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New Entire Solutions to Some Classical Semilinear Elliptic Problems

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Abstract

This paper deals with the construction of solutions to autonomous semilinear elliptic equations considered in entire space. In the absence of space dependence or explicit geometries of the ambient space, the point is to unveil internal mechanisms of the equation that trigger the presence of families of solutions with interesting concentration patterns. We discuss the connection between minimal surface theory and entire solutions of the Allen-Cahn equation. In particular, for dimensions 9 or higher, we build an example that provides a negative answer to a celebrated question by De Giorgi for this problem. We will also discuss related results for the (actually more delicate) standing wave problem in nonlinear Schrödinger equations and for sign-changing solutions of the Yamabe equation.

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1. Introduction

Understanding the entire solutions of nonlinear elliptic equations in $\mathbb{R}^N$ such as

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

is a basic problem in PDE research. This is the context of various classical results in literature like the Gidas-Ni-Nirenberg theorems on radial symmetry, Liouville type theorems, or the achievements around De Giorgi's conjecture. In those results, the geometry of level sets of the solutions turns out to be a

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posteriori very simple (planes or spheres). On the other hand, problems of the form (1.1) with nonlinearities recurrent the literature, do have solutions with more interesting patterns, and the structure of their solution sets has remained mostly a mystery.

In many studies, problems like (1.1) are considered involving explicit dependence on the space variable, or on a manifold or in a domain in $\mathbb{R}^N$ under boundary conditions. Topological and geometric features of the domain are often characteristic that trigger the presence of interesting solutions, whose precise features can be analyzed when some singular perturbation parameter is involved. In the absence of space inhomogeneity or geometry of the ambient space, as in the "clean" equation (1.1), it is less clear which internal mechanisms of the equation are behind complex patterns in the solution set, whose richness may be nearly impossible to fully grasp.

In this paper we consider specific problems of the form (1.1) and describe recent results on existence of families of solutions, depending on parameters, that exhibit interesting asymptotic patterns linked to geometric objects in entire space. We consider the following three classical problems:

1. The Allen-Cahn equation,

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N .$$

2. The standing wave problem for the (focusing) nonlinear Schrödinger equation

$$\Delta u + |u|^{p-1}u - u = 0 \quad \text{in } \mathbb{R}^N .$$

3. The Yamabe equation

$$\Delta u + |u|^{\frac{4}{N-2}} u = 0 \quad \text{in } \mathbb{R}^N , \quad N \geq 3.$$

Sections 1 to 10, will be devoted to discuss the Allen Cahn equation. We will describe a link between entire minimal surfaces and solutions to the equation which have a nodal set close to large dilations near such a surface, while approaching ± 1 away from it, in particular answering negatively a long-standing question by De Giorgi in dimensions $N \geq 9$. We shall describe in Section 13 parallels and related results for the other two problems, which are in turn more delicate.

2. The Allen-Cahn Equation

The Allen-Cahn equation in $\mathbb{R}^N$ is the semilinear elliptic problem

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N . \tag{2.1}$$

Originally formulated in the description of bi-phase separation in fluids [14] and ordering in binary alloys [3], Equation (2.1) has received extensive mathematical study. It is a prototype for the modeling of phase transition phenomena in a variety of contexts.

Introducing a small positive parameter ε and writing $v(x) := u(\varepsilon^{-1}x)$, we get the scaled version of (2.1),

$$\varepsilon^2 \Delta v + v - v^3 = 0 \quad \text{in } \mathbb{R}^N . \tag{2.2}$$

On every bounded domain $\Omega \subset \mathbb{R}^N$, (2.1) is the Euler-Lagrange equation for the action functional

$$J_\varepsilon(v) = \int_\Omega \frac{\varepsilon}{2} |\nabla v|^2 + \frac{1}{4\varepsilon}(1 - v^2)^2.$$

We observe that the constant functions $v = \pm 1$ minimize J_ε. They are idealized as two *stable phases* of a material in Ω. It is of interest to analyze configurations in which the two phases coexist. These states are represented by stationary points of J_ε, or solutions v_ε of Equation (2.2), that take values close to $+1$ in a subregion of Ω of and -1 in its complement. Modica and Mortola [64] and Modica [63], established that a family of local minimizers v_ε of J_ε for which

$$\sup_{\varepsilon > 0} J_\varepsilon(v_\varepsilon) < +\infty \tag{2.3}$$

must satisfy as $\varepsilon \to 0$, after passing to a subsequence,

$$v_\varepsilon \to \chi_\Lambda - \chi_{\Omega \setminus \Lambda} \quad \text{in } L^1_{loc}(\Omega). \tag{2.4}$$

Here Λ is an open subset of Ω with $\Gamma = \partial\Lambda \cap \Omega$ having *minimal perimeter*, being therefore a (generalized) minimal surface. Moreover,

$$J_\varepsilon(v_\varepsilon) \to \frac{2}{3}\sqrt{2}\mathcal{H}^{N-1}(\Gamma). \tag{2.5}$$

2.1. Formal asymptotic behavior of v_ε. Let us argue formally to obtain an idea on how a solution v_ε of Equation (2.2) with uniformly bounded energy (2.3) should look like near a limiting interface Γ. Let us assume that Γ is a smooth hypersurface and let ν designate a choice of its unit normal. Points δ-close to Γ can be uniquely represented as

$$x = y + z\nu(y), \quad y \in \Gamma, \ |z| < \delta \tag{2.6}$$

A well known formula for the Laplacian in these coordinates reads as follows.

$$\Delta_x = \partial_{zz} + \Delta_{\Gamma^z} - H_{\Gamma_z}\partial_z \tag{2.7}$$

Here

$$\Gamma^z := \{y + z\nu(y) \ / \ y \in \Gamma\}.$$

Δ_{Γ^z} is the Laplace-Beltrami operator on Γ^z acting on functions of the variable y, and H_{Γ^z} designates its mean curvature. Let $k_1, \ldots, k_N$ denote the principal curvatures of Γ. Then we have the validity of the expression

$$H_{\Gamma^z} = \sum_{i=1}^{N} \frac{k_i}{1 - zk_i}. \tag{2.8}$$

It is reasonable to assume that the solution has uniform smoothness in the y-direction, while in the transition direction z, elliptic estimates applied to the transformed equation (2.1) yield uniform smoothness in the variable $\zeta = \varepsilon^{-1} z$. The equation for $v_\varepsilon(y, \zeta)$ then reads

$$\varepsilon^2 \Delta_{\Gamma^{\varepsilon\zeta}} v_\varepsilon - \varepsilon H_{\Gamma^{\varepsilon\zeta}}(y)\, \partial_\zeta v_\varepsilon \ + \ \partial_\zeta^2 v_\varepsilon + v_\varepsilon - v_\varepsilon^3 \ = \ 0, \quad y \in \Gamma, \quad |\zeta| < \delta\varepsilon^{-1}. \tag{2.9}$$

We shall make two strong assumptions:

1. The zero-level set of v_ε lies within a $O(\varepsilon^2)$-neighborhood of Γ, that is on the region $|\zeta| = O(\varepsilon)$ and $\partial_\tau v_\varepsilon > 0$ on this nodal set, and
2. $v_\varepsilon(y, \zeta)$ can be expanded in powers of ε as

$$v_\varepsilon(y, \zeta) = v_0(y, \zeta) + \varepsilon v_1(y, \zeta) + \varepsilon^2 v_2(y, \zeta) + \cdots \tag{2.10}$$

for smooth coefficients bounded, with bounded derivatives. We observe also that

$$\int_\Gamma \int_{-\delta/\varepsilon}^{\delta/\varepsilon} [\, \frac{1}{2} |\partial_\zeta v_\varepsilon|^2 + \frac{1}{4}(1 - v_\varepsilon^2)^2 \,]\, d\zeta\, d\sigma(y) \ \le \ J_\varepsilon(v_\varepsilon) \le C \tag{2.11}$$

Substituting Expression (2.10) in Equation (2.9), using the first assumption, and letting $\varepsilon \to 0$, we get

$$\begin{aligned} \partial_\zeta^2 v_0 + v_0 - v_0^3 = 0, &\quad (y, \zeta) \in \Gamma \times \mathbb{R}, \\ v_0(0, y) = 0, &\quad \partial_\zeta(0, y) \ge 0, \quad y \in \Gamma. \end{aligned} \tag{2.12}$$

while from (2.11) we get

$$\int_{\mathbb{R}} [\, \frac{1}{2} |\partial_\zeta v_0|^2 + \frac{1}{4}(1 - v_0^2)^2 \,]\, d\zeta \ < \ +\infty \tag{2.13}$$

Conditions (2.13) and (2.12) force $v_0(y, \zeta) = w(\zeta)$ where w is the unique solution of the ordinary differential equation

$$w'' + w - w^3 = 0, \quad w(0) = 0, \quad w(\pm\infty) = \pm 1, \tag{2.14}$$

namely

$$w(\zeta) \ := \ \tanh(\zeta/\sqrt{2}). \tag{2.15}$$

On the other hand, substitution yields that $v_1(y,\zeta)$ satisfies

$$\partial_\zeta^2 v_1 + (1-3w(\zeta)^2)v_1 \;=\; H_\Gamma(y)\, w'(\zeta), \quad \zeta\in(-\infty,\infty) \tag{2.16}$$

Testing this equation against $w'(\zeta)$ and integrating by parts in ζ we get the relation

$$H_\Gamma(y) = 0 \quad \text{for all} \quad y\in\Gamma$$

which tells us precisely that Γ must be a minimal surface, as expected. Hence, we get $v_1 = -h_0(y)w'(\zeta)$ for a certain function $h_0(y)$. As a conclusion, from (2.10) and a Taylor expansion, we can write

$$v_\varepsilon(y,\zeta) = w(\zeta - \varepsilon h_0(y)) + \varepsilon^2 v_2 + \cdots$$

It is convenient to write this expansion in terms of the variable $t = \zeta - \varepsilon h_0(y)$ in the form

$$v_\varepsilon(y,\zeta) = w(t) + \varepsilon^2 v_2(t,y) + \varepsilon^3 v_3(t,y) + \cdots \tag{2.17}$$

Using expression (2.8) and the fact that Γ is a minimal surface, we expand

$$H_{\Gamma^{\varepsilon\zeta}}(y) = \varepsilon^2\zeta\,|A_\Gamma(y)|^2 + \varepsilon^3\zeta^2\,H_3(y) + \cdots$$

where

$$|A_\Gamma|^2 = \sum_{i=1}^{8} k_i^2, \quad H_3 = \sum_{i=1}^{8} k_i^3.$$

Thus setting $t = \zeta - \varepsilon h_0(y)$ and using (2.17), we compute

$$\begin{aligned} 0 = \Delta v_\varepsilon + v_\varepsilon + v_\varepsilon^3 \;=\; &[\partial_t^2 + (1-3w(t)^2)]\,(\varepsilon^2 v_2 + \varepsilon^3 v_3)\\ -w'(t)\,[\,\varepsilon^3\Delta_\Gamma h_0 \,+ \varepsilon^3 H_3\, t^2 + &\varepsilon^2|A_\Gamma|^2\,(t+\varepsilon h_0)\,] + O(\varepsilon^4). \end{aligned}$$

And then letting $\varepsilon\to 0$ we arrive to the equations

$$\partial_t^2 v_2 + (1-3w^2)v_2 = |A_\Gamma|^2\, tw', \tag{2.18}$$

$$\partial_t^2 v_3 + (1-3w^2)v_3 = [\Delta_\Gamma h_0 + |A_\Gamma|^2 h_0 + H_3\, t^2]\, w'. \tag{2.19}$$

Equation (2.18) has a bounded solution since $\int_{\mathbb{R}} tw'(t)^2\,dt = 0$ Instead the bounded solvability of (2.19) is obtained if and only if h_0 solves the following elliptic equation in Γ.

$$\mathcal{J}_\Gamma[h_0](y) \;:=\; \Delta_\Gamma h_0 + |A_\Gamma|^2 h_0 \;=\; c\sum_{i=1}^{8} k_i^3 \quad \text{in } \Gamma, \tag{2.20}$$

where $c = -\int_{\mathbb{R}} t^2 w'^2\,dt / \int_{\mathbb{R}} w'^2\,dt$. $\mathcal{J}_\Gamma$ is by definition the *Jacobi operator* of the minimal surface Γ.

We deal with the problem of constructing entire solutions of Equation (2.2), that exhibit the asymptotic behavior described above, around a given, fixed minimal hypersurface Γ that splits the space $\mathbb{R}^N$ into two components, and for which the coordinates (2.6) are defined for some uniform $\delta > 0$. A key element for such a construction is the precisely the question of solvability of Equation (2.20), that determines at main order the deviation of the nodal set of the solution from Γ.

In terms of the original problem (2.1), the issue is to consider a large dilation of Γ,

$$\Gamma_\varepsilon \ := \ \varepsilon^{-1}\Gamma,$$

and find an entire solution u_ε to problem (2.1) such that for a function h_ε defined on Γ with

$$\sup_{\varepsilon>0} \|h_\varepsilon\|_{L^\infty(\Gamma)} \ < \ +\infty, \tag{2.21}$$

we have

$$u_\varepsilon(x) = w(\zeta - \varepsilon h_\varepsilon(\varepsilon y)) + O(\varepsilon^2), \tag{2.22}$$

uniformly for

$$x = y + \zeta\nu(\varepsilon y), \quad |\zeta| \le \frac{\delta}{\varepsilon}, \quad y \in \Gamma_\varepsilon,$$

while

$$|u_\varepsilon(x)| \to 1 \quad \text{as dist}\,(x, \Gamma_\varepsilon) \to +\infty. \tag{2.23}$$

We shall answer affirmatively this question in some important examples for Γ. One is a nontrivial minimal graph in $\mathbb{R}^9$. The solution found provides a negative answer to to a famous question due to Ennio De Giorgi [25]. On the other hand, in $\mathbb{R}^3$ we find a broad new class of entire solutions with finite Morse index, which suggests analogs of De Giorgi's conjecture for solutions of (2.1) in parallel with known classification results for minimal surfaces.

3. From Bernstein's to De Giorgi's Conjecture

Ennio De Giorgi [25] formulated in 1978 the following celebrated conjecture concerning entire solutions of equation (2.1).

De Giorgi's Conjecture: *Let u be a bounded solution of equation* (2.1) *such that $\partial_{x_N} u > 0$. Then the level sets $[u = \lambda]$ are all hyperplanes,* **at least** *for dimension $N \le 8$.*

Equivalently, u must depend only on one Euclidean variable so that it must have the form $u(x) = w((x-p)\cdot\nu)$ for some $p \in \mathbb{R}^N$ and some ν with $|\nu| = 1$ and $\nu_N > 0$.

The condition $\partial_{x_N} u > 0$ implies that the level sets of u are all graphs of functions of the first $N-1$ variables. As we have discussed in the previous section, level sets of solutions with a transition are closely connected to minimal hypersurfaces. De Giorgi's conjecture is in fact a parallel to the following classical statement.

Bernstein's conjecture: *A minimal hypersurface in $\mathbb{R}^N$, which is also the graph of a smooth entire function of $N-1$ variables, must be a hyperplane.*

In other words, if Γ is an *entire minimal graph*, namely

$$\Gamma = \{(x', x_N) \mid x' \in \mathbb{R}^{N-1},\ x_N = F(x')\} \tag{3.1}$$

where F solves the minimal surface equation

$$H_\Gamma \equiv \nabla \cdot \left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}, \tag{3.2}$$

then Γ must be a hyperplane, hence F must be a linear affine function.

Bernstein's conjecture is known to be true up to dimension $N = 8$, see Simons [80] and references therein, while it is *false* for $N \geq 9$, as proven by Bombieri, De Giorgi and Giusti [12], by building a nontrivial solution to Equation (3.2). Let us write $x' \in \mathbb{R}^8$ as $x' = (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^4 \times \mathbb{R}^4$. Let us consider the set

$$T \ := \ \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^8 \mid \ |\mathbf{v}| > |\mathbf{u}| > 0 \ \}. \tag{3.3}$$

The set $\{|\mathbf{u}| = |\mathbf{v}|\} \in \mathbb{R}^8$ is Simons' minimal cone [80]. The solution found in [12] is radially symmetric in both variables, namely $F = F(|\mathbf{u}|, |\mathbf{v}|)$. In addition, F is positive in T and it vanishes along Simons' cone. Moreover, it satisfies

$$F(|\mathbf{u}|, |\mathbf{v}|) = -F(|\mathbf{v}|, |\mathbf{u}|)\ . \tag{3.4}$$

Let us write $(|\mathbf{u}|, |\mathbf{v}|) = (r\cos\theta, r\sin\theta)$. In [30] it is found that there is a function $g(\theta)$ with

$$g(\theta) > 0 \quad \text{in } (\pi/4, \pi/2), \quad g'(\pi/2) = 0 = g(\pi/4), \quad g'(\pi/4) > 0,$$

such that for some $\sigma > 0$,

$$F(|\mathbf{u}|, |\mathbf{v}|) \ = \ g(\theta)\, r^3 + O(r^{-\sigma}) \quad \text{in } T. \tag{3.5}$$

De Giorgi's conjecture has been established in dimensions $N = 2$ by Ghoussoub and Gui [41] and for $N = 3$ by Ambrosio and Cabré [15]. Savin [76] proved its validity for $4 \leq N \leq 8$ under the additional assumption

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \quad \text{for all} \quad x' \in \mathbb{R}^{N-1}. \tag{3.6}$$

Farina and Valdinoci [38] replaced condition (3.6) by the less restrictive assumption that the profiles at infinity are two-dimensional functions, or that their level sets are complete graphs. Condition (3.6) is related to the so-called Gibbons' Conjecture:

Gibbons' Conjecture: Let u be a bounded solution of equation (2.1) *satisfying Condition (3.6) uniformly in x'. Then the level sets of u are all hyperplanes.*

Gibbons' Conjecture has been established in all dimensions with different methods by Caffarelli and Córdoba [17], Farina [36], Barlow, Bass and Gui [10], and Berestycki, Hamel, and Monneau [11]. In [17, 10] it is proven that the conjecture is true for any solution that has one level set which is a globally Lipschitz graph.

The following result disproves De Giorgi's statement for $n \geq 9$.

Theorem 1 ([30, 31]). *Let $N \geq 9$. Then there is an entire minimal graph Γ which is not a hyperplane, such that all $\varepsilon > 0$ sufficiently small there exists a bounded solution $u_\varepsilon(x)$ of equation* (2.1) *that satisfies properties* (2.21)-(2.23). *Besides, $\partial_{x_N} u_\varepsilon > 0$ and u_ε satisfies condition* (3.6).

A counterexample to De Giorgi's conjecture in dimension $N \geq 9$ was believed to exist for a long time. Partial progress in this direction was made by Jerison and Monneau [51] and by Cabré and Terra [13]. See also the survey article by Farina and Valdinoci [37].

3.1. Outline of the proof.

For a small $\varepsilon > 0$ we look for a solution u_ε of the form (near Γ_ε),

$$u_\varepsilon(x) = w(\zeta - \varepsilon h(\varepsilon y)) + \phi(\zeta - \varepsilon h(\varepsilon y), y), \quad x = y + \zeta \nu(\varepsilon y) \tag{3.7}$$

where $y \in \Gamma_\varepsilon$, ν designates a unit normal to Γ with $\nu_N > 0$, h is a function defined on Γ, which is left as a parameter to be adjusted. Setting $r(y', y_9) = |y'|$, we assume a priori in h that

$$\|(1+r^2) D_\Gamma h\|_{L^\infty(\Gamma)} + \|(1+r)h\|_{L^\infty(\Gamma)} \ \leq \ M \tag{3.8}$$

for some large, fixed number M, also with a uniform control on $(1+r^3)D^2_\Gamma h$.

Letting $f(u) = u - u^3$ and using Expression (2.7) for the Laplacian, the equation becomes

$$\begin{aligned} S(u_\varepsilon) &:= \Delta u_\varepsilon + f(u_\varepsilon) \ = \\ &\Delta_{\Gamma^\zeta_\varepsilon} u_\varepsilon - \varepsilon H_{\Gamma^\zeta_\varepsilon}(\varepsilon y)\, \partial_\zeta u_\varepsilon \ + \\ &\qquad \partial^2_\zeta u_\varepsilon + f(u_\varepsilon) \ = 0, \quad y \in \Gamma_\varepsilon, \ |\zeta| < \delta/\varepsilon. \end{aligned} \tag{3.9}$$

Letting $t = \zeta - \varepsilon h(\varepsilon y)$, we look for u_ε of the form

$$u_\varepsilon(t, y) = w(t) + \phi(t, y)$$

for a small function ϕ. The equation in terms of ϕ becomes

$$\partial^2_t \phi + \Delta_{\Gamma_\varepsilon} \phi + B\phi + f'(w(t))\phi + N(\phi) + E \ = \ 0. \tag{3.10}$$

where B is a small linear second order operator, and

$$E = S(w(t)), \quad N(\phi) = f(w + \phi) - f(w) - f'(w)\phi \approx f''(w)\phi^2.$$

While the expression (3.10) makes sense only for $|t| < \delta\varepsilon^{-1}$, it turns out that the equation in the entire space can be reduced to one similar to (3.10) in entire $\mathbb{R} \times \Gamma_\varepsilon$, where E and the undefined coefficients in B are just cut-off far away, while the operator N is slightly modified by the addition of a small nonlinear, nonlocal operator of ϕ. Rather than solving this problem directly we carry out an infinite dimensional form of Lyapunov-Schmidt reduction, considering a projected version of it,

$$\begin{aligned} \partial_t^2 \phi + \Delta_{\Gamma_\varepsilon} \phi + B\phi + f'(w(t))\phi + N(\phi) + E \;=&\; \\ & c(y) w'(t) \quad \text{in } \mathbb{R} \times \Gamma_\varepsilon, \\ \int_{\mathbb{R}} \phi(t,y) w'(t)\, dt \;=&\; 0 \quad \text{for all} \quad y \in \Gamma_\varepsilon. \end{aligned} \tag{3.11}$$

the error of approximation E has roughly speaking a bound $O(\varepsilon^2 r(\varepsilon y)^{-2} e^{-\sigma|t|})$, and it turns out that one can find a solution $\phi = \Phi(h)$ to problem (3.11) with the same bound. We then get a solution to our original problem if h is such that $c(y) \equiv 0$. Thus the problem is reduced to finding h such that

$$c(y) \int_{\mathbb{R}} {w'}^2 = \int_{\mathbb{R}} (E + B\Phi(h) + N(\Phi(h)))\, w' dt \equiv 0.$$

A computation similar to that in the formal derivation yields that this problem is equivalent to a small perturbation of Equation (2.20)

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h \;=\; c \sum_{i=1}^{8} k_i^3 + \mathcal{N}(h) \quad \text{in } \Gamma, \tag{3.12}$$

where $\mathcal{N}(h)$ is a small operator. From an estimate by Simon [79] we know that $k_i = O(r^{-1})$. Hence $H_3 := \sum_{i=1}^{8} k_i^3 = O(r^{-3})$. A central point is to show that the unperturbed equation (2.20) has a solution $h = O(r^{-1})$, which justifies a posteriori the assumption (3.8) made originally on h. This step uses the asymptotic expression (3.5). The symmetries of the surface allow to reduce the problem to solving it in T with zero Dirichlet boundary conditions on Simons' cone. We have that $H_3 = O(g(\theta) r^{-3})$ and one gets a positive barrier of size $O(r^{-1})$. The operator $\mathcal{J}_\Gamma$ satisfies maximum principle and existence thus follows. The full nonlinear equation is then solved with the aid of contraction mapping principle. The detailed proof of this theorem is contained in [30].

The program towards the counterexample in [51] and [15] is based on an analogous one in Bernstein's conjecture: the existence of the counterexample is reduced to establishing the minimizing character of a *saddle solution* in $\mathbb{R}^8$ that vanishes on Simon's cone. Our approach of direct construction is actually applicable to build unstable solutions associated to general minimal surfaces, as we illustrate in the next section. We should mention that method of infinite dimensional reduction for the Allen Cahn equation in compact settings has precedents with similar flavor in [73], [55], [29]. Using variational approach, local minimizers were built in [54].

4. Finite Morse Index Solutions of the Allen-Cahn Equation in $\mathbb{R}^3$

The assumption of monotonicity in one direction for the solution u in De Giorgi's conjecture implies a form of stability, locally minimizing character for u when compactly supported perturbations are considered in the energy. Indeed, the linearized operator $L = \Delta + (1 - 3u^2)$, satisfies maximum principle since $L(Z) = 0$ for $Z = \partial_{x_N} u > 0$. This implies stability of u, in the sense that its associated quadratic form, namely the second variation of the corresponding energy,

$$\mathcal{Q}(\psi, \psi) := \int_{\mathbb{R}^3} |\nabla \psi|^2 + (3u^2 - 1)\, \psi^2 \tag{4.1}$$

satisfies $\mathcal{Q}(\psi, \psi) > 0$ for all $\psi \neq 0$ smooth and compactly supported. Stability of u is indeed sufficient for De Giorgi's statement to hold in dimension $N = 2$, as observed by Dancer [22]. This question is open for $3 \leq N \leq 8$. The monotonicity assumption actually implies the globally minimizing character of the solution on each compact set, subject to its own boundary conditions, see [1].

The *Morse index* $m(u)$ is defined as the maximal dimension of a vector space E of compactly supported functions such that

$$\mathcal{Q}(\psi, \psi) < 0 \quad \text{for all} \quad \psi \in E \setminus \{0\}.$$

In view of the discussion so far, it seems natural to associate complete, embedded minimal surfaces Γ with finite Morse index, and solutions of (2.1). The *Morse index* of the minimal surface Γ, $i(\Gamma)$, has a similar definition relative to the quadratic form for its Jacobi operator $\mathcal{J}_\Gamma := \Delta_\Gamma + |A_\Gamma|^2$: The number $i(\Gamma)$ is the largest dimension for a vector spaced E of compactly supported smooth functions in Γ with

$$\int_\Gamma |\nabla k|^2 \, dV \; - \int_\Gamma |A|^2 k^2 \, dV \; < 0 \quad \text{for all} \quad k \in E \setminus \{0\}.$$

We point out that for complete, embedded surfaces, finite index is equivalent to *finite total curvature,* namely

$$\int_\Gamma |K| \, dV \; < \; +\infty$$

where K denotes Gauss curvature of the manifold, see §7 of [48] and references therein.

4.1. Embedded minimal surfaces of finite total curvature.

The theory of embedded, minimal surfaces of finite total curvature in $\mathbb{R}^3$, has reached a notable development in the last 25 years. For more than a century, only two examples of such surfaces were known: the plane and the catenoid.

The first nontrivial example was found in 1981 by C. Costa, [19, 20]. The *Costa surface* is a genus one minimal surface, complete and properly embedded, which outside a large ball has exactly three components (its *ends*), two of which are asymptotically catenoids with the same axis and opposite directions, the third one asymptotic to a plane perpendicular to that axis. The complete proof of embeddedness is due to Hoffman and Meeks [49]. In [50] these authors generalized notably Costa's example by exhibiting a class of three-end, embedded minimal surface, with the same look as Costa's far away, but with an array of tunnels that provides arbitrary genus $\ell \geq 1$. This is known as the Costa-Hoffman-Meeks surface with genus ℓ.

As a special case of the main results of [32] we have the following

Theorem 2 ([32]). *Let* $\Gamma \subset \mathbb{R}^3$ *be either a catenoid or a Costa-Hoffman-Meeks surface with genus* $\ell \geq 1$. *Then for all sufficiently small* $\varepsilon > 0$ *there exists a solution* u_ε *of Problem* (2.1) *with the properties* (2.21)-(2.23). *In the case of the catenoid, the solution found is radially symmetric in two of its variables and* $m(u_\varepsilon) = 1$. *For the Costa-Hoffman-Meeks surface with genus* $\ell \geq 1$, *we have* $m(u_\varepsilon) = 2\ell + 3$.

4.2. A general statement. In what follows Γ designates a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature. Then Γ is orientable and the set $\mathbb{R}^3 \setminus \Gamma$ has exactly two components S_+, S_-, see [48]. In what follows we fix a continuous choice of unit normal field $\nu(y)$, which conventionally we take it to point towards S_+.

For $x = (x', x_3) \in \mathbb{R}^3$, we denote as before, $r = r(x) = |x'|$. It is known that after a suitable rotation of the coordinate axes, outside the infinite cylinder $r < R_0$ with sufficiently large radius R_0, Γ decomposes into a finite number m of unbounded components $\Gamma_1, \ldots, \Gamma_m$, its *ends*. From a result in [78], we know that asymptotically each end of Γ_k either resembles a plane or a catenoid. More precisely, Γ_k can be represented as the graph of a function F_k of the first two variables,

$$\Gamma_k = \{\, y \in \mathbb{R}^3 \ / \ r(y) > R_0,\ y_3 = F_k(y')\, \}$$

where F_k is a smooth function which can be expanded as

$$F_k(y') = a_k \log r + b_k + b_{ik} \frac{y_i}{r^2} + O(r^{-3}) \quad \text{as } r \to +\infty, \tag{4.2}$$

for certain constants a_k, b_k, b_{ik}, and this relation can also be differentiated. Here

$$a_1 \leq a_2 \leq \ldots \leq a_m \ , \qquad \sum_{k=1}^{m} a_k \ = \ 0\,. \tag{4.3}$$

We say that Γ has *non-parallel ends* if all the above inequalities are strict.

Let us consider the Jacobi operator of Γ

$$\mathcal{J}_\Gamma(h) \ := \ \Delta_\Gamma h + |A_\Gamma|^2 h \tag{4.4}$$

where $|A_\Gamma|^2 = k_1^2 + k_2^2 = -2K$. A smooth function $z(y)$ defined on Γ is called a *Jacobi field* if $\mathcal{J}_\Gamma(z) = 0$. Rigid motions of the surface induce naturally some bounded Jacobi fields: Associated to respectively translations along coordinates axes and rotation around the x_3-axis, are the functions

$$z_1(y) = \nu(y)\cdot e_i, \quad y \in \Gamma, \quad i = 1,2,3,$$

$$z_4(y) = (-y_2, y_1, 0)\cdot \nu(y), \quad y \in \Gamma. \tag{4.5}$$

We assume that Γ is *non-degenerate* in the sense that these functions are actually *all* the bounded Jacobi fields, namely

$$\{\ z \in L^\infty(\Gamma)\ /\ \mathcal{J}_\Gamma(z) = 0\,\}\ =\ \text{span}\,\{\, z_1, z_2, z_3, z_4\,\}\,. \tag{4.6}$$

This property is known in some important cases, most notably the catenoid and the Costa-Hoffmann-Meeks surface of any order $\ell \geq 1$. See Nayatani [67, 68] and Morabito [65].

Theorem 3 ([32]). *Let $N = 3$ and Γ be a minimal surface embedded, complete with finite total curvature and non-parallel ends, which is in addition nondegenerate. Then for all sufficiently small $\varepsilon > 0$ there exists a solution u_ε of Problem* (2.1) *with the properties* (2.21)-(2.23). *Moreover, we have*

$$m(u_\varepsilon)\ =\ i(\Gamma).$$

Besides, the solution is non-degenerate, in the sense that any bounded solution of

$$\Delta\phi\,+\,(1-3u_\varepsilon^2)\,\phi\,=\,0 \qquad in\ \mathbb{R}^3$$

must be a linear combination of the functions $Z_i,\ i = 1,2,3,4$ defined as

$$Z_i = \partial_i u_\varepsilon, \quad i = 1,2,3, \quad Z_4 = -x_2\partial_1 u_\varepsilon + x_1\partial_2 u_\varepsilon.$$

It is well-known that if Γ is a catenoid then $i(\Gamma) = 1$. Moreover, in the Costa-Hoffmann-Meeks surface it is known that $i(\Gamma) = 2\ell + 3$ where ℓ is the genus of Γ. See [67, 68, 65].

4.3. Further comments. In analogy with De Giorgi's conjecture, it seems plausible that qualitative properties of embedded minimal surfaces with finite Morse index should hold for the level sets of finite Morse index solutions of Equation (2.1), provided that these sets are embedded manifolds outside a compact set. As a sample, one may ask for the validity of the following two statements:

- *The level sets of any finite Morse index solution u of* (2.1) *in $\mathbb{R}^3$, such that $\nabla u \neq 0$ outside a compact set should have a finite, even number of catenoidal or planar ends with a common axis.*

The above fact does hold for minimal surfaces with finite total curvature and embedded ends as established by Ossermann and Schoen. On the other hand, the above statement should not hold true if the condition $\nabla u \neq 0$ outside a large ball is violated. For instance, let us consider the octant $\{x_1, x_2, x_3 \geq 0\}$. Problem (2.1) in the octant with zero boundary data can be solved by a super-subsolution scheme (similar to that in [23]) yielding a positive solution. Extending by successive odd reflections to the remaining octants, one generates an entire solution (likely to have finite Morse index), whose zero level set does not have the characteristics above: the condition $\nabla u \neq 0$ far away corresponds to *embeddedness of the ends of the level sets.*

An analog of De Giorgi's conjecture for the solutions that follow in complexity the stable ones, namely those with Morse index one, may be the following:

- *A bounded solution u of* (2.1) *in $\mathbb{R}^3$ with $i(u) = 1$, and $\nabla u \neq 0$ outside a bounded set, must be axially symmetric, namely radially symmetric in two variables.*

The solution we found, with transition on a dilated catenoid has this property. This statement would be in correspondence with results by Schoen [78] and López and Ros [58]: if $i(\Gamma) = 1$ and Γ has embedded ends, then it must be a catenoid.

5. The Allen-Cahn Equation in $\mathbb{R}^2$

5.1. Solutions with multiply connected nodal set. The only minimal surface Γ that we can consider in this case is a straight line, to which the trivial solution depending on its normal variable can be associated.

A class of solutions to (2.1) with a *finite number of transition lines*, likely to have finite Morse index, has been recently built in [34]. The location and shape of these lines is governed by the *Toda system*, a classical integrable model for scattering of particles on a line under the action of a repulsive exponential potential:

$$\frac{\sqrt{2}}{24} f_j'' = e^{-\sqrt{2}(f_j - f_{j-1})} - e^{-\sqrt{2}(f_{j+1} - f_j)}, \quad j = 1, \dots k, \tag{5.1}$$

$f_0 \equiv -\infty,\ f_{k+1} \equiv +\infty$. It is known that for a given solution there exist numbers $a_j^\pm, b_j^\pm$ such that

$$f_j(z) = a_j^\pm |z| + b_j^\pm + O(e^{-|z|}) \quad \text{as } z \to \pm\infty$$

where $a_j^\pm < a_{j+1}^\pm$, $j = 1, \dots, k-1$ (long-time scattering).

The role of this system in the construction of solutions with multiple transition lines in the Allen-Cahn equation in bounded domains was discovered in [29]. In entire space the following result holds.

Theorem 4 ([34]). *Given a solution f of* (5.1) *if we scale*

$$f_{\varepsilon,j}(z) := \sqrt{2}\,(j-\frac{k+1}{2})\log\frac{1}{\varepsilon}+f_j(\varepsilon z),$$

then for all small ε there is a solution u_ε with k transitions layers near the lines $x_2 = f_{\varepsilon,j}(x_1)$. More precisely,

$$u_\varepsilon(x_1,x_2)=\sum_{j=1}^{k}(-1)^{j-1}w(x_1-f_{\varepsilon,j}(x_2)\,)\,+\sigma_k+\,O(\varepsilon), \tag{5.2}$$

where $\sigma_k = -\frac{1}{2}(1+(-1)^k)\,.$

The transition lines are therefore nearly parallel and asymptotically straight. In particular, if $k = 2$ and f solves the ODE

$$\frac{\sqrt{2}}{24}f''(z)=e^{-2\sqrt{2}f(z)},\quad f'(0)=0,$$

and $f_\varepsilon(z) := \sqrt{2}\log\frac{1}{\varepsilon}+f(\varepsilon z)$, then there exists a solution u_ε to (2.1) in $\mathbb{R}^2$ with

$$u_\varepsilon(x_1,x_2)=w(x_1+f_\varepsilon(x_2)\,)+w(x_1-f_\varepsilon(x_2))\,-1\,+O(\varepsilon). \tag{5.3}$$

The formal reason for the appearance of the Toda system can be explained as follows: Let us consider the function

$$u_*(x_1,x_2)=\sum_{j=1}^{k}(-1)^{j-1}w(x_1-f_j(x_2)\,)\,+\sigma_k$$

and assume that the f_j's are ordered and very distant one to each other. Then the energy

$$J_S(u_*)=\frac{1}{2}\int_S|\partial_{x_2}u_*|^2+|\partial_{x_1}u_*|^2+\frac{1}{4}\int_S(1-u_*^2)^2$$

computed in a finite strip $S = \mathbb{R}\times(-\ell,\ell)$ becomes at main order, after some computation,

$$J_S(u_*)\approx 2\ell\,[\frac{1}{2}\int_{\mathbb{R}}|w'|^2+\frac{1}{4}\int_{\mathbb{R}}(1-w^2)^2\,]+c_1\sum_{j=1}^{k}\int_{-\ell}^{\ell}|f_j'|^2-c_2\sum_{i\neq j}\int_{-\ell}^{\ell}e^{-\sqrt{2}|f_i-f_j|}$$

for certain explicit constants c_1 and c_2. Assuming that the quantities $e^{-\sqrt{2}|f_i-f_j|}$ are negligible for $|i-j|\geq 2$, we obtain for the approximate equilibrium condition of the functions f_j, precisely the system (5.1).

5.2. Remarks. The solutions (5.2) show a major difference between the minimal surface problem and the Allen-Cahn equation, as it is the fact that two separate interfaces *interact*, leading to a major deformation in their asymptotic shapes. We believe that these examples should be prototypical of bounded finite Morse index solutions of (2.1). A finite Morse index solution u is stable outside a bounded set. If we follow a component of its nodal set along a unbounded sequence, translation and a standard compactness argument leads in the limit to a stable solution. Hence from the result in [22] its profile must be one-dimensional and hence its nodal set is a straight line. This makes it plausible that the ends of the nodal set of u *are asymptotically a finite, even number of straight lines*. If this is the case, those lines are not disposed in arbitrary way: Gui [46] proved that if $e_1, \dots e_{2k}$ are unit vectors in the direction of the ends of the nodal set of a solution of (2.1) in $\mathbb{R}^2$, then the balancing formula $\sum_{j=1}^{2k} e_j = 0$ holds.

As we have mentioned, another finite Morse solution is known, [23], the so-called saddle solution. It is built by positive barriers with zero boundary data on a quadrant, and then extended by odd reflections to the rest of the plane, so that its nodal set is an infinite cross, hence having 4 straight ends. The saddle solution has Morse index 1, see [77]. This is also formally the case for the solutions (5.3), which also has 4 ends.

An interesting question is whether the parameter ε of the solutions (5.3) can be continued to increase the nearly zero angle between ends up to $\frac{\pi}{2}$, the case of the saddle solution. Similarly, a saddle solutions with $2k$ ends with consecutive angles $\frac{\pi}{k}$ has been built in [2]. One may similarly ask whether this solution is in some way connected to the $2k$-end family (5.2).

6. The Stationary NLS and the Yamabe Equations

6.1. The standing wave problem for NLS. We shall discuss some results on the problem

$$\Delta u + |u|^{p-1}u - u = 0 \quad \text{in } \mathbb{R}^N \tag{6.1}$$

where $p > 1$. Equation (6.1) arises for instance as the standing-wave problem for the standard nonlinear Schrödinger equation

$$i\psi_t = \Delta\psi + |\psi|^{p-1}\psi, \tag{6.2}$$

corresponding to that of solutions of the form $\psi(y,t) = u(y)e^{-it}$. It also arises in nonlinear models in Turing's theory biological theory of pattern formation, such as the Gray-Scott or Gierer-Meinhardt systems, [44, 43]. The positive solutions of (6.1) which decay to zero at infinity are well understood. Problem

(6.1) has a radially symmetric solution $w_N(y)$ which approaches 0 at infinity provided that

$$1 < p < \begin{cases} \frac{N+2}{N-2} & \text{if } \ N \geq 3, \\ +\infty & \text{if } \ N = 1, 2, \end{cases}$$

see [81, 7]. This solution is unique [56], and actually any positive solution to (6.1) which vanishes at infinity must be radially symmetric around some point [42].

Variations of Problem (6.1), where the space homogeneity is broken by the action of an external potential or boundary conditions in a domain, have been broadly treated in the PDE literature in the last two decades, especially concerning the construction of *positive solutions*. Widely studied has been for instance a singular perturbation problem of the form

$$\varepsilon^2 \Delta - V(x)u + |u|^{p-1}u = 0 \tag{6.3}$$

where ε is a small parameter, or in a bounded domain with $V \equiv 1$, under Dirichlet or Neumann boundary conditions. Many constructions in the literature refer to "multi-bump solutions", built from a perturbation of the superposition of suitably scaled copies of the basic radial bump w_N. The location of their maxima is determined typically by a criterion related either with the potential or the geometry of the underlying domain. Among other contributions, we refer the reader to the works [4, 6, 26, 27, 57, 39, 45, 52, 69, 70, 71, 24, 75, 83] and their references. Solutions concentrating on a higher dimensional sets have been considered for instance in [60, 61, 28, 59].

It is natural to ask about positive solutions to (6.1) which do not vanish at infinity.

For instance, let us consider the solution $w := w_1$ of (6.1) in $\mathbb{R}$,

$$\begin{aligned} w'' - w + w^p = 0, \quad w > 0, \quad \text{ in } \mathbb{R}, \\ w'(0) = 0, \quad w(\pm\infty) = 0. \end{aligned} \tag{6.4}$$

Then the functions $u(x, z) := w(x - a)$, $a \in \mathbb{R}$, define a class of positive solutions on (6.1) in $\mathbb{R}^2$, which vanish in all but one space direction, corresponding to single "bump lines", very much in analogy to the trivial single transition solutions to the Allen-Cahn equation induced by (2.14). In [8], these solutions of (6.1) were found to be isolated in a uniform topology which avoids oscillations at infinity. In constrast, in [21] it is found that a there is continuum of solutions $w_\delta(x, z)$ which are periodic in z and decay exponentially in x, bifurcating from $w(x)$.

A big qualitative difference between the homoclinic solution (6.4) and the heteroclinic solution (2.14) is that the latter is stable, and that avoids these bifurcations. Instead, there is a positive eigenvalue λ_1 to with positive eigenfunction to the linearized equation

$$Z'' + (pw^{p-1} - 1)Z - \lambda_1 Z = 0 \quad \text{in } \mathbb{R}, \quad Z(\pm\infty) = 0,$$

and the bifurcating *Dancer solutions* can be expanded as .

$$w^{\delta}(x,z) = w(x) + \delta Z(x)\cos(\sqrt{\lambda_1}z) + O(\delta^2 e^{-|x|}). \tag{6.5}$$

Intuitively, as δ increases, the period becomes long and the oscillating amplitude largely varies: in fact a simple variational argument using symmetries gives also the existence of a solution $w^T(x,z)$ with a large period $T \gg 1$ whose profile is an "infinite bump array" solution like

$$w^T(x,z) \approx \sum_{k=-\infty}^{\infty} w_2(x, z - k\,T), \tag{6.6}$$

where w_2 is the radial positive solution that decays to zero of (6.1). The solutions to (6.6)

Independently in [33] and [62], positive solutions that glue together respectively bump-lines and infinite bump arrays have been built.

The result in [33] is the exact analog of Theorem 4, now with a Toda system of the form

$$c_p f_j'' = e^{-(f_j - f_{j-1})} - e^{-(f_{j+1} - f_j)}, \quad j = 1, \dots k, \tag{6.7}$$

$f_0 \equiv -\infty,\ f_{k+1} \equiv +\infty$, where c_p is a explicit positive constant.

Theorem 5 ([33]). *Given a solution f of* (6.7) *if we scale*

$$f_{\varepsilon,j}(z) \;:=\; \sqrt{2}\,(j - \frac{k+1}{2})\log\frac{1}{\varepsilon} + f_j(\varepsilon z),$$

then for all small ε there is a positive solution u_ε of (6.1) with k bump lines:

$$u_\varepsilon(x,z) = \sum_{j=1}^{k} w(x - f_{\varepsilon,j}(z)\,) \,+\, O(\varepsilon). \tag{6.8}$$

The profile of the solution (6.8) can actually be more accurately described as a superposition of bifurcating Dancer solutions (6.5) w^{δ_j}, with respective axes given at main order by the straight line asymptote of to the graphs of the f_j's, and with $\delta_j(\varepsilon) \to 0$, plus a remainder that decays away and along these lines.

In [62] a solution was built close to a given finite number of halves of infinite bump arrays (6.6), with sufficiently large T, emanating from the origin, and along three divergent rays with sufficiently large mutual angles. The solutions in [33] and those in [62] may belong to endpoints of families with opposite size in their Dancer parameters, in a way perhaps similar as the solutions in Theorem 4 are expected to connect to the symmetric saddle solutions, but this is still far from understood. Obtaining (even partial) classification of the positive solutions of (6.1) is presumably much harder than in the Allen-Cahn equation.

In particular, Morse index of the solutions built turn out to be infinite due to the oscillations along their ends.

Another interesting issue is that of understanding *sign changing solutions.* Even those with finite energy (and finite morse index) can exhibit very complex patterns. From Ljusternik-Schirelmann theory applied to the energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}$$

it is known that (6.1) possesses and infinite number of radially symmetric solutions. Nonradial solutions have been built in [66], as a small perturbation of a configuration of half arrays (6.6) symmetrically disposed, cut-off far away outside a disk of very large radius, and so that the sides of the regular polygon thus formed is filled with alternating sign, nearly equidistant bumps w_2.

6.2. The Yamabe equation in $\mathbb{R}^N$. Let us consider the equation at the critical exponent

$$\Delta u + |u|^{\frac{4}{N-2}} u = 0 \quad \text{in } \mathbb{R}^N \tag{6.9}$$

$N \geq 3$. It is known that a positive solution to this problem must be equal to one of the Aubin-Talenti extremals for Sobolev's embedding,

$$w_{\mu,\xi}(x) = \alpha_N \left(\frac{\mu}{\mu^2 + |x-\xi|^2} \right)^{\frac{N-2}{2}}, \quad \alpha_N = (N(N-2))^{\frac{N-2}{4}}. \tag{6.10}$$

See [72, 5, 82, 18].

The energy associated to Problem (6.9) is given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{N-2}{2N}\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}.$$

We consider the common value of the energy of the solitons (6.10),

$$S_N := J(w_{\mu,\xi}).$$

Concerning sign changing solutions the whole picture is still far from understood. To our knowledge, only one result is available. Ding [40] proved the existence of infinitely many solutions within a class of solutions which, when, after the equation is lifted to the sphere S^N, it is radially symmetric in two variables. The class of such functions turns out to regain the loss of compactness in Sobolev's embedding, and then Ljusternik-Schnirelmann arguments apply. No further information on the solutions is available. Understanding solutions to (6.9) and its energy levels is an major issue in the analysis of blow-up and well-prosedness for the NLS (6.2) at the critical exponent, in a program initiated in [53].

We have the following result, a special case of that in [35], which describes in precise terms a class of finite energy solutions of (6.9) which do not have the radial symmetries in [40].

Let us consider the points

$$\xi_j := (e^{2\pi i j/k}, 0) \in \mathbb{C} \times \mathbb{R}^{N-2} = \mathbb{R}^N, \quad j = 1, \dots, k$$

where

Theorem 6 ([35]). *for any k sufficiently large, there exists a solution u_k of (6.9) with the form*

$$u_k(x) = w_{1,0}(x) - \sum_{j=1}^{k} w_{\mu_j,\xi_j}(x) + o(1)$$

where for a certain number $\mu_N > 0$,

$$\mu_j = \frac{\nu_N}{k^2}$$

and $o(1) \to 0$ uniformly in $\mathbb{R}^N$ as $k \to +\infty$. Besides we have

$$J(u_k) = (k+1)S_N + O(1)$$

as $k \to \infty$.

A characteristic of this problem is the fact that eventually the concentration set becomes higher dimensional, namely a copy of S^1 in $\mathbb{R}^N$, in spite of being the context just discrete. The hidden parameter here it is of course the number of bubbles. This concentration phenomena can be regarded as somehow intermediate between point and continuum concentration. The idea of using the number of concentrating cells as a singular perturbation parameter appears already in the context of critical problems in [84].

The result o Theorem 6 extends considerably to similar patterns where the limiting concentration set, is, after stereographic projection, a submanifold of the sphere S^N with suitable rotation invariances.

Again, when the Yamabe equation is perturbed by space inhomogeneities or by exponents close to critical, many results on construction and classification of bubbling solutions are present in the literature, but we will not survey them here. The analysis of bubbling solutions has been a central tool for instance in the understanding of the Yamabe and prescribed scalar curvature problems. For changing sign solutions of equation (6.9) in dimension $N = 3$, an analysis of the topology of level sets of the associated energy for low energies is present in [9].

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