Concentration on Curves for Nonlinear Schrödinger Equations

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Abstract

We consider the problem

$$\varepsilon^2 \Delta u - V(x)u + u^p = 0, \quad u > 0, \; u \in H^1(\mathbb{R}^2),$$

where $p > 1$, $\varepsilon > 0$ is a small parameter, and V is a uniformly positive, smooth potential. Let Γ be a closed curve, nondegenerate geodesic relative to the weighted arc length $\int_\Gamma V^\sigma$, where $\sigma = (p+1)/(p-1) - 1/2$. We prove the existence of a solution u_ϵ concentrating along the whole of Γ, exponentially small in ε at any positive distance from it, provided that ε is small and away from certain critical numbers. In particular, this establishes the validity of a conjecture raised in [3] in the two-dimensional case. © 2006 Wiley Periodicals, Inc.

1 Introduction and Statement of Main Result

We consider *standing waves* for a nonlinear Schrödinger equation in $\mathbb{R}^N$ of the form

$$-i\varepsilon \frac{\partial \psi}{\partial t} = \varepsilon^2 \Delta \psi - Q(y)\psi + |\psi|^{p-1}\psi \tag{1.1}$$

where $p > 1$, namely, solutions of the form $\psi(t, y) = \exp(i\lambda\varepsilon^{-1}t)u(y)$. Assuming that the amplitude $u(y)$ is positive and vanishes at infinity, we see that this ψ satisfies (1.1) if and only if u solves the nonlinear elliptic problem

$$\varepsilon^2 \Delta u - V(y)u + u^p = 0, \quad u > 0, \; u \in H^1(\mathbb{R}^N), \tag{1.2}$$

where $V(y) = Q(y) + \lambda$. In the rest of this paper we will assume that V is a smooth function with

$$\inf_{y \in \mathbb{R}^2} V(y) > 0.$$

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Considerable attention has been paid in recent years to the problem of constructing standing waves in the so-called *semiclassical limit* of (1.1) $\varepsilon \to 0$. In the pioneering work [14], Floer and Weinstein constructed positive solutions to this problem when $p = 3$, $N = 1$, such that the concentration is taking place near a given nondegenerate critical point y_0 of $V(y)$ and the solutions are exponentially small in ϵ outside any neighborhood of y_0. More precisely, they established the existence of a solution u_ε such that

$$u_\varepsilon(y) \sim V(y_0)^{\frac{1}{p-1}} w(V(y_0)^{\frac{1}{2}} \varepsilon^{-1}(y - y_0))$$

where w is the unique solution of

$$w'' - w + w^p = 0, \quad w > 0, \ w'(0) = 0, \ w(\pm\infty) = 0. \tag{1.3}$$

Many authors have subsequently extended this result to higher dimensions to the construction of solutions exhibiting high concentration around one or more points of space under various assumptions on the potential and nonlinearity. We refer the reader, for instance, to [2, 4, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 22, 31, 32, 34, 36, 37, 38].

An important question is whether solutions exhibiting concentration on higher-dimensional sets exist. In [3], Ambrosetti, Malchiodi, and Ni have considered the case of $V = V(|y|)$, also treated in [5, 6], and constructed radial solutions $u_\varepsilon(|y|)$ exhibiting concentration on a sphere $|y| = r_0$ in the form

$$u_\varepsilon(r) \sim V(r_0)^{\frac{1}{p-1}} w(V(r_0)^{\frac{1}{2}} \varepsilon^{-1}(r - r_0))$$

under the assumption that $r_0 > 0$ is a nondegenerate critical point of

$$M(r) = r^{N-1} V^\sigma(r) \tag{1.4}$$

where

$$\sigma = \frac{p+1}{p-1} - \frac{1}{2}, \tag{1.5}$$

and w is again the unique solution of (1.3). The conjecture is raised in [3] that this type of phenomenon takes place, at least along a sequence $\varepsilon = \varepsilon_n \to 0$, whenever the sphere $|y| = r_0$ is replaced by a closed hypersurface Γ that is *stationary and nondegenerate* for the weighted area functional $\int_\Gamma V^\sigma$. In this paper we prove the validity of this conjecture in dimension $N = 2$.

For $N = 2$, the functional above defined on closed Jordan curves Γ has a simple geometrical meaning: it corresponds to the arc length of Γ measured with respect to the metric $V^\sigma(dy_1^2 + dy_2^2)$ in $\mathbb{R}^2$. Thus we will establish the concentration phenomenon on Γ provided that this curve is a nondegenerate closed geodesic for this metric in $\mathbb{R}^2$.

We do not prove the result for all small values $\varepsilon > 0$ but only for those which lie away from certain critical numbers. More precisely, there is an explicit number

$\lambda_* > 0$ such that given $c > 0$, if ε is sufficiently small and satisfies the gap condition

$$|k^2\varepsilon^2 - \lambda_*| \geq c\varepsilon \quad \text{for all } k \in \mathbb{N}, \tag{1.6}$$

then a solution u_ε with the required concentration property indeed exists. In other words, this will be the case whenever ε is small and away from the critical numbers $\sqrt{\lambda_*}/k$ in the sense that for fixed and arbitrarily small $c < \sqrt{\lambda_*}$,

$$\varepsilon \notin \left[\frac{\sqrt{\lambda_*}}{k} - \frac{c}{k^2}, \frac{\sqrt{\lambda_*}}{k} + \frac{c}{k^2}\right] \quad \text{for all } k \in \mathbb{N}.$$

To state our main result, we need to make precise the concept of a curve Γ being stationary and nondegenerate for the weighted length functional $\int_\Gamma V^\sigma$.

Let Γ be a closed smooth curve in $\mathbb{R}^2$ and $\ell = |\Gamma|$ its total length. We consider the natural parametrization $\gamma(\theta)$ of Γ with positive orientation, where θ denotes an arc length parameter measured from a fixed point of Γ. Let $\nu(\theta)$ denote the outer unit normal to Γ. Points y that are δ_0-close to Γ for sufficiently small δ_0 can be represented in the form

$$y = \gamma(\theta) + t\nu(\theta), \quad |t| < \delta_0, \ \ \theta \in [0, \ell), \tag{1.7}$$

where map $y \mapsto (t, \theta)$ is a local diffeomorphism. Any curve sufficiently close to Γ can be parametrized as

$$\gamma_g(\theta) = \gamma(\theta) + g(\theta)\nu(\theta).$$

where g is a smooth, l-periodic function with small L^∞-norm. Call Γ_g the curve defined this way. By a slight abuse of notation, we denote $V(t, \theta)$ to actually mean $V(y)$ for y in (1.7). Then the weighted length of this curve is given by the functional of g

$$\begin{aligned} J(g) \equiv \int_{\Gamma_g} V^\sigma &= \int_0^\ell V^\sigma(\gamma_g(\theta))|\gamma_g'(\theta)| d\theta \\ &= \int_0^\ell V^\sigma(g(\theta), \theta)|\gamma' + g\nu' + g'\nu| d\theta. \end{aligned}$$

Since $|\gamma'| = 1$ and $\nu' = k(\theta)\gamma'$, where $k(\theta)$ denotes the curvature of Γ, we get that the above quantity becomes

$$J(g) = \int_0^\ell V^\sigma(g(\theta), \theta)[(1 + kg)^2 + (g')^2]^{\frac{1}{2}}\, d\theta. \tag{1.8}$$

Γ is said to be *stationary* for the weighted length $\int_\Gamma V^\sigma$ if the first variation of the functional (1.8) at $g = 0$ is equal to zero. That is, for any smooth, ℓ-periodic function $h(\theta)$

$$0 = J'(0)[h] = \int_0^\ell [(V^\sigma)_t h + V^\sigma k h] d\theta,$$

which is equivalent to the relation

$$\sigma V_t(0,\theta) = -k(\theta) V(0,\theta) \quad \text{for all } \theta \in (0,\ell). \tag{1.9}$$

We assume the validity of this relation at Γ.

Let us consider now the second variation quadratic form

$$\begin{aligned} J''(0)[h,h] &= \frac{1}{2}\int_0^\ell [V^\sigma |h'|^2 + [(V^\sigma)_{tt} + 2(V^\sigma)_t k]h^2] d\theta \\ &= \frac{1}{2}\int_0^\ell [V^\sigma |h'|^2 + [(V^\sigma)_{tt} - 2V^\sigma k^2]h^2] d\theta. \end{aligned}$$

We say that Γ is *nondegenerate* if this quadratic form is in the space of all functions $h \in H^1(0,\ell)$ with $h(0) = h(\ell)$. This is equivalent to the statement that the differential equation

$$(V^\sigma h')' - [(V^\sigma)_{tt} - 2V^\sigma k^2]h = 0$$

has only the ℓ-periodic solution $h \equiv 0$, or using (1.9), that the boundary value problem

$$\begin{gathered} h'' + \sigma V^{-1} V_\theta h' - [\sigma V^{-1} V_{tt} - (\sigma^{-1}+1)k^2]h = 0, \\ h(0) = h(\ell), \quad h'(0) = h'(\ell), \end{gathered} \tag{1.10}$$

has only the trivial solution.

As an example, let us consider the radial case $V = V(r)$. Then we see that $\Gamma = \{r = r_0\}$ is stationary precisely if $M'(r_0) = 0$ where M is defined by (1.4). If in addition $M''(r_0) > 0$, namely, if r_0 is a nondegenerate local minimizer, we have that $(V^\sigma)_{tt} + 2(V^\sigma)_t k > 0$. This automatically makes the quadratic form $J''(0)[h,h]$ positive definite, hence nonsingular. A nondegenerate stationary curve close to Γ will still be present if the radial potential is modified by small nonradial perturbations. The geometric interpretation allows the construction of other examples. If $V^\sigma(y) \sim 1/(1+|y|^2)^2$ up to a large value of $|y|$, then the metric $V^\sigma\, dy^2$ represents approximately that of a sphere embedded in $\mathbb{R}^3$. If eventually V increases so that $V(y) \sim 1$ for very large $|y|$, the whole metric will resemble that of a "globe attached to a plane," the presence of at least two geodesics thus being clear: one on the globe, the other on the connecting neck. Nondegeneracy of these geodesics is not true in general, but may be generically expected in a suitably strong C^m-topology.

We need another element to describe the gap condition (1.6). Let w denote the unique positive solution of problem (1.3). We consider the associated linearized eigenvalue problem

$$h'' - h + pw^{p-1}h = \lambda h \quad \text{in } \mathbb{R}, \ \ h(\pm\infty) = 0. \tag{1.11}$$

It is well-known that this equation possesses a unique positive eigenvalue λ_0 in $H^1(\mathbb{R})$, with associated eigenfunction even and positive Z, which we normalize so that $\int_{\mathbb{R}} Z^2 = 1$ (this follows, for instance, from the analysis in [30]). In fact, a simple computation shows that

$$\lambda_0 = \frac{1}{4}(p-1)(p+3), \quad Z = \frac{1}{\sqrt{\int_{\mathbb{R}} w^{p+1}}} w^{\frac{p+1}{2}}. \tag{1.12}$$

We define the number λ_* as

$$\lambda_* = \lambda_0 \frac{1}{4\pi^2} \left(\int_0^{\ell} V(0,\theta)^{\frac{1}{2}}\, d\theta \right)^2. \tag{1.13}$$

Now we can state our main result.

THEOREM 1.1 *Let Γ be a nondegenerate, stationary curve for the weighted length functional $\int_\Gamma V^\sigma$, as described above. Then given $c > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ satisfying the gap condition*

$$|\varepsilon^2 k^2 - \lambda_*| \ge c\varepsilon \quad \forall k \in \mathbb{N}, \tag{1.14}$$

where $\lambda_* > 0$ is the number in (1.13)*, problem* (1.2) *has a positive solution u_ϵ, which near Γ, for y given by* (1.7)*, takes the form*

$$u_\epsilon(y) = V(0,\theta)^{\frac{1}{p-1}} w\left(V(0,\theta)^{\frac{1}{2}} \frac{t}{\varepsilon} \right)(1 + o(1)). \tag{1.15}$$

For some number $c_0 > 0$, u_ε satisfies globally

$$u_\varepsilon(y) \le \exp(-c_0 \varepsilon^{-1} \operatorname{dist}(y, \Gamma)).$$

To explain in a few words the difficulties encountered in constructing these solutions, let us assume for the moment that $V \equiv 1$ on Γ and that $\ell = 2\pi$. Then in terms of the stretched coordinates $(s, z) = \varepsilon^{-1}(t, \theta)$, the equation would look near the curve *approximately* like

$$v_{zz} + v_{ss} + v^p - v = 0, \quad (s, z) \in \mathbb{R}^2,$$

where v is $2\pi\varepsilon^{-1}$-periodic in the z-variable. The effect of curvature and of variations of V are here neglected. The linearization of this problem around the profile $w(s)$ thus becomes

$$\phi_{zz} + \phi_{ss} + p w^{p-1}\phi - \phi = 0, \quad (s, z) \in \mathbb{R}^2,$$

with ϕ being $2\pi\varepsilon^{-1}$-periodic in z. Functions of the form

$$\phi^1 = w_s(s)[a \sin k\varepsilon z + b \cos k\varepsilon z], \quad \phi^2 = Z(s)[a \sin k\varepsilon z + b \cos k\varepsilon z],$$

are eigenfunctions associated to eigenvalues $-k^2\varepsilon^2$ and $\lambda_0 - k^2\varepsilon^2$, respectively. Many of these numbers are small, and "near noninvertibility" of the linear operator thus occurs. Therefore the use of a fixed-point argument after inverting the linear operator in the actual nonlinear problem is a very delicate matter. Worse than this, these two effects combined, in principle orthogonal because of the L^2-orthogonality of Z and w_s, are actually coupled through the smaller-order terms neglected. In [1, 20, 19, 33] related singular perturbation problems involving the

Allen-Cahn equation in phase transitions exhibiting only the translation effect ϕ^1 have been successfully treated through successive improvements of the approximation and fine spectral analysis of the actual linearized operator. The principle is simple: the better the approximation, the higher the chances of a correct inversion of the linearized operator to obtain a contraction mapping formulation of the problem. In [26, 27, 28, 35] resonance phenomena similar to the "ϕ^2 effect" have been faced in related problems. In [26, 27] a Neumann problem involving whole boundary concentration, widely treated for point concentration after the works [23, 29, 30], has been considered. Recently in [24, 25] this boundary concentration on a geodesic of the boundary in the three-dimensional case has been treated via arbitrarily high-order approximations. Our method, closer in spirit to that of Floer and Weinstein, provides substantial simplification and flexibility to deal with larger noise and coupling of the two effects inherent in this problem.

The solution to the full problem in the above idealized situation is roughly decomposed in the form

$$v(s,z) = w(s - f(\varepsilon z)) + \varepsilon e(\varepsilon z) Z(s - f(\varepsilon z)) + \tilde{\phi}(s,z)$$

where f and e are 2π-periodic functions left as parameters, while $\tilde{\phi}(s,z)$ is $L^2(ds)$-orthogonal for each z both to $w_s(s - f(\varepsilon z))$ and to $Z(s - f(\varepsilon z))$. Solving first in $\tilde{\phi}$ a natural projected problem where the linear operator is uniformly invertible, the resolution of the full problem becomes reduced to a nonlinear, nonlocal second-order system of differential equations in (f, e), which turns out to be directly solvable thanks to the assumptions made. This approach is familiar when the parameters (f, e) lie in a finite-dimensional space, corresponding this time to adjusting infinitely many parameters. To stress the difference with the radial case: the parameter e is not present, and f is just a single number. The analysis we make takes special advantage through Fourier analysis of the fact that the objects to be adjusted are one-variable functions, while we still believe that the current approach may be modified to the higher-dimensional case. We also believe the gap condition may be improved to size $s\varepsilon^q$ for any $q > 1$.

In the rest of the paper we carry out the program outlined above that leads to the proof of Theorem 1.1.

2 Setup near the Curve

Let Γ be the curve in the statement of the theorem. We shall use the notation introduced in the previous section.

Stretching variables, absorbing ε from Laplace's operator and replacing $u(y)$ by $u(\varepsilon y)$, equation (1.2) becomes

$$\Delta u - V(\varepsilon y)u + u^p = 0, \quad u > 0, \quad u \in H^1(\mathbb{R}^2). \tag{2.1}$$

Let $(s,z)=\varepsilon^{-1}(t,\theta)$ be natural stretched coordinates associated to the curve $\Gamma_\varepsilon = \varepsilon^{-1}\Gamma$, now defined for

$$z\in[0,\varepsilon^{-1}\ell), \quad s\in(-\varepsilon^{-1}\delta_0, \varepsilon^{-1}\delta_0). \tag{2.2}$$

Equation (2.1) for u expressed in these coordinates becomes

$$u_{zz}+u_{ss}+B_1(u)-V(\varepsilon s,\varepsilon z)u+u^p=0 \tag{2.3}$$

in the region (2.2), where

$$B_1(u)=u_{zz}\left[1-\frac{1}{(1+\varepsilon k(\varepsilon z)s)^2}\right]+\frac{\varepsilon k(\varepsilon z)u_s}{1+\varepsilon k(\varepsilon z)s}+\frac{\varepsilon^2 s k'(\varepsilon z)u_z}{(1+\varepsilon k(\varepsilon z)s)^3}.$$

For further reference, it is convenient to expand this operator in the form

$$B_1(u)=(\varepsilon k(\varepsilon z)-\varepsilon^2 s k^2(\varepsilon z))u_s+B_0(u), \tag{2.4}$$

where

$$B_0(u)=\varepsilon^2 s a_1(\varepsilon s,\varepsilon z)u_z+\varepsilon s a_2(\varepsilon s,\varepsilon z)u_{zz}+\varepsilon^3 s^2 a_3(\varepsilon s,\varepsilon z)u_s \tag{2.5}$$

for certain smooth functions $a_j(t,\theta)$, $j=1,2,3$. Observe that all terms in the operator B_1 have ε as a common factor.

We now consider a further change of variables in equation (2.3) with the property that it replaces at main order the potential V by 1. Let

$$\alpha(\theta)=V(0,\theta)^{\frac{1}{p-1}}, \quad \beta(\theta)=V(0,\theta)^{\frac12}, \tag{2.6}$$

and fix a twice differentiable, ℓ-periodic function $f(\theta)$. We define $v(x,z)$ by the relation

$$u(s,z)=\alpha(\varepsilon z)v(x,z), \quad x=\beta(\varepsilon z)(s-f(\varepsilon z)). \tag{2.7}$$

We want to express equation (2.3) in terms of these new coordinates. We compute:

$$u_s=\alpha\beta v_x, \quad u_{ss}=\alpha\beta^2 v_{xx}, \tag{2.8}$$

$$u_z=\varepsilon\alpha' v+\alpha v_x(\beta(s-f))_z+\alpha v_z, \tag{2.9}$$

$$\begin{aligned} u_{zz}=\varepsilon^2\alpha'' v &+ 2\varepsilon\alpha'[v_x(\beta(s-f))_z+v_z] \\ &+\alpha\big[v_{xx}|(\beta(s-f))_z|^2+2v_{xz}(\beta(s-f))_z \\ &\quad + v_x(\beta(s-f))_{zz}+v_{zz}\big]. \end{aligned} \tag{2.10}$$

We also have

$$\begin{aligned} (\beta(s-f))_z &= \varepsilon[\beta'(s-f)-\beta f'], \\ (\beta(s-f))_{zz} &= \varepsilon^2[\beta''(s-f)-2\beta' f'-\beta f'']. \end{aligned}$$

In order to write down the equation, it is also convenient to expand

$$V(\varepsilon s,\varepsilon z)=V(0,\varepsilon z)+V_t(0,\varepsilon z)\varepsilon s+\frac12 V_{tt}(0,\varepsilon z)\varepsilon^2 s^2+a_4(\varepsilon s,\varepsilon z)\varepsilon^3 s^3 \tag{2.11}$$

for a smooth function $a_4(t,\theta)$. It turns out that u solves (2.3) if and only if v defined by (2.7) solves

$$S(v) \equiv B_3(v) + \beta^{-2} v_{zz} + v_{xx} + v^p - v = 0, \tag{2.12}$$

where $B_3(v)$ is a linear differential operator defined by

$$\begin{aligned}
B_3(v) = {} & \beta^{-1}\left[\varepsilon k - \varepsilon^2 k^2\left(\frac{x}{\beta} + f\right)\right] v_x \\
& + \beta^{-2}\left[\varepsilon^2\left|\frac{\beta'}{\beta}x - \beta f'\right|^2 v_{xx} + 2\varepsilon\left(\frac{\beta'}{\beta}x - \beta f'\right) v_{xz} \right.\\
& \qquad \left. + \varepsilon^2\left(\frac{\beta''}{\beta}x - 2\beta' f' - \beta f''\right) v_x\right] \\
& + \frac{\varepsilon^2}{\alpha\beta^2}\alpha'' v + \frac{2\varepsilon\alpha'}{\alpha\beta^2}\left[\varepsilon\left(\frac{\beta'}{\beta}x - \beta f'\right) v_x + v_z\right] \\
& - \left[\varepsilon\beta^{-2} V_t\left(\frac{x}{\beta} + f\right) + \frac{\varepsilon^2}{2}\beta^{-2} V_{tt}\left(\frac{x}{\beta} + f\right)^2\right] v + B_2(v)
\end{aligned}$$

and

$$B_2(v) = (\alpha\beta^2)^{-1} B_0(u) + (\alpha\beta^2)^{-1} a_4(\varepsilon s, \varepsilon z)\varepsilon^3 s^3. \tag{2.13}$$

$B_0(u)$ is the operator in (2.5) where derivatives are expressed in terms of formulas (2.8) through (2.10), a_4 is given by (2.11), and s is replaced by $\beta^{-1}x + f$.

Let $w(x)$ denote the unique positive solution of (1.3). Then, taking $w(x)$ as a first approximation, the error produced is ε times a function with exponential decay. Let us be more precise. We need to identify the terms of order ε and those of order ε^2:

$$\begin{aligned}
S(w) = B_3(w) = {} & \beta^{-1}\left[\varepsilon k - \varepsilon^2 k^2\left(\frac{x}{\beta} + f\right)\right] w_x \\
& + \beta^{-2}\left[\varepsilon^2\left|\frac{\beta'}{\beta}x - \beta f'\right|^2 w_{xx} + \varepsilon^2\left(\frac{\beta''}{\beta}x - 2\beta' f' - \beta f''\right) w_x\right] \\
& + \frac{\varepsilon^2}{\alpha\beta^2}\alpha'' w + \frac{2\varepsilon\alpha'}{\alpha\beta^2}\left[\varepsilon\left(\frac{\beta'}{\beta}x - \beta f'\right) w_x\right] \\
& - \left[\beta^{-2}\varepsilon V_t\left(\frac{x}{\beta} + f\right) + \frac{\varepsilon^2}{2}\beta^{-2} V_{tt}\left(\frac{x}{\beta} + f\right)^2\right] w + B_2(w).
\end{aligned}$$

$B_2(w)$ turns out to be of size ε^3. Gathering terms of order ε and ε^2, we get

$$\begin{aligned} S(w) &= \varepsilon\beta^{-1}\Big[kw_x - \frac{1}{\beta^2}V_t(0,\varepsilon z)xw\Big] - \varepsilon\beta^{-2}V_t(0,\varepsilon z)fw \\ &\quad - \varepsilon^2\Big[\frac{k^2}{\beta}fw_x + \frac{f''}{\beta}w_x + \frac{2\beta'}{\beta^2}f'w_x + \frac{2\alpha'}{\alpha\beta}f'w_x \\ &\quad + \frac{2\beta'}{\beta^2}f'xw_{xx} + \frac{V_{tt}}{\beta^3}fxw\Big] \\ &\quad + \varepsilon^2\beta^{-2}\Big[-k^2xw_x + \frac{|\beta'|^2}{\beta^2}x^2w_{xx} + \beta^2|f'|^2w_{xx} + \frac{\beta''}{\beta}xw_x \\ &\quad + \frac{\alpha''}{\alpha}w + \frac{2\alpha'}{\alpha\beta}xw_x - \frac{1}{2\beta^2}V_{tt}x^2w - \frac{1}{2}V_{tt}f^2w\Big] + B_2(w) \\ &= \varepsilon S_1 + \varepsilon S_2 + \varepsilon^2 S_3 + \varepsilon^2 S_4 + B_2(w). \end{aligned}$$

Let us observe that grouped this way, the quantities S_1 and S_3 are odd functions of x while S_2 and S_4 are even. We now want to construct a further approximation to a solution that eliminates the terms of order ε in the error. We see that

$$S(w+\phi) = S(w) + L_0(\phi) + B_3(\phi) + N_0(\phi),$$

where

$$L_0(\phi) = \beta^{-2}\phi_{zz} + \phi_{xx} + pw^{p-1}\phi - \phi \tag{2.14}$$

and

$$N_0(\phi) = (w+\phi)^p - w^p - pw^{p-1}\phi. \tag{2.15}$$

We write

$$\begin{aligned} S(w+\phi) &= [\varepsilon(S_1+S_2) + \phi_{xx} + pw^{p-1}\phi - \phi] + \varepsilon^2 S_3 \\ &\quad + \varepsilon^2 S_4 + B_2(w) + \beta^{-2}\phi_{zz} + B_3(\phi) + N_0(\phi). \end{aligned} \tag{2.16}$$

We choose $\phi = \phi_1$ in order to eliminate the term between brackets in the above expression. Namely, for fixed z, we need a solution of

$$-\phi_{xx} + \phi - pw^{p-1}\phi = \varepsilon(S_1+S_2), \quad \phi(\pm\infty) = 0.$$

As is well-known, this problem is solvable provided that

$$\int_{-\infty}^{\infty}(S_1+S_2)w_x\,dx = 0. \tag{2.17}$$

Furthermore, the solution is unique under the constraint

$$\int_{-\infty}^{\infty}\phi w_x\,dx = 0. \tag{2.18}$$

We compute

$$\int_{-\infty}^{\infty}(S_1+S_2)w_x\,dx = \int_{-\infty}^{\infty} S_1 w_x\,dx$$
$$= \beta^{-1}\left[k\int_{-\infty}^{\infty} w_x^2 - V^{-1}V_t\int_{-\infty}^{\infty} x w w_x\right].$$

The assumption that Γ is stationary (1.9) amounts to $k = -\sigma V^{-1}V_t$ where σ is the constant that makes the amount between brackets identically zero. In fact, we have the validity of the identity $\int_{\mathbb{R}} w^2\,dx = 2\sigma\int_{\mathbb{R}} w_x^2$. The solution has the form

$$\phi_1 = \phi_{11} + \phi_{12}, \tag{2.19}$$

where

$$\phi_{11} = \varepsilon a_{11}(\varepsilon z)w_1(x), \quad \phi_{12} = \varepsilon f(\varepsilon z)a_{12}(\varepsilon z)w_2, \tag{2.20}$$

with

$$a_{11} = \beta^{-1}k, \quad a_{12} = -\beta^{-2}V_t(0,\theta) = \sigma^{-1}k. \tag{2.21}$$

Function w_1 is the unique odd function satisfying

$$-w_{1,xx} + w_1 - pw^{p-1}w_1 = w_x + \frac{1}{\sigma}xw, \quad \int_{\mathbb{R}} w_1 w_x\,dx = 0, \tag{2.22}$$

and w_2 is the unique even solution satisfying

$$-w_{2,xx} + w_2 - pw^{p-1}w_2 = w. \tag{2.23}$$

In fact, we can write

$$w_2 = -\frac{1}{p-1}w - \frac{1}{2}xw_x. \tag{2.24}$$

Substituting $\phi = \phi_1$ into (2.16), we can compute the new error $S(w+\phi_1)$,

$$S(w+\phi_1) = \varepsilon^2 S_3 + \varepsilon^2 S_4 + B_2(w) + \beta^{-2}(\phi_1)_{zz} + B_3(\phi_1) + N_0(\phi_1). \tag{2.25}$$

Observe that since ϕ_1 is of size $O(\varepsilon)$, all terms above carry ε^2 in front. We compute, for instance,

$$B_3(\phi_1) = \varepsilon\beta^{-1}[k(\phi_1)_x - \beta^{-2}V_t(0,\varepsilon z)x\phi_1] - \varepsilon\beta^{-2}V_t(0,\varepsilon z)f\phi_1 + \varepsilon^3 a_6. \tag{2.26}$$

Observe that all functions involved are expressed in (x,z)–variables, and the natural domain for those variables is the infinite strip

$$\mathcal{S} = \left\{-\infty < x < \infty,\ 0 < z < \frac{\ell}{\varepsilon}\right\}.$$

We now want to measure the size of the error in the $L^2(\mathcal{S})$-norm. A rather delicate term in the cubic remainder is the one carrying f'', since in reality we shall only assume a uniform bound on $\|f''\|_{L^2(0,\ell)}$. Observe that a similar term arises from the computation of $(\phi_1)_{zz}$. Both of those terms have a similar form. For instance, the one arising from $(\phi_1)_{zz}$ can be written as $R = \varepsilon^3 f''(\varepsilon z) a_{12}(\varepsilon z) w_{2,x}(x)$ with a_{12} smooth. Observe that

$$\int_{\mathcal{S}} |R|^2 \le C\varepsilon^6 \int_0^{\frac{\ell}{\varepsilon}} |f''(\varepsilon z)|^2\, dz = \varepsilon^5 \|f''\|_{L^2(0,\ell)}.$$

Hence

$$\|R\|_{L^2(\mathcal{S})} \le C\varepsilon^{\frac{5}{2}} \|f''\|_{L^2(0,\ell)}.$$

On the other hand, we have

$$\|B_3(\phi_1)\|_{L^2(\mathcal{S})} \le C\varepsilon^{\frac{3}{2}}.$$

Let us consider the term $N_0(\phi_1)$. Since ϕ_1 can be bounded by $C\varepsilon|x|^2 w(x)$ for large $|x|$, we obtain that

$$\begin{aligned} |N_0(\phi_1)| &= |(w+\phi_1)^p - w^p - p w^{p-1}\phi_1| \\ &= p(w + t\phi_1)^{p-2}|\phi_1|^2 \le C\varepsilon^2(1+|x|^p) w(x)^p; \end{aligned}$$

hence

$$\|N_0(\phi_1)\|_{L^2(\mathcal{S})} \le C\varepsilon^{\frac{3}{2}}.$$

In summary, we have

$$\|S(w+\phi_1)\|_{L^2(\mathcal{S})} \le \varepsilon^{\frac{3}{2}}. \tag{2.27}$$

To improve the approximation for a solution still keeping terms of order ε^2, we need to introduce a new parameter in addition to f. We let $Z(x)$ be the first eigenfunction of the problem

$$Z'' + p w^{p-1} Z - Z = \lambda_0 Z, \quad Z(\pm\infty) = 0.$$

Then, as is well-known, $\lambda_0 > 0$, and $Z(x)$ is one signed and even in x. We now consider our basic approximation to a solution to the problem near the curve Γ_ε to be

$$\mathtt{w} = w + \phi_1 + \varepsilon e(\varepsilon z) Z. \tag{2.28}$$

In all that follows, we will assume the validity of the following constraints on the parameters f and e:

$$\|f\|_a \equiv \|f\|_{L^\infty(0,\ell)} + \|f'\|_{L^\infty(0,\ell)} + \|f''\|_{L^2(0,\ell)} \le \varepsilon^{\frac{1}{2}}, \tag{2.29}$$

$$\|e\|_b \equiv \varepsilon^2 \|e''\|_{L^2(0,\ell)} + \varepsilon\|e'\|_{L^2(0,\ell)} + \|e\|_{L^\infty(0,\ell)} \le \varepsilon^{\frac{1}{2}}. \tag{2.30}$$

In reality, a posteriori, these parameters will turn out to be smaller than stated here.

We set up the full problem in the form $S(\mathtt{w}+\phi) = 0$, which can be expanded in the following way:

$$\begin{aligned} S(\mathtt{w}+\phi) &= \beta^{-2}\phi_{zz} + \phi_{xx} - \phi + p\mathtt{w}^{p-1}\phi \\ &\quad + S(\mathtt{w}) + B_3(\phi) + (\mathtt{w}+\phi)^p - \mathtt{w}^p - p\mathtt{w}^{p-1}\phi = 0. \end{aligned}$$

The new error of approximation is

$$
\begin{aligned}
E_1 = S(\mathtt{w}) &= S(w+\phi_1) + \varepsilon L_0(eZ) \\
&+ \varepsilon[p((w+\phi_1)^{p-1} - w^{p-1})(eZ) + B_3(eZ)] \\
&+ (w+\phi_1+\varepsilon eZ)^p - (w+\phi_1)^p - p(w+\phi_1)^{p-1}\varepsilon eZ
\end{aligned}
$$

where $S(w+\phi_1)$ is given at (2.25). We decompose

$$E_1 = E_{11} + E_{12}$$

where

$$E_{11} \equiv \varepsilon L_0(eZ) = \beta^{-2}\varepsilon^3 e'' Z + \lambda_0 \varepsilon e Z \tag{2.31}$$

and

$$E_{12} = E_1 - E_{11} \tag{2.32}$$

In summary, near the curve the problem takes the form

$$L_1(\phi) + B_3(\phi) + E_1 + N_1(\phi) = 0 \tag{2.33}$$

where E_1 was just described, and

$$L_1(\phi) = \beta^{-2}\phi_{zz} + \phi_{xx} + p\mathtt{w}^{p-1}\phi - \phi, \tag{2.34}$$

$$N_1(\phi) = (\mathtt{w}+\phi)^p - \mathtt{w}^p - p\mathtt{w}^{p-1}\phi. \tag{2.35}$$

We recall that the description made here is only local. We will be able however to reduce the problem to one qualitatively similar to that of the above form in the infinite strip.

3 The Gluing Procedure

Let $\mathtt{w}(y)$ denote the first approximation constructed near the curve in the coordinate y in $\mathbb{R}^2$. Let $\delta < \delta_0/100$ be a fixed number. We consider a smooth cutoff function $\eta_\delta(t)$ where $t \in \mathbb{R}$ such that $\eta_\delta(t) = 1$ if $t < \delta$ and $= 0$ if $t > 2\delta$. Denote as well $\eta_\delta^\varepsilon(s) = \eta_\delta(\varepsilon|s|)$, where s is the normal coordinate to Γ_ε. We define our first global approximation to be simply

$$\mathbf{w}(y) = \eta_{3\delta}^\varepsilon(s)\mathtt{w}$$

extended globally as 0 beyond the $6\delta/\varepsilon$-neighborhood of Γ_ε. Denote $S(u) = \Delta u - V(\varepsilon y)u + u^p$ for $u = \mathbf{w} + \tilde{\phi}$, now $\tilde{\phi}$ globally defined in $\mathbb{R}^2$. Then $S(\mathbf{w}+\tilde{\phi}) = 0$ if and only if

$$\tilde{L}(\tilde{\phi}) = \tilde{E} + \tilde{N}(\tilde{\phi}), \tag{3.1}$$

where

$$\tilde{E} = S(\mathbf{w}), \quad \tilde{L}(\tilde{\phi}) = \Delta\tilde{\phi} + p\mathbf{w}^{p-1}\tilde{\phi} - V\tilde{\phi},$$

and

$$\tilde{N}(\tilde{\phi}) = (\mathbf{w}+\tilde{\phi})^p - \mathbf{w}^p - p\mathbf{w}^{p-1}\tilde{\phi}.$$

We further separate $\tilde{\phi}$ in the following form:

$$\tilde{\phi} = \eta_{3\delta}^{\varepsilon}\phi + \psi$$

where, in coordinates (x, z), we assume that ϕ is defined in the whole strip $\mathcal{S}$ so that we want

$$\tilde{L}(\eta_{\delta}^{\varepsilon}\phi) + \tilde{L}(\psi) = \tilde{E} + \tilde{N}(\eta_{\delta}^{\varepsilon}\phi).$$

We achieve this if the pair (ψ, ϕ) satisfies the following nonlinear coupled system:

$$\tilde{L}(\phi) = \eta_{\delta}^{\varepsilon}\tilde{N}(\phi + \psi) + \eta_{\delta}^{\varepsilon}\tilde{E} + \eta_{\delta}^{\varepsilon}p\mathbf{w}^{p-1}\psi, \tag{3.2}$$

$$\begin{aligned}&\Delta\psi - V\psi + (1 - \eta_{\delta}^{\varepsilon})p\mathbf{w}^{p-1}\psi \\ &\quad = (1 - \eta_{\delta}^{\varepsilon})\tilde{E} + 2\varepsilon\nabla\eta_{3\delta}^{\varepsilon}\nabla\phi + 2\varepsilon^2(\Delta\eta_{3\delta}^{\varepsilon})\phi + (1 - \eta_{\delta}^{\varepsilon})\tilde{N}(\eta_{3\delta}^{\varepsilon}\phi + \psi).\end{aligned} \tag{3.3}$$

Notice that the operator $\tilde{L}$ in the strip $\mathcal{S}$ may be taken as any compatible extension outside the $6\delta/\varepsilon$-neighborhood of the curve.

What we want to do next is to reduce the problem to a problem in the strip. To do this, we solve, given a small ϕ, problem (3.3) for ψ. This can be done in an elementary way: Let us observe first that since V is uniformly positive and $\mathbf{w}$ is exponentially small for $|s| > \delta\varepsilon^{-1}$, where s is the normal coordinate to Γ_ε, then the problem

$$\Delta\psi - (V - (1 - \eta_{\delta}^{\varepsilon})p\mathbf{w}^{p-1})\psi = h \tag{3.4}$$

has a unique bounded solution ψ whenever $\|h\|_\infty < +\infty$. Moreover,

$$\|\psi\|_\infty \le C\|h\|_\infty.$$

Assume now that ϕ satisfies the following decay condition:

$$|\nabla\phi(y)| + |\phi(y)| \le e^{-\frac{\gamma}{\varepsilon}} \quad \text{for } |s| > \frac{\delta}{\varepsilon} \tag{3.5}$$

for a certain constant $\gamma > 0$. Since $\tilde{N}$ has a powerlike behavior with power greater than 1, a direct application of the contraction mapping principle yields that problem (3.3) has a unique (small) solution $\psi = \psi(\phi)$ with

$$\|\psi(\phi)\|_\infty \le C\varepsilon[\|\phi\|_{L^\infty(|s|>\delta\varepsilon^{-1})} + \|\nabla\phi\|_{L^\infty(|s|>\delta\varepsilon^{-1})}],$$

where with some abuse of notation by $\{|s| > \delta/\varepsilon\}$ we denote the complement of a δ/ε-neighborhood of Γ_ε. The nonlinear operator ψ satisfies a Lipschitz condition of the form

$$\|\psi(\phi_1) - \|\psi(\phi_2)\|_\infty \le C\varepsilon[\|\phi_1 - \phi_2\|_{L^\infty(|s|>\delta\varepsilon^{-1})} + \|\nabla(\phi_1 - \phi_2)\|_{L^\infty(|s|>\delta\varepsilon^{-1})}].$$

The full problem has been reduced to solving the (nonlocal) problem in the infinite strip $\mathcal{S}$

$$L_2(\phi) = \eta_{\delta}^{\varepsilon}\tilde{N}(\phi + \psi(\phi)) + \eta_{\delta}^{\varepsilon}\tilde{E} + \eta_{\delta}^{\varepsilon}p\mathrm{w}^{p-1}\psi(\phi) \tag{3.6}$$

for a $\phi \in H^2(\mathcal{S})$ satisfying condition (3.5). Here L_2 denotes a linear operator that coincides with $\tilde{L}$ on the region $|s| < 10\delta/\varepsilon$.

We shall define this operator next. The operator $\tilde{L}$ for $|s| < 10\delta/\varepsilon$ is given in coordinates (x, z) by formula (2.34). We extend it for functions ϕ defined in the entire strip $\mathcal{S}$, in terms of (x, z), as follows:

$$L_2(\phi) = L_1(\phi) + \chi(\varepsilon|x|)B_3(\phi) \tag{3.7}$$

where $\chi(r)$ is a smooth cutoff function that equals 1 for $r < 10\delta$ and vanishes identically for for $r > 20\delta$, and L_1 is the operator defined in (2.34).

Rather than solving problem (3.1) directly, we shall do it in steps. We consider the following projected problem in $H^2(\mathcal{S})$: given f and e satisfying bounds (2.29)–(2.30), find functions $\phi \in H^2(\mathcal{S})$, $c, d \in L^2(0, \ell)$, such that

$$L_2(\phi) = \chi E_1 + N_2(\phi) + c(\varepsilon z)\chi w_x + d(\varepsilon z)\chi Z \quad \text{in } \mathcal{S}, \tag{3.8}$$

$$\phi(x, 0) = \phi(x, \ell/\varepsilon), \quad \phi_z(x, 0) = \phi_z\left(x, \frac{\ell}{\varepsilon}\right), \quad -\infty < x < +\infty, \tag{3.9}$$

$$\int_{-\infty}^{\infty} \phi(x, z)w_x(x)dx = \int_{-\infty}^{\infty} \phi(x, z)Z(x)dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon}. \tag{3.10}$$

Here $N_2(\phi) = \eta_\delta^\varepsilon \tilde{N}(\phi + \psi(\phi)) + \eta_\delta^\varepsilon p \mathrm{w}^{p-1}\psi(\phi)$.

We will prove that this problem has a unique solution whose norm is controlled by the L^2-norm, not of the whole E_1 but rather that of E_{12}.

After this has been done, our task is to adjust the parameters f and e in such a way that c and d are identically zero. As we will see, this turns out to be equivalent to solving a nonlocal, nonlinear coupled second-order system of differential equations for the pair (e, d) under periodic boundary conditions. As we will see, this system is solvable in a region where the bounds (2.29) and (2.30) hold.

We will carry out this program in the following sections. To solve (3.8)–(3.10), we need to investigate the invertibility of L_2 in an L^2-H^2 setting under periodic boundary and orthogonality conditions.

4 Invertibility of L_2

Let L_2 be the operator defined in $H^2(\mathcal{S})$ by (3.7). In this section we study the linear problem

$$L_2(\phi) = h + c(\varepsilon z)\chi w_x + d(\varepsilon z)\chi Z \quad \text{in } \mathcal{S}, \tag{4.1}$$

$$\phi(x, 0) = \phi\left(x, \frac{\ell}{\varepsilon}\right), \quad \phi_z(x, 0) = \phi_z\left(x, \frac{\ell}{\varepsilon}\right), \quad -\infty < x < +\infty, \tag{4.2}$$

$$\int_{-\infty}^{\infty} \phi(x, z)w_x(x)dx = \int_{-\infty}^{\infty} \phi(x, z)Z(x)dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon}, \tag{4.3}$$

for given $h \in L^2(\mathcal{S})$. Here $\chi(\varepsilon|x|)$ is the cutoff introduced in the definition of L_2 in (3.7). Our main result in this section is the following:

PROPOSITION 4.1 *If δ in the definition of L_2 is chosen sufficiently small, then there exists a constant $C > 0$, independent of ε, such that for all small ε problem* (4.1)–(4.3) *has a unique solution $\phi = T(h)$ that satisfies the estimate*

$$\|\phi\|_{H^2(\mathcal{S})} \leq C\|h\|_{L^2(\mathcal{S})}.$$

For the proof of this result we need to show the validity of the corresponding assertion for a simpler operator that does not depend on δ. Let us consider the problem

$$\mathrm{L}(\phi) = -\Delta\phi + \phi - pw^{p-1}\phi = h \quad \text{in } \mathcal{S}, \tag{4.4}$$

$$\phi(x, 0) = \phi\left(x, \frac{\ell}{\varepsilon}\right), \quad \phi_z(x, 0) = \phi_z\left(x, \frac{\ell}{\varepsilon}\right), \quad -\infty < x < +\infty, \tag{4.5}$$

$$\int_{-\infty}^{\infty} \phi(x, z) w_x(x) dx = \int_{-\infty}^{\infty} \phi(x, z) Z(x) dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon}. \tag{4.6}$$

LEMMA 4.2 *There exists a constant $C > 0$, independent of ε, such that solutions of* (4.4)–(4.6) *satisfy the a priori estimate*

$$\|\phi\|_{H^2(\mathcal{S})} \leq C\|h\|_{L^2(\mathcal{S})}.$$

PROOF: Let us consider Fourier series decompositions for h and ϕ of the form

$$\phi(x, z) = \sum_{k=0}^{\infty} \left[\phi_{1k}(x) \cos\left(\frac{2\pi k}{\ell}\, \varepsilon z\right) + \phi_{2k}(x) \sin\left(\frac{2\pi k}{\ell}\, \varepsilon z\right)\right],$$

$$h(x, z) = \sum_{k=0}^{\infty} \left[h_{1k}(x) \cos\left(\frac{2\pi k}{\ell}\, \varepsilon z\right) + h_{2k}(x) \sin\left(\frac{2\pi k}{\ell}\, \varepsilon z\right)\right].$$

Then we have the validity of the equations

$$k^2\varepsilon^2\phi_{lk} + \mathrm{L}_0(\phi_{lk}) = h_{lk}, \quad x \in \mathbb{R}, \tag{4.7}$$

with orthogonality conditions

$$\int_{-\infty}^{\infty} \phi_{lk} w_x \, dx = \int_{-\infty}^{\infty} \phi_{lk} Z \, dx = 0. \tag{4.8}$$

We have denoted here

$$\mathrm{L}_0(\phi_{lk}) = -\phi_{lk,xx} + \phi_{lk} - pw^{p-1}\phi_{lk}.$$

Let us consider the bilinear form in $H^1(\mathbb{R})$ associated to the operator L_0, namely

$$B(\psi, \psi) = \int_{-\infty}^{\infty} [|\psi_x|^2 + (1 - pw^{p-1})|\psi|^2] dx.$$

Since (4.8) holds, we conclude that

$$C[\|\phi_{lk}\|^2_{L^2(\mathbb{R})} + \|(\phi_{lk})_x\|^2_{L^2(\mathbb{R})}] \leq B(\phi_{lk}, \phi_{lk}) \tag{4.9}$$

for a constant $C > 0$ independent of l and k. Using this fact and equation (4.7), we conclude the estimate

$$(1 + k^4\varepsilon^4)\|\phi_{lk}\|^2_{L^2(\mathbb{R})} + \|\phi_{lk,x}\|^2_{L^2(\mathbb{R})} \le C\|h_{lk}\|^2_{L^2(\mathbb{R})}.$$

In particular, we see from (4.7) that ϕ_{lk} satisfies an equation of the form

$$-\phi_{lk,xx} + \phi_{lk} = \tilde{h}_{lk}, \quad x \in \mathbb{R},$$

where $\|\tilde{h}_{lk}\|_{L^2(\mathbb{R})} \le C\|h_{lk}\|_{L^2(\mathbb{R})}$. Hence it follows that we have the estimate

$$\|(\phi_{lk,xx}\|^2_{L^2(\mathbb{R})} \le C\|h_{lk}\|^2_{L^2(\mathbb{R})}. \tag{4.10}$$

Adding up estimates (4.9) and (4.10) in k and l, we conclude that

$$\|D^2\phi\|^2_{L^2(\mathcal{S})} + \|D\phi\|^2_{L^2(\mathcal{S})} + \|\phi\|^2_{L^2(\mathcal{S})} \le C\|h\|^2_{L^2(\mathcal{S})},$$

which ends the proof. □

We consider now the following problem: given $h \in L^2(\mathcal{S})$, find functions $\phi \in H^2(\mathcal{S})$, $c, d \in L^2(0,\ell)$, such that

$$\mathrm{L}(\phi) = h + c(\varepsilon z)w_x + d(\varepsilon z)Z \quad \text{in } \mathcal{S}, \tag{4.11}$$

$$\phi_(x,0) = \phi\left(x, \frac{\ell}{\varepsilon}\right), \quad \phi_z(x,0) = \phi_z\left(x, \frac{\ell}{\varepsilon}\right), \quad -\infty < x < +\infty, \tag{4.12}$$

$$\int_{-\infty}^{\infty} \phi(x,z)w_x(x)dx = \int_{-\infty}^{\infty} \phi(x,z)Z(x)dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon}. \tag{4.13}$$

LEMMA 4.3 *Problem* (4.11)–(4.13) *possesses a unique solution. Moreover,*

$$\|\phi\|_{H^2(\mathcal{S})} \le C\|h\|_{L^2(\mathcal{S})}.$$

PROOF: To establish existence, we assume that

$$h(x,z) = \sum_{k=0}^{\infty}\left[h_{1k}(x)\cos\left(\frac{2\pi k}{\ell}\varepsilon z\right) + h_{2k}(x)\sin\left(\frac{2\pi k}{\ell}\varepsilon z\right)\right],$$

and consider the problem of finding $\phi_{lk} \in H^1(\mathbb{R})$ and constants c_{lk} and d_{lk} such that

$$k^2\varepsilon^2\phi_{lk} + L_0(\phi_{lk}) = h_{lk} + c_{lk}w_x + d_{lk}Z, \quad x \in \mathbb{R},$$

and

$$\int_{-\infty}^{\infty} \phi_{lk}w_x\,dx = \int_{-\infty}^{\infty} \phi_{lk}Z\,dx = 0.$$

Fredholm's alternative yields that this problem is solvable with the choices

$$c_{lk} = -\frac{\int_{-\infty}^{\infty} h_{lk}w_x\,dx}{\int_{-\infty}^{\infty} w_x^2\,dx}, \quad d_{lk} = -\frac{\int_{-\infty}^{\infty} h_{lk}Z\,dx}{\int_{-\infty}^{\infty} Z^2\,dx}.$$

Observe in particular that

$$\sum_{k=0}^{\infty} |c_{lk}|^2 + |d_{lk}|^2 \le C\varepsilon \|h\|^2_{L^2(\mathcal{S})}. \tag{4.14}$$

Finally, define

$$\phi(x,z) = \sum_{k=0}^{\infty} \left[\phi_{1k}(x) \cos\left(\frac{2\pi k}{\ell}\varepsilon z\right) + \phi_{2k}(x) \sin\left(\frac{2\pi k}{\ell}\varepsilon z\right)\right],$$

and correspondingly

$$c(z) = \sum_{k=0}^{\infty} \left[c_{1k} \cos\left(\frac{2\pi k}{\ell} z\right) + c_{2k} \sin\left(\frac{2\pi k}{\ell} z\right)\right],$$
$$d(z) = \sum_{k=0}^{\infty} \left[d_{1k} \cos\left(\frac{2\pi k}{\ell} z\right) + d_{2k} \sin\left(\frac{2\pi k}{\ell} z\right)\right].$$

Estimate (4.14) gives that $c(\varepsilon z) w_x$ and $d(\varepsilon z) Z$ have their $L^2(\mathcal{S})$ norms controlled by that of h. The a priori estimates of the previous lemma tell us that the series for ϕ is convergent in $H^2(\mathcal{S})$ and defines a unique solution for the problem with the desired bounds. □

PROOF OF PROPOSITION 4.1: We will reduce problem (4.1)–(4.3) to a small perturbation of a problem of the form (4.4)–(4.6) in which Lemma 4.2 is applicable. We will achieve this by introducing a change of variables that eliminates the weight β^{-2} in front of ϕ_{zz}.

We let

$$\phi(x,z) = \varphi(x, a(z)), \quad a(z) = \varepsilon^{-1} \int_0^{\varepsilon z} \beta(r)\,dr.$$

The map $a : [0, \ell/\varepsilon) \to [0, \hat{l}/\varepsilon)$ is a diffeomorphism, where $\hat{l} = \int_0^\ell \beta(r)\,dr$. We denote then

$$\phi_z = \beta \varphi_{z'}, \quad \phi_{zz} = \beta^2(\varepsilon z)\varphi_{z'z'} + \varepsilon\beta'(\varepsilon z)\varphi_{z'},$$

while differentiation in x does not change. The equation in terms of φ now reads

$$\begin{aligned}
&\Delta\varphi - p w^{p-1}\varphi + \varphi + \chi \hat{B}_3(\varphi) + p(\mathrm{w}^{p-1} - w^{p-1})\varphi + \varepsilon\beta'\varphi_{z'} \\
&\quad = \hat{h} + \hat{c}(\varepsilon z') w_x + \hat{d}(\varepsilon z') Z \quad \text{in } \hat{\mathcal{S}}, \\
&\varphi(x,0) = \varphi\left(x, \frac{\hat{l}}{\varepsilon}\right), \quad \varphi_{z'}(x,0) = \varphi_{z'}\left(x, \frac{\hat{l}}{\varepsilon}\right), \quad -\infty < x < +\infty, \\
&\int_{-\infty}^{\infty} \varphi(x,z') w_x(x)\,dx = \int_{-\infty}^{\infty} \varphi(x,z') Z(x)\,dx = 0, \quad 0 < z' < \frac{\hat{l}}{\varepsilon}.
\end{aligned}$$

Here $\hat{h}(x, z') = h(x, a^{-1}(z'))$ and the operator $\hat{B}_3$ is defined by using the above formulas to replace the z-derivatives by z'-derivatives and the variable z by $a^{-1}(z')$ in the operator B_3. The key point is the following: the operator

$$B_4(\varphi) = \chi \hat{B}_3(\varphi) + \varepsilon\beta'\varphi_{z'} + p(\mathtt{w}^{p-1} - w^{p-1})\varphi$$

is small in the sense that

$$\|B_4(\varphi)\|_{L^2(\hat{\mathcal{S}})} \leq C\delta\|\varphi\|_{H^2(\hat{\mathcal{S}})}.$$

This last estimate is a rather straightforward consequence of the fact that $|\varepsilon s| < 20\delta\varepsilon^{-1}$ wherever the operator $\hat{B}_3$ is supported, and the other terms are even smaller when ε is small. Thus by reducing δ if necessary, we apply the invertibility result of Lemma 4.2. The result thus follows by transforming the estimate for φ into a similar one for ϕ via a change of variables. This concludes the proof. □

5 Solving the Nonlinear Intermediate Problem

In this section we will solve problem (3.8)–(3.10),

$$L_2(\phi) + B_3(\phi) = \chi E_1 + N_2(\phi) + c(\varepsilon z)\chi w_x + d(\varepsilon z)\chi Z,$$

under periodic boundary and orthogonality conditions in $\mathcal{S}$. Here

$$N_2(\phi) = \chi N_1(\phi + \psi(\phi))$$

whenever this operator is well-defined, namely, for ϕ satisfying (3.5). A first elementary but crucial observation is that the term

$$E_{11} = [\varepsilon^3\beta^{-2}e'' + \varepsilon\lambda_0 e]Z$$

in the decomposition of E_1, (2.31)–(2.32), has precisely the form $d(\varepsilon z)Z$ and can therefore be absorbed for now in that term. Thus, the equivalent problem we will look at is

$$L_2(\phi) + B_3(\phi) = \chi E_{12} + N_2(\phi) + c(\varepsilon z)\chi\, w_x + d(\varepsilon z)\chi Z.$$

The big difference between the terms E_{11} and E_{12} is their sizes. Notice that

$$\|E_{12}\|_{L^2(\mathcal{S})} \leq C\varepsilon^{\frac{3}{2}},$$

while E_{11} is a priori only of size $O(\varepsilon^{1/2})$. We call $E_2 \equiv \chi E_{12}$.

For further reference, it is useful to point out the Lipschitz dependence of the term of error E_2 on the parameters f and e for the norms defined in (2.29)–(2.30). We have the validity of the estimate

$$\|E_{12}(f_1, e_1) - E_{12}(f_2, e_2)\|_{L^2(\mathcal{S})} \leq C\varepsilon^{\frac{3}{2}}[\|f_1 - f_2\|_a + \|e_1 - e_2\|_b]. \tag{5.1}$$

Let T be the operator defined by Proposition 4.1. Then the equation is equivalent to the fixed-point problem

$$\phi = T(E_2 + N_2(\phi)) \equiv \mathcal{A}(\phi). \tag{5.2}$$

The operator T has a useful property: Assume h has support contained in $|x| \leq 20\delta/\varepsilon$. Then $\phi = T(h)$ satisfies the estimate

$$|\phi(x,z)| + |\nabla\phi(x,z)| \leq \|\phi\|_\infty e^{-\frac{2\delta}{\varepsilon}} \quad \text{for } |x| > \frac{40\delta}{\varepsilon}. \tag{5.3}$$

In fact, since B_3 is supported on $|x| < 20\delta/\varepsilon$ and so do the terms involving c and d, then ϕ satisfies for $|x| \geq 20\delta/\varepsilon$ an equation of the form

$$\beta^{-2}\phi_{zz} + \phi_{xx} - (1 + o(1))\phi = 0$$

with $o(1) \to 0$ uniformly as $\varepsilon \to 0$. For $|x| \geq 20\delta/\varepsilon$ we can then use a barrier of the form $\varphi(x,z) = \|\phi\|_\infty e^{-(1/2)(x-20\delta/\varepsilon)}$ to conclude that for $|x| > 40\delta/\varepsilon$, we have

$$\phi(x,z) \leq \|\phi\|_\infty e^{-\frac{10\delta}{\varepsilon}}.$$

The remaining inequalities for ϕ are found in the same way.

The bound for $\nabla\phi$ follows simply by local elliptic estimates. Now we recall that the operator $\psi(\phi)$ satisfies, as seen directly from its definition,

$$\|\psi(\phi)\|_{L^\infty} \leq C[\||\nabla\phi| + |\phi|\|_{L^\infty(|x| > \frac{20\delta}{\varepsilon})} + e^{-\frac{\delta}{\varepsilon}}], \tag{5.4}$$

and also the Lipschitz condition

$$\|\psi(\phi_1) - \psi(\phi_2)\|_{L^\infty} \leq C[\||\nabla(\phi_1 - \phi_2)| + |\phi_1 - \phi_2|\|_{L^\infty(|x| > \frac{20}{\varepsilon})}]. \tag{5.5}$$

These facts will allow us to construct a region where the contraction mapping principle applies. As we have said,

$$\|E_2\|_{L^2(\mathcal{S})} \leq C_* \varepsilon^{\frac{3}{2}}$$

for a certain constant $C_* > 0$. We consider the following closed, bounded subset of $H^2(\mathcal{S})$:

$$\mathcal{B} = \left\{ \phi \in H^2(\mathcal{S}) \,\middle|\, \begin{array}{l} \|\phi\|_{H^2(\mathcal{S})} \leq D\varepsilon^{3/2} \\ |\phi| + |\nabla\phi| \;\|_{L^\infty(|x| > 40\delta/\varepsilon)} \leq \|\phi\|_{H^2(\mathcal{S})} e^{-\delta/\varepsilon} \end{array} \right\}.$$

We claim that if the constant D is fixed sufficiently large, then the map $\mathcal{A}$ defined in (5.2) is a contraction from B into itself.

Let us analyze the Lipschitz character of the nonlinear operator involved in $\mathcal{A}$ for functions in $\mathcal{B}$:

$$N_2(\phi) = \chi N_1(\phi + \psi(\phi))$$

where

$$N_1(\phi) = p[(\mathtt{w} + t\phi)^{p-1} - \mathtt{w}^{p-1}]\phi^2$$

for $t \in (0,1)$. From here it follows that

$$|N_1(\phi)| \leq C[|\phi|^p + |\phi|^2],$$

so that denoting $\mathcal{S}_\delta = \mathcal{S} \cap \{|x| < 10\delta/\varepsilon\}$, we have that for $\phi \in \mathcal{B}$,

$$\|N_2(\phi)\|_{L^2(\mathcal{S})} \leq C[\|\phi\|^p_{L^{2p}(\mathcal{S})} + \|\phi\|^2_{L^4(\mathcal{S})} + \|\psi(\phi)\|^p_{L^{2p}(\mathcal{S}_\delta)} + \|\psi(\phi)\|^2_{L^4(\mathcal{S}_\delta)}].$$

Using Sobolev's embedding we get

$$\|\phi\|^p_{L^{2p}(\mathcal{S})} + \|\phi\|^2_{L^4(\mathcal{S})} \le C\,[\|\phi\|^p_{H^2(\mathcal{S})} + \|\phi\|^2_{H^2(\mathcal{S})}],$$

while using estimate (5.4), $\phi \in \mathcal{B}$, (5.3), the fact that the area of $\mathcal{S}_\delta$ is of order $O(\delta/\varepsilon)$, and Sobolev's embedding, we get

$$\|\psi(\phi)\|^p_{L^{2p}(\mathcal{S}_\delta)} + \|\psi(\phi)\|^2_{L^4(\mathcal{S}_\delta)} \le Ce^{-\frac{\delta}{4\varepsilon}}[1 + \|\phi\|^p_{H^2(\mathcal{S})} + \|\phi\|^2_{H^2(\mathcal{S})}].$$

It thus follows that

$$\|N_2(\phi)\|_{L^2(\mathcal{S})} \le C\,(\varepsilon^{\frac{3p}{2}} + \varepsilon^3). \tag{5.6}$$

Besides, as for the Lipschitz condition, we find after a direct estimate,

$$\begin{aligned}\|N_1(\phi_1) - N_1(\phi_2)\|_{L^2(\mathcal{S})} \le [\|\phi_1\|^{p-1}_{L^{2p}(\mathcal{S})} + \|\phi_1\|_{L^4(\mathcal{S})} + \|\phi_2\|_{L^4(\mathcal{S})} + \|\phi_2\|^{p-1}_{L^{2p}(\mathcal{S})}]\\ \times [\|\phi_2 - \phi_1\|_{L^{2p}(\mathcal{S})} + \|\phi_2 - \phi_1\|_{L^4(\mathcal{S})}],\end{aligned}$$

with constants C independent of the bound a priori assumed on $\|f\|_{H^2(0,\ell)}$. We then conclude for N_2,

$$\begin{aligned}&\|N_2(\phi_1) - N_2(\phi_2)\|_{L^2(\mathcal{S})}\\ &\quad\le \|N_1(\phi_1 + \psi(\phi_1)) - N_1(\phi_2 + \psi(\phi_1))\|_{L^2(\mathcal{S}_\delta)}\\ &\qquad + \|N_1(\phi_2 + \psi(\phi_1)) - N_1(\phi_2 + \psi(\phi_2))\|_{L^2(\mathcal{S}_\delta)}\\ &\quad\le A[\|\phi_1 - \phi_2\|_{L^4(\mathcal{S}_\delta)} + \|\phi_1 - \phi_2\|_{L^{2p}(\mathcal{S}_\delta)}]\\ &\qquad + A[\|\psi(\phi_1) - \psi(\phi_2)\|_{L^4(\mathcal{S}_\delta)} + \|\psi(\phi_1) - \psi(\phi_2)\|_{L^{2p}(\mathcal{S}_\delta)}]\end{aligned}$$

where

$$A = A_1 + A_2$$

with

$$A_l = \|\phi_l\|^{p-1}_{L^{2p}(\mathcal{S}_\delta)} + \|\psi(\phi_l)\|^{p-1}_{L^{2p}(\mathcal{S}_\delta)} + \|\phi_l\|^{p-1}_{L^{2p}(\mathcal{S}_\delta)} + \|\psi(\phi_l)\|_{L^4(\mathcal{S}_\delta)}, \quad l = 1, 2.$$

Arguing as before, the conclusion of these estimates is

$$\|N_2(\phi_1) - N_2(\phi_2)\|_{L^2(\mathcal{S})} \le C(\varepsilon^{\frac{3}{2}(p-1)} + \varepsilon^{\frac{3}{2}})\|\phi_1 - \phi_2\|_{H^2(\mathcal{S})}. \tag{5.7}$$

Now, let $\phi \in B$; then $\varphi = \mathcal{A}(\phi)$ satisfies, thanks to (5.6),

$$\|\varphi\|_{H^2(\mathcal{S})} \le \|T\|\{C_*\varepsilon^{\frac{3}{2}} + CD^p\varepsilon^{\frac{3}{2}p}\}.$$

Choosing any number $D > C_*\|T\|$ we get that for small ε

$$\|\varphi\|_{H^2(\mathcal{S})} \le D\varepsilon^{\frac{3}{2}}.$$

On the other hand, we have

$$\|\varphi\|_{L^\infty(\mathcal{S})} \le C\|\varphi\|_{H^2(\mathcal{S})}.$$

But φ satisfies an equation of the form $L_2(\varphi) = h$ with h compactly supported. Hence φ belongs to $\mathcal{B}$ thanks to the discussion above. $\mathcal{A}$ is clearly a contraction mapping thanks to (5.7). We conclude that (5.2) has a unique fixed point in $\mathcal{B}$.

We recall that the error E_2 and the operator T itself carry the functions f and e as parameters. A tedious but straightforward analysis of all terms involved in the differential operator and in the error yield that this dependence is indeed Lipschitz with respect to the H^2-norm (for each fixed ε).

In the operator, consider, for instance, the following only term involving f'':

$$B_f(\phi) = \varepsilon^2 f''(\varepsilon z)\phi_x.$$

Then we have

$$\|B_f(\phi)\|^2_{L^2(\mathcal{S})} \le \varepsilon^3 \int_0^\ell |f''(\theta)|^2\, d\theta \left(\sup_z \int_{-\infty}^{\infty} |\phi_x(x,z)|^2\, dx \right).$$

Let $\varphi(z) = \int_{-\infty}^{\infty} |\phi_x(x,z)|^2\, dx$. Then

$$\sup_z \varphi(z) \le \varepsilon \int_S |\phi_x|^2 + 2\int_S |\phi_x|\,|\phi_{xz}| \le \frac{1}{2}\sup_z \varphi(z) + 4\varepsilon^{-1}\int_S |\phi_{xz}|^2.$$

Hence

$$\varphi(z) \le C\varepsilon^{-1}\|\phi\|^2_{H^2(\mathcal{S})}, \tag{5.8}$$

so that finally

$$\|B_f(\phi)\|_{L^2(\mathcal{S})} \le \varepsilon\|f\|_a.$$

For the other terms the analysis follows in a simpler way. Emphasizing the dependence on f, we find for the linear operator T the Lipschitz dependence

$$\|T_{f_1} - T_{f_2}\| \le C\varepsilon\|f_1 - f_2\|_a.$$

We recall that we have the Lipschitz dependence (5.1). Moreover, the operator N also has Lipschitz dependence on (f, e). It is easily checked that for $\phi \in \mathcal{B}$ we have, with obvious notation,

$$\|N_{(f_1,e_1)}(\phi) - N_{(f_2,e_2)}(\phi)\|_{L^2(\mathcal{S})} \le C\varepsilon^{\frac{5}{2}}[\|f_1 - f_2\|_a + \|e_1 - e_2\|_b].$$

Hence from the fixed-point characterization we then see that

$$\|\phi_{(f_1,e_1)} - \phi_{(f_2,e_2)}\|_{H^2(\mathcal{S})} \le C\varepsilon^{\frac{3}{2}}[\|f_1 - f_2\|_a + \|e_1 - e_2\|_b]. \tag{5.9}$$

We summarize the result we have obtained in the following:

PROPOSITION 5.1 *There is a number $D > 0$ such that for all sufficiently small ε and all (f, e) satisfying* (2.29)–(2.30), *problem* (3.8)–(3.9) *has a unique solution $\phi = \phi(f, e)$ that satisfies*

$$\|\phi\|_{H^2(\mathcal{S})} \le D\varepsilon^{\frac{3}{2}},$$
$$\||\phi| + |\nabla\phi|\|_{L^\infty(|x| > \frac{40\delta}{\varepsilon})} \le \|\phi\|_{H^2(\mathcal{S})} e^{-\frac{\delta}{\varepsilon}}.$$

Besides, ϕ depends Lipschitz-continuously on f and e in the sense of estimate (5.9).

Next we carry out the second part of the program, which is to set up equations for f and d that are equivalent to making c and d identically zero. These equations are obtained by simply integrating the equation (only in x) against w_x and Z, respectively. It is therefore of crucial importance to carry out computations of the terms $\int_{\mathbb{R}} E_1 w_x \, dx$ and $\int_{\mathbb{R}} E_1 Z \, dx$. We do that in the next section.

6 Estimates for Projections of the Error

In this section we carry out some estimates for the terms $\int_{\mathbb{R}} E_1 w_x \, dx$ and $\int_{\mathbb{R}} E_1 Z \, dx$, where E_1, we recall, was defined in (2.31)–(2.32) and w_x is an odd function. Integration against all even terms in E_1 therefore just vanish. We have

$$\begin{aligned}\int_{\mathbb{R}} E_1 w_x &= \int_{\mathbb{R}} E_{12} w_x \\ &= \int_{\mathbb{R}} S(w+\phi_1) w_x + \int_{\mathbb{R}} w_x [\varepsilon p [(w+\phi_1)^{p-1} - w^{p-1}](eZ) + \varepsilon B_3(eZ)] \\ &\quad + \int_{\mathbb{R}} w_x [(w+\phi_1+\varepsilon e Z)^p - (w+\phi_1)^p - p(w+\phi_1)^{p-1} \varepsilon e Z].\end{aligned}$$

We recall

$$\begin{aligned}S(w+\phi_1) = \varepsilon^2 S_3 + \varepsilon^2 S_4 + B_2(w) + B_3(\phi_1) \\ + [(w+\phi_1)^p - w^p - p w^{p-1} \phi_1] + \beta^{-2} (\phi_1)_{zz}\end{aligned}$$

where S_2 is an odd function, S_4 is an even function, and $B_2(w)$ is of order ε^3. Thus we see that

$$\begin{aligned}&\int_{\mathbb{R}} S(w+\phi_1) w_x \\ &\quad = -\varepsilon^2 \Big\{ \beta^{-1} f'' \int_{\mathbb{R}} w_x^2 + 2 f' \beta^{-2} \beta' \int_{\mathbb{R}} (w_x^2 + x w_{xx} w_x) \\ &\qquad\quad + 2\alpha^{-1} \beta^{-1} \alpha' f' \int_{\mathbb{R}} w_x^2 + f \Big(\beta^{-1} k^2 \int_{\mathbb{R}} w_x^2 + \beta^{-3} V_{tt} \int_{\mathbb{R}} x w w_x \Big) \Big\} \\ &\qquad + \int_{\mathbb{R}} w_x [(w+\phi_1)^p - w^p - p w^{p-1} \phi_1] \\ &\qquad + \int_{\mathbb{R}} w_x B_3(\phi_1) + \varepsilon^3 b_{1\varepsilon} f'' + \varepsilon^3 b_{2\varepsilon}.\end{aligned} \tag{6.1}$$

Here and below we denote by $b_{l\varepsilon}$, $l = 1, 2$, generic, uniformly bounded continuous functions of the form

$$b_{l\varepsilon} = b_{l\varepsilon}(z, f(\varepsilon z), e(\varepsilon z), f'(\varepsilon z), \varepsilon e'(\varepsilon z))$$

where additionally $b_{1\varepsilon}$ is uniformly Lipschitz in its four last arguments.

The coefficient in front of $f(\varepsilon z)$ in (6.1) can be computed as

$$f\left(\beta^{-1}k^2\int_{\mathbb{R}} w_x^2+\beta^{-3}V_{tt}\int_{\mathbb{R}} xww_x\right)=f\left(\beta^{-1}k^2\int_{\mathbb{R}} w_x^2-\sigma\beta^{-3}V_{tt}\int_{\mathbb{R}} w_x^2\right)$$

where we have used the fact that $\int_{\mathbb{R}} xww_x=-\frac{1}{2}\int_{\mathbb{R}} w^2=-\sigma\int_{\mathbb{R}} w_x^2$.

We see that the term $[(w+\phi_1)^p-w^p-pw^{p-1}\phi_1]$ is to main order of the form

$$\frac{p(p-1)}{2}w^{p-2}\phi_1^2,$$

and it is therefore of the order $O(\varepsilon^2)$. We have $\phi_1=\phi_{11}+\phi_{12}$ where we recall ϕ_{11} is odd and ϕ_{12} is even, and with their sizes proportional to ε and εf, respectively. Thus in the expansion of the square term $(p(p-1)/2)w^{p-2}\phi_1^2$ asymptotically, only the mixed product between ϕ_{11} and ϕ_{12} gives rise to a nonzero term after the integration against w_x. We have

$$\begin{aligned}
&\int_{\mathbb{R}} w_x[(w+\phi_1)^p-w^p-pw^{p-1}\phi_1]dx\\
&=\frac{p(p-1)}{2}\int_{\mathbb{R}} w_x w^{p-2}\phi_1^2\,dx+\varepsilon^3 b_{2\varepsilon}\\
&=\varepsilon^2 a_{11}(\varepsilon z)a_{12}(\varepsilon z)f(\varepsilon z)p(p-1)\int_{\mathbb{R}} w_x w^{p-2}w_1w_2\,dx+\varepsilon^3 b_{2\varepsilon}.
\end{aligned}\tag{6.2}$$

Now, let us consider

$$\varphi(\varepsilon z)=\int_{\mathbb{R}} B_3(\phi_1)w_x.$$

All terms in this expression carry in the L^2-norm as functions of $\theta=\varepsilon z$ powers 3 or higher with the exception of the terms of size ε in B_3. Thus we find

$$\begin{aligned}
\varphi(\varepsilon z)=\varepsilon\beta^{-1}&\int_{\mathbb{R}}[k(\phi_1)_x-\beta^{-2}V_t(0,\varepsilon z)x\phi_1]w_x\\
&-\varepsilon\beta^{-2}V_t(0,\varepsilon z)f\int_{\mathbb{R}}\phi_1 w_x+O(\varepsilon^3).
\end{aligned}$$

Since

$$\phi_1=\varepsilon a_{11}(\varepsilon z)w_1(x)+\varepsilon f(\varepsilon z)a_{12}(\varepsilon z)w_2(x)$$

with $w_1(x)$ odd and $w_2(x)$ even, we obtain

$$\begin{aligned} \varphi(\varepsilon z) &= \varepsilon^2 f(\varepsilon z) a_{12}(\varepsilon z)\left\{\beta^{-1}\int_{\mathbb{R}} [k(w_2)_x - \beta^{-2} V_t(0,\varepsilon z) x w_2] w_x\right\} \\ &\quad - \varepsilon^2 f(\varepsilon z)\beta^{-2} a_{11}(\varepsilon z) V_t(0,\varepsilon z)\int_{\mathbb{R}} w_1 w_x + O(\varepsilon^3) \\ &= \varepsilon^2 a_{11}(\varepsilon z) a_{12}(\varepsilon z) f(\varepsilon z)\int_{\mathbb{R}}\left[w_{2,x} w_x + \frac{1}{\sigma} x w_x w_2 + w_1 w_x\right]. \end{aligned} \tag{6.3}$$

Note that by differentiating equation (2.23) and using equation (2.22), we obtain

$$\int_{\mathbb{R}} p(p-1) w^{p-2} w_x w_1 w_2 = -\int_{\mathbb{R}} w_x w_1 + \int_{\mathbb{R}}\left(w_x + \frac{1}{\sigma} x w\right) w_{2,x}. \tag{6.4}$$

Adding (6.2) and (6.3) and using (6.4), we have

$$\begin{aligned} &\int_{\mathbb{R}} w_x[(w+\phi_1)^p - w^p - p w^{p-1}\phi_1] + \int_{\mathbb{R}} w_x B_3(\phi_1) \\ &= \varepsilon^2 a_{11}(\varepsilon z) a_{12}(\varepsilon z) f(\varepsilon z)\int_{\mathbb{R}}\Big[p(p-1) w^{p-2} w_x w_1 w_2 \\ &\quad + w_{2,x} w_x + \frac{1}{\sigma} x w_x w_2 + w_1 w_x\Big] + \varepsilon^3 [b_{1\varepsilon} f'' + b_{2\varepsilon}] \\ &= \varepsilon^2 \beta^{-1}\sigma^{-1} k^2 f\int_{\mathbb{R}} [2 w_{2,x} w_x + \sigma^{-1} x (w_2 w)_x] + \varepsilon^3 [b_{1\varepsilon} f'' + b_{2\varepsilon}] \\ &= -\varepsilon^2 \beta^{-1}\sigma^{-1} k^2 f\int_{\mathbb{R}} w_x^2 + \varepsilon^3 [b_{1\varepsilon} f'' + b_{2\varepsilon}] \end{aligned} \tag{6.5}$$

where we have used (1.9) and the following integral identities:

$$\int_{\mathbb{R}} w_{2,x} w_x = -\left(\frac{2}{p-1} + \frac{1}{2}\right)\int_{\mathbb{R}} w_x^2, \qquad \sigma^{-1}\int_{\mathbb{R}} w_2 w = \left(\frac{1}{2} - \frac{2}{p-1}\right)\int_{\mathbb{R}} w_x^2.$$

In summary, we have established that

$$\int_{\mathbb{R}} S(w+\phi_1) w_x \, dx = \varepsilon^2 [f''(\varepsilon z) + \gamma_1(\varepsilon z) f' + \gamma_2(\varepsilon z) f] + \varepsilon^3 [b_{1\varepsilon} f'' + b_{2\varepsilon}] \tag{6.6}$$

where γ_1 is given by

$$\gamma_1(\theta) = \beta(\beta^{-2}\beta' + 2\alpha^{-1}\beta^{-1}\alpha') = \sigma V^{-1} V_\theta, \tag{6.7}$$

and γ_2 is given by

$$\gamma_2(\theta) = -\sigma V^{-1} V_{tt} + (\sigma^{-1} + 1) k^2. \tag{6.8}$$

Let us now estimate the term

$$\int_{\mathbb{R}} w_x\big[\varepsilon p[(w+\phi_1)^{p-1}-w^{p-1}](eZ)+\varepsilon B_3(eZ)\big]$$

$$\int_{\mathbb{R}} w_x\big[(w+\phi_1+\varepsilon eZ)^p-(w+\phi_1)^p-p(w+\phi_1)^{p-1}\varepsilon eZ\big].$$

We find now that

$$\begin{aligned}
&\int_{\mathbb{R}} w_x\{\varepsilon p[(w+\phi_1)^{p-1}-w^{p-1}]\}(eZ)\\
&\qquad+\int_{\mathbb{R}} w_x\big\{(w+\phi_1+\varepsilon eZ)^p-(w+\phi_1)^p-p(w+\phi_1)^{p-1}\varepsilon eZ]\big\}\\
&\qquad=\varepsilon p(p-1)\int_{\mathbb{R}} w^{p-2}\phi_1 eZw_x+\varepsilon^2 p(p-1)\int_{\mathbb{R}} w^{p-2}e^2Z^2w_x\\
&\qquad\quad+\varepsilon^3 b_{2\varepsilon}.
\end{aligned}\tag{6.9}$$

The second integral vanishes, while in the first only the term carrying the odd part of ϕ_1 is nonzero. Thus we find that this term equals

$$\varepsilon^2 a_{11}(\varepsilon z)e\int_{\mathbb{R}} Zw_1w_x\,dx+\varepsilon^3 b_{2\varepsilon}.$$

Let us now compute $\varepsilon\int_{\mathbb{R}} w_xB_3(eZ)dx$. In this term, we have to consider components of order $O(\varepsilon)$ in the coefficients of B_3, which are odd functions. We obtain

$$\begin{aligned}
&\varepsilon\int_{\mathbb{R}} w_xB_3(eZ)dx\\
&\quad=\varepsilon^2 ek(\varepsilon z)\int_{\mathbb{R}} w_x[Z_x+c_1xZ]+2\varepsilon^4 e''k(\varepsilon z)\int_{\mathbb{R}} xw_xZ\\
&\qquad+\varepsilon^3[b^1_{1\varepsilon}e''+b^2_{1\varepsilon}f''+b_{2\varepsilon}].
\end{aligned}$$

Summarizing, we have proven that

$$\begin{aligned}
\int_{\mathbb{R}} E_1w_x\,dx&=\varepsilon^2[f''+\gamma_1 f'+\gamma_2(\varepsilon z)f]+\varepsilon^2[\gamma_3(\varepsilon z)e+\varepsilon^2\gamma_4 e'']\\
&\qquad+\varepsilon^3[b^1_{1\varepsilon}e''+b^2_{1\varepsilon}f''+b_{2\varepsilon}].
\end{aligned}\tag{6.10}$$

The next computations, which are rather analogous, correspond to the projection onto Z of the error. We compute now $\int_{\mathbb{R}} E_1Z$. The main component in this expression is given by

$$\varepsilon[\varepsilon^2\beta^{-2}e'' + \lambda_0 e]\int_{\mathbb{R}} Z^2.$$

We also have a term like

$$\begin{aligned}
\varphi(\varepsilon z) &= \varepsilon\beta^{-1}\int_{\mathbb{R}}[k(\phi_1)_x - \beta^{-2}V_t(0,\varepsilon z)x\phi_1]Z + \varepsilon\beta^{-2}V_t(0,\varepsilon z)f\int_{\mathbb{R}}\phi_1 Z \\
&= \varepsilon^2 k\beta^{-1}a_{11}(\varepsilon z)\int_{\mathbb{R}}[(w_1)_x + c_1 x w_1]Z \\
&\quad + \varepsilon^2 k\beta^{-2}V_t(0,\varepsilon z)f^2 a_{12}(\varepsilon z)\int_{\mathbb{R}} w_2 Z \\
&= \varepsilon^2 k\beta^{-1}a_{11}(\varepsilon z)\int_{\mathbb{R}}[(w_1)_x + c_1 x w_1]Z + \varepsilon^3 b_{2\varepsilon}
\end{aligned}$$

because of the assumption on f.

There are also terms of order ε^2 coming from the second-order expansion of $S(w)$. We recall from the decomposition (2.31)–(2.32) that the error carries either terms accompanied by ε^2 as a factor or by ε^3. The terms with ε^3 produce functions of $\theta = \varepsilon z$ with $L^2(0,\ell)$–norms of order ε^3. The terms of order ε^2 in the decomposition of E_1 are either even or odd in the variable x. Those that are odd do not contribute to the integral since the function Z is even. Taking also into account that f and f' are uniformly controlled by $\varepsilon^{1/2}$, we just need to consider

$$\begin{aligned}
d_4(\varepsilon z) &= \varepsilon\beta^{-2}V_t(0,\varepsilon z)f(\varepsilon z)\int_{\mathbb{R}}\phi_1 Z\,dx, \\
d_5(\varepsilon z) &= \varepsilon^2 a_{11}^2(\varepsilon z)p(p-1)\int_{\mathbb{R}}[w^{p-2}w_1^2]Z\,dx, \\
d_6(\varepsilon z) &= \varepsilon^2\int_{\mathbb{R}}\Big[-\beta^{-2}k^2 x w_x + \beta^{-4}|\beta'|^2 x^2 w_{xx} + \beta^{-1}\beta'' x w_x \\
&\qquad + \alpha^{-1}\beta^{-2}\alpha'' w + 2\alpha^{-1}\beta^{-3}\alpha' x w_x - \frac{1}{2}\beta^{-4}V_{tt}x^2 w\Big]Z\,dx.
\end{aligned}$$

It is easy to see that also $d_4 = O(\varepsilon^3)$. The common pattern of d_5 and d_6 is that even though they have size ε^2 in the L^2-norm, they define smooth functions of $\theta = \varepsilon z$, which is a very important fact to obtain the desired result.

For the term parallel to (6.9), we get

$$\varepsilon p(p-1)\int_{\mathbb{R}} w^{p-2}\phi_1 eZZ\,dx + \varepsilon^2 p(p-1)\int_{\mathbb{R}} w^{p-2}e^2 Z^3\,dx + \varepsilon^3 b_{1\varepsilon} = \varepsilon^3 b_{1\varepsilon}.$$

We consider now another component:

$$\varepsilon \int_{\mathbb{R}} B_3(eZ)Z = -\varepsilon^2 e(\varepsilon z) f(\varepsilon z)\beta^{-2} V_t \int_{\mathbb{R}} Z^2 + \varepsilon^3 = O(\varepsilon^3).$$

Additionally, we also need to consider some higher-order terms in e. The ones involving the first derivative are

$$\varepsilon^3 e' \frac{2\alpha'}{\alpha\beta^2} \int_{\mathbb{R}} Z^2 + 2\varepsilon^3 e' \frac{\beta'}{\beta^3} \int_{\mathbb{R}} x Z_x Z.$$

Only one term (in $B_2(eZ)$) involving e'' also contains ε^3. But this term is also accompanied by $\int_{\mathbb{R}} Z(x)^2 x\, dx = 0$. There also is a term of the form

$$\varepsilon^3[\varepsilon f \beta^{-2} \gamma_5(\varepsilon z) + O(\varepsilon^2)] e''(\varepsilon z)$$

with $O(\varepsilon^2)$ uniform in ε.

In summary, we have established that, as a function of $\theta = \varepsilon z$,

$$\begin{aligned} \int_{\mathbb{R}} E_1 Z\, dx = \varepsilon^3[1 + \varepsilon f \gamma_5 + O(\varepsilon^2)]\beta^{-2} e'' \\ + \varepsilon^3 \gamma_6 e' + \varepsilon \lambda_0 e - \varepsilon^2 \gamma_7(\varepsilon z) + O(\varepsilon^3), \end{aligned} \tag{6.11}$$

where γ_i, $i = 5, \ldots, 7$, are smooth functions of their argument. An explicit expression for the coefficient γ_6, which we will need later, is

$$\gamma_6 = \frac{2\alpha'}{\alpha\beta^2} - \frac{\beta'}{\beta^3}. \tag{6.12}$$

7 Projections of Terms Involving ϕ

We will estimate next the terms that involve ϕ in (3.8)–(3.10) integrated against w_x and Z. Concerning w_x, we call the sum of them $\varphi(\phi)$, which can be decomposed as $\varphi = \sum_{i=1}^3 \varphi_i$ below.

Let $\varphi_1(\varepsilon z) = \int_{\mathbb{R}} B_3(\phi) w_x$. We make the following observation: all terms in $B(\phi)$ carry ε and involve powers of x times derivatives of 0, 1, or two orders of ϕ. The conclusion is that since w_x has exponential decay then

$$\int_0^\ell |\varphi(\theta)|^2\, d\theta \le C\varepsilon^3 \|\phi\|^2_{H^2(S)}.$$

Hence

$$\|\varphi_1\|_{L^2(0,\ell)} \le C\varepsilon^3.$$

In $B_3(\phi)$ we single out two less regular terms. The one whose coefficient depends on f'' explicitly has the form

$$\varphi_{1*} = \varepsilon^2 f'' \int_{\mathbb{R}} \phi_x Z(1 + k\varepsilon\beta(x - f))^{-2}$$

$$= -\varepsilon^2 f'' \int_{\mathbb{R}} \phi\{Z(1+k\varepsilon\beta(x-f))^{-2}\}_x .$$

Since ϕ has Lipschitz dependence on (f, e) in the form (5.9), we see that this is transmitted from Sobolev's embedding into

$$\|\phi_{(f_1,e_1)} - \phi_{(f_2,e_2)}\|_{L^\infty(\mathcal{S})} \le C\varepsilon^{\frac{3}{2}}[\|f_1 - f_2\|_a + \|e_1 - e_2\|_b],$$

from where it follows

$$\|\varphi_{1*}(f_1, e_1) - \varphi_{1*}(f_2, e_2)\|_{L^2(0,\ell)} \le C\varepsilon^{3+\frac{1}{2}}[\|f_1 - f_2\|_a + \|e_1 - e_2\|_b].$$

The one arising from a second derivative in z for ϕ is

$$\varphi_{1**} = \int_{\mathbb{R}} \phi_{zz} Z[1 - (1+k\varepsilon\beta(x-f))^{-2}]dx.$$

We readily see that

$$\|\varphi_{1**}(f_1, e_1) - \varphi_{1**}(f_2, e_2)\|_{L^2(0,\ell)} \le C\varepsilon^3[\|f_1 - f_2\|_a + \|e_1 - e_2\|_b].$$

The remainder $\varphi_1 - \varphi_{1*} - \varphi_{1**}$ actually defines for fixed ε a compact operator of the pair (f, e) into $L^2(0, \ell)$. This is a consequence of the fact that weak convergence in $H^2(\mathcal{S})$ implies local strong convergence in $H^1(\mathcal{S})$, and the same is the case for $H^2(0, \ell)$ and $C^1[0, \ell]$. If f_j and e_j are weakly convergent sequences in $H^2(0, \ell)$, then clearly the functions $\phi_{(f_j,e_j)}$ constitute a bounded sequence in $H^1(\mathcal{S})$. In the above remainder, one can integrate by parts if necessary once in x. Averaging against w_x, which decays exponentially, localizes the situation and the desired fact follows. We observe also that $\varphi_2(\varepsilon z) = \int_{\mathbb{R}} \tilde{N}(\phi) w_x$ can be estimated similarly. Using the definition of $\tilde{N}(\phi)$ and the exponential decay of w_x, we obtain

$$\|\varphi_2\|_{L^2(0,\ell)} \le C\varepsilon^{\frac{1}{2}}\|\phi\|^2_{H^2(\mathcal{S})} \le C\varepsilon^3.$$

Let us now consider

$$\varphi_3(\varepsilon z) = \int_{\mathbb{R}} p[\mathtt{w}^{p-1} - w^{p-1}]\phi w_x .$$

Since $\mathtt{w} = w + \phi_1 + \varepsilon e Z$ and ϕ_1 can be estimated as

$$\varepsilon|eZ(x)| + |\phi_1(x, z)| \le C\varepsilon(|x|^2 + 1)e^{-|x|},$$

we easily see that for some $\sigma > 0$, we have the uniform bound

$$|\mathtt{w}^{p-1} - w^{p-1}|\,|w_x| \le C\varepsilon e^{-\sigma|x|}.$$

From here we readily find that

$$\|\varphi_3\|_{L^2(0,\ell)} \le C\varepsilon^{\frac{3}{2}}\|\phi\|_{H^2(\mathcal{S})} \le C\varepsilon^3.$$

These terms define compact operators similarly as before. We observe that exactly the same estimates can be carried out in the terms obtained from integration after multiplying Z.

8 System for (f, e): Proof of Theorem

In this section we set up equations relating f and e such that for the solution ϕ of (3.8)–(3.9) predicted by Proposition 5.1 one has that the coefficient $c(\varepsilon z)$ is identically zero. To achieve this, we first multiply the equation against w_x and integrate only in x. The equation $c = 0$ is then equivalent to the relation

$$\int_{\mathbb{R}} E_1 w_x \, dx + \int_{\mathbb{R}} (N_2(\phi) + B_3(\phi) + p[\mathtt{w}^{p-1} - w^{p-1}]\phi) w_x \, dx = 0.$$

Similarly, $d = 0$ if and only if

$$\int_{\mathbb{R}} E_1 Z \, dx + \int_{\mathbb{R}} (N_2(\phi) + B_3(\phi) + p[\mathtt{w}^{p-1} - w^{p-1}]\phi) Z \, dx = 0.$$

Using the estimates in the previous sections, we then find that these relations are equivalent to the following nonlinear, nonlocal system of differential equations for the pair (f, e):

$$\mathcal{L}_1(f) \equiv f'' + \gamma_1 f' + \gamma_2 f = \gamma_3 e + \varepsilon^2 \gamma_4 e'' + \varepsilon M_{1\varepsilon}, \tag{8.1}$$

$$\mathcal{L}_2(e) \equiv \varepsilon^2(\beta^{-2} e'' + \gamma_6 e') + \lambda_0 e = \varepsilon^3 f \gamma_5 e'' + \varepsilon \gamma_7 + \varepsilon^2 M_{2\varepsilon}. \tag{8.2}$$

The operators $M_{l\varepsilon} = M_{l\varepsilon}(f, e)$ can be decomposed into the following form:

$$M_{l\varepsilon}(f, e) = A_{l\varepsilon}(f, e) + K_{l\varepsilon}(f, e)$$

where $K_{l\varepsilon}$ is uniformly bounded in $L^2(0, \ell)$ for (f, ε) satisfying constraints (2.29)–(2.30) and is also compact. The operator $A_{l\varepsilon}$ is Lipschitz in this region,

$$\|A_{l\varepsilon}(f_1, e_1) - A_{l\varepsilon}(f_2, e_2)\|_{L^2(0,\ell)} \le C\varepsilon[\|f_1 - f_2\|_a + \|e_1 - e_2\|_b].$$

The functions γ_i, $i = 1, \ldots, 7$, are smooth.

We will now solve system (8.1)–(8.2). The first observation is that the operator $\mathcal{L}_1$ is invertible under ℓ-periodic boundary conditions. This follows from the assumed nondegeneracy condition (1.10): if $g \in L^2(0, \ell)$, then there is a unique solution $f \in H^2(0, \ell)$ of $\mathcal{L}_1(f) = g$ that is ℓ-periodic and satisfies

$$\|f''\|_{L^2(0,\ell)} + \|f'\|_{L^\infty(0,\ell)} + \|f\|_{L^\infty(0,\ell)} \le C\|g\|_{L^2(0,\ell)}.$$

We now use assumption (1.14) to deal with the invertibility of $\mathcal{L}_2$. We have the following:

LEMMA 8.1 *Assume that condition* (1.14) *holds. If $d \in L^2(0, \ell)$, then there is a unique solution $e \in H^2(0, \ell)$ of $\mathcal{L}_2(e) = d$ that is ℓ-periodic and satisfies*

$$\varepsilon^2\|e''\|_{L^2(0,\ell)} + \varepsilon\|e'\|_{L^2(0,\ell)} + \|e\|_{L^\infty(0,\ell)} \le C\varepsilon^{-1}\|d\|_{L^2(0,\ell)}.$$

Moreover, if d is in $H^2(0, \ell)$, then

$$\begin{aligned}&\varepsilon^2\|e''\|_{L^2(0,\ell)} + \|e'\|_{L^2(0,\ell)} + \|e\|_{L^\infty(0,\ell)} \\ &\quad \le C[\|d''\|_{L^2(0,\ell)} + \|d'\|_{L^2(0,\ell)}] + C\|d\|_{L^2(0,\ell)}.\end{aligned}$$

Let us accept for the moment the validity of this result and conclude the proof of the theorem.

We first solve $\mathcal{L}_2(e_0) = \varepsilon\gamma_7(\theta)$ and replace $e = e_0 + \tilde{e}$. Observe that by Lemma 8.1 we have

$$\varepsilon^2 \|e_0''\|_{L^2(0,\ell)} + \|e_0\|_{L^\infty(0,\ell)} \le C\varepsilon.$$

The system resulting on $(f, \tilde{e})$ has the same form as (8.1)–(8.2) except that now the term $\varepsilon\gamma_7$ disappears. Let us observe now that the linear operator

$$\mathcal{L}(f, e) = (\mathcal{L}_1(f) - \gamma_3 e - \varepsilon^2\gamma_4 e'',\ \mathcal{L}_2(e))$$

is invertible with bounds for $\mathcal{L}(f, e) = (g, d)$ given by

$$\|f\|_a + \|e\|_b \le C\|g\|_2 + \varepsilon^{-1}\|d\|_2.$$

It then follows from the contraction mapping principle that the problem

$$[\mathcal{L} + (\varepsilon A_{1\varepsilon}, \varepsilon^2 A_{2\varepsilon})](f, e) = (g, d)$$

is uniquely solvable for (f, e) satisfying (2.29)–(2.30) if $\|g\|_2 < \varepsilon^{1/2+\rho}$ and $\|d\|_2 < \varepsilon^{3/2+\rho}$ for some $\rho > 0$. The desired result for the full problem (8.1)–(8.2) then follows directly from Schauder's fixed-point theorem. In fact, refining the fixed-point region, we can actually get $\|e\|_b + \|f\|_a = O(\varepsilon)$ for the solution. □

PROOF OF LEMMA 8.1: We consider then the boundary value problem

$$\mathcal{L}_2(e) = d, \quad e(0) = e(\ell),\ \ e'(0) = e'(\ell). \tag{8.3}$$

We make the following *Liouville transformation* (cf. [21]):

$$\ell_0 = \int_0^\ell \beta(\theta)d\theta, \quad t = \frac{\int_0^\theta \beta(\theta)d\theta}{\ell_0}\pi, \quad \tilde{\lambda}_0 = \frac{\ell_0^2}{\pi^2}\lambda_0$$

$$\Psi(\theta) = \beta^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\int_0^\theta \beta^2\gamma_6(\theta)d\theta\right),$$

$$y(t) = \Psi^{-1}(\theta)e(\theta), \quad q(t) = \frac{\ell_0^2}{\pi^2}\frac{\Psi''}{\beta^2\Psi}.$$

Observe that Ψ is ℓ-periodic thanks to the explicit formula (6.12) for the coefficient γ_6. Then (8.3) gets transformed into

$$\tilde{\mathcal{L}}_2(y) = \varepsilon^2(y'' + q(t)y) + \tilde{\lambda}_0 y = \tilde{d}, \quad y(0) = y(\pi),\ \ y'(0) = y'(\pi), \tag{8.4}$$

and it then suffices to establish the estimates in Lemma 8.1 for the solution of this problem in terms of the corresponding norms of $\tilde{d}$. It is standard that the eigenvalue problem

$$y'' + q(t)y + \lambda y = 0, \quad y(0) = y(\pi),\ \ y'(0) = y'(\pi), \tag{8.5}$$

has an infinite sequence of eigenvalues λ_k, $k \ge 0$, with an associated orthonormal basis in $L^2(0,\pi)$, $\{y_k\}$, constituted by eigenfunctions. A result in [21] provides asymptotic expressions as $k \to +\infty$ for these eigenvalues and eigenfunctions, which turn out to correspond to those for $q \equiv 0$. We have

$$\sqrt{\lambda_k} = 2k + O\left(\frac{1}{k^3}\right). \tag{8.6}$$

Problem (8.4) is then solvable if and only if $\lambda_k \varepsilon^2 \neq \tilde{\lambda}_0$ for all k. In such a case, the solution to (8.3) can then be written as

$$y(t) = \sum_{k=0}^{\infty} \frac{\tilde{d}_k}{\tilde{\lambda}_0 - \lambda_k \varepsilon^2}\, y_k(t)$$

with this series convergent in L^2. Hence

$$\|y\|^2_{L^2(0,\pi)} = \sum_{k=0}^{\infty} \frac{|\tilde{d}_k|^2}{(\tilde{\lambda}_0 - \lambda_k \varepsilon^2)^2}.$$

We then choose ε such that

$$|4k^2\varepsilon^2 - \tilde{\lambda}_0| \geq c\varepsilon \tag{8.7}$$

for all k, where c is small. This corresponds precisely to the condition (1.14) in the statement of the theorem. From (8.6) we then find that $|\tilde{\lambda}_0 - \lambda_k \varepsilon^2| \geq (c/2)\varepsilon$ if ε is also sufficiently small. It follows that $\|y\|_{L^2(0,\pi)} \leq C\varepsilon^{-1}\|\tilde{d}\|_{L^2(0,\pi)}$. Observe also that

$$\|y'\|^2_{L^2(0,\pi)} \leq C \sum_{k=0}^{\infty} |\tilde{d}_k|^2 \frac{1 + |\lambda_k|}{(\tilde{\lambda}_0 - \lambda_k \varepsilon^2)^2} \leq C \sum_{k=0}^{\infty} (1 + k^4)|\tilde{d}_k|^2.$$

Hence

$$\varepsilon\|y'\|_{L^2(0,\pi)} + \|y\|_{L^\infty(0,\pi)} \leq C\varepsilon^{-1}\|\tilde{d}\|_{L^2(0,\pi)}.$$

Besides, if d is in $H^2(0,\pi)$ with $d(0) = d(\pi)$ and $d'(0) = d'(\pi)$, then the sum $\sum_k k^4 d_k^2$ is finite and bounded by the H^2-norm of d. This automatically implies

$$\varepsilon^2\|y''\|_{L^2(0,\pi)} + \|y'\|_{L^2(0,\pi)} + \|y\|_{L^\infty(0,\pi)} \leq C\|\tilde{d}\|_{H^2(0,\pi)},$$

and the proof is complete. □

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