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A Strong Maximum Principle for the Laplace Equation with Mixed Boundary Condition

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In this work we present a comparison result for two solutions of the Laplace equation in a smooth bounded domain, satisfying the same mixed boundary condition (zero Dirichlet data on part of the boundary and zero Neumann data on the rest). The result is in some sense a generalization of the Hopf lemma to the case of mixed boundary conditions, where the barrier function is not given explicitly, but as the solution of the Laplace equation with a constant right hand side and mixed boundary condition

1. INTRODUCTION

In this work we consider the following situation. Let $\Omega \subset \mathbf{R}^N$, $N \geqslant 2$, be a bounded smooth domain and let Γ_1, Γ_2 be a partition of $\partial\Omega$, with $\Gamma_1 \neq \emptyset$. For simplicity we assume that Γ_1 is relatively closed, with a smooth boundary $\Sigma = \partial\Gamma_1$. Let $f \in C_0^\infty(\Omega)$, $f \geqslant 0$, $f \not\equiv 0$, and let u denote the solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2, \end{cases} \tag{1}$$

where ν is the outer unit normal vector to $\partial\Omega$. By this we mean that u is the unique element in $H = \{w \in H^1(\Omega) \mid w|_{\Gamma_1} = 0\}$ that satisfies

$$\int_\Omega \nabla u \, \nabla \varphi = \int_\Omega f\varphi \qquad \text{for all} \quad \varphi \in H.$$

Let v denote the solution of

$$\begin{cases} -\Delta v = 1 & \text{in } \Omega \\ v = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma_2. \end{cases} \tag{2}$$

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The maximum principle implies that

$$u \leqslant Cv \quad \text{in } \Omega \qquad \text{where} \quad C = \sup_{\Omega} f.$$

The question that motivates this work is: does there exist a constant C_1 such that

$$v \leqslant C_1 u \qquad \text{in } \Omega? \tag{3}$$

We remark that in the case $\Gamma_1 = \partial\Omega$, such an estimate follows from the Hopf lemma applied to u and a Lipschitz estimate for v. In this case (3) can be made more precise,

$$u(x) \geqslant c \left(\int_{\Omega} f(y)\, \delta(y)\, dy \right) \delta(x), \qquad x \in \Omega, \tag{4}$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ and $c > 0$ depends only on Ω. This estimate appeared first in unpublished work by Morel and Oswald [11], and a nice proof of it can be found in the work of Brezis and Cabré [4, Lemma 3.2].

The main result is the following:

THEOREM 1. *Let u denote the solution of (1) with $f \in C_0^\infty(\Omega)$, $f \geqslant 0$, and let v be the solution of (2). Then there exists a constant $c > 0$ depending only on Ω, Γ_1, Γ_2 such that*

$$u(x) \geqslant c \left(\int_{\Omega} f(y)\, v(y)\, dy \right) v(x), \qquad x \in \Omega.$$

This theorem will be derived as a consequence of the following result.

LEMMA 2. *Suppose that v satisfies*

$$\begin{cases} -\Delta v = g & \text{in } \Omega \\ v = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma_2, \end{cases} \tag{5}$$

where $g \in L^p(\Omega)$ with $p > N$. Let u be the solution of (1) where $f \in L^\infty(\Omega)$, $f \geqslant 0$, $f \not\equiv 0$. Then there exists a constant $C > 0$ such that

$$\left\| \frac{v}{u} \right\|_{L^\infty(\Omega)} \leqslant C \, \|g\|_{L^p(\Omega)}.$$

The constant $C > 0$ depends on Ω, Γ_1, Γ_2, N, p, $\|u\|_\infty$, $\|f\|_\infty$ and $1/(\int_\Omega f\delta\, dy)$.

In Section 2 we present the proofs of Lemma 2 and Theorem 1, and in Section 3 we mention some generalizations of these results.

2. PROOF OF THE RESULTS

Before giving the proofs, let us mention that the solution v of (5) is bounded, and for this it suffices that $g \in L^p(\Omega)$, $p > \frac{N}{2}$. This is standard and can be shown by the same technique of Hartman and Stampacchia [9, Lemma 7.3]. Similarly, the solution u of (1) is bounded.

We remark also that the solution u of (1) satisfies

$$u(x) \geqslant c \left(\int_\Omega f\delta \, dy \right) \delta(x).$$

Indeed, $u \geqslant 0$ by (1), and therefore $u \geqslant \tilde{u}$, where

$$\begin{cases} -\Delta \tilde{u} = f & \text{in } \Omega \\ \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

But by [4, Lemma 3.2]

$$u(x) \geqslant \tilde{u}(x) \geqslant c \left(\int_\Omega f\delta \, dy \right) \delta(x), \qquad x \in \Omega.$$

Proof of Lemma 2. First note that by working with $g^+ = \max(g, 0)$ we can assume that $g \geqslant 0$ and $v \geqslant 0$. Let $w = \frac{v}{u}$. We want to prove that w is bounded, and to do so we note that w satisfies (formally)

$$\begin{cases} -\operatorname{div}(u^2 \, \nabla w) = gu - vf & \text{in } \Omega \\ u^2 \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

To show that w is bounded, we follow the same idea as in the work of P. Hartman and G. Stampacchia [9, Lemma 7.3]. We multiply the equation by $(w-k)^+ = \max(w-k, 0)$ where $k \geqslant 0$ and then integrate by parts to obtain (still formally)

$$\int_\Omega u^2 \, |\nabla (w-k)^+|^2 \, dx \leqslant \int_\Omega gu(w-k)^+ \, dx.$$

Then we use the following Sobolev's inequality with weight functions.

LEMMA 3. *Let u denote the solution of* (1) *where* $f \in L^\infty(\Omega)$, $f \geqslant 0$, $f \not\equiv 0$. *Then there exists* $C > 0$ *such that for all* $\varphi \in H^1(\Omega)$

$$\left(\int_\Omega u^r \, |\varphi|^q \, dx \right)^{1/q} \leqslant C \left(\int_\Omega u^2 \, |\nabla \varphi|^2 + u^2 \varphi^2 \, dx \right)^{1/2},$$

where $0 \leqslant r \leqslant 2^*$ ($0 \leqslant r < \infty$ *if* $N = 2$), 2^* *is the classical Sobolev exponent defined by* $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$, *and* q *is given by* $\frac{q}{2} = 1 + \frac{r}{N}$. *The constant* $C > 0$ *depends on* Ω, N, $\|u\|_\infty$, $\|f\|_\infty$, $1/(\int_\Omega f\delta \, dy)$, *and* r *if* $N = 2$.

Proof of Lemma 2 *completed.* First note that for any $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ we have $u\varphi$, $v\varphi \in H$, and therefore we can multiply the equation in (1) by $v\varphi$ and the equation in (5) by $u\varphi$ and integrate by parts to obtain

$$\int_\Omega \varphi \, \nabla u \, \nabla v + v \, \nabla u \, \nabla \varphi \, dx = \int_\Omega f v \varphi \, dx$$

$$\int_\Omega \varphi \, \nabla v \, \nabla u + u \, \nabla v \, \nabla \varphi \, dx = \int_\Omega g u \varphi \, dx$$

and hence

$$\int_\Omega (u \, \nabla v - v \, \nabla u) \, \nabla \varphi \, dx = \int_\Omega (gu - fv) \, \varphi \, dx. \tag{6}$$

Set $\varphi_\varepsilon = (\frac{v}{u+\varepsilon} - k)^+ \in H^1(\Omega) \cap L^\infty(\Omega)$, where $\varepsilon > 0$ and $k \geqslant 0$. Note that

$$\nabla \varphi_\varepsilon = \frac{u \, \nabla v - v \, \nabla u}{(u+\varepsilon)^2} 1_{\{v > k(u+\varepsilon)\}} + \varepsilon \frac{\nabla v}{(u+\varepsilon)^2} 1_{\{v > k(u+\varepsilon)\}}. \tag{7}$$

So, multiplying (7) on both sides by $(u+\varepsilon)^2 \, \nabla \varphi_\varepsilon$ we find

$$(u+\varepsilon)^2 \, |\nabla \varphi_\varepsilon|^2 = (u \, \nabla v - v \, \nabla u) \, \nabla \varphi_\varepsilon + \varepsilon \, \nabla v \, \nabla \varphi_\varepsilon. \tag{8}$$

Using (6) with φ_ε and (8), we obtain

$$\int_\Omega (u+\varepsilon)^2 \, |\nabla \varphi_\varepsilon|^2 \, dx \leqslant \varepsilon \int_\Omega \nabla v \, \nabla \varphi_\varepsilon \, dx + \int_\Omega g u \varphi_\varepsilon \, dx \tag{9}$$

because $f \geqslant 0$, $v \geqslant 0$ and $\varphi_\varepsilon \geqslant 0$.

On the other hand, $0 \leqslant \varphi_\varepsilon \leqslant \frac{v}{\varepsilon}$, and so $\varphi_\varepsilon \in H$. Hence, multiplying (5) by φ_ε and integrating by parts we get:

$$\int_\Omega \nabla v \, \nabla \varphi_\varepsilon \, dx = \int_\Omega g \varphi_\varepsilon \, dx. \tag{10}$$

Combining (10) and (9) we find

$$\int_\Omega (u+\varepsilon)^2\,|\nabla\varphi_\varepsilon|^2\,dx \leqslant \int_\Omega g(u+\varepsilon)\,\varphi_\varepsilon\,dx. \tag{11}$$

At this point we use Lemma 3, which combined with (11) yields

$$\left(\int_\Omega u^r\,|\varphi_\varepsilon|^q\,dx\right)^{2/q} \leqslant C\left(\int_\Omega g(u+\varepsilon)\,\varphi_\varepsilon\,dx + u^2\varphi_\varepsilon^2\,dx\right).$$

But as $\varepsilon\to 0$, $(u+\varepsilon)\,\varphi_\varepsilon = (v-k(u+\varepsilon))^+ \nearrow (v-ku)^+ = u(w-k)^+$ and $\varphi_\varepsilon \nearrow (w-k)^+$. So by monotone convergence we obtain

$$\left(\int_\Omega u^r(w-k)^{+q}\,dx\right)^{2/q} \leqslant C\int_\Omega gu(w-k)^+ + u^2(w-k)^{+2}\,dx.$$

Now we choose $r=\frac{p}{p-1}\in(1,2^*)$. Then, by Hölder's inequality we have

$$\begin{aligned}&\left(\int_\Omega u^r(w-k)^{+q}\,dx\right)^{2/q}\\ &\quad\leqslant C\left(\int_\Omega |g|^p\,dx\right)^{1/p}\left(\int_\Omega u^r(w-k)^{+q}\,dx\right)^{1/q}\left(\int_{\{w>k\}} u^r\,dx\right)^{1-1/p-1/q}\\ &\qquad + C\left(\int_\Omega u^r(w-k)^{+q}\,dx\right)^{2/q}\left(\int_{\{w>k\}} u^{2(q-r)/(q-2)}\,dx\right)^{1-2/q}.\end{aligned} \tag{12}$$

Note that since $r\in(1,2^*)$ we have $q>2$ and $2\,\frac{q-r}{q-2}>0$.

Set $s=2\frac{q-r}{q-2}$. To estimate the last term in (12), observe that

$$k^s\int_{\{w>k\}} u^s\,dx \leqslant \int_\Omega v^s\,dx \leqslant C\,\|v\|_\infty^s.$$

Hence

$$C\left(\int_{\{w>k\}} u^s\,dx\right)^{1-2/q} \leqslant C\left(\frac{\|v\|_\infty}{k}\right)^{s(1-2/q)}. \tag{13}$$

We set

$$k_0=(2C)^{q/(s(q-2))}\,\|v\|_\infty \tag{14}$$

so that for $k\geqslant k_0$, from (12) and (13) we have

$$\left(\int_\Omega u^r(w-k)^{+q}\,dx\right)^{1/q} \leqslant C\,\|g\|_p\left(\int_{\{w>k\}} u^r\,dx\right)^{1-1/p-1/q}.$$

Using Hölder's inequality once more we get

$$\int_\Omega u^r(w-k)^+\,dx \leqslant \left(\int_\Omega u^r(w-k)^{+q}\,dx\right)^{1/q}\left(\int_{\{w>k\}} u^r\,dx\right)^{1-1/q}$$
$$\leqslant C\,\|g\|_p\left(\int_{\{w>k\}} u^r\,dx\right)^{2-2/q-1/p}$$

so finally we obtain

$$\int_\Omega u^r(w-k)^+\,dx \leqslant C\,\|g\|_p\left(\int_{\{w>k\}} u^r\,dx\right)^{\gamma}, \tag{15}$$

where $\gamma = 2-\frac{2}{q}-\frac{1}{p} > 1$, because $p > N$. We set

$$a(k) = \int_\Omega u^r(w-k)^+\,dx = \int_k^\infty \int_{\{w>t\}} u^r\,dx\,dt$$

so that

$$a'(k) = -\int_{\{w>k\}} u^r\,dx$$

and therefore by (15), a satisfies the differential inequality

$$a(k) \leqslant C\,\|g\|_p\,(-a'(k))^{\gamma}, \qquad k \geqslant k_0. \tag{16}$$

As in [9], since $\gamma > 1$ this implies that $a(k) = 0$ for some $k > k_0$. Indeed, integrating (16) between k_0 and k we obtain

$$a^{1-1/\gamma}(k) \leqslant a^{1-1/\gamma}(k_0) - \frac{k-k_0}{C\,\|g\|_p^{1/\gamma}}.$$

Since $a(k)$ is nonnegative, we must have $a(k) = 0$ for some $k \leqslant C\,\|g\|_p^{1/\gamma}\,a^{1-1/\gamma}(k_0) + k_0$, and hence

$$w \leqslant C\,\|g\|_p^{1/\gamma}\,a^{1-1/\gamma}(k_0) + k_0. \tag{17}$$

But by (15)

$$a(k_0) = \int_\Omega u^r(w-k)^+\,dx \leqslant C\,\|g\|_p\left(\int_{\{w>k_0\}} u^r\,dx\right)^{\gamma} \leqslant C\,\|g\|_p$$

and so using (14) and (17)

$$w \leqslant C(\|g\|_p + \|v\|_\infty) \leqslant C\,\|g\|_p. \quad \blacksquare$$

Proof of Lemma 3. We can assume that φ is smooth.

Step 1 (*Case* $r=0$). There exists $C>0$ depending only on Ω such that

$$\int_\Omega \varphi^2\,dx \leqslant C\int_\Omega \delta^2\,|\nabla\varphi|^2+\delta^2\varphi^2\,dx \tag{18}$$

(recall that $\delta(x)=\operatorname{dist}(x,\partial\Omega)$). Since $\delta\leqslant Cu$ we also have

$$\int_\Omega \varphi^2\,dx \leqslant C\int_\Omega u^2\,|\nabla\varphi|^2+u^2\varphi^2\,dx. \tag{19}$$

Proof. We use here Hardy's inequality, which states that there is a constant $C>0$ depending only on Ω such that for all $\psi\in H_0^1(\Omega)$ we have

$$\int_\Omega \frac{\psi^2}{\delta^2}\,dx \leqslant C\int_\Omega |\nabla\psi|^2\,dx. \tag{20}$$

See a proof of this for example in [6].

Now, let λ_1, $\chi_1>0$ be the first eigenvalue and eigenfunction of $-\Delta$ with zero Dirichlet boundary condition, that is

$$\begin{cases} -\Delta\chi_1=\lambda_1\chi_1 & \text{in } \Omega\\ \chi_1=0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

Note that there is a constant $c>0$ (depending only on Ω) such that $\chi_1\geqslant c\delta$. We will establish (18) with δ replaced by χ_1.

Indeed, take $\psi=\varphi\chi_1$ in (20). Then

$$\begin{aligned}\int_\Omega \varphi^2\,dx \leqslant C\int_\Omega \frac{\varphi^2\chi_1^2}{\delta^2} &\leqslant C\int_\Omega |\nabla(\varphi\chi_1)|^2\,dx\\ &= C\int_\Omega \chi_1^2\,|\nabla\varphi|^2+2\chi_1\varphi\,\nabla\chi_1\,\nabla\varphi+\varphi^2\,|\nabla\chi_1|^2\,dx\\ &= C\int_\Omega \chi_1^2\,|\nabla\varphi|^2+\nabla\chi_1\,\nabla(\chi_1\varphi^2)\,dx. \end{aligned}\tag{22}$$

But multiplying Eq. (21) by $\chi_1\varphi^2$ and integrating by parts we find

$$\int_\Omega \nabla\chi_1\,\nabla(\chi_1\varphi^2)\,dx=\lambda_1\int_\Omega \chi_1^2\varphi^2\,dx. \tag{23}$$

Combining (22) with (23) we obtain

$$\int_\Omega \varphi^2\,dx \leqslant C\int_\Omega \chi_1^2\,|\nabla\varphi|^2+\chi_1^2\varphi^2\,dx.$$

Step 2 (*Case* $r=2^*$). We have

$$\left(\int_\Omega |u\varphi|^{2^*}\,dx\right)^{1/2^*}\leqslant C\left(\int_\Omega u^2\,|\nabla\varphi|^2+u^2\varphi^2\,dx\right)^{1/2}, \tag{24}$$

where $\frac{1}{2^*}=\frac{1}{2}-\frac{1}{N}$ is the classical Sobolev exponent (if $N=2$ we take $2^*\in(2,\infty)$).

Proof. By the standard Sobolev inequality, we have

$$\begin{aligned}\left(\int_\Omega |u\varphi|^{2^*}\,dx\right)^{2/2^*}&\leqslant C\int_\Omega |\nabla(u\varphi)|^2+u^2\varphi^2\,dx\\&=C\int_\Omega u^2\,|\nabla\varphi|^2+2u\varphi\,\nabla u\,\nabla\varphi+\varphi^2\,|\nabla u|^2+u^2\varphi^2\,dx\\&=C\int_\Omega u^2\,|\nabla\varphi|^2+\nabla u\,\nabla(u\varphi^2)+u^2\varphi^2\,dx.\end{aligned}$$

To estimate $\int_\Omega \nabla u\,\nabla(u\varphi^2)\,dx$ we multiply Eq. (1) by $u\varphi^2$ and integrate by parts (note that $u\varphi^2\in H$):

$$\int_\Omega \nabla u\,\nabla(u\varphi)^2\,dx=\int_\Omega fu\varphi^2\,dx.$$

Therefore

$$\begin{aligned}\left(\int_\Omega |u\varphi|^{2^*}\,dx\right)^{2/2^*}&\leqslant C\int_\Omega u^2\,|\nabla\varphi|^2+fu\varphi^2+u^2\varphi^2\,dx\\&\leqslant C\int_\Omega u^2\,|\nabla\varphi|^2\,dx+\varphi^2+u^2\varphi^2\,dx\\&\leqslant C\int_\Omega u^2\,|\nabla\varphi|^2+u^2\varphi^2\,dx\end{aligned}$$

by (19).

Step 3. We now combine estimates (19) and (24) to obtain the conclusion. By Hölder's inequality, for any $0<\lambda<1$ we have

$$\int_\Omega u^r\,|\varphi|^q\,dx \leqslant \left(\int_\Omega |\varphi|^2\,dx\right)^{1-\lambda}\left(\int_\Omega u^{r/\lambda}\,|\varphi|^{(q-2(1-\lambda))/\lambda}\,dx\right)^{\lambda}.$$

Imposing $\frac{r}{\lambda}=\frac{q-2(1-\lambda)}{\lambda}=2^*$ we find the relations $1+\frac{r}{N}=\frac{q}{2}$ and $r=\lambda 2^*\in(0,2^*)$. Thus by (19) and (24) we have

$$\int_\Omega u^r\,|\varphi|^q\,dx \leqslant C\left(\int_\Omega u^2\,|\nabla\varphi|^2+u^2\varphi^2\,dx\right)^{1+\lambda((2^*/2)-1)}.$$

Finally note that $1+\lambda(\frac{2^*}{2}-1)=\frac{q}{2}$. ∎

Proof of Theorem 1. This proof is essentially the same as the one of Lemma 3.2 in Brezis and Cabré [4], where they derive (4) from the standard Hopf lemma. Here, instead of $\delta(x)=\operatorname{dist}(x,\partial\Omega)$ we use the function v which is the solution of (2). u denotes the solution of (1).

Fix a point $x_0\in\Omega$ and let $r=\frac{1}{3}\operatorname{dist}(x_0,\partial\Omega)$. Let $B=B_r(x_0)$, $f_0\in C_0^\infty(B)$, $0\leqslant f_0\leqslant 1$, $f_0\not\equiv 0$, and let u_0 denote the solution of (1) with right hand side f_0.

Then by Lemma 2 we have $u_0\geqslant cv$ where $c>0$ and depends only on Ω, $\|u_0\|_\infty$ and $\|v\|_\infty$.

Now, for $x\in\bar{B}$ we have $\operatorname{supp}(f_0)\subset B_{2r}(x)\subset\Omega$. So, since u_0 is superharmonic in Ω we have for $x\in\bar{B}$

$$u(x)\geqslant\frac{1}{|B_{2r(x)}|}\int_{B_{2r}(x)}u\,dy\geqslant c'\int_\Omega uf_0\,dy=c'\int_\Omega fu_0\,dy$$

$$\geqslant c''\int_\Omega fv\,dy\geqslant c'''\left(\int_\Omega fv\,dy\right)u_0(x)$$

Set $\lambda=c'''\int_\Omega fv\,dy$. Then $u-\lambda u_0\geqslant 0$ on ∂B, $u-\lambda u_0=0$ on Γ_1, $\frac{\partial}{\partial\nu}(u-\lambda u_0)=0$ on Γ_2, and $-\Delta(u-\lambda u_0)\geqslant 0$ on $\Omega\backslash\bar{B}$. Thus we obtain

$$u(x)\geqslant c'''\left(\int_\Omega fv\,dy\right)u_0(x)\qquad\text{in }\Omega$$

and therefore

$$u(x)\geqslant c\left(\int_\Omega fv\,dy\right)v(x)\qquad\text{in }\Omega.\ ∎$$

3. VARIOUS GENERALIZATIONS

Theorem 1 admits a generalization to weak solutions and weak supersolutions.

Definition. We say that $u \in L^1(\Omega)$ is a weak supersolution of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2 \end{cases} \tag{25}$$

if

$$\int_\Omega u(-\Delta \zeta)\, dx \geqslant 0$$

for all $\zeta \geqslant 0$ solution of

$$\begin{cases} -\Delta \zeta = \varphi & \text{in } \Omega \\ \zeta = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial \zeta}{\partial \nu} = 0 & \text{on } \Gamma_2 \end{cases}$$

for some $\varphi \in C_0^\infty(\Omega)$.

Lemma 4. *Let $u \in L^1(\Omega)$ be a weak supersolution of* (25). *Then either $u \equiv 0$ or there exists a constant $c > 0$ such that*

$$u(x) \geqslant c v(x) \qquad \textit{a.e. in } \Omega$$

where v is the solution of (2).

Proof. Note that since u is a weak supersolution of (25) we have $u \geqslant 0$ in Ω. Indeed, if $\varphi \in C_0^\infty(\Omega)$, and $\varphi \geqslant 0$, let ζ be the solution of

$$\begin{cases} -\Delta \zeta = \varphi & \text{in } \Omega \\ \zeta = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial \zeta}{\partial \nu} = 0 & \text{on } \Gamma_2 \end{cases}$$

Note that $\zeta \geqslant 0$, so by definition of weak supersolution $0 \leqslant \int_\Omega u(-\Delta \zeta)\, dx = \int_\Omega u\varphi\, dx$.

Since $u \geqslant 0$ and u is superharmonic, we have either $u \equiv 0$ or $\operatorname{ess\,inf}_K u > 0$ for any compact set $K \subset \Omega$. Then the proof proceeds as in Lemma 1. ▮

Remark. The original interest in obtaining (3) is its application to a generalization of a result of Martel [10], concerning the uniqueness of the extremal solution for a nonlinear problem. More precisely, let g: $[0, \infty) \to [0, \infty)$ be a smooth, nondecreasing, convex function with $g(0) > 0$ and $\int_0^\infty \frac{ds}{g(s)} < \infty$. For $\lambda > 0$ we consider the nonlinear problem

$$(P_\lambda) \begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2. \end{cases}$$

The case with zero Dirichlet boundary condition has been extensively studied (see for example [3, 5, 10] and the references therein), and one of the basic results in this case is the existence of an extremal parameter $\lambda^* \in (0, \infty)$, such that for $0 < \lambda \leqslant \lambda^*$ (P_λ) admits a solution and for $\lambda > \lambda^*$ (P_λ) has no solution. The result of Martel [10] is that for $\lambda = \lambda^*$ (P_λ) has a unique solution. An analog statement holds for the case of a mixed boundary condition, and Lemma 4 plays a crucial role in its proof.

Remark. We note that all the results stated before are valid if instead of a mixed boundary condition we work with a Robin boundary condition

$$\frac{\partial u}{\partial \nu} + \sigma u = 0 \qquad \text{on } \partial\Omega,$$

where $0 \leqslant \sigma(x) \leqslant \infty$, $x \in \partial\Omega$, and σ is only assumed to be Borel measurable. The proofs remain almost unchanged.

Remark. For our results we assumed that $\partial\Omega$ is smooth, and from the proofs we see that C^2 is enough regularity. It is then a natural question to ask whether or not (3) is still true in domains with only a Lipschitz boundary. The answer is negative, as the next example shows. Let us mention that the failure of (3) in domains with corners was already noticed by Berestycki, Nirenberg and Varadhan [1] (see Remark 5.1 in page 70). There is also some relation with the failure of the so called “anti-maximum principle”; see Birindelli [2]. The anti-maximum principle holds in smooth domains and was discovered by Clément and Peletier [7].

EXAMPLE. Consider a truncated cone in the complex plane of angle $\alpha \in (0, \pi)$ and radius 1

$$\Omega = \{z \in \mathbf{C} \mid \arg(z) \in (0, \alpha),\ |z| < 1\},$$

where $\arg(re^{i\theta}) = \theta$, $\theta \in [0, 2\pi)$. Let v denote the solution of

$$\begin{cases} -\Delta v = 1 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

and u be the solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $f \in C_0^\infty(\Omega)$, $f \geqslant 0$, $f \not\equiv 0$.

Claim 1. If $0 < \alpha \leqslant \pi/2$ there is no constant C_1 such that

$$v \leqslant C_1 u. \tag{26}$$

Proof. We use the change of variables $z \to w = z^{\pi/\alpha}$, which maps $z \in \Omega$ to $w \in B_+ = \{w \in \mathbf{C} \mid \operatorname{Im}(w) > 0, |w| < 1\}$. Let $\tilde{u}(w) = u(z)$, $\tilde{v}(w) = v(z)$ and $\tilde{f}(w) = f(z)$. Then $\tilde{u}$ satisfies the equation

$$\begin{cases} -\Delta \tilde{u} = \left(\dfrac{\alpha}{\pi}\right)^2 |w|^{2(\alpha/\pi - 1)} \tilde{f}(w) & \text{in } B_+ \\ \tilde{u} = 0 & \text{on } \partial B_+ \end{cases}$$

and similarly $\tilde{v}$ satisfies

$$\begin{cases} -\Delta \tilde{v} = \left(\dfrac{\alpha}{\pi}\right)^2 |w|^{2(\alpha/\pi - 1)} & \text{in } B_+ \\ \tilde{v} = 0 & \text{on } \partial B_+. \end{cases} \tag{27}$$

Since f has compact support in Ω we see that $\tilde{u}$ is smooth in a neighborhood of 0. On the other hand, the regularity of $\tilde{v}$ depends on α. First note that the right hand side of (27),

$$g(w) = \left(\frac{\alpha}{\pi}\right)^2 |w|^{2(\alpha/\pi - 1)},$$

belongs to $L^p(B_+)$ only for $p < \frac{\pi}{\pi - \alpha}$. In particular $g \in L^p(B_+)$ for some $p > 2$ only if $\alpha > \pi/2$. From here we expect $\tilde{v}$ not to be Lipschitz at 0 if $0 < \alpha \leqslant \pi/2$.

To prove Claim 1 let

$$g_m(w) = \min(m, g(w)), \qquad m > 0$$

and $\tilde{v}_m$ the solution of

$$\begin{cases} -\Delta \tilde{v}_m = g_m & \text{in } B_+ \\ \tilde{v}_m = 0 & \text{on } \partial B_+. \end{cases}$$

Note that since g_m is bounded, $\tilde{v}_m$ is $C^{1,\alpha}$ in a neighborhood of 0. Now, if (26) holds for some constant C_1, then since $\tilde{v}_m \leqslant v$ we have

$$\left| \frac{\partial \tilde{v}_m}{\partial x_2}(0) \right| \leqslant C$$

with C independent of m. But

Claim 2. If $0 < \alpha \leqslant \pi/2$

$$\frac{\partial \tilde{v}_m}{\partial x_2}(0) \to \infty$$

as $m \to \infty$, and this shows that (26) is impossible.

Indeed, consider the Newtonian potential (see for example [8])

$$\bar{v}_m(x) = -\frac{1}{2\pi} \int_{B_+} (\log |x-w| - \log |x^* - w|)\, g_m(w)\, dw, \tag{28}$$

where $x^* = (x_1, x_2)^* = (x_1, -x_2)$. Then $\bar{v}_m - \tilde{v}_m$ is harmonic in B_+, vanishes on $\Sigma = \partial B_+ \cap \{x_2 = 0\}$ and is bounded independently of m on $\partial B_+ \backslash \Sigma$. Hence,

$$\left| \frac{\partial (\bar{v}_m - \tilde{v}_m)}{\partial x_2}(0) \right| \leqslant C$$

with C independent of m. But a calculation using (28) shows that

$$\frac{\partial \bar{v}_m}{\partial x_2}(0) = \frac{1}{\pi} \int_{B_+} \frac{w_2}{|w|^2}\, g_m(w)\, dw \tag{29}$$

and since $g_m(w)$ increases to $g(w) = (\frac{\alpha}{\pi})^2 \, |w|^{2(\alpha/\pi - 1)}$, the expression on the right hand side of (29) increases to

$$\frac{\alpha^2}{\pi^3} \int_{B_+} w_2 \, |w|^{2(\alpha/\pi - 1) - 2}\, dw.$$

Finally note that this integral is finite only for $\alpha > \pi/2$.

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REFERENCES

1. H. Berestycki, L. Nirenberg, and S. R. S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, *Comm. Pure Appl. Math.* **47** (1994), 47–92.
2. I. Birindelli, Hopf's lemma and anti-maximum principle in general domains, *J. Differential Equations* **119** (1995), 450–472.
3. H. Brezis, Is there failure of the inverse function theorem?, *in* "Proceedings of the Workshop held at the Morningside Center of Mathematics, Chinese Academy of Science, Beijing, June 1999."
4. H. Brezis and X. Cabré, Some simple nonlinear pde's without solutions, *Boll. Unione Mat. Ital. Sez. B* **1** (1998), 223–262.
5. H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, Blow up for $u_t - \Delta u = g(u)$ revisited, *Adv. Differential Equations* **1** (1996), 73–90.
6. H. Brezis and M. Marcus, Hardy's inequalities revisited, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **25** (1997), 217–237.
7. Ph. Clément and L. A. Peletier, An anti-maximum principle for second-order elliptic operators, *J. Differential Equations* **34** (1979), 218–229.
8. D. Gilbarg and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order," second ed., Springer-Verlag, Berlin, 1983.
9. P. Hartman and G. Stampacchia, On some non-linear elliptic differential-functional equations, *Acta Math.* **115** (1996), 271–310.
10. Y. Martel, Uniqueness of weak extremal solutions of nonlinear elliptic problems, *Houston J. Math.* **23** (1997), 161–168.
11. J. M. Morel and L. Oswald, A uniform formulation for the Hopf maximum principle, preprint, 1985.