -\frac{1}{2}\int_{B_\tau^+} |\nabla(W-w)|^2 + C\int_{\Gamma_\tau^1} \left(W^{-1-\beta} + \tau\right)(W-w)^2.$$

But $W \geq v_\tau \geq cs_0 \operatorname{dist}(y, \Gamma_\tau^2)$ by (11) and therefore

$$\psi_\tau(W) - \psi_\tau(w) \leq -\frac{1}{2} \int_{B_\tau^+} |\nabla(W-w)|^2 + C \int_{\Gamma_\tau^1} \left(s_0^{-1-\beta} \operatorname{dist}(\Gamma_\tau^2)^{-1-\beta} + \tau \right) (W-w)^2.$$

By Hardy's (Proposition 2.2) and Sobolev's inequality

$$\psi_\tau(W) - \psi_\tau(w) \leq \left(\frac{C}{s_0^{1+\beta}} + C\tau - \frac{1}{2} \right) \int_{B_\tau^+} |\nabla(W-w)|^2. \tag{40}$$

□

Proof of Proposition 3.3. For $\tau > 0$ sufficiently small (31) has a unique solution. Therefore for τ small u_τ is the solution of (31) and the minimizer of ψ_τ.

We claim that if w is any minimizer of ψ_τ then $w \geq v_\tau$ in B_τ^+. Indeed take $v = v_\tau$ in Lemma 3.5 and observe that since $w = u_\tau$ on Γ_τ^i, we have by Lemma 3.2 $w \geq v_\tau$ on Γ_τ^i. Thus we can apply Lemma 3.5. Let us look at (40). We can choose s_0 larger and τ_0 smaller if necessary in order to make $\frac{C}{s_0^{1+\beta}} + C\tau - \frac{1}{2} < 0$. Thus $\psi_\tau(\max(w, v_\tau)) < \psi_\tau(w)$ unless $\max(w, v_\tau) \equiv w$, which is equivalent to assert $v_\tau \leq w$ in B_τ^+.

Let us see now that for $0 < \tau \leq \tau_1$ ψ_τ has a unique minimizer. Indeed, consider w_1, w_2 minimizers of ψ_τ. By the previous claim they satisfy $w_j \geq v_\tau$, $j = 1, 2$. Then from Lemma 3.5 it follows that $w_1 = w_2$. From now on w_τ denotes the unique minimizer of ψ_τ. We claim that the operator $D^2\psi_\tau(w_\tau)$ is coercive on the space $E_\tau = \{w \in H^1(B_\tau^+) \mid w = 0 \text{ on } \Gamma^i \cup \Gamma^2\}$ in the sense that

$$\int_{B_\tau^+} (|\nabla\varphi|^2 + \tau^2\varphi^2) - \int_{\Gamma_\tau^1} \frac{\partial g_\tau^\varepsilon}{\partial u}(x, w_\tau)\varphi^2 \geq \sigma \int_{B_\tau^+} |\nabla\varphi|^2 \tag{41}$$

for some $\sigma > 0$ independent of $0 < \tau \leq \tau_1$ and all $\varphi \in H^1(B_\tau^+)$ with $\varphi = 0$ on $\Gamma_\tau^i \cup \Gamma_\tau^2$. This follows from the behavior of $\frac{\partial g_\tau^\varepsilon}{\partial u}$ as given in (39), the estimate $w_\tau \geq v_\tau \geq cs_0 \operatorname{dist}(y, \Gamma_\tau^2)^{\frac{1}{1+\beta}}$ and Hardy's inequality, Proposition 2.2. We will use this to show that u_τ is the minimizer of ψ_τ. We know that this is true for small $\tau > 0$. Assume this fails for some $0 < \tau < \tau_1$ and set

$$\mu = \inf\{\tau \in (0, \tau_1) \mid \ u_\tau \text{ is not the minimizer of } \psi_\tau\}.$$

Then by continuity u_μ is the minimizer of ψ_μ. Thus $D^2\psi_\mu(u_\mu)$ is coercive in the sense above. On the other hand, for a sequence (τ_j) such that $\mu < \tau_j < \tau_1$, $\tau_j \to \mu$ there are at least two solutions of (31), one being u_τ and the other one the minimizer w_τ of ψ_τ. Both of them are uniformly bounded as $\tau_j \to \mu$. Set

$$z_j = \frac{u_{\tau_j} - w_{\tau_j}}{\|u_{\tau_j} - w_{\tau_j}\|_{L^2(B_{\tau_j}^+)}}.$$

Then

$$\begin{cases} -\Delta z_j + \tau^2 z_j = 0 & \text{in } B^+_{\tau_j} \\ z_j = 0 & \text{on } \Gamma^i_{\tau_j} \cup \Gamma^2_{\tau_j} \\ \dfrac{\partial z_j}{\partial \nu} = \dfrac{\partial g^\varepsilon_{\tau_j}}{\partial u}(y, \xi_j(y)) z_j & \text{on } \Gamma^1_{\tau_j}, \end{cases}$$

where ξ_j is between u_{τ_j} and w_{τ_j}. Multiplying by z_j and integrating we find

$$\int_{B^+_{\tau_j}} (|\nabla z_j|^2 + \tau_j^2 z_j^2) = \int_{\Gamma^1_{\tau_j}} \frac{\partial g^\varepsilon_{\tau_j}}{\partial u}(y, \xi_j(y)) z_j^2.$$

Since z_j is bounded in $L^2(B^+_{\tau_j})$ and for fixed $\varepsilon > 0$ $\frac{\partial g^\varepsilon_{\tau_j}}{\partial u}(y, \xi_j(y))$ is continuous and bounded, we see that z_j is bounded in $H^1(B^+_{\tau_j})$. Thus we can extract a subsequence for which $z_j \rightharpoonup z$ weakly in $H^1(B^+_{\tau_j})$ and strongly in $L^2(B^+_{\tau_j})$. In particular $\|z\|_{L^2(B^+_\mu)} = 1$ which shows that $z \not\equiv 0$. Taking $j \to \infty$ we find

$$\int_{B^+_\mu} (|\nabla z|^2 + \mu^2 z^2) \le \int_{\Gamma^1_\mu} \frac{\partial g^\varepsilon_\mu}{\partial u}(y, u_\mu(y)) z^2,$$

and since $z \not\equiv 0$ we have a contradiction with (41). □

Finally let us show that estimate (30) is enough to obtain the desired result.

Proposition 3.6. *Let $x_1 \in \Omega$ and assume we are in Case 2, i.e.,* (20) *holds. Then*

$$|\nabla u(x_1)| \le C u(x_1)^{-\beta},$$

with a constant that depends on Ω, n, β, f and $\|u\|_{L^\infty(\Omega)}$.

Proof. Recall x_0 given by (20), the definition of τ_1 in (23) and u_{τ_1}, c.f. (24). Let $y_1 = \frac{1}{\tau_1}(x_1 - x_0)$ which satisfies

$$|y_1| \le \frac{1}{6} \tag{42}$$

by (17), (19), (20). A direct calculation shows that it is sufficient to establish

$$|\nabla u_{\tau_1}(y_1)| \le C. \tag{43}$$

By (30) and (11) we have the estimate

$$u_{\tau_1}(y) \ge c s_0 \operatorname{dist}(y, \Gamma^2_{\tau_1})^{\frac{1}{1+\beta}} \quad \forall y \in \Gamma^1_{\tau_1}. \tag{44}$$

Using this in the boundary condition in (31) we deduce that

$$\left|\frac{\partial u_{\tau_1}}{\partial \nu}\right| \le C \operatorname{dist}(y, \Gamma^2_{\tau_1})^{-\frac{\beta}{1+\beta}} + \tau^{\frac{\beta}{1+\beta}} M \quad \text{on } \Gamma^1_{\tau_1}, \tag{45}$$

and therefore, on a smaller set we obtain an estimate

$$\left|\frac{\partial u_{\tau_1}}{\partial \nu}\right| \le C \quad \text{on } B_{1/3} \cap \partial\Omega_{\tau_1}, \tag{46}$$

with a constant C independent of ε.

Let us prove (43). For this purpose choose $p > n$ and take $n < r < \frac{np}{n-1}$. By Lemma 2.4

$$\|u_{\tau_1}\|_{W^{1,r}(B_{1/4}\cap\Omega_{\tau_1})} \le C\left(\left\|\frac{\partial u_{\tau_1}}{\partial \nu}\right\|_{L^p(B_{1/3}\cap\partial\Omega_{\tau_1})} + \|u_{\tau_1}\|_{L^1(B_{1/3}\cap\Omega_{\tau_1})}\right),$$

and by the embedding $W^{1,r} \subset C^\mu$ we have for some $0 < \mu < 1$

$$\|u_{\tau_1}\|_{C^\mu(B_{1/4}\cap\Omega_{\tau_1})} \le C\left(\left\|\frac{\partial u_{\tau_1}}{\partial \nu}\right\|_{L^p(B_{1/3}\cap\partial\Omega_{\tau_1})} + \|u_{\tau_1}\|_{L^1(B_{1/3}\cap\Omega_{\tau_1})}\right).$$

By the assumption (2) and the lower bound (44) we see that the right-hand side of the boundary condition in (31) satisfies

$$\|g^\varepsilon_\tau(y, u_{\tau_1})\|_{C^\mu(B_{1/4}\cap\partial\Omega_{\tau_1})} \le C\left(\left\|\frac{\partial u_{\tau_1}}{\partial \nu}\right\|_{L^p(B_{1/3}\cap\partial\Omega_{\tau_1})} + \|u_{\tau_1}\|_{L^1(B_{1/3}\cap\Omega_{\tau_1})}\right).$$

Using Schauder estimates (see, e.g., [8]) we deduce

$$\|u_{\tau_1}\|_{C^{1,\mu}(B_{1/5}\cap\Omega_{\tau_1})} \le C\left(\left\|\frac{\partial u_{\tau_1}}{\partial \nu_{\tau_1}}\right\|_{L^p(B_{1/3}\cap\partial\Omega_{\tau_1})} + \|u_{\tau_1}\|_{L^1(B_{1/3}\cap\Omega_{\tau_1})}\right).$$

Recalling that $|y_1| \le \frac{1}{6}$ by (42) we obtain

$$|\nabla u_{\tau_1}(y_1)| \le C\left(\left\|\frac{\partial u_{\tau_1}}{\partial \nu}\right\|_{L^p(B_{1/3}\cap\partial\Omega_{\tau_1})} + \|u_{\tau_1}\|_{L^1(B_{1/3}\cap\Omega_{\tau_1})}\right).$$

By (46) we can assert that

$$\left\|\frac{\partial u_{\tau_1}}{\partial \nu}\right\|_{L^p(B_{1/3}\cap\partial\Omega_{\tau_1})} \le C$$

with C independent of ε. It suffices then to find an estimate for $\|u_{\tau_1}\|_{L^1(B_{1/3}\cap\Omega_{\tau_1})}$. Using (45) we see that

$$\left|\frac{\partial u_{\tau_1}}{\partial \nu}\right| \le C \quad \text{on } B_{5/12}\cap\partial\Omega_{\tau_1}$$

and therefore, using Lemma 2.5 we find

$$\int_{B_{1/3}\cap\Omega_{\tau_1}} u_{\tau_1} \le C(u_{\tau_1}(y) + 1), \quad \forall y \in B_{1/2}\cap\Omega_{\tau_1}. \tag{47}$$

Remark that by the choice of τ_1 (cf. 23) we have

$$u_{\tau_1}(y_1) = \widetilde{C}.$$

Thus, selecting $y = y_1$ in (47) (recall (42)) we obtain the desired conclusion. □

4. Proof of Theorem 1.3

We consider the approximating scheme (4) with $f(x,u) = u^p$ and $1 < p < \frac{n}{n-2}$:

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = -\dfrac{u}{(u+\varepsilon)^{1+\beta}} + u^p & \text{on } \partial\Omega. \end{cases} \tag{48}$$

Let Φ_ε be defined as in (6) with

$$g^\varepsilon(u) = \begin{cases} -\frac{u}{(u+\varepsilon)^{1+\beta}} + u^p & \text{if } u \geq 0 \\ |u|^p & \text{if } u < 0. \end{cases}$$

We will show that for fixed $\varepsilon > 0$ (48) has a nontrivial solution, using the mountain pass theorem of Ambrosetti and Rabinowitz [1, 10] in the space $H^1(\Omega)$ with the usual norm $\|u\|_{H^1}^2 = \int_\Omega |\nabla u|^2 + u^2$. We have

$$g^\varepsilon(u)u \geq \theta G^\varepsilon(u) \qquad \forall u \geq u_0$$

for some $\theta > 2$ and some $u_0 > 0$ and this together with the subcritical exponent $1 < p < \frac{n}{n-2}$ implies that the Palais-Smale condition holds for Φ_ε. Also, if $\|u\|_{H^1} = \rho$ we have by the trace embedding theorem

$$\begin{aligned} \int_{\partial\Omega} G^\varepsilon(u) \leq C \int_{\partial\Omega} |u|^{p+1} &\leq a \int_{\partial\Omega} u^2 + C_a \int_{\partial\Omega} |u|^{\frac{2(n-1)}{n-2}} \\ &\leq Ca\|u\|_{H^1}^2 + C_a\|u\|_{H^1}^{p+1} \end{aligned}$$

with $a > 0$ as small as we like. Thus if $\|u\|_{H^1} = \rho$ then

$$\Phi_\varepsilon(u) \geq \frac{1}{2}\rho^2 - Ca\rho^2 - C_a\rho^{p+1} \geq \alpha > 0$$

choosing $\rho >$ small. Notice that ρ and $\alpha > 0$ are independent of ε. Let u_ε denote the mountain pass solution to (48). We will show that $\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C$ for some C independent of ε employing the blow-up method of [4]. Suppose that for a sequence $\varepsilon \to 0$ we have $m_\varepsilon \equiv \|u_\varepsilon\|_{L^\infty(\Omega)} \to \infty$ and let x_ε be a point where the maximum of u_ε in $\overline{\Omega}$ is attained. Then necessarily $x_\varepsilon \in \partial\Omega$ and we can assume that $x_\varepsilon \to x_0 \in \partial\Omega$. Define

$$v_\varepsilon(y) = \frac{1}{m_\varepsilon} u(m_\varepsilon^{1-p} y + x_\varepsilon).$$

Then $\Delta v_\varepsilon + m_\varepsilon^{2(1-p)} v_\varepsilon = 0$ in the domain $\Omega_\varepsilon \equiv (\Omega - x_\varepsilon)/m_\varepsilon^{1-p}$ and

$$\frac{\partial v_\varepsilon}{\partial \nu} = -m_\varepsilon^{-p-\beta} v_\varepsilon^{-\beta} + v_\varepsilon^p \qquad \text{on } \partial\Omega_\varepsilon.$$

The proof of Theorem 1.1 can be adapted to yield a uniform Hölder estimate locally for v_ε:

$$\|v_\varepsilon\|_{C^\gamma(\overline{\Omega}_\varepsilon \cap B_R)} \leq C \qquad \forall \varepsilon > 0$$

for some constant C depending on R but independent of ε. For a subsequence we find that $v_\varepsilon \to v$ uniformly on compact sets with v a nontrivial, nonnegative solution to the problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^n_+ \\ \dfrac{\partial v}{\partial \nu} = v^p & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

where $\mathbb{R}^n_+$ is a half-space. But this is impossible, see, e.g., [5] and also [7]. This shows that u_ε is uniformly bounded in $L^\infty(\Omega)$. Corollary 1.2 implies that $u = \lim_{\varepsilon\to 0} u_\varepsilon$ is a solution to (7). This solution is nontrivial because $\Phi_\varepsilon(u_\varepsilon) \geq \alpha > 0$ for all $\varepsilon > 0$.

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