Some extremal singular solutions of a nonlinear elliptic equation

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1 Introduction

Let Ω ⊂ IR^n be a smooth, bounded domain, and let f be a smooth function on Ω, f ≥ 0, f ≠ 0. Let p > 1 and consider the semi-linear elliptic equation

\[ \begin{aligned}
(P_t) \quad & \left\{ \begin{array}{ll}
-\Delta u = u^p + tf & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{array} \right.
\end{aligned} \]

where t ≥ 0 is a parameter.

We are concerned with weak solutions of \((P_t)\), and we use the definition introduced in [BCMR]: a weak solution of \((P_t)\) is a function \(u \in L^1(\Omega), u \geq 0\), such that \(u^p \delta \in L^1(\Omega)\), where \(\delta(x) = \text{dist}(x, \partial \Omega)\), and such that

\[-\int_{\Omega} u \Delta \zeta \, dx = \int_{\Omega} (u^p + tf) \zeta \, dx\]

for all \(\zeta \in C^2(\overline{\Omega})\) with \(\zeta = 0\) on \(\partial \Omega\).

We start by mentioning some well known facts (see for example [BCMR], [BC], [Ma]).

Theorem 1 There exists \(0 < t^* < \infty\) such that for \(0 < t < t^*\) \((P_t)\) has a unique minimal solution \(u(\cdot, t)\) (which is smooth), for \(t = t^*\) \((P_{t^*})\) has a unique solution \(u^*\) (possibly unbounded), and for \(t > t^*\) there is no solution of \((P_t)\) (even in the weak sense). Moreover \(u(\cdot, t)\) depends smoothly on \(t \in (0, t^*)\), increases as \(t\) increases, and

\(u(\cdot, t) \nearrow u^* \quad \text{a.e. in } \Omega, \text{ as } t \nearrow t^*\).

We call \(u^*\) the extremal solution.

An important feature of the minimal solution \(u\) is that the linearized operator at \(u\)

\[-\Delta - pu^{p-1}\]

has a positive first eigenvalue for all \(0 < t < t^*\). This property can be used as in [CR] or [MP], to prove the following

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Theorem 2  If

\[ n < 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}} \]  

then there exists a constant C independent of t such that

\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for any } 0 < t < t^*. \]

This is equivalent to the statement that the extremal solution \( u^* \) is bounded.

We note that if the extremal solution \( u^* \) is bounded, then by elliptic regularity it is also smooth, and in this case the first eigenvalue of \(-\Delta - p(u^*)^{p-1}\) is zero.

In the present work we are interested in the case \( n \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}} \). First in Section 2 we show that Theorem 2 is sharp, i.e. assuming \( n \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}} \), for any domain \( \Omega \) there exists a smooth function \( f \geq 0, f \not\equiv 0 \) for which the extremal solution \( u^* \) is unbounded (or singular). Then in Section 3 we study the radially symmetric case with \( \Omega \) the open unit ball in \( IR^n \), and we show that assuming \( n \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}} \) the extremal solution is singular for any smooth radially symmetric function \( f, f \geq 0, f \not\equiv 0 \). We also give a precise description of the singularity of \( u^* \) in this case.

A problem related to \((P_t)\) that has received much attention in the literature is the following:

\[ \begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases} \]  

where \( \lambda \geq 0 \) is a parameter and \( g : [0, \infty) \to (0, \infty) \) is a \( C^1 \) convex, positive, nondecreasing function with \( g(0) > 0 \) and \( g(u)/u \to \infty \) as \( u \to \infty \). Typical examples are \( g(u) = e^u \) and \( g(u) = (1 + u)^p, p > 1 \). For this equation there is again an extremal parameter \( \lambda^* < \infty \), such that for \( 0 < \lambda < \lambda^* \) there is a minimal solution, for \( \lambda = \lambda^* \) there is a unique weak solution (called the extremal solution), and for \( \lambda > \lambda^* \) there is no solution (see for example [BCMR],[BV],[MP] and their references for results on this problem).

Several very interesting open problems for \((2)\) were proposed in [BV], and we mention some of them in the context of problem \((P_t)\).

1) Assume \( n \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}} \). If \( \Omega \) is convex, is it true that for any smooth \( f \geq 0, f \not\equiv 0 \) the extremal solution is singular? Is it true for \( f \equiv 1 \)? We note that some restriction on \( \Omega \) has to be imposed, as shown by the example \( \Omega = B_2 \setminus B_1 \) and \( f \equiv 1 \) (or any radially symmetric positive smooth \( f \)). In this case it can be easily shown that the extremal solution is always smooth, without any restriction on \( n \) and \( p \). (See Problem 3 in [BV]).

2) Concerning problem \((2)\) in some cases the extremal solution is bounded and in others it is singular (see [BV]). In a recent work, G. Nedev [Ne] has shown that in dimension 2, for any nonlinearity \( g \) satisfying the hypothesis above the extremal solution of \((2)\) is bounded. His argument can be adapted to show that the same is true for a more general version of \((P_t)\), where
the nonlinearity \( u^p \) is replaced by \( g(u) \), and \( g \) is a \( C^1 \) positive, convex, increasing function with \( g(0) = 0 \), and \( g(u)/u \to \infty \) as \( u \to \infty \) (we note that Theorem 1 is still true for this more general problem). In dimension 3, it is not known whether or not there exist nonlinearities \( g \) for which the extremal solution is singular.

2 Is condition (1) sharp?

**Theorem 3** Let \( \Omega \subset \mathbb{R}^n \) be any bounded, smooth domain. If

\[
 n \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}} \quad (3)
\]

then there exists a smooth function \( f \geq 0, f \neq 0 \) so that the extremal solution \( u^* \) is singular.

The idea of the construction is the same as in [BV], that is, to find a smooth function \( f \geq 0, f \neq 0 \), a number \( 0 < t < \infty \) and an unbounded function \( u \in H^1_0(\Omega) \) which is a weak solution of (P), and such that the operator

\[
 -\Delta - pu^{p-1}
\]

has a nonnegative first eigenvalue, in the sense that

\[
 \int_{\Omega} |\nabla \varphi|^2 \, dx \geq p \int_{\Omega} u^{p-1}\varphi^2 \, dx \quad \forall \varphi \in C^1_0(\Omega).
\]

(4)

Then we can conclude using the following lemma (similar to Theorem 3 in [BV]).

**Lemma 4** Suppose that \( u \in H^1_0(\Omega) \) is an unbounded weak solution of (P) such that the operator

\[
 -\Delta - pu^{p-1}
\]

has a nonnegative first eigenvalue (in the sense of (4)). Then \( t = t^* \) and \( u = u^* \).

**Proof.** Since there is no solution for \( t > t^* \) we must have \( t \leq t^* \). Let \( \underline{u} = u(\cdot, t) \) denote the minimal solution of (P), and let \( g(u) = u^p \). The inequality

\[
 \int_{\Omega} |\nabla \varphi|^2 \, dx \geq \int_{\Omega} g'(u)\varphi^2 \, dx
\]

holds by assumption for all \( \varphi \in C^1_0(\Omega) \) and by approximation also for \( \varphi \in H^1_0(\Omega) \). We take \( \varphi = u - \underline{u} \in H^1_0(\Omega) \) (note that by assumption \( u \in H^1_0(\Omega) \) and by the estimates in the appendix, \( \underline{u} \in H^1_0(\Omega) \) even for \( t = t^* \), i.e. \( \underline{u} = u^* \)). We have

\[
 \int_{\Omega} g'(u)(u - \underline{u})^2 \, dx \leq \int_{\Omega} |\nabla(u - \underline{u})|^2 \, dx \\
 = \int_{\Omega} -\Delta(u - \underline{u})(u - \underline{u}) \, dx \\
 = \int_{\Omega} (g(u) + tf - g(\underline{u}) - f)(u - \underline{u}) \, dx
\]
so that
\[ \int_\Omega (u - u)(g(u) + g'(u)(u - u) - g(u)) \, dx \geq 0. \]
Because of the convexity of \( g \) and since \( u \geq u \), the integrand is non-positive and we conclude that
\[ g(u) = g(u) + g'(u)(u - u). \]
Since \( g(u) = u^p \) is strictly convex we conclude that \( u = u \). But \( u \) is unbounded and this forces \( t = t^* \).

Consider the function
\[ v(x) = v(|x|) = \lambda |x|^\alpha \quad (5) \]
where
\[ \lambda = \lambda_{n,p} = \left( \frac{2}{p-1} \left( n - \frac{2p}{p-1} \right) \right)^{\frac{1}{p-1}} \]
and
\[ \alpha = \alpha_p = -\frac{2}{p - 1}. \]
Then \( v \in H^1(\Omega) \) for \( n > 2 + 4/(p - 1) \), and
\[ -\Delta v = v^p \text{ in } \mathbb{R}^n. \]
From now on we assume that \( 0 \in \Omega \), and we will construct \( u \) with a singularity at the origin so that it satisfies the requirements in Lemma 4. We look for a function \( u \) of the form \( u = v - \psi \).

**Lemma 5** There exists a smooth function \( \psi \) defined on \( \bar{\Omega} \) with the properties:

1. \( \psi \geq 0 \) and is smooth in \( \bar{\Omega} \),
2. \( \Delta \psi \geq 0 \) in \( \Omega \),
3. \( \psi \equiv 0 \) in a neighborhood of \( 0 \), and
4. \( \psi = v \) on \( \partial \Omega \).

**Proof of Theorem 3.** Let \( u = v - \psi \). Then
\[
-\Delta u = -\Delta v + \Delta \psi \\
= v^p + \Delta \psi \\
\geq 0
\]
and \( u = 0 \) on \( \partial \Omega \), so \( u \geq 0 \). Taking
\[
f = \Delta \psi + v^p - u^p
\]
we then have

$$-\Delta u = u^p + f.$$  

Note that \( f \geq 0 \) and is smooth, because \( u \leq v \) and \( u \equiv v \) in a neighborhood of 0. The only condition that still needs to be checked to apply Lemma 4 is the non-negativity of the first eigenvalue of the operator \(-\Delta - pu^{p-1} \). Here enters into play condition (3). Recall the Hardy inequality (see [BV] for example):

$$\frac{(n-2)^2}{4} \int_\Omega \frac{1}{|x|^2} \varphi^2 \, dx \leq \int_\Omega |\nabla \varphi|^2 \, dx$$

for any \( \varphi \in C^1_0(\Omega) \), when \( n \geq 3 \). Note that \( u \leq v \) so that for any \( \varphi \in C^1_0(\Omega) \)

$$\int_\Omega pu^{p-1} \varphi^2 \, dx \leq \int_\Omega pv^{p-1} \varphi^2 \, dx = \frac{2p}{p-1} \left(n - \frac{2p}{p-1}\right) \int_\Omega \frac{1}{|x|^2} \varphi^2 \, dx \leq \frac{(n-2)^2}{4} \int_\Omega \frac{1}{|x|^2} \varphi^2 \, dx \leq \int_\Omega |\nabla \varphi|^2 \, dx$$

where the third inequality is a consequence of (3).

\[ \square \]

**Proof of Lemma 5.** Let \( r = \text{dist}(0, \partial \Omega)/2 \), and let \( \psi_1 \) be the solution of the following problem

\[
\begin{cases}
\Delta \psi_1 = 0 & \text{in } \Omega \setminus \overline{B}_r \\
\psi_1 = v & \text{on } \partial \Omega \\
\psi_1 = 0 & \text{on } \partial B_r
\end{cases}
\]

Then \( \psi_1 \) is smooth and positive in \( \Omega \setminus \overline{B}_r \) and by the Hopf boundary lemma \( \frac{\partial \psi_1}{\partial \nu} > 0 \) on \( \partial B_r \), where \( \nu \) is the normal vector, pointing away from the origin. Let \( \psi_1 \) be extended by 0 in \( B_r \). Then \( \Delta \psi_1 \geq 0 \) in \( D(\Omega)' \).

Now we regularize \( \psi_1 \) by convolution to get a smooth function \( \psi \):

$$\psi = \psi_1 * \rho_\varepsilon$$

where \( \rho_\varepsilon \) is a standard mollifier \( (\rho_\varepsilon(x) = \varepsilon^{-n} \rho(x/\varepsilon), \rho \in C_0^\infty(\mathbb{R}^n), \rho \geq 0, \text{supp}(\rho) \subset B_1, \int \rho \, dx = 1) \). \( \psi(x) \) is well defined and subharmonic on the set

$$\{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \varepsilon \}.$$

If \( \rho_\varepsilon \) is radially symmetric, then \( \psi_1 = \psi_1 * \rho_\varepsilon = \psi \) on

$$\{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \varepsilon, \text{dist}(x, B_r) > \varepsilon \}.$$

By fixing \( \varepsilon > 0 \) but small enough we can consider \( \psi \) to be defined and smooth up to \( \partial \Omega \). \[ \square \]
3 The radially symmetric case

**Theorem 6** Assume now that $\Omega$ is the open unit ball $B_1(0)$ in $\mathbb{R}^n$, and that $f \geq 0$, $f \not\equiv 0$ is any smooth, radially symmetric function. If $n \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}$ then the extremal solution $u^*$ is singular.

First we give a short proof of this theorem, but actually more can be said about the extremal solution $u^*$ than merely $u^* \notin L^\infty(\Omega)$.

**Theorem 7** Assume $\Omega$ is the open unit ball $B_1(0)$ in $\mathbb{R}^n$, and $f \geq 0$, $f \not\equiv 0$ is a radially symmetric function $f(x) = f(|x|)$ with $f \in C^2([0,1])$. Suppose $n \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}$. Let $v$ be the function defined by (5) and set $w = r^{-2}(u - v)$, $r = |x|$. Then $w$ is $C^2([0,1])$, and if moreover $f'(0) = 0$ (i.e. $f \in C^2(\bar{\Omega})$), then the same is true for $w$.

Before giving the proofs we note that if $\Omega$ and $f$ are radially symmetric, then the minimal solution $u$ of $(P_t)$ is also radially symmetric.

**Proof of Theorem 6.** Let $v$ denote the function defined by (5). We use the improved Hardy inequality, proved in [BV]: for all $\varphi \in C^1_0(\Omega)$ we have

$$\int_\Omega |\nabla \varphi|^2 \, dx \geq \frac{(n-2)^2}{4} \int_\Omega \frac{\varphi^2}{|x|^2} \, dx + c \int_\Omega \varphi^2 \, dx$$

where $c = H_2(w_n/|\Omega|)^{2/n} > 0$, $H_2$ is the first eigenvalue of the Laplacian with Dirichlet boundary condition in the unit ball in dimension 2, and $w_n$ is the measure of the unit ball in $\mathbb{R}^n$. This inequality implies that if (3) holds, then the operator $-\Delta - pv^{p-1}$ has a positive first eigenvalue, and although $pv^{p-1} = C/|x|^2$ is not in $L^{n/2}(\Omega)$, the maximum principle can be applied to it.

**Claim:** for all $0 < t \leq t^*$ we have $u(\cdot,t) \leq v$, and the inequality is strict for $0 < t < t^*$.

Indeed let $0 < t < t^*$ and suppose that there exists some $0 < r < 1$ such that $u(r,t) \geq v(r)$. Then

$$u - v \geq 0 \quad \text{on } \partial B_r$$

and by the convexity of $u \to u^p$ we have

$$-\Delta(u - v) = u^p - v^p + tf \geq pv^{p-1}(u - v) + tf$$

so that

$$-\Delta(u - v) - pv^{p-1}(u - v) \geq 0.$$

By the maximum principle we conclude that $u \geq v$ on $B_r$, which is impossible, because $u$ is bounded for $0 < t < t^*$. The conclusion for $t = t^*$ is obtained by taking the limit as $t \to t^*$. 

6
Since $u^* \leq v$ we conclude that the first eigenvalue for the operator $-\Delta - pu^{p-1}$ is
\[
\inf_{\|\varphi\|_{L^2}} \int_{\Omega} |\nabla \varphi|^2 - pu^{p-1} \varphi^2 \, dx \geq \inf_{\|\varphi\|_{L^2}} \int_{\Omega} |\nabla \varphi|^2 - pv^{p-1} \varphi^2 \, dx > 0.
\]
This shows that $u^*$ cannot be bounded.

**Proof of Theorem 7.** This proof involves again the same idea as in [BV], using Lemma 4. We set
\[ u = v + r^2 w \]
Then a calculation shows that the equation $-\Delta u = u^p + f$ is equivalent to
\[-(r^2 w'' + (n + 3)rw' + 2nw) = |v + r^2 w|^p - v^p + f, \quad 0 < r < 1 \]
It is convenient to rewrite this equation as
\[ w'' + \frac{n + 3}{r} w' + \frac{2n + pv^{p-1}r^2}{r^2} w = -\frac{1}{r^2} \left( |v + r^2 w|^p - v^p - pv^{p-1}r^2 w \right) - r^{-2} f \]
or
\[ Lw = -g(r, w) - r^{-2} f \tag{6} \]
where
\[ Lw = w'' + \frac{n + 3}{r} w' + \frac{2n + pv^{p-1}r^2}{r^2} w \]
and
\[ g(r, w) = \frac{1}{r^2} \left( |v + r^2 w|^p - v^p - pv^{p-1}r^2 w \right) \]
Note that $pv^{p-1}r^2 = p\lambda^{p-1}$ is a constant, and that $g(r, w) \geq 0$ by convexity.

The aim is the to find a solution $w$ of (6), that behaves nicely near 0 and such that $w(1) = -v(1)$. It turns out that a nice behavior of $w$ near 0 can be imposed for example by the requirement that
\[ w(r), rw'(r) \text{ are bounded near 0} \tag{7} \]
We show in Proposition 8 that if $f$ is a continuous function on $[0, \infty)$ then (6) together with (7) has a unique solution $w$, which is defined on an open maximal interval. We also prove that $w \leq 0$ if $f \geq 0$. Then, in Proposition 15, we show that if we replace $f$ by $tf$ in (6), where $t \geq 0$, $f \geq 0$, $f \not\equiv 0$ in $[0, 1]$, then there exists $t$ such that the solution $w$ to (6)-(7) is defined on $[0, 1]$ and $w(1) = -v(1)$. We also show in Lemma 12 that if $f$ is smooth enough, then $w$ has the regularity stated in Theorem 7.

Accepting these results for a moment, we see that
\[ u = v + r^2 w \]
satisfies the requirements in Lemma 4, the non negativity of the first eigenvalue of the operator $-\Delta - pu^{p-1}$ following again from $u \leq v$, the Hardy inequality and condition (3).

From now on until the end of this section we assume that condition (3) holds.
Proposition 8

a) Let $K > 0$. Then there exists $R > 0$ such that for any continuous function $f$ on $[0, R]$ with $\|f\|_{C[0,R]} \leq K$, (6)-(7) has a unique solution on $(0, R)$. Moreover, the solution depends continuously on $f$. More precisely, there exists a constant $C > 0$, such that for any continuous functions $f_1, f_2$ on $[0, R]$, $\|f_i\|_{C[0,R]} \leq K$, $i = 1, 2$, if $w_1, w_2$ are the corresponding solutions of (6)-(7), then
\[ \|w_1 - w_2\|_{C[0,R]} \leq C\|f_1 - f_2\|_{C[0,R]} \]

b) If $f$ is a continuous function on $[0, \infty)$, the (6)-(7) has a unique solution $w$ defined on an open maximal interval. The solution depends continuously on $f$.

We need some preparatory lemmas.

Lemma 9 There exists $C > 0$ depending only on $n, p$ such that if
\[ M > 0, \ R > 0 \text{ and } 2MR^{2p} \leq \lambda \]
then
\[ |g(r, w)| \leq Cr^{2p-1}|w|^2 \] (8)
for any $|w| \leq M$ and $0 < r < R$ and
\[ |g(r, w_1) - g(r, w_2)| \leq CMr^{2p-1}|w_1 - w_2| \] (9)
for any $|w_1|, |w_2| \leq M$ and $0 < r < R$.

Proof. Let $a(x) = x^p$, which is a convex functions (recall that $p > 1$). Let $|w| \leq M$ and $0 < r < R$. Then, using $2MR^{2p} \leq \lambda$, we obtain $|r^2w| \leq \frac{1}{2}\lambda r^{-2p+1}$. With $v = v(r) = \lambda r^{-2p+1}$, we have $\frac{1}{2}v \leq v + r^2w \leq \frac{3}{2}v$. Notice that
\[
g(r, w) = \frac{1}{r^2} \left( a(v + r^2w) - a(v) - a'(v)r^2w \right) = \frac{1}{2} a''(\xi)r^2w^2 \]
where $\xi$ is in the interval with endpoints $v$ and $v + r^2w$. Using that $a''$ is monotone, we thus have
\[
|g(r, w)| \leq \frac{1}{2} p(p - 1)r^2|w|^2 \max\{(1/2)^{p-2}, (3/2)^{p-2}\} v^{p-2} \\
\leq C(p)r^2|w|^2\lambda^{p-2}r^{-2p+1}(p-2) \\
\leq C(n, p)r^{2p-1}|w|^2
\]
We now prove estimate (9):

\[ |g(r, w_1) - g(r, w_2)| = \frac{1}{r^2} \left| (v + r^2 w_1)^p - (v + r^2 w_2)^p - p v^{p-1} r^2 (w_1 - w_2) \right| \]

\[ = \frac{1}{r^2} \left| \int_0^1 \frac{d}{dt} (v + r^2 (tw_1 + (1 - t)w_2))^p - p v^{p-1} r^2 (w_1 - w_2) \, dt \right| \]

\[ \leq p \int_0^1 \left| (v + r^2 (tw_1 + (1 - t)w_2))^{p-1} - v^{p-1} \right| |w_1 - w_2| \, dt \]

But

\[ \left| (v + r^2 (tw_1 + (1 - t)w_2))^{p-1} - v^{p-1} \right| = (p - 1) |\xi|^{p-2} r^2 |tw_1 + (1 - t)w_2| \]

where \( \xi \) is in the interval with endpoints \( v \) and \( v + r^2 (tw_1 + (1 - t)w_2) \). Therefore

\[ \left| (v + r^2 (tw_1 + (1 - t)w_2))^{p-1} - v^{p-1} \right| \leq (p - 1) \max \{(1/2)^{p-2}, (3/2)^{p-2}\} v^{p-2} r^2 M \]

\[ \leq C(n, p) r^{\frac{2}{p-1}} M \]

\[ \square \]

**Lemma 10** Let \( w \) be a solution of (6) in \((0, R)\) (i.e. \( w \in C^2(0, R) \) and satisfies the equation) and let \( 0 < r_0 < R \). Then

\[ w(r) = w_h(r) - \int_{r_0}^r k(s/r) \left( s g(s, w(s)) + s^{-1} f(s) \right) ds, \quad 0 < r < R \quad (10) \]

where \( w_h \) is the solution of the linear homogeneous equation

\[ \left\{ \begin{array}{l}
Lw_h = 0 \quad \text{in } (0, R) \\
w_h(r_0) = w(r_0) \\
w_h'(r_0) = w'(r_0)
\end{array} \right. \quad (11) \]

and \( k \) is the continuous function on \([0, 1]\) given by:

\[ k(t) = \begin{cases} 
  t^{-\beta} \ln(1/t) & \text{if } n = 6 + \frac{4}{p-1} + 4 \sqrt{\frac{p}{p-1}} \quad \text{where } \beta = -\frac{n+2}{2} \\
  t^{-\beta_1 - t^{-\beta_2}} & \text{if } n > 6 + \frac{4}{p-1} + 4 \sqrt{\frac{p}{p-1}} \quad \text{where } \beta_{1,2} = -\frac{n+2}{2} \pm \sqrt{\left(\frac{n+2}{2}\right)^2 - 2n - p\lambda^{p-1}}
\end{cases} \]

We note that (3) implies that \( \beta_1, \beta_2 \) are real, and that \( k > 0 \) on \((0, 1)\), \( k(0) = k(1) = 0 \).

**Proof.** We use the variation of parameters formula, noting that two linearly independent solutions of the homogeneous equation \( Ly = 0 \) on \((0, \infty)\) are:

\[ y_1 = s^\beta, \quad y_2 = \ln(s)^s s^\beta \quad \text{if } n = 6 + \frac{4}{p-1} + 4 \sqrt{\frac{p}{p-1}} \]

\[ y_1 = s^{\beta_1}, \quad y_2 = s^{\beta_2} \quad \text{if } n > 6 + \frac{4}{p-1} + 4 \sqrt{\frac{p}{p-1}} \]
Lemma 11 Let $w$ be a solution of (6) in $(0, R)$ and suppose it satisfies (7). The

$$w(r) = -\int_0^r k(s/r) \left( sg(s, w(s)) + s^{-1} f(s) \right) \, ds$$  \hspace{1cm} (12)

Proof. A direct computation gives the following expression for the solution $w_h$ of the homogeneous equation (11):

\[
\begin{align*}
\text{case } n &= 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}: & w_h(r) &= c_1 r^\beta + c_2 \ln(r) r^\beta \\
& & c_1 &= w(r_0) r_0^{-\beta} (\beta \ln(r_0) + 1) - w'(r_0) r_0^{-\beta+1} \ln(r_0) \\
& & c_2 &= -\beta w(r_0) r_0^{-\beta} + w'(r_0) r_0^{-\beta+1} \\
\text{case } n &> 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}: & w_h(r) &= c_1 r_0^{-\beta_1} + c_2 \beta_2 \\
& & c_1 &= (\beta_2 w(r_0) r_0^{-\beta_1} - w'(r_0) r_0^{-\beta_2+1})/(\beta_2 - \beta_1) \\
& & c_2 &= -\beta_1 w(r_0) r_0^{-\beta_1} + w'(r_0) r_0^{-\beta_2+1})/(\beta_2 - \beta_1)
\end{align*}
\]

In both cases we see that under the assumption (7) we have $c_1, c_2 \to 0$ as $r_0 \to 0^+$, and that we can take the limit as $r_0 \to 0^+$ in (10).

Lemma 12 Let $R > 0$ and $w \in L^\infty(0, R)$ satisfy (12) for $0 < r < R$, where $f \in C([0, R])$. Then $w \in C^2(0, R)$ and is a solution of (6)-(7). If moreover $f \in C^2([0, R])$ then the same is true for $w$, and if $f''(0) = 0$ then $w'(0) = 0$.

Proof. We differentiate under the integral sign and check that the equation (6) is satisfied. Set $w = w_1 + w_2$ where

$$w_1(r) = -\int_0^r k(s/r) s g(s, w(s)) \, ds \quad w_2(r) = -\int_0^r k(s/r) s^{-1} f(s) \, ds$$

It is easy to see that if $f$ is smooth the $w_2$ is also smooth and that

$$w_2'(0) = -f'(0) \int_0^1 k(t) \, dt$$

so that $f'(0) = 0$ implies $w_2'(0) = 0$. It is also easy to check that if $f$ is only continuous then $rw_2'(r) \to 0$ as $r \to 0^+$. To estimate $w_1$ and its derivatives, consider $M = \|w\|_{L^\infty(0, R)}$ and let $r_0 > 0$ be small enough so that $2MR_0^{\frac{2p}{p-1}} \leq \lambda$. Then by (8), for $0 < r < r_0$ we have

$$|w_1(r)| \leq CM^2 \int_0^r k(s/r) s^{\frac{p+1}{p-1}} \, ds \leq CM^2 r_0^{\frac{2p}{p-1}} \int_0^1 k(t) t^{\frac{p+1}{p-1}} \, dt \quad \to 0 \quad \text{as } r \to 0^+$$

In a similar way one proves that $w_1'(r), w_1''(r) \to 0$ as $r \to 0^+$.
Proof of Proposition 8. To prove part b) of the proposition we use part a) to obtain the conclusions on some interval \((0, R), R > 0\), and then we can quote standard results for ODE’s (see for example [CL]).

The proof of part a) consists in applying the Banach fixed point theorem to the operator suggested by (12). Let \(R > 0\) (to be specified later) and let \(f \in C[0, R]\). Consider the operator \(T : C[0, R] \to C[0, R]\) defined by

\[ Tw(r) = - \int_0^r k(s/r) \left( sg(s, w(s)) + s^{-1} f(s) \right) \, ds \]

Let \(M > 0\) (also to be chosen later) and let \(X_M\) be the closed ball of \(C[0, R]\) centered at 0 of radius \(M\). Then, for \(w \in X_M\) and if \(2MR^{2p-1} \leq \lambda\), using (8) we have

\[ |Tw(r)| \leq \int_0^r |k(s/r)| \left(Cs^{\frac{p+1}{p}}|w(s)|^2 + s^{-1}|f(s)| \right) \, ds \leq CM^2 \int_0^r |k(s/r)|s^{\frac{p+1}{p-1}} \, ds + \|f\|_{C[0,R]} \int_0^r |k(s/r)|s^{-1} \, ds \]

But

\[ \int_0^r |k(s/r)|s^q \, ds = r^{q+1} \int_0^1 |k(t)|t^q \, dt \]

and using the expression for \(k\) one can check that the integrals in the right hand side are finite for \(q = \frac{p+1}{p-1}\) and \(q = -1\). We obtain thus

\[ \|Tw\|_{C[0,R]} \leq C \left(M^2 R^{\frac{2p}{p-1}} + K \right) \]

if \(\|f\|_{C[0,R]} \leq K\). Also, for \(w_1, w_2 \in X_M\), by (9) we have

\[ |Tw_1(r) - Tw_2(r)| \leq \int_0^r |k(s/r)|s|g(s, w_1(s)) - g(s, w_2(s))| \, ds \leq CM \int_0^r |k(s/r)|s^{\frac{p+1}{p-1}}|w_1(s) - w_2(s)| \, ds \leq CMR^{\frac{2p}{p-1}} \|w_1 - w_2\|_{C[0,R]} \]

So, given \(K > 0\) we choose \(M\) so that \(2CK \leq M\) and then we take \(R\) small enough so that

\[ MR^{\frac{2p}{p-1}} \leq \min\{\lambda/2, 1/2C\} \]

With these choices \(T\) is a contraction (with Lipschitz constant 1/2) that maps \(X_M\) into \(X_M\). Therefore it has a fixed point (unique in \(X_M\)), which is a solution of (6)-(7) by Lemma 12.

To prove uniqueness, suppose that \(w_1, w_2\) are two solutions of (6)-(7) on \((0, R)\). Then choose \(M'\) so that

\[ M' \geq \max\{2CK, \|w_1\|_{C[0,R]}, \|w_2\|_{C[0,R]}\} \]

and \(R'\) so that

\[ M'R^{\frac{2p}{p-1}} \leq \min\{\lambda/2, 1/2C\} \quad \text{and} \quad R' \leq R \]

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Then $w_1, w_2$ are in $\{w \in C[0, R] \mid \|w\|_{C[0, R]} \leq M\}$ and are fixed points of $T$. Hence $w_1 \equiv w_2$ on $(0, R')$. The equality on $(0, R)$ is obtained by a standard uniqueness result for ODE’s.

Regarding continuous dependence, let $f_1, f_2 \in C[0, R]$ be such that $\|f_i\|_{C[0, R]} \leq K, i = 1, 2,$ and let $w_1, w_2$ be the corresponding solutions to (6)-(7), i.e., the fixed points of $T_i$ in $X_M$, where

$$T_i w(r) = - \int_0^r k(s/r) \left(s g(s, w(s)) + s^{-1} f_i(s)\right) \, ds$$

Recall that $T_i$ maps $X_M$ into $X_M$ and that it has a Lipschitz constant of $1/2$. Then,

$$\|w_1 - w_2\|_{C[0, R]} \leq \|T_1(w_1) - T_1(w_2)\| + \|T_1(w_2) - T_2(w_2)\| \leq \frac{1}{2} \|w_1 - w_2\| + \sup_{0 \leq r \leq R} \int_0^r |k(s/r)| s^{-1} |f_1(s) - f_2(s)| \, ds \leq \frac{1}{2} \|w_1 - w_2\| + C \|f_1 - f_2\|.$$

Remark that by part a) of Proposition 8, given a continuous function $f$ on $[0, \infty)$ there exists $R > 0$ such that the sequence

$$\begin{cases}
    w_0 \equiv 0 \\
    w_{k+1} = T(w_k)
\end{cases}$$

converges in $C[0, R]$ to the solution of (6)-(7).

**Lemma 13** Assume now that $f \geq 0$ is a continuous function on $[0, \infty)$ and let $w$ be the corresponding solution of (6)-(7) with maximal domain $(0, R)$. Then

$$w_k \searrow w \text{ on } (0, R)$$

**Proof.** Indeed, first note that $w_k \leq 0$ for all $k$, because $g(r, w) \geq 0$ and $f \geq 0$. In particular $w_1 \leq w_0 \equiv 0$. Then observe that for fixed $r$, $g(r, w)$ is non increasing in $w$ for $w \leq 0$. This implies that $T(w_1) \leq T(w_0)$, i.e. $w_2 \leq w_1$, and by induction $w_{k+1} \leq w_k$ for all $k$. Note also that since $w$ is a fixed point of $T$, from $w \leq w_0 \equiv 0$ follows that $w \leq T(w_0) = w_1$, and again an induction argument shows that $w \leq w_k$ for all $k$. It follows that $w_k \searrow \bar{w}$ pointwise, and taking the limit in the recurrence relation

$$w_{k+1}(r) = - \int_0^r k(s/r) \left(s g(s, w_k(s)) + s^{-1} f(s)\right) \, ds$$

we obtain that $\bar{w}$ is a fixed point of $T$, and hence a solution of (6)-(7). By uniqueness $w = \bar{w}$. □

**Lemma 14** Let $f_1, f_2$ be continuous functions on $[0, R]$ and suppose that the corresponding solutions $w^{(1)}, w^{(2)}$ to (6)-(7) are defined on $(0, R)$. Assume that $f_1 \geq f_2 \geq 0$. Then $w^{(1)} \leq w^{(2)} \leq 0$ on $(0, R)$.
Proof. For $i = 1, 2$ define the operators $T_i$ corresponding to $f_i$ as before, and consider the sequences
\[
\begin{aligned}
    w_0^{(i)} &\equiv 0 \\
    w_{k+1}^{(i)} &= T_i(w_k^{(i)})
\end{aligned}
\]
Then $w_k^{(i)} \searrow w^{(i)}$, $i = 1, 2$. But since $f_1 \geq f_2 \geq 0$ we have (inductively)
\[
\begin{aligned}
    w_1^{(1)} &= T_1(w_1^{(1)}) \\
    &\leq T_1(w_2^{(2)}) \\
    &\leq T_2(w_1^{(2)}) \\
    &= w_2^{(2)}
\end{aligned}
\]
Therefore $w^{(1)} \leq w^{(2)}$.

Proposition 15 Let $f$ be a continuous function on $[0, \infty)$, $f \geq 0$, $f \not\equiv 0$ on $[0, 1]$. For each $t \geq 0$ let $w_t$ be the solution of
\[
\begin{aligned}
    Lw_t &= -g(r, w_t) - r^{-2}tf \\
    w_t(r), rw'_t(r) &\text{ are bounded near 0}
\end{aligned}
\]
which is defined on a maximal interval $(0, R_t)$. Then the set
\[
\{w_t(1) / t \geq 0 \text{ and } w_t(1) \text{ is defined}\}
\]
is the whole interval $(-\infty, 0]$. In particular there exists $\bar{t} \geq 0$ such that $w_{\bar{t}}(1)$ is defined and is equal to $-v(1)$.

Proof. Define
\[
A = \{w_t(1) / t \geq 0 \text{ and } w_t(1) \text{ is defined}\}
\]
and note that $0 \in A$. Next we show that $A$ is connected. Consider $\varphi(t) = w_t(1)$ with domain $\text{dom}(\varphi) = \{t \geq 0 / w_t(1) \text{ is defined}\}$. Then $\varphi$ is continuous, and to conclude that $A$ is connected we only need to check that $\text{dom}(\varphi)$ is connected. So let $0 \leq t_2 \leq t_1$ and suppose that $R_{t_1} > 1$ (i.e. $t_1 \in \text{dom}(\varphi)$). By monotonicity with respect to $t$ we have $0 \geq w_{t_2} \geq w_{t_1}$ on $(0, R_{t_1}) \cap (0, R_{t_2})$. If $R_{t_2} < R_{t_1}$ then we have an apriori bound for $w_{t_2}$ on $(0, R_{t_2})$, so that $w_{t_2}$ can be continued beyond $R_{t_2}$. This contradiction shows that $R_{t_2} \geq R_{t_1} > 1$ and therefore $t_2 \in \text{dom}(\varphi)$.

Now we prove that $A$ is open in $(-\infty, 0]$. Let $a \in A$ and $t \geq 0$ be such that $w_t(1) = a$. By the continuous dependence of $w_t$ in $t$, we have that $w_{t'}(1)$ is defined for $t'$ close to $t$. Take $t' > t$ but close enough. Then
\[
\begin{aligned}
w_{t'}(1) &= -\int_0^{t'} k(s)(sg(s, w_{t'}(s)) + s^{-1}t'h(s)) \, ds \\
&< w_t(1)
\end{aligned}
\]
because \( f \not\equiv 0 \) on \([0, 1]\). Hence \( A \) contains an interval of the form \((a - \varepsilon, a]\) for some \( \varepsilon > 0 \).

Suppose now that \( A \) is bounded and let \( a = \inf A \notin A \). Then there exists a sequence \( a_n \searrow a \), \( a_n \in A \). Let \( t_n \geq 0 \) be such that \( w_{t_n}(1) = a_n \). Then, if \( t_n < t_m \) we must have \( a_m < a_n \), and we conclude that \( (t_n) \) is increasing. If \( t_n \not\to \infty \), then

\[
a_n = -\int_0^1 k(s)(sg(s, w_{t_n}(s)) + s^{-1}t_n f(s)) \, ds \to -\infty \text{ as } n \to \infty
\]

which contradicts the assumption that \( A \) is bounded. Hence we may assume that \( t_n \not\to t < \infty \). Note that \( w_{t_n}(r) \) is decreasing, so that \( w(r) = \lim_{n} w_{t_n}(r) \) exists for \( 0 < r < 1 \). Taking the limit as \( n \to \infty \) in

\[
w_{t_n}(r) = -\int_0^r k(s/r) \left( sg(s, w_{t_n}(s)) + s^{-1}t_n f(s) \right) \, ds
\]

we obtain by monotone convergence

\[
w(r) = -\int_0^r k(s/r) \left( sg(s, w(s)) + s^{-1}f(s) \right) \, ds \tag{13}
\]

Claim: \( w \) is the solution of

\[
\begin{align*}
L w_t &= -g(t, w_t) - r^{-2}tf \\
(w_t, re^{-t}w'_t) &\text{ are bounded near } 0
\end{align*}
\]

and \( w(1) = a \). Thus \( a \in A \) and from this contradiction we conclude that \( A = (-\infty, 0] \).

Proof of the claim. We need to prove that \( w \in L^\infty(0, 1) \) so that we can apply Lemma 12, and then show that \( \lim_{r \to 1^-} w(r) = a \).

Note that by Proposition 8 there exists \( M > 0 \) and \( 0 < R < 1 \) such that \( |w_{t_n}(r)| \leq M \) for \( 0 < r < R \), and therefore \( |w(r)| \leq M \) for \( 0 < r < R \). Let’s estimate \( w(r) \) for \( r \in [R, 1] \). Let \( \lambda = sg(s, w(s)) + s^{-1}f(s) \) and let’s use the convention that \( k(t) = 0 \) for \( t \geq 1 \). So

\[
w(r) = -\int_0^R k(s/r)sg(s, w(s)) \, ds - \int_0^R k(s/r)s^{-1}f(s) \, ds - \int_1^R k(s/r)m(s) \, ds \tag{14}
\]

We may take \( R \) smaller if necessary so that \( 2MR^2 \lambda \leq \lambda \) and therefore by (8) we find as in the proof of Lemma 12 that the first 2 terms are bounded independently of \( r \in [R, 1] \). To estimate

\[
\int_R^1 k(s/r)m(s) \, ds
\]

note that by taking \( r = 1 \) in (13) we get \( km \in L^1(0, 1) \). But there exists \( C > 0 \) such that for \( R \leq s, r \leq 1, k(s/r) \leq Ck(s) \), which shows that \( w \) is bounded in \((0, 1)\).

Finally, because of the same estimates as before we can use dominated convergence in (14) to find that \( w(r) \to a \) as \( r \to 1^- \).
Appendix

Here we give a proof of Theorem 2. Let $f \geq 0$ be smooth, and let $u$ denote here the minimal solution of $(P_t)$, which we know is smooth for $0 < t < t^*$. We omit from the notation the explicit dependence of $u$ in $t$.

We know that the first eigenvalue of $-\Delta - pu^{p-1}$ is non-negative, so for all $\varphi \in C^1_0(\Omega)$ we have

$$\int_\Omega |\nabla \varphi|^2 \, dx \geq p \int_\Omega u^{p-1} \varphi^2 \, dx$$

Let $j \geq 1$ and take $\varphi = u^j$. We then get

$$j^2 \int_\Omega u^{2j-2} |\nabla u|^2 \, dx \geq p \int_\Omega u^{p+2j-1} \, dx.$$

Now multiply $(P_t)$ by $\frac{j^2}{2j-1} u^{2j-1}$ and integrate by parts to obtain

$$j^2 \int_\Omega u^{2j-2} |\nabla u|^2 \, dx = \frac{j^2}{2j-1} \int_\Omega u^{p+2j-1} + tfu^{2j-1} \, dx$$

Combining these two we obtain

$$\frac{j^2}{2j-1} \int_\Omega u^{p+2j-1} + tfu^{2j-1} \, dx \geq p \int_\Omega \Omega u^{p+2j-1} \, dx$$

If $\frac{j^2}{2j-1} < p$ we see that there is a constant $C$ independent of $t$ such that

$$\|u\|_{L^{p+2j-1}} \leq C$$

(recall that $t < t^*$). From now on we denote by $C$ different numbers independent of $t$. The restriction on $j$ can be rewritten as $1 \leq j < p + \sqrt{p^2 - p}$. Hence for $q = p + 2j - 1$ we find a bound for $\|u\|_{L^q}$ independent of $t$, for $q < 3p + 2\sqrt{p^2 - p} - 1$, and hence

$$\|u^p + tf\|_{L^r} \leq C$$

for $r < 3 + 2\sqrt{1 - 1/p} - 1/p$. Now we use the equation and the $L^p$ theory to improve this estimate. Let $1 < r_0 < 3 + 2\sqrt{1 - 1/p} - 1/p$. By $L^p$ estimates

$$\|u\|_{W^{2,r_0}} \leq C$$

and if $1/r_0 - 2/n > 0$, by Sobolev embedding we get

$$\|u^p + tf\|_{L^{r_1}} \leq C$$

with $1/r_1 = p(1/r_0 - 2/n)$. If on the other hand $1/r_0 - 2/n \leq 0$ we conclude that

$$\|u\|_{C(\overline{\Omega})} \leq C$$
(If $1/r_0 - 2/n < 0$, we use Sobolev embedding, and if $1/r_0 - 2/n = 0$ we apply once more the $L^p$ estimates and the Sobolev embedding). Continuing in this way we define a sequence $r_k$ by $1/r_{k+1} = p(1/r_k - 2/n)$, and we would like to find some $k$ for which $r_k \leq 0$, so that as before we obtain a bound for $u$ in $C(\bar{\Omega})$. To compute $r_k$ we introduce $a_k = 1/r_k - 2/n$ which satisfies then $a_{k+1} = pa_k - 2/n$. Therefore

$$a_k = p^k \left( a_0 - \frac{2}{n(p-1)} \right) + \frac{2}{n(p-1)}$$

We want to find some $k$ for which $a_k \leq 0$ and this occurs for some $k$ iff

$$a_0 - \frac{2}{n(p-1)} < 0$$

Going back to $r_0$ this requires $r_0 > \frac{n}{2}(1 - 1/p)$. But we had already the restriction $r_0 < 3 + 2\sqrt{1 - 1/p - 1/p}$, so that the argument works if

$$\frac{n}{2}(1 - 1/p) < 3 + 2\sqrt{1 - 1/p - 1/p}$$

which is equivalent to

$$n < 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}.$$  

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**References**


