Some extremal singular solutions of a nonlinear elliptic equation

Juan Dávila[1]

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain, and let f be a smooth function on Ω, $f \geq 0$, $f \not\equiv 0$. Let $p > 1$ and consider the semi-linear elliptic equation

$$(P_t) \begin{cases} -\Delta u &= u^p + tf & \text{in } \Omega \\ u &> 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$

where $t \geq 0$ is a parameter.

We are concerned with weak solutions of (P_t), and we use the definition introduced in [BCMR]: a weak solution of (P_t) is a function $u \in L^1(\Omega)$, $u \geq 0$, such that $u^p \delta \in L^1(\Omega)$, where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$, and such that

$$-\int_\Omega u \Delta\zeta \, dx = \int_\Omega (u^p + tf)\zeta \, dx$$

for all $\zeta \in C^2(\bar{\Omega})$ with $\zeta = 0$ on $\partial\Omega$.

We start by mentioning some well known facts (see for example [BCMR], [BC], [Ma]).

Theorem 1 *There exists $0 < t^* < \infty$ such that for $0 < t < t^*$ (P_t) has a unique minimal solution $\underline{u}(\cdot, t)$ (which is smooth), for $t = t^*$ (P_{t^*}) has a unique solution u^* (possibly unbounded), and for $t > t^*$ there is no solution of (P_t) (even in the weak sense). Moreover $\underline{u}(\cdot, t)$ depends smoothly on $t \in (0, t^*)$, increases as t increases, and*

$$\underline{u}(\cdot, t) \nearrow u^* \quad \text{a.e. in } \Omega, \text{ as } t \nearrow t^*.$$

We call u^* the extremal solution.

An important feature of the minimal solution $\underline{u}$ is that the linearized operator at $\underline{u}$

$$-\Delta - p\underline{u}^{p-1}$$

has a positive first eigenvalue for all $0 < t < t^*$. This property can be used as in [CR] or [MP], to prove the following

[1]Dept. of Mathematics, Rutgers University, New Brunswick, NJ 08903, U. S. A., davila@math.rutgers.edu

Theorem 2 *If*

$$n < 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}} \tag{1}$$

then there exists a constant C *independent of* t *such that*

$$\|\underline{u}(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for any } 0 < t < t^*.$$

This is equivalent to the statement that the extremal solution u^* is bounded.

We note that if the extremal solution u^* is bounded, then by elliptic regularity it is also smooth, and in this case the first eigenvalue of $-\Delta - p(u^*)^{p-1}$ is zero.

In the present work we are interested in the case $n \ge 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}$. First in Section 2 we show that Theorem 2 is sharp, i.e. assuming $n \ge 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}$, for any domain Ω there exists a smooth function $f \ge 0$, $f \not\equiv 0$ for which the extremal solution u^* is unbounded (or singular). Then in Section 3 we study the radially symmetric case with Ω the open unit ball in $I\!R^n$, and we show that assuming $n \ge 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}$ the extremal solution is singular for any smooth radially symmetric function f, $f \ge 0$, $f \not\equiv 0$. We also give a precise description of the singularity of u^* in this case.

A problem related to (P_t) that has received much attention in the literature is the following:

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

where $\lambda \ge 0$ is a parameter and $g : [0,\infty) \to (0,\infty)$ is a C^1 convex, positive, nondecreasing function with $g(0) > 0$ and $g(u)/u \to \infty$ as $u \to \infty$. Typical examples are $g(u) = e^u$ and $g(u) = (1+u)^p$, $p > 1$. For this equation there is again an extremal parameter $\lambda^* < \infty$, such that for $0 < \lambda < \lambda^*$ there is a minimal solution, for $\lambda = \lambda^*$ there is a unique weak solution (called the extremal solution), and for $\lambda > \lambda^*$ there is no solution (see for example [BCMR],[BV],[MP] and their references for results on this problem).

Several very interesting open problems for (2) were proposed in [BV], and we mention some of them in the context of problem (P_t).

1) Assume $n \ge 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}$. If Ω is convex, is it true that for any smooth $f \ge 0$, $f \not\equiv 0$ the extremal solution is singular? Is it true for $f \equiv 1$? We note that some restriction on Ω has to be imposed, as shown by the example $\Omega = B_2 \setminus \bar{B}_1$ and $f \equiv 1$ (or any radially symmetric positive smooth f). In this case it can be easily shown that the extremal solution is always smooth, without any restriction on n and p. (See Problem 3 in [BV]).

2) Concerning problem (2) in some cases the extremal solution is bounded and in others it is singular (see [BV]). In a recent work, G. Nedev [Ne] has shown that in dimension 2, for any nonlinearity g satisfying the hypothesis above the extremal solution of (2) is bounded. His argument can be adapted to show that the same is true for a more general version of (P_t), where

the nonlinearity u^p is replaced by $g(u)$, and g is a C^1 positive, convex, increasing function with $g(0) = 0$, and $g(u)/u \to \infty$ as $u \to \infty$ (we note that Theorem 1 is still true for this more general problem). In dimension 3, it is not known whether or not there exist nonlinearities g for which the extremal solution is singular.

2 Is condition (1) sharp?

Theorem 3 *Let $\Omega \subset \mathbb{R}^n$ be any bounded, smooth domain. If*

$$n \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}} \tag{3}$$

then there exists a smooth function $f \geq 0$, $f \not\equiv 0$ so that the extremal solution u^* is singular.

The idea of the construction is the same as in [BV], that is, to find a smooth function $f \geq 0$, $f \not\equiv 0$, a number $0 < t < \infty$ and an unbounded function u in $H_0^1(\Omega)$ which is a weak solution of (P_t), and such that the operator

$$-\Delta - pu^{p-1}$$

has a nonnegative first eigenvalue, in the sense that

$$\int_\Omega |\nabla \varphi|^2 \, dx \geq p \int_\Omega u^{p-1} \varphi^2 \, dx \quad \forall \varphi \in C_0^1(\Omega). \tag{4}$$

Then we can conclude using the following lemma (similar to Theorem 3 in [BV]).

Lemma 4 *Suppose that $u \in H_0^1(\Omega)$ is an unbounded weak solution of (P_t) such that the operator*

$$-\Delta - pu^{p-1}$$

has a nonnegative first eigenvalue (in the sense of (4). Then $t = t^*$ and $u = u^*$.

Proof. Since there is no solution for $t > t^*$ we must have $t \leq t^*$. Let $\underline{u} = \underline{u}(\cdot, t)$ denote the minimal solution of (P_t), and let $g(u) = u^p$. The inequality

$$\int_\Omega |\nabla \varphi|^2 \, dx \geq \int_\Omega g'(u) \varphi^2 \, dx$$

holds by assumption for all $\varphi \in C_0^1(\Omega)$ and by approximation also for $\varphi \in H_0^1(\Omega)$. We take $\varphi = u - \underline{u} \in H_0^1(\Omega)$ (note that by assumption $u \in H_0^1(\Omega)$ and by the estimates in the appendix, $\underline{u} \in H_0^1(\Omega)$ even for $t = t^*$, i.e. $\underline{u} = u^*$). We have

$$\begin{aligned} \int_\Omega g'(u)(u - \underline{u})^2 \, dx &\leq \int_\Omega |\nabla (u - \underline{u})|^2 \, dx \\ &= \int_\Omega -\Delta(u - \underline{u})(u - \underline{u}) \, dx \\ &= \int_\Omega (g(u) + tf - g(\underline{u}) - tf)(u - \underline{u}) \, dx \end{aligned}$$

so that

$$\int_\Omega (u-\underline{u})(g(u)+g'(u)(\underline{u}-u)-g(\underline{u}))\ dx \geq 0.$$

Because of the convexity of g and since $u \geq \underline{u}$, the integrand is non-positive and we conclude that

$$g(\underline{u}) = g(u) + g'(u)(\underline{u}-u).$$

Since $g(u) = u^p$ is strictly convex we conclude that $u = \underline{u}$. But u is unbounded and this forces $t = t^*$. □

Consider the function

$$v(x) = v(|x|) = \lambda |x|^\alpha \tag{5}$$

where

$$\lambda = \lambda_{n,p} = \left(\frac{2}{p-1}\left(n - \frac{2p}{p-1}\right)\right)^{\frac{1}{p-1}}$$

and

$$\alpha = \alpha_p = -\frac{2}{p-1}.$$

Then $v \in H^1(\Omega)$ for $n > 2 + 4/(p-1)$, and

$$-\Delta v = v^p \quad \text{in } \mathbb{R}^n.$$

From now on we assume that $0 \in \Omega$, and we will construct u with a singularity at the origin so that it satisfies the requirements in Lemma 4. We look for a function u of the form $u = v - \psi$.

Lemma 5 *There exists a smooth function ψ defined on $\bar{\Omega}$ with the properties:*

1. $\psi \geq 0$ *and is smooth in* $\bar{\Omega}$,
2. $\Delta\psi \geq 0$ *in* Ω,
3. $\psi \equiv 0$ *in a neighborhood of 0, and*
4. $\psi = v$ *on* $\partial\Omega$.

Proof of Theorem 3. Let $u = v - \psi$. Then

$$\begin{aligned} -\Delta u &= -\Delta v + \Delta\psi \\ &= v^p + \Delta\psi \\ &\geq 0 \end{aligned}$$

and $u = 00$ on $\partial\Omega$, so $u \geq 0$. Taking

$$f = \Delta\psi + v^p - u^p$$

we then have

$$-\Delta u = u^p + f.$$

Note that $f \geq 0$ and is smooth, because $u \leq v$ and $u \equiv v$ in a neighborhood of 0. The only condition that still needs to be checked to apply Lemma 4 is the non-negativity of the first eigenvalue of the operator $-\Delta - pu^{p-1}$. Here enters into play condition (3). Recall the Hardy inequality (see [BV] for example):

$$\frac{(n-2)^2}{4}\int_\Omega \frac{1}{|x|^2}\varphi^2\,dx \leq \int_\Omega |\nabla\varphi|^2\,dx$$

for any $\varphi \in C_0^1(\Omega)$, when $n \geq 3$. Note that $u \leq v$ so that for any $\varphi \in C_0^1(\Omega)$

$$\begin{aligned}
\int_\Omega pu^{p-1}\varphi^2\,dx &\leq \int_\Omega pv^{p-1}\varphi^2\,dx \\
&= \frac{2p}{p-1}\left(n - \frac{2p}{p-1}\right)\int_\Omega \frac{1}{|x|^2}\varphi^2\,dx \\
&\leq \frac{(n-2)^2}{4}\int_\Omega \frac{1}{|x|^2}\varphi^2\,dx \\
&\leq \int_\Omega |\nabla\varphi|^2\,dx
\end{aligned}$$

where the third inequality is a consequence of (3). □

Proof of Lemma 5. Let $r = \mathrm{dist}(0, \partial\Omega)/2$, and let ψ_1 be the solution of the following problem

$$\begin{cases}
\Delta\psi_1 = 0 & \text{in } \Omega \setminus \bar{B}_r \\
\psi_1 = v & \text{on } \partial\Omega \\
\psi_1 = 0 & \text{on } \partial B_r
\end{cases}$$

Then ψ_1 is smooth and positive in $\Omega \setminus \bar{B}_r$ and by the Hopf boundary lemma $\frac{\partial\psi_1}{\partial\nu} > 0$ on ∂B_r, where ν is the normal vector, pointing away from the origin. Let ψ_1 be extended by 0 in B_r. Then $\Delta\psi_1 \geq 0$ in $\mathcal{D}(\Omega)'$.

Now we regularize ψ_1 by convolution to get a smooth function ψ:

$$\psi = \psi_1 * \rho_\varepsilon$$

where ρ_ε is a standard mollifier ($\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$, $\rho \in C_0^\infty(\mathbb{R}^n)$, $\rho \geq 0$, $\mathrm{supp}(\rho) \subset B_1$, $\int \rho\,dx = 1$). $\psi(x)$ is well defined and subharmonic on the set

$$\{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \varepsilon\}.$$

If ρ_ε is radially symmetric, then $\psi_1 = \psi_1 * \rho_\varepsilon = \psi$ on

$$\{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \varepsilon, \mathrm{dist}(x, B_r) > \varepsilon\}.$$

By fixing $\varepsilon > 0$ but small enough we can consider ψ to be defined and smooth up to $\partial\Omega$. □

3 The radially symmetric case

Theorem 6 *Assume now that Ω is the open unit ball $B_1(0)$ in $I\!R^n$, and that $f \geq 0$, $f \not\equiv 0$ is any smooth, radially symmetric function. If $n \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}$ then the extremal solution u^* is singular.*

First we give a short proof of this theorem, but actually more can be said about the extremal solution u^* than merely $u^* \notin L^\infty(\Omega)$.

Theorem 7 *Assume Ω is the open unit ball $B_1(0)$ in $I\!R^n$, and $f \geq 0$, $f \not\equiv 0$ is a radially symmetric function $f(x) = f(|x|)$ with $f \in C^2([0,1])$. Suppose $n \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}$. Let v be the function defined by (5) and set $w = r^{-2}(u - v)$, $r = |x|$. Then w is $C^2([0,1])$, and if moreover $f'(0) = 0$ (i.e. $f \in C^2(\bar{\Omega})$), then the same is true for w.*

Before giving the proofs we note that if Ω and f are radially symmetric, then the minimal solution $\underline{u}$ of (P_t) is also radially symmetric.

Proof of Theorem 6. Let v denote the function defined by (5). We use the improved Hardy inequality, proved in [BV]: for all $\varphi \in C_0^1(\Omega)$ we have

$$\int_\Omega |\nabla\varphi|^2 \, dx \geq \frac{(n-2)^2}{4} \int_\Omega \frac{\varphi^2}{|x|^2} \, dx + c \int_\Omega \varphi^2 \, dx$$

where $c = H_2(w_n/|\Omega|)^{2/n} > 0$, H_2 is the first eigenvalue of the Laplacian with Dirichlet boundary condition in the unit ball in dimension 2, and w_n is the measure of the unit ball in $I\!R^n$. This inequality implies that if (3) holds, then the operator $-\Delta - pv^{p-1}$ has a positive first eigenvalue, and although $pv^{p-1} = C/|x|^2$ is not in $L^{n/2}(\Omega)$, the maximum principle can be applied to it.
Claim: for all $0 < t \leq t^*$ we have $\underline{u}(\cdot, t) \leq v$, and the inequality is strict for $0 < t < t^*$.
Indeed let $0 < t < t^*$ and suppose that there exists some $0 < r < 1$ such that $\underline{u}(r,t) \geq v(r)$. Then

$$\underline{u} - v \geq 0 \quad \text{on} \partial B_r$$

and by the convexity of $u \to u^p$ we have

$$\begin{aligned} -\Delta(\underline{u} - v) &= \underline{u}^p - v^p + tf \\ &\geq pv^{p-1}(\underline{u} - v) + tf \end{aligned}$$

so that

$$-\Delta(\underline{u} - v) - pv^{p-1}(\underline{u} - v) \geq 0.$$

By the maximum principle we conclude that $\underline{u} \geq v$ on B_r, which is impossible, because $\underline{u}$ is bounded for $0 < t < t^*$. The conclusion for $t = t^*$ is obtained by taking the limit as $t \to t^*$.

Since $u^* \leq v$ we conclude that the first eigenvalue for the operator $-\Delta - pu^{*p-1}$ is

$$\inf_{\|\varphi\|_{L^2}} \int_\Omega |\nabla\varphi|^2 - pu^{*p-1}\varphi^2 \, dx \geq \inf_{\|\varphi\|_{L^2}} \int_\Omega |\nabla\varphi|^2 - pv^{p-1}\varphi^2 \, dx > 0.$$

This shows that u^* cannot be bounded. □

Proof of Theorem 7. This proof involves again the same idea as in [BV], using Lemma 4. We set

$$u = v + r^2 w$$

Then a calculation shows that the equation $-\Delta u = u^p + f$ is equivalent to

$$-(r^2 w'' + (n+3)rw' + 2nw) = |v + r^2 w|^p - v^p + f, \quad 0 < r < 1$$

It is convenient to rewrite this equation as

$$w'' + \frac{n+3}{r}w' + \frac{2n + pv^{p-1}r^2}{r^2}w = -\frac{1}{r^2}\Big(|v + r^2 w|^p - v^p - pv^{p-1}r^2 w\Big) - r^{-2}f$$

or

$$Lw \;=\; -g(r,w) - r^{-2}f \tag{6}$$

where

$$Lw = w'' + \frac{n+3}{r}w' + \frac{2n + pv^{p-1}r^2}{r^2}w$$

and

$$g(r,w) = \frac{1}{r^2}\Big(|v + r^2 w|^p - v^p - pv^{p-1}r^2 w\Big)$$

Note that $pv^{p-1}r^2 = p\lambda^{p-1}$ is a constant, and that $g(r,w) \geq 0$ by convexity.

The aim is the to find a solution w of (6), that behaves nicely near 0 and such that $w(1) = -v(1)$. It turns out that a nice behavior of w near 0 can be imposed for example by the requirement that

$$w(r), rw'(r) \text{ are bounded near } 0 \tag{7}$$

We show in Proposition 8 that if f is a continuous function on $[0,\infty)$ then (6) together with (7) has a unique solution w, which is defined on an open maximal interval. We also prove that $w \leq 0$ if $f \geq 0$. Then, in Proposition 15, we show that if we replace f by tf in (6), where $t \geq 0$, $f \geq 0$, $f \not\equiv 0$ in $[0,1]$, then there exists t such that the solution w to (6)-(7) is defined on $[0,1]$ and $w(1) = -v(1)$. We also show in Lemma 12 that if f is smooth enough, then w has the regularity stated in Theorem 7.

Accepting these results for a moment, we see that

$$u = v + r^2 w$$

satisfies the requirements in Lemma 4, the non negativity of the first eigenvalue of the operator $-\Delta - pu^{p-1}$ following again from $u \leq v$, the Hardy inequality and condition (3). □

From now on until the end of this section we assume that condition (3) holds.

Proposition 8
a) Let $K > 0$. Then there exists $R > 0$ such that for any continuous function f on $[0, R]$ with $\|f\|_{C[0,R]} \le K$, (6)-(7) has a unique solution on $(0, R)$. Moreover, the solution depends continuously on f. More precisely, there exists a constant $C > 0$, such that for any continuous functions f_1, f_2 on $[0, R]$, $\|f_i\|_{C[0,R]} \le K$, $i = 1, 2$, if w_1, w_2 are the corresponding solutions of (6)-(7), then

$$\|w_1 - w_2\|_{C[0,R]} \le C\|f_1 - f_2\|_{C[0,R]}$$

b) If f is a continuous function on $[0, \infty)$, the (6)-(7) has a unique solution w defined on an open maximal interval. The solution depends continuously on f.

We need some preparatory lemmas.

Lemma 9 *There exists $C > 0$ depending only on n, p such that if*

$$M > 0, \; R > 0 \text{ and } 2MR^{\frac{2p}{p-1}} \le \lambda$$

then

$$|g(r, w)| \le C r^{\frac{2}{p-1}} |w|^2 \tag{8}$$

for any $|w| \le M$ and $0 < r < R$, and

$$|g(r, w_1) - g(r, w_2)| \le CMr^{\frac{2}{p-1}} |w_1 - w_2| \tag{9}$$

for any $|w_1|, |w_2| \le M$ and $0 < r < R$.

Proof. Let $a(x) = x^p$, which is a convex functions (recall that $p > 1$). Let $|w| \le M$ and $0 < r < R$. Then, using $2MR^{\frac{2p}{p-1}} \le \lambda$, we obtain $|r^2 w| \le \frac{1}{2}\lambda r^{-\frac{2}{p-1}}$. With $v = v(r) = \lambda r^{-\frac{-2}{p-1}}$, we have $\frac{1}{2}v \le v + r^2 w \le \frac{3}{2}v$. Notice that

$$\begin{aligned} g(r, w) &= \frac{1}{r^2}\left(a(v + r^2 w) - a(v) - a'(v) r^2 w\right) \\ &= \frac{1}{2} a''(\xi) r^2 w^2 \end{aligned}$$

where ξ is in the interval with endpoints v and $v + r^2 w$. Using that a'' is monotone, we thus have

$$\begin{aligned} |g(r, w)| &\le \frac{1}{2} p(p-1) r^2 |w|^2 \max\{(1/2)^{p-2}, (3/2)^{p-2}\} v^{p-2} \\ &\le C(p) r^2 |w|^2 \lambda^{p-2} r^{-\frac{2}{p-1}(p-2)} \\ &\le C(n, p) r^{\frac{2}{p-1}} |w|^2 \end{aligned}$$

We now prove estimate (9):

$$\begin{aligned}
|g(r,w_1)-g(r,w_2)| &= \frac{1}{r^2}\left|(v+r^2w_1)^p-(v+r^2w_2)^p-pv^{p-1}r^2(w_1-w_2)\right| \\
&= \frac{1}{r^2}\left|\int_0^1 \frac{d}{dt}(v+r^2(tw_1+(1-t)w_2))^p-pv^{p-1}r^2(w_1-w_2)\,dt\right| \\
&\leq p\int_0^1 \left|(v+r^2(tw_1+(1-t)w_2))^{p-1}-v^{p-1}\right||w_1-w_2|\,dt
\end{aligned}$$

But

$$\left|(v+r^2(tw_1+(1-t)w_2))^{p-1}-v^{p-1}\right| = (p-1)|\xi|^{p-2}r^2|tw_1+(1-t)w_2|$$

where ξ is in the interval with endpoints v and $v+r^2(tw_1+(1-t)w_2)$. Therefore

$$\begin{aligned}
\left|(v+r^2(tw_1+(1-t)w_2))^{p-1}-v^{p-1}\right| &\leq (p-1)\max\{(1/2)^{p-2},(3/2)^{p-2}\}v^{p-2}r^2M \\
&\leq C(n,p)r^{\frac{2}{p-1}}M
\end{aligned}$$

□

Lemma 10 *Let w be a solution of (6) in $(0,R)$ (i.e. $w\in C^2(0,R)$ and satisfies the equation) and let $0<r_0<R$. Then*

$$w(r)=w_h(r)-\int_{r_0}^r k(s/r)\left(sg(s,w(s))+s^{-1}f(s)\right)\,ds, \quad 0<r<R \tag{10}$$

where w_h is the solution of the linear homogeneous equation

$$\begin{cases} Lw_h = 0 & \text{in } (0,R) \\ w_h(r_0) = w(r_0) & \\ w_h'(r_0) = w'(r_0) & \end{cases} \tag{11}$$

and k is the continuous function on $[0,1]$ given by:

$$k(t)=\begin{cases} t^{-\beta}\ln(1/t) & \text{if } n=6+\frac{4}{p-1}+4\sqrt{\frac{p}{p-1}} \quad \text{where } \beta=-\frac{n+2}{2} \\ \frac{t^{-\beta_1}-t^{-\beta_2}}{\beta_1-\beta_2} & \text{if } n>6+\frac{4}{p-1}+4\sqrt{\frac{p}{p-1}} \quad \text{where } \beta_{1,2}=-\frac{n+2}{2}\pm\sqrt{\left(\frac{n+2}{2}\right)^2-2n-p\lambda^{p-1}} \end{cases}$$

We note that (3) implies that β_1,β_2 are real, and that $k>0$ on $(0,1)$, $k(0)=k(1)=0$.

Proof. We use the variation of parameters formula, noting that two linearly independent solutions of the homogeneous equation $Ly=0$ on $(0,\infty)$ are:

$$\begin{aligned}
&y_1=s^\beta, \quad y_2=\ln(s)s^\beta && \text{if } n=6+\frac{4}{p-1}+4\sqrt{\frac{p}{p-1}} \\
&y_1=s^{\beta_1}, \quad y_2=s^{\beta_2} && \text{if } n>6+\frac{4}{p-1}+4\sqrt{\frac{p}{p-1}}
\end{aligned}$$

Lemma 11 *Let w be a solution of (6) in $(0, R)$ and suppose it satisfies (7). The*

$$w(r) = -\int_0^r k(s/r)\left(sg(s, w(s)) + s^{-1}f(s)\right)\, ds \tag{12}$$

Proof. A direct computation gives the following expression for the solution w_h of the homogeneous equation (11):

$$\text{case } n = 6 + \tfrac{4}{p-1} + 4\sqrt{\tfrac{p}{p-1}}: \qquad \begin{aligned} w_h(r) &= c_1 r^{\beta} + c_2 \ln(r) r^{\beta} \\ c_1 &= w(r_0) r_0^{-\beta}(\beta \ln(r_0) + 1) - w'(r_0) r_0^{-\beta+1} \ln(r_0) \\ c_2 &= -\beta w(r_0) r_0^{-\beta} + w'(r_0) r_0^{-\beta+1} \end{aligned}$$

$$\text{case } n > 6 + \tfrac{4}{p-1} + 4\sqrt{\tfrac{p}{p-1}}: \qquad \begin{aligned} w_h(r) &= c_1 r^{\beta_1} + c_2 \beta_2 \\ c_1 &= (\beta_2 w(r_0) r_0^{-\beta_1} - w'(r_0) r_0^{-\beta_2+1})/(\beta_2 - \beta_1) \\ c_2 &= (-\beta_1 w(r_0) r_0^{-\beta_1} + w'(r_0) r_0^{-\beta_2+1})/(\beta_2 - \beta_1) \end{aligned}$$

In both cases we see that under the assumption (7) we have $c_1, c_2 \to 0$ as $r_0 \to 0^+$, and that we can take the limit as $r_0 \to 0^+$ in (10). □

Lemma 12 *Let $R > 0$ and $w \in L^\infty(0, R)$ satisfy (12) for $0 < r < R$, where $f \in C([0, R])$. Then $w \in C^2(0, R)$ and is a solution of (6)-(7). If moreover $f \in C^2([0, R))$ then the same is true for w, and if $f'(0) = 0$ then $w'(0) = 0$.*

Proof. We differentiate under the integral sign and check that the equation (6) is satisfied. Set $w = w_1 + w_2$ where

$$w_1(r) = -\int_0^r k(s/r) s g(s, w(s))\, ds \quad w_2(r) = -\int_0^r k(s/r) s^{-1} f(s)\, ds$$

It is easy to see that if f is smooth the w_2 is also smooth and that

$$w_2'(0) = -f'(0) \int_0^1 k(t)\, dt$$

so that $f'(0) = 0$ implies $w_2'(0) = 0$. It is also easy to check that if f is only continuous then $r w_2'(r) \to 0$ as $r \to 0^+$. To estimate w_1 and its derivatives, consider $M = \|w\|_{L^\infty(0,R)}$ and let $r_0 > 0$ be small enough so that $2MR_0^{\frac{2p}{p-1}} \le \lambda$. Then by (8), for $0 < r < r_0$ we have

$$\begin{aligned} |w_1(r)| &\le CM^2 \left|\int_0^r k(s/r) s^{\frac{p+1}{p-1}}\, ds\right| \\ &\le CM^2 r^{\frac{2p}{p-1}} \int_0^1 k(t) t^{\frac{p+1}{p-1}}\, dt \\ &\to 0 \quad \text{as } r \to 0^+ \end{aligned}$$

In a similar way one proves that $w_1'(r), w_1''(r) \to 0$ as $r \to 0^+$. □

Proof of Proposition 8. To prove part b) of the proposition we use part a) to obtain the conclusions on some interval $(0,R)$, $R>0$, and then we can quote standard results for ODE's (see for example [CL]).

The proof of part a) consists in applying the Banach fixed point theorem to the operator suggested by (12). Let $R>0$ (to be specified later) and let $f\in C[0,R]$. Consider the operator $T: C[0,R]\to C[0,R]$ defined by

$$Tw(r) = -\int_0^r k(s/r)\left(sg(s,w(s)) + s^{-1}f(s)\right)\, ds$$

Let $M>0$ (also to be chosen later) and let X_M be the closed ball of $C[0,R]$ centered at 0 of radius M. Then, for $w\in X_M$ and if $2MR^{\frac{2p}{p-2}}\le\lambda$, using (8) we have

$$\begin{aligned}|Tw(r)| &\le \int_0^r |k(s/r)|\left(Cs^{\frac{p+1}{p-1}}|w(s)|^2 + s^{-1}|f(s)|\right)\, ds\\ &\le CM^2\int_0^r |k(s/r)|s^{\frac{p+1}{p-1}}\, ds + \|f\|_{C[0,R]}\int_0^r |k(s/r)|s^{-1}\, ds\end{aligned}$$

But

$$\int_0^r |k(s/r)|s^q\, ds = r^{q+1}\int_0^1 |k(t)|t^q\, dt$$

and using the expression for k one can check that the integrals in the right hand side are finite for $q=\frac{p+1}{p-1}$ and $q=-1$. We obtain thus

$$\|Tw\|_{C[0,R]} \le C\left(M^2R^{\frac{2p}{p-1}} + K\right)$$

if $\|f\|_{C[0,R]}\le K$. Also, for $w_1, w_2\in X_M$, by (9) we have

$$\begin{aligned}|Tw_1(r) - Tw_2(r)| &\le \int_0^r |k(s/r)|s|g(s,w_1(s)) - g(s,w_2(s))|\, ds\\ &\le CM\int_0^r |k(s/r)|s^{\frac{p+1}{p-1}}|w_1(s)-w_2(s)|\, ds\\ &\le CMR^{\frac{2p}{p-1}}\|w_1-w_2\|_{C[0,R]}\end{aligned}$$

So, given $K>0$ we choose M so that $2CK\le M$ and then we take R small enough so that

$$MR^{\frac{2p}{p-1}} \le \min\{\lambda/2, 1/2C\}$$

With these choices T is a contraction (with Lipschitz constant 1/2) that maps X_M into X_M. Therefore it has a fixed point (unique in X_M), which is a solution of (6)-(7) by Lemma 12.

To prove uniqueness, suppose that w_1, w_2 are two solutions of (6)-(7) on $(0,R)$. Then choose M' so that

$$M' \ge \max\{2CK, \|w_1\|_{C[0,R]}, \|w_2\|_{C[0,R]}\}$$

and R' so that

$$M'R'^{\frac{2p}{p-1}} \le \min\{\lambda/2, 1/2C\} \quad \text{and} \quad R'\le R$$

Then w_1, w_2 are in $\{w \in C[0, R'] \ / \ \|w\|_{C[0.R']} \leq M'\}$ and are fixed points of T. Hence $w_1 \equiv w_2$ on $(0, R')$. The equality on $(0, R)$ is obtained by a standard uniqueness result for ODE's.

Regarding continuous dependence, let $f_1, f_2 \in C[0, R]$ be such that $\|f_i\|_{C[0.R']} \leq K, i = 1, 2$, and let w_1, w_2 be the corresponding solutions to (6)-(7), i.e., the fixed points of T_i in X_M, where

$$T_i w(r) = -\int_0^r k(s/r)\left(sg(s, w(s)) + s^{-1} f_i(s)\right)\, ds$$

Recall that T_i maps X_M into X_M and that it has a Lipschitz constant of 1/2. Then,

$$\begin{aligned}
\|w_1 - w_2\|_{C[0,R]} &\leq \|T_1(w_1) - T_1(w_2)\| + \|T_1(w_2) - T_2(w_2)\| \\
&\leq \frac{1}{2}\|w_1 - w_2\| + \sup_{0 \leq r \leq R} \int_0^r |k(s/r)| s^{-1} |f_1(s) - f_2(s)|\, ds \\
&\leq \frac{1}{2}\|w_1 - w_2\| + C\|f_1 - f_2\|
\end{aligned}$$

□

Remark that by part a) of Proposition 8, given a continuous function f on $[0, \infty)$ there exists $R > 0$ such that the sequence

$$\begin{cases} w_0 \equiv 0 \\ w_{k+1} = T(w_k) \end{cases}$$

converges in $C[0, R]$ to the solution of (6)-(7).

Lemma 13 *Assume now that $f \geq 0$ is a continuous function on $[0, \infty)$ and let w be the corresponding solution of (6)-(7) with maximal domain $(0, R)$. Then*

$$w_k \searrow w \quad \text{on } (0, R)$$

Proof. Indeed, first note that $w_k \leq 0$ for all k, because $g(r, w) \geq 0$ and $f \geq 0$. In particular $w_1 \leq w_0 \equiv 0$. Then observe that for fixed r, $g(r, w)$ is non increasing in w for $w \leq 0$. This implies that $T(w_1) \leq T(w_0)$, i.e. $w_2 \leq w_1$, and by induction $w_{k+1} \leq w_k$ for all k. Note also that since w is a fixed point of T, from $w \leq w_0 \equiv 0$ follows that $w \leq T(w_0) = w_1$, and again an induction argument shows that $w \leq w_k$ for all k. It follows that $w_k \searrow \tilde{w}$ pointwise, and taking the limit in the recurrence relation

$$w_{k+1}(r) = -\int_0^r k(s/r)\left(sg(s, w_k(s)) + s^{-1} f(s)\right)\, ds$$

we obtain that $\tilde{w}$ is a fixed point of T, and hence a solution of (6)-(7). By uniqueness $w = \tilde{w}$. □

Lemma 14 *Let f_1, f_2 be continuous functions on $[0, R]$ and suppose that the corresponding solutions $w^{(1)}, w^{(2)}$ to (6)-(7) are defined on $(0, R)$. Assume that $f_1 \geq f_2 \geq 0$. Then $w^{(1)} \leq w^{(2)} \leq 0$ on $(0, R)$.*

Proof. For $i = 1, 2$ define the operators T_i corresponding to f_i as before, and consider the sequences

$$\begin{cases} w_0^{(i)} \equiv 0 \\ w_{k+1}^{(i)} = T_i(w_k^{(i)}) \end{cases}$$

Then $w_k^{(i)} \searrow w^{(i)}$, $i = 1, 2$. But since $f_1 \geq f_2 \geq 0$ we have (inductively)

$$\begin{aligned} w_{k+1}^{(1)} &= T_1(w_k^{(1)}) \\ &\leq T_1(w_k^{(2)}) \\ &\leq T_2(w_k^{(2)}) \\ &= w_{k+1}^{(2)} \end{aligned}$$

Therefore $w^{(1)} \leq w^{(2)}$. □

Proposition 15 *Let f be a continuous function on $[0, \infty)$, $f \geq 0$, $f \not\equiv 0$ on $[0, 1]$. For each $t \geq 0$ let w_t be the solution of*

$$\begin{cases} Lw_t = -g(r, w_t) - r^{-2} t f \\ w_t(r), r w_t'(r) \quad \text{are bounded near } 0 \end{cases}$$

which is defined on a maximal interval $(0, R_t)$. Then the set

$$\{w_t(1) \ / \ t \geq 0 \text{ and } w_t(1) \text{ is defined}\}$$

is the whole interval $(-\infty, 0]$. In particular there exists $\bar{t} \geq 0$ such that $w_{\bar{t}}(1)$ is defined and is equal to $-v(1)$.

Proof. Define

$$A = \{w_t(1) \ / \ t \geq 0 \text{ and } w_t(1) \text{ is defined}\}$$

and note that $0 \in A$. Next we show that A is connected. Consider $\varphi(t) = w_t(1)$ with domain $dom(\varphi) = \{t \geq 0 \ / \ w_t(1) \text{ is defined}\}$. Then φ is continuous, and to conclude that A is connected we only need to check that $dom(\varphi)$ is connected. So let $0 \leq t_2 \leq t_1$ and suppose that $R_{t_1} > 1$ (i.e. $t_1 \in dom(\varphi)$). By monotonicity with respect to t we have $0 \geq w_{t_2} \geq w_{t_1}$ on $(0, R_{t_1}) \cap (0, R_{t_2})$. If $R_{t_2} < R_{t_1}$ then we have an apriori bound for w_{t_2} on $(0, R_{t_2})$, so that w_{t_2} can be continued beyond R_{t_2}. This contradiction shows that $R_{t_2} \geq R_{t_1} > 1$ and therefore $t_2 \in dom(\varphi)$.

Now we prove that A is open in $(-\infty, 0]$. Let $a \in A$ and $t \geq 0$ be such that $w_t(1) = a$. By the continuous dependence of w_t in t, we have that $w_{t'}(1)$ is defined for t' close to t. Take $t' > t$ but close enough. Then

$$\begin{aligned} w_{t'}(1) &= -\int_0^1 k(s)(sg(s, w_{t'}(s)) + s^{-1} t' h(s)) \, ds \\ &< w_t(1) \end{aligned}$$

because $f \not\equiv 0$ on $[0,1]$. Hence A contains an interval of the form $(a-\varepsilon, a]$ for some $\varepsilon > 0$.

Suppose now that A is bounded and let $a = \inf A \notin A$. Then there exists a sequence $a_n \searrow a$, $a_n \in A$. Let $t_n \geq 0$ be such that $w_{t_n}(1) = a_n$. Then, if $t_n < t_m$ we must have $a_m < a_n$, and we conclude that (t_n) is increasing. If $t_n \nearrow \infty$, then

$$\begin{aligned} a_n &= -\int_0^1 k(s)(sg(s, w_{t_n}(s)) + s^{-1} t_n f(s))\ ds \\ &\to -\infty \quad \text{as} n \to \infty \end{aligned}$$

which contradicts the assumption that A is bounded. Hence we may assume that $t_n \nearrow t < \infty$. Note that $w_{t_n}(r)$ is decreasing, so that $w(r) = \lim_n w_{t_n}(r)$ exists for $0 < r < 1$. Taking the limit as $n \to \infty$ in

$$w_{t_n}(r) = -\int_0^r k(s/r) \left(sg(s, w_{t_n}(s)) + s^{-1} t_n f(s) \right)\ ds$$

we obtain by monotone convergence

$$w(r) = -\int_0^r k(s/r) \left(sg(s, w(s)) + s^{-1} t f(s) \right)\ ds \tag{13}$$

Claim: w is the solution of

$$\begin{cases} Lw_t = -g(r, w_t) - r^{-2} t f \\ w_t(r), rw_t'(r) \text{ are bounded near } 0 \end{cases}$$

and $w(1) = a$. Thus $a \in A$ and from this contradiction we conclude that $A = (-\infty, 0]$.

Proof of the claim. We need to prove that $w \in L^\infty(0,1)$ so that we can apply Lemma 12, and then show that $\lim_{r \to 1^-} w(r) = a$.

Note that by Proposition 8 there exists $M > 0$ and $0 < R < 1$ such that $|w_{t_n}(r)| \leq M$ for $0 < r < R$, and therefore $|w(r)| \leq M$ for $0 < r < R$. Let's estimate $w(r)$ for $r \in [R, 1]$. Let $m(s) = sg(s, w(s)) + s^{-1} f(s)$ and let's use the convention that $k(t) = 0$ for $t \geq 1$. So

$$w(r) = -\int_0^R k(s/r) s g(s, w(s))\ ds - \int_0^R k(s/r) s^{-1} f(s)\ ds - \int_R^1 k(s/r) m(s)\ ds \tag{14}$$

We may take R smaller if necessary so that $2MR^{\frac{2p}{p-1}} \leq \lambda$ and therefore by (8) we find as in the proof of Lemma 12 that the first 2 terms are bounded independently of $r \in [R, 1]$. To estimate

$$\int_R^1 k(s/r) m(s)\ ds$$

note that by taking $r = 1$ in (13) we get $km \in L^1(0,1)$. But there exists $C > 0$ such that for $R \leq s, r \leq 1$, $k(s/r) \leq Ck(s)$, which shows that w is bounded in $(0,1)$.

Finally, because of the same estimates as before we can use dominated convergence in (14) to find that $w(r) \to a$ as $r \to 1^-$. □

Appendix

Here we give a proof of Theorem 2. Let $f \geq 0$ be smooth, and let u denote here the minimal solution of (P_t), which we know is smooth for $0 < t < t^*$. We omit from the notation the explicit dependence of u in t.

We know that the first eigenvalue of $-\Delta - pu^{p-1}$ is non-negative, so for all $\varphi \in C_0^1(\Omega)$ we have

$$\int_\Omega |\nabla \varphi|^2 \, dx \geq p \int_\Omega u^{p-1} \varphi^2 \, dx$$

Let $j \geq 1$ and take $\varphi = u^j$. We then get

$$j^2 \int_\Omega u^{2j-2} |\nabla u|^2 \, dx \geq p \int_\Omega u^{p+2j-1} \, dx.$$

Now multiply (P_t) by $\frac{j^2}{2j-1} u^{2j-1}$ and integrate by parts to obtain

$$j^2 \int_\Omega u^{2j-2} |\nabla u|^2 \, dx = \frac{j^2}{2j-1} \int_\Omega u^{p+2j-1} + tfu^{2j-1} \, dx$$

Combining these two we obtain

$$\frac{j^2}{2j-1} \int_\Omega u^{p+2j-1} + tfu^{2j-1} \, dx \geq p \int \Omega u^{p+2j-1} \, dx$$

If $\frac{j^2}{2j-1} < p$ we see that there is a constant C independent of t such that

$$\|u\|_{L^{p+2j-1}} \leq C$$

(recall that $t < t^*$). From now on we denote by C different numbers independent of t. The restriction on j can be rewritten as $1 \leq j < p + \sqrt{p^2 - p}$. Hence for $q = p + 2j - 1$ we find a bound for $\|u\|_{L^q}$ independent of t, for $q < 3p + 2\sqrt{p^2 - p} - 1$, and hence

$$\|u^p + tf\|_{L^r} \leq C$$

for $r < 3 + 2\sqrt{1 - 1/p} - 1/p$. Now we use the equation and the L^p theory to improve this estimate. Let $1 < r_0 < 3 + 2\sqrt{1 - 1/p} - 1/p$. By L^p estimates

$$\|u\|_{W^{2,r_0}} \leq C$$

and if $1/r_0 - 2/n > 0$, by Sobolev embedding we get

$$\|u^p + tf\|_{L^{r_1}} \leq C$$

with $1/r_1 = p(1/r_0 - 2/n)$. If on the other hand $1/r_0 - 2/n \leq 0$ we conclude that

$$\|u\|_{C(\bar{\Omega})} \leq C$$

(If $1/r_0 - 2/n < 0$, we use Sobolev embedding, and if $1/r_0 - 2/n = 0$ we apply once more the L^p estimates and the Sobolev embedding). Continuing in this way we define a sequence r_k by $1/r_{k+1} = p(1/r_k - 2/n)$, and we would like to find some k for which $r_k \leq 0$, so that as before we obtain a bound for u in $C(\bar{\Omega})$. To compute r_k we introduce $a_k = 1/r_k - 2/n$ which satisfies then $a_{k+1} = pa_k - 2/n$. Therefore

$$a_k = p^k \left(a_0 - \frac{2}{n(p-1)} \right) + \frac{2}{n(p-1)}$$

We want to find some k for which $a_k \leq 0$ and this occurs for some k iff

$$a_0 - \frac{2}{n(p-1)} < 0$$

Going back to r_0 this requires $r_0 > \frac{n}{2}(1 - 1/p)$. But we had already the restriction $r_0 < 3 + 2\sqrt{1 - 1/p} - 1/p$, so that the argument works if

$$\frac{n}{2}(1 - 1/p) < 3 + 2\sqrt{1 - 1/p} - 1/p$$

which is equivalent to

$$n < 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}.$$

Acknowledgments. The author wishes to thank Prof. H. Brezis for introducing the problem and useful discussions concerning this work.

References

[BC] H. Brezis and X. Cabré, *Some simple nonlinear pde's without solutions*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **1** (1998), no. 2, 223–262.

[BCMR] H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, *Blow up for $u_t - \Delta u = g(u)$ revisited*, Adv. Differential Equations **1** (1996), no. 1, 73–90.

[BV] H. Brezis and J. L. Vázquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), no. 2, 443–469.

[CL] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[CR] M. G. Crandall and P. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rational Mech. Anal. **58** (1975), no. 3, 207–218.

[Ma] Y. Martel, *Uniqueness of weak extremal solutions of nonlinear elliptic problems*, Houston J. Math. **23** (1997), no. 1, 161–168.

[MP] F. Mignot and J.P. Puel, *Sur une classe de problèmes non linéaires avec non linéairité positive, croissante, convexe*, Comm. Partial Differential Equations **5** (1980), no. 8, 791–836.

[Ne] G. Nedev, Preprint, 1999.