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A nonlinear elliptic equation with rapidly oscillating boundary conditions

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1. Introduction

1.1. Motivation

Let $\Omega \subset \mathbf{R}^n,\ n \geqslant 2$, be a bounded, smooth domain, and consider a partition $\{\Gamma_1, \Gamma_2\}$ of the boundary $\partial\Omega$, that is $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, with $\Gamma_1 \neq \emptyset$.

Consider the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2, \end{cases} \tag{1}$$

where ν is the unit outward normal vector to $\partial\Omega,\ \lambda$ is a positive parameter, and $f : [0, \infty) \to [0, \infty)$ is a C^1 nondecreasing, strictly convex function, with $f(0) > 0$ and

$$\int_0^\infty \frac{\mathrm{d}s}{f(s)} < \infty. \tag{2}$$

Typical examples are $f(u) = \mathrm{e}^u$ and $f(u) = (1+u)^p$ where $p > 1$. This type of nonlinear problems arises, for example, from a model of exothermic reaction, and was originally formulated on a disk in $\mathbf{R}^2$ with zero boundary condition. Barenblatt et al. [1] introduced a modification of the original model by considering a mixed boundary condition as in (1).

The case of a zero Dirichlet condition has been well studied, see, for example, Fujita [13], Gelfand [14], Brezis et al. [4], Brezis [2], Martel [16], Brezis and Vázquez [5]. Some of the basic properties described in these works still hold for (1): there is a value $\lambda^* \in (0, \infty)$ such that for $\lambda < \lambda^*$ problem (1) has a solution, and for $\lambda > \lambda^*$ (1) has no solution. For $\lambda = \lambda^*$ there is a unique solution u^* (see Section 3.3 and also Proposition 1.5 below). We call λ^* the extremal parameter associated to $\Gamma_1,\ \Gamma_2$, and u^* the extremal solution. In the original model, λ is a constant depending on physical parameters, and the relevance of λ^* is that a nonexplosive reaction is possible only if $\lambda \leqslant \lambda^*$.

We consider now a family $\{\Gamma_1^\varepsilon, \Gamma_2^\varepsilon\}_{\varepsilon>0}$ of partitions of the boundary, that is, $\Gamma_1^\varepsilon, \Gamma_2^\varepsilon \subset \partial\Omega,\ \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon = \partial\Omega,\ \Gamma_1^\varepsilon \cap \Gamma_2^\varepsilon = \emptyset$, and we assume $|\Gamma_1^\varepsilon| > 0$ for all ε. Here ε is a positive index approaching zero, and we denote by λ_ε^* the corresponding extremal parameter. There are several ways in which we want this family to behave as $\varepsilon \to 0$, but the general idea is that the partition Γ_1^ε, Γ_2^ε "becomes finer" as $\varepsilon \to 0$. For example we can consider the case in which Ω is the unit disk in $\mathbf{R}^2$, $\partial\Omega$ is subdivided in segments of length ε, and we impose homogeneous Dirichlet and Neumann conditions on alternate segments. In this particular case, Barenblatt suggested to study the asymptotic behavior of the extremal parameters λ_ε^* as $\varepsilon \to 0$. A numerical study is presented in [1].

The main goal in this work is to study the asymptotic behavior of the extremal parameters and solutions of (1). More precisely, we show that the limit $\lim_{\varepsilon\to 0} \lambda_\varepsilon^*$ exists (at least for a sequence $\varepsilon_i \to 0$), and we identify it as the extremal parameter of some limit problem. Similarly, we prove that the extremal solutions u_ε^* converge in some sense, to the extremal solution of a limit problem.

1.2. *Definitions and main results*

When dealing with the nonlinear problem (1) it is important to know the asymptotic behavior of solutions of a linear equation with the same boundary condition as in (1), namely

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = h & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \Gamma_1^\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \Gamma_2^\varepsilon, \end{cases} \tag{3}$$

where $h \in L^2(\Omega)$.

It turns out that a convenient class of linear problems to consider, is

$$\begin{cases} -\Delta u + u + \sigma u = h & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $h \in L^2(\Omega)$ and σ belongs to a certain class of Borel measures. The main reference that we use here for the linear problem (4) and questions on the asymptotic behavior of their solutions is Buttazzo et al. [6]. Other references are [10,8,9].

Definition 1.1.

(a) $\mathcal{M}$ denotes the collection of Borel measures on $\mathbf{R}^n$ with values into $[0, \infty]$ that vanish on Borel sets of capacity zero and have support in $\overline{\Omega}$.

(b) For $\sigma \in \mathcal{M}$ we set $H_\sigma = H^1(\Omega) \cap L^2(\overline{\Omega}, \sigma)$ which is a Hilbert space with the inner product

$$\langle u, \varphi \rangle = \int_\Omega \nabla u \nabla \varphi + u\varphi \, \mathrm{d}x + \int_{\overline{\Omega}} \widetilde{u}\widetilde{\varphi} \, \mathrm{d}\sigma,$$

where $\widetilde{u}$ and $\widetilde{\varphi}$ are quasi-continuous representatives of u and φ.

(c) We say that u is an H^1-solution of (4) if $u \in H_\sigma$ and

$$\int_\Omega \nabla u \nabla \varphi + u\varphi \, \mathrm{d}x + \int_{\overline{\Omega}} \widetilde{u}\widetilde{\varphi} \, \mathrm{d}\sigma = \int_\Omega h\varphi \, \mathrm{d}x \quad \text{for all } \varphi \in H_\sigma.$$

Remarks.

1) We note here that the integrals with respect to the measure σ are well defined for $u, \varphi \in H_\sigma$ because σ vanishes on sets of capacity zero, and quasi-continuous representatives of an element in $H^1(\Omega)$ agree up to sets of capacity zero (see [6]). From now on we drop the "$\sim$" in $\tilde{u}$, $\tilde{\varphi}$ and always use quasi-continuous representatives in integrals with respect to a measure $\sigma \in \mathcal{M}$.
2) Problem (4) has a unique solution, which is also the minimizer of

$$\int_\Omega |\nabla u|^2 + u^2 \,\mathrm{d}x + \int_{\overline{\Omega}} u^2 \,\mathrm{d}\sigma - 2\int_\Omega hu \,\mathrm{d}x.$$

A trivial case which can occur is when for all Borel sets B, $\sigma(B) = \infty$ if $B \cap \overline{\Omega}$ has positive capacity, and $\sigma(B) = 0$ otherwise. Then $H_\sigma = \{0\}$, and in this case 0 is the solution of (4) for any h.
3) A mixed boundary condition as in (3) can be obtained by taking

$$\sigma_\varepsilon(B) = \begin{cases} \infty & \text{if } B \cap \Gamma_1^\varepsilon \text{ has positive capacity,} \\ 0 & \text{otherwise} \end{cases}$$

for all Borel sets B.
4) If supp$(\sigma) \subset \partial\Omega$, then (4) can also be rewritten in the form

$$\begin{cases} -\Delta u + u = h & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma u = 0 & \text{on } \partial\Omega. \end{cases}$$

5) Here is an example in which the measures have support inside Ω. Consider a union of disjoint balls $T = \bigcup_i B_i$, and let $\widetilde{\Omega} = \Omega \setminus T$ (this is usually called a perforated domain, and the balls are usually taken in a periodic arrangement). Taking $\sigma(B) = \infty$ if $B \cap (T \cup \partial\Omega)$ has positive capacity, and $\sigma(B) = 0$ otherwise, (4) can be written as

$$\begin{cases} -\Delta u + u = h & \text{in } \widetilde{\Omega}, \\ u = 0 & \text{on } \partial\widetilde{\Omega}. \end{cases}$$

We consider the following notion of convergence for measures in $\mathcal{M}$.

Definition 1.2. If $(\sigma_i) \subset \mathcal{M}$ is a sequence of measures we write $\sigma_i \overset{B}{\rightharpoonup} \sigma_\infty$ where $\sigma_\infty \in \mathcal{M}$ if for all $h \in L^2(\Omega)$, the solutions u_i of

$$\begin{cases} -\Delta u_i + u_i + \sigma_i u_i = h & \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial \nu} + \sigma_i u_i = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

satisfy $u_i \rightharpoonup u_\infty$ in $H^1(\Omega)$ weakly as $i \to \infty$, where u_∞ is the solution of

$$\begin{cases} -\Delta u_\infty + u_\infty + \sigma_\infty u_\infty = h & \text{in } \Omega, \\ \dfrac{\partial u_\infty}{\partial \nu} + \sigma_\infty u_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that we formulate this definition for the operator $-\Delta + I$ instead of $-\Delta$, which would be more natural for the nonlinear problem (1). The advantage of this formulation is that the solution u_i of (5) is bounded in $H^1(\Omega)$ without any assumption on σ_i or h.

As an example, in the case in which Ω is the unit disk in $\mathbf{R}^2$, $\partial\Omega$ is subdivided in segments of length ε and the boundary condition is zero Dirichlet and zero Neumann on alternate segments, the limit boundary condition in the sense of Definition 1.2 is a zero Dirichlet condition. This is shown in Example 1 of Section 2.2. That section contains also some other examples.

The following compactness theorem is a consequence of the results in [6].

Theorem 1.3. *If* $(\sigma_i) \subset \mathcal{M}$ *is a sequence, then there is a subsequence* (σ_{i_j}) *and* $\sigma_\infty \in \mathcal{M}$ *such that* $\sigma_{i_j} \overset{B}{\rightharpoonup} \sigma_\infty$. *Moreover, if* $\operatorname{supp}(\sigma_i) \subset \partial\Omega$ *for all* i*, then* $\operatorname{supp}(\sigma_\infty) \subset \partial\Omega$.

Next we consider the nonlinear problem

$$\begin{cases} -\Delta u + \sigma u = \lambda f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\sigma \in \mathcal{M}$, $\sigma \not\equiv 0$. Recall that $f(u) > 0$ and we are interested in $\lambda > 0$. If $\sigma \equiv 0$ then (6) has no solution for $\lambda > 0$. On the other hand, the hypothesis $\sigma \not\equiv 0$ implies that for any $\varphi \in L^2(\Omega)$ there is a unique solution $\zeta \in H_\sigma$ of

$$\begin{cases} -\Delta \zeta + \sigma \zeta = \varphi & \text{in } \Omega, \\ \dfrac{\partial \zeta}{\partial \nu} + \sigma \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

We use the notation

$$\zeta = T_\sigma(\varphi) \tag{7}$$

and this defines a bounded linear operator $T_\sigma : L^2(\Omega) \to H_\sigma$.

Definition 1.4. Let $\sigma \in \mathcal{M}$ with $\sigma \not\equiv 0$. We say that $u \in L^1(\Omega)$ is a weak solution of (6) if $\int_\Omega f(u)\chi < \infty$ where $\chi = T_\sigma(1)$, and for all $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega u\varphi \, \mathrm{d}x = \lambda \int_\Omega f(u) T_\sigma(\varphi) \, \mathrm{d}x.$$

Remark. In the case of the zero Dirichlet boundary condition, this is the same notion of weak solution introduced by Brezis et al. [4]. In this case, the test functions $\zeta = T_\sigma(\varphi)$ belong to $C^2(\overline{\Omega})$ and vanish on the boundary in the usual sense. But for a general $\sigma \in \mathcal{M}$ it is hard to describe the precise regularity of ζ.

Proposition 1.5. *Assume* $\sigma \in \mathcal{M}$ *is not identically zero and that* $H_\sigma \neq \{0\}$. *Then there exits* $\lambda^* \in (0, \infty)$ *such that for* $0 < \lambda < \lambda^*$ *problem* (6) *has an* H^1*-solution which is bounded, and for* $\lambda > \lambda^*$ (6) *has no solution even in the weak sense of Definition* 1.4. *If furthermore* $\operatorname{supp}(\sigma) \subset \partial\Omega$, *then for* $\lambda = \lambda^*$ (6) *has a unique weak solution* $u^* \in L^1(\Omega)$.

See Section 3 and specially Theorem 3.14 for more properties of (6).

Important notation. In order to state the main results, for a given $\sigma \in \mathcal{M}$ with $\sigma \not\equiv 0$ and $H_\sigma \neq \{0\}$, we let $\lambda^*(\sigma)$ denote the corresponding extremal parameter of (6). If additionally $\mathrm{supp}(\sigma) \subset \partial\Omega$ we let $u^*(\sigma)$ be the extremal solution of (6). Note that if $\sigma \equiv 0$ then (6) has no solution for any $\lambda > 0$, so we use the convention $\lambda^*(\sigma) = 0$. On the other hand, if $H_\sigma = \{0\}$ we use the convention $\lambda^*(\sigma) = \infty$.

Theorem 1.6. *If $(\sigma_i) \subset \mathcal{M}$ is a sequence such that $\sigma_i \overset{B}{\rightharpoonup} \sigma_\infty$ then*

$$\lim_i \lambda^*(\sigma_i) = \lambda^*(\sigma_\infty). \tag{8}$$

In particular we find $\lim_{\varepsilon\to 0} \lambda^*_\varepsilon$ in the example where $\Omega \subset \mathbf{R}^2$, $\partial\Omega$ is subdivided in segments of length ε, with zero Dirichlet and Neumann conditions on alternate segments. The result states that $\lim_{\varepsilon\to 0} \lambda^*_\varepsilon$ is the extremal parameter for the same nonlinear equation but with zero Dirichlet boundary condition.

On the asymptotic behavior of the extremal solution, we have the following result:

Theorem 1.7. *Let $(\sigma_i) \subset \mathcal{M}$ be sequence such that* $\mathrm{supp}(\sigma_i) \subset \partial\Omega$ *for all i and $\sigma_i \overset{B}{\rightharpoonup} \sigma_\infty$, where $\sigma_\infty \not\equiv 0$. Then*

$$u^*(\sigma_i) \to u^*(\sigma_\infty), \quad \text{as } i \to \infty, \tag{9}$$

in $L^p(\Omega)$ for $1 \leqslant p < n/(n-1)$. Moreover, if $u^*(\sigma_\infty)$ is unbounded then

$$\|u_i^*\|_\infty \to \infty$$

and if $u^*(\sigma_\infty) \in L^\infty(\Omega)$ then

$$\limsup \|u^*(\sigma_i)\|_\infty < \infty.$$

In the latter case the convergence $u^*(\sigma_i) \to u^*(\sigma_\infty)$ takes place also in $C^k_{\mathrm{loc}}(\Omega)$ for any $k \geqslant 0$.

This work is organized as follows. In Section 2 we give a proof of Theorem 1.3 and some examples of the convergence $\sigma_i \overset{B}{\rightharpoonup} \sigma_\infty$. In Section 3 we collect some preliminary results that are needed later. Then in Section 4 we prove Theorem 1.6 and in Section 5 we prove Theorem 1.7.

2. Asymptotics for a linear problem

2.1. A compactness result

In this section we give a proof of Theorem 1.3, using the results of [6].

Proof of Theorem 1.3. Fix (ε_i) a sequence of positive numbers such that $\varepsilon_i \to 0$, and let L^{ε_i} be the operator

$$L^{\varepsilon_i} = \begin{cases} \varepsilon_i \Delta & \text{in } \mathbf{R}^n \setminus \Omega, \\ \Delta & \text{in } \Omega. \end{cases}$$

Let $g \in C^\infty(\mathbf{R}^n),\ g > 0$ in $\Omega,\ g = 0$ in $\mathbf{R}^n \setminus \Omega$ and let v_i denote the solution of

$$\begin{cases} -L^{\varepsilon_i} v_i + v_i + \sigma_i v_i = g & \text{in } \mathbf{R}^n, \\ v_i \in H^1(\mathbf{R}^n). \end{cases} \tag{10}$$

The variational formulation of (10) is

$$\int_{\mathbf{R}^n} (\varepsilon_i 1_{\mathbf{R}^n\setminus\Omega} + 1_\Omega)\nabla v_i \nabla\varphi + v_i\varphi \,\mathrm{d}x + \int_{\overline{\Omega}} v_i\varphi \,\mathrm{d}\sigma_i = \int_{\mathbf{R}^n} g\varphi \,\mathrm{d}x \tag{11}$$

for all $\varphi \in H^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n, \sigma_i)$.

By Theorems 4.1 and 5.1 of [6] we have that there is a subsequence σ_{i_j} and a measure $\sigma_\infty \in \mathcal{M}$ supported in $\overline{\Omega}$ such that $v_{i_j} \to v_\infty$ in $L^2(\mathbf{R}^n)$ where $v_\infty = g = 0$ in $\mathbf{R}^n \setminus \overline{\Omega}$ and $v_\infty|_\Omega = v_0$ where v_0 is the solution of

$$\begin{cases} -\Delta v_0 + v_0 + \sigma_\infty v_0 = g & \text{in } \Omega, \\ \dfrac{\partial v_0}{\partial \nu} + \sigma_\infty v_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

We mention here that if $\mathrm{supp}(\sigma_i) \subset \partial\Omega$ for all i, then by [6, Lemma 6.2] we have $\mathrm{supp}(\sigma_\infty) \subset \partial\Omega$.

Let $h \in L^2(\Omega)$, let u_i denote the solution of

$$\begin{cases} -\Delta u_i + u_i + \sigma_i u_i = h & \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial \nu} + \sigma u_i = 0 & \text{on } \partial\Omega \end{cases} \tag{13}$$

and u_∞ denote the solution of

$$\begin{cases} -\Delta u_\infty + u_\infty + \sigma_\infty u_\infty = h & \text{in } \Omega, \\ \dfrac{\partial u_\infty}{\partial \nu} + \sigma_\infty u_\infty = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

Note that (13) implies that u_i is bounded in $H^1(\Omega)$, so that for a further subsequence we can assume that $u_i \rightharpoonup u$ in $H^1(\Omega)$ weakly. From now on we will just use the index i for all subsequences. To conclude, we only need to show that $u = u_\infty$ where u_∞ is the solution of (14). We start with the case $h \in L^\infty(\Omega)$. The general case can then be obtained by a density argument.

Let $\zeta \in C_0^\infty(\mathbf{R}^n)$ and let us use ζv_i as a test function in the variational formulation of (13). Note that v_i is bounded, so that $\zeta v_i \in H^1(\Omega)$ and also note that $\zeta v_i \in L^2(\overline{\Omega}, \sigma_i)$. Thus we obtain

$$\int_\Omega \zeta\nabla u_i \nabla v_i + v_i \nabla u_i \nabla\zeta + u_i v_i \zeta \,\mathrm{d}x + \int_{\overline{\Omega}} u_i v_i \zeta \,\mathrm{d}\sigma_i = \int_\Omega h v_i \zeta \,\mathrm{d}x. \tag{15}$$

Now we need to extend $u_i \in H^1(\Omega)$ to $\mathbf{R}^n$. We denote by $E: H^1(\Omega) \to H^1(\mathbf{R}^n)$ a linear bounded extension operator, with the property that $\|Ew\|_{L^\infty(\mathbf{R}^n)} \leqslant C\|w\|_{L^\infty(\Omega)}$. Set now $\bar{u}_i = Eu_i$. We want to use $\varphi = \bar{u}_i\zeta$ in (11). Remark that since we assume $h \in L^\infty(\Omega)$ we have that $u_i \in L^\infty(\Omega)$ and so

$\bar{u}_i \in L^\infty(\mathbf{R}^n)$. Therefore $\bar{u}_i\zeta \in H^1(\mathbf{R}^n)$ and we also have $\bar{u}_i\zeta \in L^2(\mathbf{R}^n, \sigma_i)$. Hence we obtain

$$\varepsilon_i \int_{\mathbf{R}^n\setminus\Omega} \nabla v_i \nabla(\bar{u}_i\zeta)\,\mathrm{d}x + \int_\Omega \zeta\nabla v_i\nabla\bar{u}_i + \bar{u}_i\nabla v_i\nabla\zeta\,\mathrm{d}x + \int_{\mathbf{R}^n} v_i\bar{u}_i\zeta\,\mathrm{d}x + \int_{\overline{\Omega}} v_i\bar{u}_i\zeta\,\mathrm{d}\sigma_i$$
$$= \int_\Omega g u_i\zeta\,\mathrm{d}x. \tag{16}$$

We now subtract (15) from (16):

$$\varepsilon_i \int_{\mathbf{R}^n\setminus\Omega} \nabla v_i \nabla(\bar{u}_i\zeta)\,\mathrm{d}x + \int_\Omega (u_i\nabla v_i - v_i\nabla u_i)\nabla\zeta\,\mathrm{d}x + \int_{\mathbf{R}^n} v_i\bar{u}_i\zeta\,\mathrm{d}x - \int_\Omega v_i u_i\zeta\,\mathrm{d}x$$
$$= \int_\Omega (g\bar{u}_i - hv_i)\zeta\,\mathrm{d}x. \tag{17}$$

We want now to pass to the limit as $i \to \infty$. For this observe that from (11) (with $\varphi = v_i$) we find

$$\int_{\mathbf{R}^n} (\varepsilon_i 1_{\mathbf{R}^n\setminus\Omega} + 1_\Omega)|\nabla v_i|^2 + v_i^2\,\mathrm{d}x + \int_{\overline{\Omega}} v_i^2\,\mathrm{d}\sigma_i = \int_\Omega g v_i\,\mathrm{d}x. \tag{18}$$

This shows that $v_i|_\Omega$ is bounded in $H^1(\Omega)$ and therefore converges weakly in $H^1(\Omega)$ to v_0, which is the solution of (12). But also from (18) we find that

$$\varepsilon_i \int_{\mathbf{R}^n\setminus\Omega} |\nabla v_i|^2\,\mathrm{d}x \leqslant C$$

with C independent of i. We use this to estimate the first term on the left-hand side of (17):

$$\varepsilon_i \int_{\mathbf{R}^n\setminus\Omega} \nabla v_i\nabla(\bar{u}_i\zeta)\,\mathrm{d}x \leqslant \varepsilon_i^{1/2}\left(\varepsilon_i\int_{\mathbf{R}^n\setminus\Omega} |\nabla v_i|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbf{R}^n\setminus\Omega} \left|\nabla(\bar{u}_i\zeta)\right|^2\mathrm{d}x\right)^{1/2} \to 0$$

as $i \to \infty$. So, taking the limit as $i \to \infty$ in (17) we arrive at

$$\int_\Omega (u\nabla v_0 - v_0\nabla u)\nabla\zeta\,\mathrm{d}x = \int_\Omega (gu - hv_0)\zeta\,\mathrm{d}x. \tag{19}$$

We note that (19) is also satisfied if we replace u by u_∞. This can be seen by using $v_0\zeta$ in the variational formulation of (14), then taking $\varphi = u_\infty\zeta$ in the variational formulation of (12) and subtracting. Hence, if we set $\tilde{u} = u - u_\infty$, we obtain

$$\int_\Omega (\tilde{u}\nabla v_0 - v_0\nabla\tilde{u})\nabla\zeta\,\mathrm{d}x = \int_\Omega g\tilde{u}\zeta\,\mathrm{d}x \tag{20}$$

for all $\zeta \in C_0^\infty(\mathbf{R}^n)$ and hence for all $\zeta \in C^\infty(\overline{\Omega})$. Remark that u_i is bounded in $L^\infty(\Omega)$ and therefore $\tilde{u} \in L^\infty(\Omega)$. Also $v_0 \in L^\infty(\Omega)$, so (20) is valid for all $\zeta \in H^1(\Omega)$. We take $\zeta = \tilde{u}$ in (20) and obtain

$$\int_\Omega \frac{1}{2}\nabla v_0\nabla(\tilde{u})^2 - v_0|\nabla\tilde{u}|^2\,\mathrm{d}x = \int_\Omega g\tilde{u}^2\,\mathrm{d}x. \tag{21}$$

But taking $\varphi = \tilde{u}^2$ in the variational formulation of (12) we find

$$\int_\Omega \nabla v_0 \nabla(\tilde{u})^2 + v_0 \tilde{u}^2 \, \mathrm{d}x + \int_{\overline{\Omega}} v_0 \tilde{u}^2 \, \mathrm{d}\sigma_\infty = \int_\Omega g \tilde{u}^2 \, \mathrm{d}x. \tag{22}$$

Combining (21) and (22) we obtain

$$\int_\Omega g\tilde{u}^2 + 2v_0 |\nabla \tilde{u}|^2 + v_0 \tilde{u}^2 \, \mathrm{d}x + \int_{\overline{\Omega}} v_0 \tilde{u}^2 \, \mathrm{d}\sigma_\infty = 0.$$

Since $g > 0$ in Ω we conclude that $\tilde{u} = 0$, and therefore $u = u_\infty$. □

2.2. Some examples

There are many examples in the literature.

Example 1. This example includes the one mentioned in the introduction, in which Ω is the unit disk in $\mathbf{R}^2$, $\partial\Omega$ is divided in segments of length ε and a zero Dirichlet and Neumann condition is applied on alternate segments.

More generally, suppose that $\Gamma_1^\varepsilon,\ \Gamma_2^\varepsilon$ is a family of partitions of $\partial\Omega$ that satisfies the following conditions:

$$\lim_{\varepsilon \to 0} \sup_{x \in \partial\Omega} \operatorname{dist}(x, \overline{\Gamma_1^\varepsilon}) = 0 \tag{23}$$

(with this we want to capture the notion that the partition becomes finer as $\varepsilon \to 0$), and

$$\begin{cases} \text{there exist } \rho_0 > 0,\ \nu_0 > 0 \text{ such that for all } y \in \overline{\Gamma_1^\varepsilon} \text{ and all } 0 < \rho < \rho_0 \text{ we have} \\ \dfrac{|B_\rho(y) \cap \Gamma_1^\varepsilon|}{|B_\rho(y) \cap \partial\Omega|} \geqslant \nu_0 \end{cases} \tag{24}$$

(this condition says, roughly speaking, that the local proportion of Γ_1^ε stays away from zero around points of $\overline{\Gamma_1^\varepsilon}$). Set

$$\sigma_\varepsilon(B) = \begin{cases} \infty & \text{if } B \cap \Gamma_1^\varepsilon \text{ has positive capacity,} \\ 0 & \text{otherwise.} \end{cases}$$

Claim. Then

$$\sigma_\varepsilon \xrightarrow{B} \sigma_D, \tag{25}$$

where $\sigma_D(B) = \infty$ if $B \cap \partial\Omega$ has positive capacity, and 0 otherwise, that is σ_D is the measure that gives a zero Dirichlet boundary condition. The point of this example is that there are no regularity requirements on the partitions $\Gamma_1^\varepsilon,\ \Gamma_2^\varepsilon$.

Proof of (25). Fix some $h \in L^\infty(\Omega)$ and let u_ε be the solution of

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = h & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \Gamma_1^\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \Gamma_2^\varepsilon. \end{cases} \tag{26}$$

Since the partitions Γ_1^ε, Γ_2^ε satisfy (24) with constants independent of ε, by Theorem 3.4 u_ε is bounded in $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$. Hence, by taking a subsequence we can assume that $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$. But then, by (23) $u|_{\partial\Omega} = 0$. Now let $\zeta \in C^2(\overline{\Omega})$ with $\zeta|_{\partial\Omega} = 0$. By (26) we have

$$\int_\Omega u_\varepsilon(-\Delta\zeta + \zeta)\,\mathrm{d}x + \int_{\partial\Omega} u_\varepsilon \frac{\partial \zeta}{\partial \nu}\,\mathrm{d}s = \int_\Omega h\zeta\,\mathrm{d}x$$

and taking the limit as $\varepsilon \to 0$ we find that u is the solution of

$$\begin{cases} -\Delta u + u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Example 2. There are some examples by Cioranescu and Murat [7], where the measures in question have support inside Ω. We refer to their article for the detailed description of the results.

Example 3. This example is a consequence of the results of Damlamian for the Neumann sieve [11]. We mention it in connection with Example 1, to show what happens if the local proportion Γ_1^ε (the part of $\partial\Omega$ where we set $u_\varepsilon = 0$) goes to zero at a certain speed.

More concretely, suppose that a portion Σ of the boundary $\partial\Omega$ is contained in the hyperplane $\{x_n = 0\}$ (we use the standard notation $x = (x', x_n) \in \mathbf{R}^n$ with $x' \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$), and that $\Omega \subset \mathbf{R}^n_+ = \{x_n > 0\}$.

Let $\{\Gamma_1^\varepsilon, \Gamma_2^\varepsilon\}$ denote a family of partitions of $\partial\Omega$ such that:

1) $\Gamma_1^\varepsilon \cap \Sigma$ is a periodic arrangement with period εY, $Y = (0,1)^{n-1}$, of sets $\mathcal{O}_\varepsilon^i$. Each $\mathcal{O}_\varepsilon^i$ is assumed to be, up to a translation, equal to $b_\varepsilon\mathcal{O}$, where $\mathcal{O} \subset \mathbf{R}^{n-1}$ is the reference set, and $b_\varepsilon > 0$ is the "size" of $\mathcal{O}_\varepsilon^i$, to be defined later as a function of ε.
2) $\partial\Omega \setminus \Sigma \subset \Gamma_1^\varepsilon$.

Let $h \in L^2(\Omega)$ and let u_ε be the solution of

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = h & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \Gamma_1^\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \Gamma_2^\varepsilon. \end{cases}$$

Claim. Assume that $\mathcal{O}$ (the reference set) is a bounded, open, smooth subset of $\mathbf{R}^{n-1}$, $n \geqslant 3$, and $b_\varepsilon = \varepsilon^{(n-1)/(n-2)}$. Then

$$u_\varepsilon \rightharpoonup u \quad \text{in } H^1(\Omega) \text{ weakly}, \tag{27}$$

where u is the solution of

$$\begin{cases} -\Delta u + u = h & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \dfrac{c}{2} u = 0 & \text{on } \Sigma, \\ u = 0 & \text{on } \partial\Omega \setminus \Sigma \end{cases}$$

and $c > 0$ is the capacity in $\mathbf{R}^n$ of $\mathcal{O} \times \{0\}$. We highlight here the boundary condition on Σ:

$$\frac{\partial u}{\partial \nu} + \frac{c}{2} u = 0 \quad \text{on } \Sigma.$$

This can be rephrased in terms of measures as well.

From the work in [11] one can also see that if $b_\varepsilon = \mathrm{o}(\varepsilon^{(n-1)/(n-2)})$ in the case $n \geqslant 3$, then the limit boundary condition on Σ is a zero Neumann condition.

Sketch of the proof of (27). Define

$$\tilde{u}_\varepsilon(x', x_n) = \begin{cases} u_\varepsilon(x', x_n) & \text{if } x_n > 0, \\ -u_\varepsilon(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

By [11, Theorem 1] $\tilde{u}_\varepsilon \rightharpoonup \tilde{u}$ in V, where V is the Hilbert space $H^1(\Omega) \times H^1(\Omega^-)$, Ω^- is the reflection of Ω across $\{x_n = 0\}$, and $\tilde{u}$ solves

$$\begin{cases} -\Delta \tilde{u} + \tilde{u} = \tilde{h} & \text{in } \Omega \cup \Omega^-, \\ \tilde{u} = 0 & \text{on } \partial\Omega \cup (\partial\Omega^-) \setminus \Sigma, \\ \dfrac{\partial \tilde{u}}{\partial \nu^-} = \dfrac{\partial \tilde{u}}{\partial \nu^+} = -\dfrac{c}{4}[\tilde{u}] & \text{on } \Sigma. \end{cases} \tag{28}$$

Here $\partial/\partial\nu^-$ and $\partial/\partial\nu^+$ are the normal derivatives of $\tilde{u}$ coming from Ω^- and Ω, respectively (recall that ν points to the outside of Ω, so $\partial/\partial\nu = -\partial/\partial x_n$), and $[\tilde{u}] = \tilde{u}^+ - \tilde{u}^-$; $\tilde{u}^+$, $\tilde{u}^-$ being the values of $\tilde{u}$ on Σ when coming from Ω and Ω^-, respectively.

But $\tilde{u}$ is odd across Σ, so the jump condition in (28) may be written as

$$\frac{\partial \tilde{u}}{\partial \nu} + \frac{c}{2}\tilde{u}^+ = 0 \quad \text{on } \Sigma.$$

3. Preliminaries

In this section we collect a number of preliminary results that are needed later. We denote by σ a fixed element in $\mathcal{M}$ with $\sigma \not\equiv 0$.

Recall that we defined $H_\sigma = H^1(\Omega) \cap L^2(\overline{\Omega}, \sigma)$ which is a Hilbert space with the inner product

$$\langle u, v\rangle_\sigma = \int_\Omega \nabla u \nabla v + uv \,\mathrm{d}x + \int_{\overline{\Omega}} uv \,\mathrm{d}\sigma.$$

The assumption $\sigma \not\equiv 0$ implies that there is a constant $C > 0$ (depending on σ and Ω) such that for all $\varphi \in H_\sigma$

$$\int_\Omega \varphi^2 \,\mathrm{d}x \leqslant C\left(\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x + \int_{\overline{\Omega}} \varphi^2 \,\mathrm{d}\sigma\right)$$

or equivalently, that the first eigenvalue of $-\Delta + \sigma|_\Omega$, with the generalized Robin boundary condition $\partial\varphi/\partial\nu + \sigma\varphi = 0$ on $\partial\Omega$, is positive:

$$\lambda_1(\sigma) = \inf_{\varphi\in H_\sigma} \frac{\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x + \int_{\overline{\Omega}} \varphi^2 \,\mathrm{d}\sigma}{\int_\Omega \varphi^2 \,\mathrm{d}x} > 0. \tag{29}$$

Note that it can happen that $H_\sigma = \{0\}$. In this case we adopt the convention $\lambda_1(\sigma) = \infty$.

If $\sigma \in \mathcal{M}$ and $\lambda_1(\sigma) < \infty$, then the infimum in (29) is attained at some nonnegative, nonzero function $\varphi_1 \in H_\sigma$ which we call the first eigenfunction associated to σ. It satisfies the equation

$$\begin{cases} -\Delta\varphi_1 + \sigma\varphi_1 = \lambda_1(\sigma)\varphi_1 & \text{in } \Omega, \\ \dfrac{\partial\varphi_1}{\partial\nu} + \sigma\varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

We remark here that in many elliptic estimates in this and later sections, we will say that the constants depend on σ only through $\lambda_1(\sigma)$, meaning that these constants remain bounded as long as $\lambda_1(\sigma)$ is bounded away from zero.

3.1. Some elliptic estimates

The first result we mention here is an L^∞ bound. Its proof is standard, and follows that of Lemma 7.3 of Hartman and Stampacchia [15].

Proposition 3.1. *Let $p > n/2$. Then there exists a constant $C > 0$ depending only on Ω, n, p and $\lambda_1(\sigma)$ such that for any solution u of*

$$\begin{cases} -\Delta u + \sigma u = h & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu} + \sigma u = 0 & \text{on } \partial\Omega \end{cases}$$

with $h \in L^p(\Omega)$ we have

$$\|u\|_\infty \leqslant C\|h\|_p.$$

The next result is also important (see [12]).

Lemma 3.2. *Assume that $\sigma \in \mathcal{M}$ has support on $\partial\Omega$. Let χ be the H^1-solution of*

$$\begin{cases} -\Delta\chi = 1 & \text{in } \Omega, \\ \dfrac{\partial\chi}{\partial\nu} + \sigma\chi = 0 & \text{on } \partial\Omega. \end{cases}$$

Suppose that ζ *is the* H^1*-solution of*

$$\begin{cases} -\Delta\zeta = \varphi & \text{in } \Omega, \\ \dfrac{\partial\zeta}{\partial\nu} + \sigma\zeta = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\varphi \in L^p(\Omega),\ p > n$. *Then there exists* C *such that*

$$\left\| \frac{\zeta}{\chi} \right\|_\infty \leqslant C\|\varphi\|_p. \tag{30}$$

The constant C *depends on* $\Omega,\ n,\ p$ *and* $\lambda_1(\sigma)$.

Remark 3.3. We mention that the assumption $\operatorname{supp}(\sigma) \subset \partial\Omega$ is not absolutely necessary. It is enough that the support of σ is contained in $\partial\Omega \cup K$ where K is a compact smooth $n-1$ dimensional manifold contained in Ω.

Another observation is that in [12] the result is stated for a mixed boundary condition, but the proof given there works also for a measure $\sigma \in \mathcal{M}$ with $\operatorname{supp}(\sigma) \subset \partial\Omega$.

Under some extra assumptions on σ it is possible to establish the Hölder continuity of the solutions (this is an adaptation of a result of Stampacchia [17]).

Theorem 3.4. *Suppose* u *is a solution of*

$$\begin{cases} -\Delta u = h & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial\nu} + \sigma u = g & \text{on } \Gamma_2, \end{cases}$$

where $\Gamma_1,\ \Gamma_2$ *is a partition of* $\partial\Omega,\ h \in L^p(\Omega),\ p > n/2$, *and* $\sigma, g \in L^q(\Gamma_2),\ q > n-1$. *We assume the following "regularity" condition*:

there exists $\rho_0 > 0,\ \nu_0 > 0$ *such that for all* $y \in \overline{\Gamma_1}$ *and all* $0 < \rho < \rho_0$ *we have*

$$\frac{|B_\rho(y) \cap \Gamma_1|}{|B_\rho(y) \cap \partial\Omega|} \geqslant \nu_0. \tag{31}$$

Then there exists $\alpha \in (0,1)$ *and* $C > 0$ *such that*

$$\|u\|_{C^\alpha(\overline{\Omega})} \leqslant C\big(\|u\|_\infty + \|h\|_p + \|g\|_{q,\Gamma_2}\big).$$

The constants $\alpha,\ C$ *depend only on* $\Omega,\ n,\ p,\ q,\ \|\sigma\|_{q,\Gamma_2},\ \rho_0$ *and* ν_0.

3.2. Weak solutions of the linear problem

Throughout this section $\sigma \in \mathcal{M}$ is not identically zero. We first introduce an analog for the function $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ used in [4] for the Dirichlet boundary condition, and a definition of weak solution of

$$\begin{cases} -\Delta u + \sigma u = h & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu} + \sigma u = 0 & \text{on } \partial\Omega. \end{cases} \tag{32}$$

Definition 3.5.

(a) Let $\chi = T_\sigma(1)$ (T_σ was defined in (7)).
(b) We introduce $L^1_\chi = L^1(\Omega, \chi\,\mathrm{d}x)$ and $\|h\|_{L^1_\chi} = \int_\Omega |h|\chi$.
(c) Let $h \in L^1_\chi$. We say that $u \in L^1(\Omega)$ is a weak solution of (32) if

$$\int_\Omega u\varphi\,\mathrm{d}x = \int_\Omega hT_\sigma(\varphi)\,\mathrm{d}x \tag{33}$$

for any $\varphi \in C_0^\infty(\Omega)$.

Remarks.

1) The functions $\zeta = T_\sigma(\varphi) \in H_\sigma$ as in the previous definition play the role of the test functions $\zeta \in C^2(\overline{\Omega})$ with $\zeta|_{\partial\Omega} = 0$ in the case of a Dirichlet boundary condition (see [4]).
2) Observe also that any H^1-solution is a weak solution.
3) Note that $\int_\Omega |hT_\sigma(\varphi)|\,\mathrm{d}x < \infty$ for $h \in L^1_\chi$ and $\varphi \in C_0^\infty(\Omega)$.

Lemma 3.6. *Given $h \in L^1_\chi$ there exists a unique weak solution $u \in L^1(\Omega)$ of* (32), *and*

$$\|u\|_{L^1} \leqslant \|h\|_{L^1_\chi}. \tag{34}$$

Moreover, if $h \geqslant 0$ then $u \geqslant 0$.

The proof is like the one of Lemma 1 in [4], where instead of $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ we use χ.
If $\mathrm{supp}(\sigma) \subset \partial\Omega$, then the estimate (34) can be improved using Lemma 3.2.

Lemma 3.7. *Assume* $\mathrm{supp}(\sigma) \subset \partial\Omega$. *Then given $1 \leqslant p < n/(n-1)$ there is a constant $C > 0$ depending only Ω, n, p and $\lambda_1(\sigma)$ such that if u is the weak solution of* (32) *then*

$$\|u\|_p \leqslant C\|h\|_{L^1_\chi}.$$

Proof. We use a duality argument. Let p' denote the conjugate exponent of p (that is $1/p + 1/p' = 1$) and let $\varphi \in C_0^\infty(\Omega)$ and $\zeta = T_\sigma(\varphi)$. Then from (33) we find

$$\int_\Omega u\varphi\,\mathrm{d}x = \int_\Omega h\zeta\,\mathrm{d}x \leqslant \|h\|_{L^1_\chi}\left\|\frac{\zeta}{\chi}\right\|_\infty \leqslant C\|h\|_{L^1_\chi}\|\varphi\|_{p'},$$

where the last inequality is a consequence of (30) (note that since $1 \leqslant p < n/(n-1)$ we have $p' > n$). □

Remark. Again, we can relax the assumption on the support of σ as in Remark 3.3.

Definition 3.8. Let $h \in L^1_\chi$. We say that $u \in L^1(\Omega)$ is a weak supersolution of (32), which we denote by

$$\begin{cases} -\Delta u + \sigma u \geqslant h & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma u \geqslant 0 & \text{on } \partial\Omega \end{cases}$$

if for any $\varphi \in C_0^\infty(\Omega)$ such that $T_\sigma(\varphi) \geqslant 0$ we have

$$\int_\Omega u\varphi \, \mathrm{d}x \geqslant \int_\Omega hT_\sigma(\varphi) \, \mathrm{d}x.$$

The two following versions of the strong maximum principle for $-\Delta$ with Robin boundary condition are consequences of Lemma 3.2 (see [12]).

Theorem 3.9. *Assume* $\operatorname{supp}(\sigma) \subset \partial\Omega$. *Then there exists* $c > 0$ *depending only on* Ω *and* $\lambda_1(\sigma)$ *such that if* $h \in L^1_\chi$ *and* u *is a solution of* (32) *then*

$$u(x) \geqslant c\left(\int_\Omega h\chi\right)\chi(x) \quad \text{a.e. in } \Omega.$$

Lemma 3.10. *Assume* $\operatorname{supp}(\sigma) \subset \partial\Omega$ *and suppose that* u *is a supersolution of* (32) *with* $h = 0$. *Then either* $u \equiv 0$ *or there exists* $c > 0$ *such that*

$$u \geqslant c\chi \quad \text{a.e. in } \Omega.$$

Finally, an important tool is the following result (see the case of zero Dirichlet condition in [4,3]).

Lemma 3.11 (Kato's inequality). Let $h \in L^1_\sigma$ and $u \in L^1(\Omega)$ a weak solution of (32). Let $\Phi : \mathbf{R} \to \mathbf{R}$ be a C^2 concave function with $\Phi' \in L^\infty$ and $\Phi(0) = 0$. Then

$$\begin{cases} -\Delta\Phi(u) + \sigma\Phi(u) \geqslant \Phi'(u)h & \text{in } \Omega, \\ \dfrac{\partial\Phi(u)}{\partial\nu} + \sigma\Phi(u) \geqslant 0 & \text{on } \partial\Omega. \end{cases}$$

For completeness we give a proof in the appendix.

3.3. The nonlinear problem

In this section we consider the nonlinear problem

$$\begin{cases} -\Delta u + \sigma u = \lambda f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu} + \sigma u = 0 & \text{on } \partial\Omega. \end{cases} \tag{35}$$

Definition 3.12. We say that $u \in L^1(\Omega)$ is a weak solution of (35) if $f(u) \in L^1_\chi$ and

$$\int_\Omega u\varphi \, \mathrm{d}x = \lambda \int_\Omega f(u) T_\sigma(\varphi) \, \mathrm{d}x$$

for any $\varphi \in C_0^\infty(\Omega)$.

We also say that $\overline{U} \in L^1(\Omega)$ is a weak supersolution of (35) if $f(\overline{U}) \in L^1_\chi$ and

$$\int_\Omega \overline{U}\varphi \, \mathrm{d}x \geqslant \int_\Omega f(\overline{U}) T_\sigma(\varphi) \, \mathrm{d}x$$

for any $\varphi \in C_0^\infty(\Omega)$ such that $T_\sigma(\varphi) \geqslant 0$.

Lemma 3.13. *Suppose that* $\overline{U}$ *is a weak supersolution of* (35). *Then* (35) *has a minimal solution* $0 \leqslant \underline{u} \leqslant \overline{U}$.

The proof is analog to the case of zero Dirichlet boundary condition. See [4] for example.

The following theorem summarizes some of the properties of (35).

Theorem 3.14. *Let* $\sigma \in \mathcal{M}$ *with* $\sigma \not\equiv 0$ *and suppose that* $H_\sigma \neq \{0\}$. *Then*:

(i) *There exists* $0 < \lambda^* < \infty$ *such that Eq.* (35) *has a weak solution for* $0 < \lambda < \lambda^*$ *and has no weak solution for* $\lambda > \lambda^*$. λ^* *is called the extremal parameter.*

(ii) *We denote by* u_λ *the minimal solution of* (35), *for* $0 < \lambda < \lambda^*$. *We have that* u_λ *is bounded for* $\lambda < \lambda^*$, *and hence is a* H^1*-solution. Moreover, the map* $\lambda \in (0, \lambda^*) \to u_\lambda$ *is monotone increasing and continuous in the* L^∞ *norm.*

(iii) *The minimal solution* u_λ *is stable, that is, for* $0 < \lambda < \lambda^*$

$$\inf_{\varphi \in H_\sigma} \frac{\int_\Omega |\nabla \varphi|^2 + \int_{\overline{\Omega}} \varphi^2 \,\mathrm{d}\sigma - \lambda \int_\Omega f'(u_\lambda)\varphi^2}{\int_\Omega \varphi^2} > 0.$$

(iv) *If* $\sigma_i \in \mathcal{M}, \ \sigma_i \not\equiv 0$ *for* $i = 1, 2$ *let us denote by* $\lambda^*(\sigma_i)$ *the extremal parameter for* (35) *with* σ *replaced by* σ_i. *Then, if* $\sigma_1 \leqslant \sigma_2$ *we have*

$$\lambda^*(\sigma_1) \leqslant \lambda^*(\sigma_2).$$

For the rest of the properties we assume that $\operatorname{supp}(\sigma) \subset \partial\Omega$.

(v) *For* $\lambda = \lambda^*$, (35) *has a unique weak solution* u^* *which coincides with the monotone limit* $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$. *Moreover, for* $\lambda = \lambda^*$ (35) *has no strict supersolutions, that is, if* u *is a supersolution of* (35) *for* $\lambda = \lambda^*$ *then* $u = u^*$.

(vi) *There exists* C *depending only on* $\Omega, \ f$ *and* $\lambda_1(\sigma)$ *such that*

$$\lambda^* \int_\Omega f(u^*)\chi \leqslant C.$$

(vii) *The map* $\lambda \in (0, \lambda^*] \to \sup_\Omega u_\lambda \in [0, \infty]$ *is continuous.*

(viii) *The extremal solution satisfies*

$$\int_\Omega |\nabla \varphi|^2 + \int_{\overline{\Omega}} \varphi^2 \,\mathrm{d}\sigma \geqslant \lambda^* \int_\Omega f'(u^*)\varphi^2 \quad \textit{for all } \varphi \in H_\sigma.$$

(ix) (Stability characterizes the minimal solutions). *Suppose that* $u \in H_\sigma$ *is a weak solution of* (35) *for some* $\lambda > 0$ *and it satisfies*

$$\int_\Omega |\nabla \varphi|^2 + \int_{\overline{\Omega}} \varphi^2 \,\mathrm{d}\sigma \geqslant \lambda \int_\Omega f'(u)\varphi^2 \quad \textit{for all } \varphi \in H_\sigma. \tag{36}$$

Then $u = u_\lambda$.

Remarks.

1) Most of these results are adaptations of the analog statements for the Dirichlet boundary condition using mainly Lemmas 3.13 and 3.14, and we refer to the literature [4,2,5]. The proof of (v), on the other hand, requires a new result: a strong maximum principle with Robin boundary condition which is given in Lemma 3.10. With it is possible to adapt the argument given by Martel [16] for the case of zero Dirichlet boundary condition.
2) If σ is not supported on $\partial\Omega$, but on $\partial\Omega \cup K$ with K a compact smooth $n-1$ dimensional manifold contained in Ω, then the conclusions of the theorem still hold.
3) In the general case we can always consider the monotone limit $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$. It can be shown to exist pointwise, and it satisfies

$$\int_\Omega u^* \varphi_1 \, \mathrm{d}x < \infty, \qquad \int_\Omega f(u^*) \varphi_1 \, \mathrm{d}x < \infty,$$

where φ_1 is the first eigenfunction associated to σ. We still can regard u^* as a solution of (35) for $\lambda = \lambda^*$ in the following sense. Recall the bounded linear operator $T_\sigma : L^2(\Omega) \to H_\sigma$ (defined in (7)). In Definition 3.5 and by Lemma 3.6 we have extended $T_\sigma : L^1_\chi \to L^1(\Omega)$. But is easy to check that $\|T_\sigma(h)\|_{L^1_{\varphi_1}} \leqslant C\|h\|_{L^1_{\varphi_1}}$ where $\|h\|_{L^1_{\varphi_1}} = \int_\Omega |h|\varphi_1 \, \mathrm{d}x$. So T_σ can be extended as a bounded linear map $T_\sigma : L^1_{\varphi_1} \to L^1_{\varphi_1}$ where $L^1_{\varphi_1} = L^1(\Omega, \varphi_1 \, \mathrm{d}x)$. Then u^* is a solution of (35) for $\lambda = \lambda^*$ in the sense that $u^*, \ f(u^*) \in L^1_{\varphi_1}$ and $T_\sigma(\lambda^* f(u^*)) = u^*$. Is is also the minimal one among these solutions. But for a general σ it is not known whether or not it is unique, or if there exists a strict supersolution of (35) for $\lambda = \lambda^*$.

3.4. Two preliminary lemmas

Lemma 3.15. *Assume that $(\sigma_i)_i \subset \mathcal{M}$ is a sequence such that $\sigma_i \xrightarrow{B} \sigma_\infty \in \mathcal{M}$. Then $\lambda_1(\sigma_i) \to \lambda_1(\sigma_\infty)$. In particular, if $\sigma_\infty \not\equiv 0$ then $\lambda_1(\sigma_i)$ stays away from zero for i large.*

Proof of Lemma 3.15. We use the notations λ_i and λ_∞ for the first eigenvalues associated to σ_i and σ_∞, and also we denote by φ_i and φ_∞ the first eigenfunctions associated to σ_i and σ_∞. We use the convention that $\varphi_i = 0$ whenever $H_{\sigma_i} = \{0\}$, and recall that φ_i satisfies

$$\begin{cases} -\Delta\varphi_i + \sigma_i \varphi_i = \lambda_i \varphi_i & \text{in } \Omega, \\ \dfrac{\partial \varphi_i}{\partial \nu} + \sigma_i \varphi_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{37}$$

Step 1. If $\lambda_\infty = \infty$ *then* $\lambda_i \to \infty$.

Proof. Suppose not, so that for a subsequence we have $\lambda_i \leqslant C$ for some constant C. We normalize the eigenfunctions φ_i so that $\|\varphi_i\|_{L^2} = 1$. Testing (37) with φ_i we see that φ_i is bounded in $H^1(\Omega)$, so we extract a new subsequence such that $\varphi_i \rightharpoonup \varphi$ in $H^1(\Omega)$ weakly. Note that $\|\varphi\|_{L^2} = 1$.

Let $h \in L^2(\Omega)$ and let ζ_i be the solution of

$$\begin{cases} -\Delta\zeta_i + \zeta_i + \sigma_i \zeta_i = h & \text{in } \Omega, \\ \dfrac{\partial \zeta_i}{\partial \nu} + \sigma_i \zeta_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{38}$$

By assumption of $\sigma_i \overset{B}{\rightharpoonup} \sigma_\infty$ and since $\lambda_1(\sigma_\infty) = \infty$ we have $\zeta_i \rightharpoonup 0$ in $H^1(\Omega)$ weakly.

Now we multiply (37) by ζ_i and integrate by parts, multiply (38) by φ_i and integrate by parts, and take the difference to obtain

$$\int_\Omega \zeta_i \varphi_i \,\mathrm{d}x = \int_\Omega h\varphi_i - \lambda_i \varphi_i \zeta_i \,\mathrm{d}x.$$

But $\zeta_i \rightharpoonup 0$ and $\varphi_i \rightharpoonup \varphi$ in $H^1(\Omega)$ weakly, so

$$\int_\Omega h\varphi \,\mathrm{d}x = 0.$$

Since $h \in L^2(\Omega)$ was arbitrary we conclude that $\varphi = 0$, but this is in contradiction with $\|\varphi\|_{L^2} = 1$.

Step 2. *If $\lambda_\infty < \infty$ then there exists $C < \infty$ such that $\lambda_i \leqslant C$ for i large.*

Proof. Since $\lambda_\infty < \infty$ we have $H_{\sigma_\infty} \neq \{0\}$. Fix $h \in H_{\sigma_\infty} \setminus \{0\}$ and let ζ_i be the solution of (38). By the assumption $\sigma_i \overset{B}{\rightharpoonup} \sigma_\infty$ we have $\zeta_i \rightharpoonup \zeta_\infty$ in $H^1(\Omega)$ weakly, where ζ_∞ is the solution of

$$\begin{cases} -\Delta\zeta_\infty + \zeta_\infty + \sigma_\infty\zeta_\infty = h & \text{in } \Omega, \\ \dfrac{\partial\zeta_\infty}{\partial\nu} + \sigma_\infty\zeta_\infty = 0 & \text{on } \partial\Omega. \end{cases} \tag{39}$$

Note that $\zeta_\infty \neq 0$. Indeed, since $h \in H_{\sigma_\infty}$, testing (39) with h we find

$$\int_\Omega \nabla\zeta_\infty \nabla h + \zeta_\infty h \,\mathrm{d}x + \int_{\overline{\Omega}} \zeta_\infty h \,\mathrm{d}\sigma_\infty = \int_\Omega h^2 \,\mathrm{d}x \neq 0$$

and therefore ζ_∞ cannot be zero. Hence

$$\lambda_i \leqslant \frac{\int_\Omega |\nabla\zeta_i|^2 \,\mathrm{d}x + \int_{\overline{\Omega}} \zeta_i^2 \,\mathrm{d}\sigma_i}{\int_\Omega \zeta_i^2 \,\mathrm{d}x} = \frac{\int_\Omega (h\zeta_i - \zeta_i^2) \,\mathrm{d}x}{\int_\Omega \zeta_i^2 \,\mathrm{d}x} \leqslant C$$

because ζ_i is bounded in $L^2(\Omega)$ and $\int_\Omega \zeta_i^2 \,\mathrm{d}x \to \int_\Omega \zeta_\infty^2 \,\mathrm{d}x \neq 0$.

Step 3. *If $\lambda_\infty < \infty$ then $\lambda_i \to \lambda_\infty$.*

Proof. By Step 2 λ_i is bounded so for a subsequence we can assume that $\lambda_i \to \lambda$.

Let φ_i denote the first eigenfunction associated to σ_i, normalized so that $\|\varphi_i\|_{L^2} = 1$. Then φ_i is bounded in $H^1(\Omega)$, so we take a new subsequence so that $\varphi_i \rightharpoonup \varphi$ in $H^1(\Omega)$ weakly. Note that $\varphi_i \geqslant 0$ for all i, so $\varphi \geqslant 0$, and $\|\varphi\|_{L^2} = 1$.

Let $h \in L^2(\Omega)$, with $\int_\Omega h = 0$ if $\sigma_\infty \equiv 0$, and let ζ be a solution of

$$\begin{cases} -\Delta\zeta + \sigma_\infty\zeta = h & \text{in } \Omega, \\ \dfrac{\partial\zeta}{\partial\nu} + \sigma_\infty\zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that if $\sigma_\infty \not\equiv 0$ then ζ is uniquely defined, and otherwise ζ is defined up to constant. Let ζ_i denote the solution of

$$\begin{cases} -\Delta\zeta_i + \zeta_i + \sigma_i\zeta_i = h + \zeta & \text{in } \Omega, \\ \dfrac{\partial\zeta_i}{\partial\nu} + \sigma_i\zeta_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{40}$$

Claim.

$$\zeta_i \rightharpoonup \zeta \quad \textit{in } H^1(\Omega) \textit{ weakly}. \tag{41}$$

Proof of Lemma 3.15 completed. Multiplying (40) by φ_i, integrating by parts and using (37) we find

$$\int_\Omega \lambda_i\varphi_i\zeta_i + \zeta_i\varphi_i = \int_\Omega h\varphi_i + \zeta\varphi_i$$

so that by letting $i \to \infty$ we have

$$\lambda \int_\Omega \varphi\zeta = \int_\Omega h\varphi. \tag{42}$$

In the case $\sigma_\infty \equiv 0$, since we could replace ζ by $\zeta + c$ in (42), we conclude that $\lambda = 0 = \lambda_1(\sigma_\infty)$.

In the case $\sigma_\infty \not\equiv 0$, from (42) we deduce that φ satisfies

$$\begin{cases} -\Delta\varphi + \sigma_\infty = \lambda\varphi & \text{in } \Omega, \\ \dfrac{\partial\varphi}{\partial\nu} + \sigma_\infty\varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{43}$$

Since $\varphi \not\equiv 0$, $\varphi \geqslant 0$, (43) implies that $\lambda = \lambda_1(\sigma_\infty)$.

Proof of (41). By definition of $\sigma_i \overset{B}{\rightharpoonup} \sigma_\infty$ we have $\zeta_i \rightharpoonup \tilde\zeta$ in $H^1(\Omega)$ weakly, where $\tilde\zeta$ is the solution of

$$\begin{cases} -\Delta\tilde\zeta + \tilde\zeta + \sigma_\infty\tilde\zeta = h + \zeta & \text{in } \Omega, \\ \dfrac{\partial\tilde\zeta}{\partial\nu} + \sigma_\infty\tilde\zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

But $-\Delta\zeta + \sigma_\infty\zeta = h$ so that

$$\begin{cases} -\Delta(\tilde\zeta - \zeta) + (\tilde\zeta - \zeta) + \sigma_\infty(\tilde\zeta - \zeta) = 0 & \text{in } \Omega, \\ \left(\dfrac{\partial}{\partial\nu} + \sigma_\infty\right)(\tilde\zeta - \zeta) = 0 & \text{on } \partial\Omega \end{cases}$$

so that $\zeta = \tilde\zeta$. □

Lemma 3.16. *Assume* $\sigma_i \overset{B}{\rightharpoonup} \sigma_\infty$ *where* $\sigma_\infty \not\equiv 0$. *By Lemma* 3.15 *we have that* $\lambda_1(\sigma_i)$ *is bounded away from zero for* i *large. Let* $\varphi \in L^2(\Omega)$ *and* ζ_i *be the solution of*

$$\begin{cases} -\Delta\zeta_i + \sigma_i\zeta_i = \varphi & \text{in } \Omega, \\ \dfrac{\partial\zeta_i}{\partial\nu} + \sigma_i\zeta_i = 0 & \text{on } \Omega. \end{cases} \tag{44}$$

Then $\zeta_i \rightharpoonup \zeta_\infty$ *in* $H^1(\Omega)$ *weakly where* ζ_∞ *is the solution of*

$$\begin{cases} -\Delta\zeta_\infty + \sigma_\infty\zeta_\infty = \varphi & \text{in } \Omega, \\ \dfrac{\partial\zeta_\infty}{\partial\nu} + \sigma_\infty\zeta_\infty = 0 & \text{on } \Omega. \end{cases} \tag{45}$$

Proof. Since $\lambda_1(\sigma_i)$ is bounded away from zero, we have that $\|\zeta_i\|_{H^1} \leqslant C$ for some C independent of i, and therefore up to subsequence $\zeta_i \rightharpoonup \zeta$ in $H^1(\Omega)$ weakly. We let v_i denote the solution of

$$\begin{cases} -\Delta v_i + v_i + \sigma_i v_i = \varphi + \zeta & \text{in } \Omega, \\ \dfrac{\partial v_i}{\partial\nu} + \sigma_i v_i = 0 & \text{on } \Omega \end{cases} \tag{46}$$

so that by definition $v_i \rightharpoonup v$ in $H^1(\Omega)$ weakly to v_∞ which is the solution of

$$\begin{cases} -\Delta v_\infty + v_\infty + \sigma_\infty v_\infty = \varphi + \zeta & \text{in } \Omega, \\ \dfrac{\partial v_\infty}{\partial\nu} + \sigma_\infty v_\infty = 0 & \text{on } \Omega. \end{cases} \tag{47}$$

Then by (44) and (46) we have

$$\|v_i - \zeta_i\|_{H^1} \leqslant \|\zeta - \zeta_i\|_{L^2} \to 0$$

and this implies that $v_\infty = \zeta$. But then, by (47) we see that ζ satisfies (45) and by uniqueness of the solution of this problem we have $\zeta = \zeta_\infty$. □

4. Convergence of the extremal parameter

Throughout this section $(\sigma_i)_i$ is a sequence in $\mathcal{M}$ such that $\sigma_i \overset{B}{\rightharpoonup} \sigma_\infty$, and we use the notation $\lambda_i^* = \lambda^*(\sigma_i)$, $\lambda_\infty^* = \lambda^*(\sigma_\infty)$.

We divide the proof of Theorem 1.6 in two steps.

Step 1. *If* $\sigma_i \overset{B}{\rightharpoonup} \sigma_\infty$, *then*

$$\limsup_i \lambda_i^* \leqslant \lambda_\infty^*.$$

Proof. If $\lambda_\infty^* = \infty$ there is nothing to prove, so we assume that $\lambda_\infty^* < \infty$. Suppose that the conclusion is not true, and take a subsequence (which we denote the same) such that $\lambda_i^* \to \lambda$ with $\lambda_\infty^* < \lambda \leqslant \infty$. Fix λ' such that $\lambda_\infty^* < \lambda' < \lambda$ and for i large enough let v_i denote the minimal solution of

$$\begin{cases} -\Delta v_i + \sigma_i v_i = \lambda' f(v_i) & \text{in } \Omega, \\ \dfrac{\partial v_i}{\partial \nu} + \sigma_i v_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{48}$$

Claim. *There is a constant C independent of i such that*

$$\|v_i\|_{L^\infty(\Omega)} \leqslant C.$$

Indeed fix $\lambda'' \in (\lambda', \lambda)$ and let $\tilde{v}_i$ be the minimal solution of (48) but with parameter λ''. For $\varepsilon > 0$ consider the concave function Φ_ε defined by

$$\int_0^{\Phi_\varepsilon(u)} \frac{\mathrm{d}s}{f(s)} = (1-\varepsilon)\int_0^u \frac{\mathrm{d}s}{f(s)}.$$

Using Kato's inequality (Lemma 3.11), a calculation as in [4] shows that if $(1-\varepsilon)\lambda'' \geqslant \lambda'$, then

$$v_i \leqslant \Phi_\varepsilon(\tilde{v}_i) \leqslant C_\varepsilon.$$

We fix then ε so that $(1-\varepsilon)\lambda'' \geqslant \lambda'$ for i large. Hence $\|v_i\|_{H^1(\Omega)}$ is bounded independently of i. (Note: by (48) and since v_i is bounded in $L^\infty(\Omega)$ we find that ∇v_i is bounded in $L^2(\Omega)$. This and the L^∞ bound for v_i imply that v_i is bounded in $H^1(\Omega)$.) So after taking a new subsequence we can assume that $v_i \rightharpoonup v$ in $H^1(\Omega)$ weakly.

We claim that v is a solution of

$$\begin{cases} -\Delta v + \sigma_\infty v = \lambda' f(v) & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} + \sigma_\infty v = 0 & \text{on } \partial\Omega. \end{cases} \tag{49}$$

If this is true, then we have contradicted the maximality of λ_∞^* in the case $\sigma_\infty \not\equiv 0$, and in the case $\sigma_\infty = 0$ we arrive at a contradiction because v satisfies a zero Neumann boundary condition, but the right-hand side of (49) is strictly positive.

To show that v is a solution of (49), consider w_i the solution of

$$\begin{cases} -\Delta w_i + w_i + \sigma_i w_i = \lambda' f(v) + v & \text{in } \Omega, \\ \dfrac{\partial w_i}{\partial \nu} + \sigma_i w_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{50}$$

Then by hypothesis $w_i \rightharpoonup w_\infty$ in $H^1(\Omega)$ weakly where w_∞ solves

$$\begin{cases} -\Delta w_\infty + w_\infty + \sigma_\infty w_\infty = \lambda' f(v) + v & \text{in } \Omega, \\ \dfrac{\partial w_\infty}{\partial \nu} + \sigma_\infty w_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$

But subtracting (48) from (50) we get:

$$\|w_i - v_i\|_{H^1(\Omega)} \leqslant C\|\lambda' f(v) - \lambda' f(v_i) + v - v_i\|_{L^2(\Omega)} \to 0.$$

Hence we must have $v = w$. □

Step 2.

$$\liminf_i \lambda_i^* \geqslant \lambda_\infty^*.$$

Proof. If the conclusion is not true, then there exists a subsequence (denoted the same) such that $\lambda_i^* \to \lambda < \lambda_\infty^*$. Fix λ' such that $\lambda < \lambda' < \lambda_\infty^*$ and let u' denote the minimal solution of

$$\begin{cases} -\Delta u' + \sigma_\infty u' = \lambda' f(u') & \text{in } \Omega, \\ \dfrac{\partial u'}{\partial \nu} + \sigma_\infty u' = 0 & \text{on } \partial\Omega. \end{cases} \tag{51}$$

Then $u' \in L^\infty(\Omega)$. To arrive at a contradiction, we want to find a supersolution for the nonlinear problem with measure σ_i and a parameter λ'', with $\lambda < \lambda'' < \lambda' < \lambda^*$. Consider then v_i the solution of

$$\begin{cases} -\Delta v_i + v_i + \sigma_i v_i = \lambda' f(u') + u' & \text{in } \Omega, \\ \dfrac{\partial v_i}{\partial \nu} + \sigma_i v_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{52}$$

By definition of $\sigma_i \overset{B}{\rightharpoonup} \sigma_\infty$ we have $v_i \rightharpoonup v_\infty$ in H^1-weakly, where v_∞ is the solution of

$$\begin{cases} -\Delta v_\infty + v_\infty + \sigma_\infty v_\infty = \lambda' f(u') + u' & \text{in } \Omega, \\ \dfrac{\partial v_\infty}{\partial \nu} + \sigma_\infty v_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$

But from here and (51) we deduce that $v_\infty = u'$. Now consider w_i the solution of

$$\begin{cases} -\Delta w_i + w_i + \sigma_i w_i = \lambda' f(v_i) + v_i & \text{in } \Omega, \\ \dfrac{\partial w_i}{\partial \nu} + \sigma_i w_i = 0 & \text{on } \partial\Omega \end{cases} \tag{53}$$

and note the following:

$$\begin{aligned} -\Delta w_i + \sigma_i w_i &= \lambda' f(v_i) + v_i - w_i \\ &= \lambda'' f(w_i) + (\lambda' - \lambda'') f(v_i) + \lambda'' \big(f(v_i) - f(w_i)\big) + v_i - w_i \\ &\geqslant \lambda'' f(w_i) + (\lambda' - \lambda'') f(0) + \lambda'' \big(f(v_i) - f(w_i)\big) + v_i - w_i. \end{aligned} \tag{54}$$

Since $f(0) > 0$, if we can show that

$$w_i - v_i \to 0 \quad \text{uniformly} \tag{55}$$

then we have shown that w_i is a supersolution for the problem

$$\begin{cases} -\Delta u + \sigma_i u = \lambda'' f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma_i u = 0 & \text{on } \partial\Omega \end{cases}$$

and this contradicts the fact that λ_i^* is the maximal parameter for this nonlinear problem.

Proof of (55). Subtracting (52) from (53) and using Proposition 3.1 we find that

$$\|w_i - v_i\|_\infty \leqslant C\|\lambda' f(u') + u' - \lambda' f(v_i) - v_i\|_p,$$

where we fix some $n/2 < p < \infty$. The constant C depends only on Ω, n and p (not on $\lambda_1(\sigma_i)$). But $v_i \rightharpoonup u'$ in $H^1(\Omega)$ weakly, and v_i is bounded in $L^\infty(\Omega)$, therefore

$$\|\lambda' f(u') + u' - \lambda' f(v_i) - v_i\|_p \to 0 \quad \text{as } i \to \infty. \quad \square$$

5. Convergence of the extremal solution

Throughout this section we use the following notation: $(\sigma_i)_i$ is a sequence in $\mathcal{M}$ of measures with support in $\partial\Omega$ such that $\sigma_i \xrightarrow{B} \sigma_\infty$. We assume that $\sigma_i \not\equiv 0$ for each i, and that $\sigma_\infty \not\equiv 0$. This assumption implies, by Lemma 3.15 that $\lambda_1(\sigma_i)$ stays away from zero. Therefore all of the estimates in Sections 3.1, 3.2 and 3.3 which depend on $\lambda_1(\sigma_i)$, will hold uniformly in i.

We write $\lambda_i^* = \lambda^*(\sigma_i)$, $\lambda_\infty^* = \lambda^*(\sigma_\infty)$, $u_i^* = u^*(\sigma_i)$ and $u_\infty^* = u^*(\sigma_\infty)$, and we let χ_i ($i = 1, \dots, \infty$) denote the solution of

$$\begin{cases} -\Delta \chi_i = 1 & \text{on } \Omega, \\ \dfrac{\partial \chi_i}{\partial \nu} + \sigma_i \chi_i = 0 & \text{on } \partial\Omega. \end{cases}$$

(Note that since we assume that σ_i has support on the boundary, the term $\sigma_i\chi_i$ does not appear in the equation.)

5.1. Convergence in L^p

Lemma 5.1. *Assume that $\sigma_i \rightharpoonup \sigma_\infty$ and that $\sigma_\infty \not\equiv 0$. Then there exists a subsequence i_j and $u \in L^1(\Omega)$ such that $u_{i_j}^* \to u$ in $L^p(\Omega)$ for $1 \leqslant p < n/(n-1)$.*

Proof. Note that since $\lambda_1(\sigma_i)$ stays away from zero, by Theorem 3.14 property (vi) we have

$$\lambda_i^* \int_\Omega f(u_i^*)\chi_i \, \mathrm{d}x \leqslant C \tag{56}$$

which C independent of i. Therefore, by Lemma 3.7 we have also

$$\|u_i^*\|_p \leqslant C, \tag{57}$$

where $1 \leqslant p < n/(n-1)$, and C is independent of i.

Since Δu_i^* is bounded in $L^1_{\mathrm{loc}}(\Omega)$ and u_i^* is bounded in $L^1(\Omega)$, we have that u_i^* is bounded in $W^{1,1}_{\mathrm{loc}}(\Omega)$. So we can extract a subsequence (which we denote the same) such that $u_i^* \to u$ in $L^q_{\mathrm{loc}}(\Omega)$ and a.e., where we fix $1 < q < n/(n-1)$.

Let $\varepsilon > 0$ and let U be an open neighborhood of $\partial\Omega$ in $\overline{\Omega}$ such that $\|1_U\|_{q'} < \varepsilon$, where q' is the conjugate exponent of q, that is, $1 = 1/q + 1/q'$. Let ζ_i denote the solution of

$$\begin{cases} -\Delta\zeta_i = 1_U & \text{in } \Omega, \\ \dfrac{\partial\zeta_i}{\partial\nu} + \sigma_i\zeta_i = 0 & \text{in } \partial\Omega. \end{cases}$$

Then

$$\int_U u_i^* \,\mathrm{d}x = \int_\Omega u_i^*(-\Delta\zeta_i)\,\mathrm{d}x = \lambda_i^* \int_\Omega f(u_i^*)\zeta_i \,\mathrm{d}x \leqslant C\left\|\frac{\zeta_i}{\chi_i}\right\|_\infty \lambda_i^* \int_\Omega f(u_i^*)\chi_i. \tag{58}$$

But by Lemma 3.2

$$\left\|\frac{\zeta_i}{\chi_i}\right\|_\infty \leqslant C\|1_U\|_{q'} \leqslant C\varepsilon. \tag{59}$$

So, from (56), (58) and (59) we find that

$$\int_U u_i^* \,\mathrm{d}x \leqslant C\varepsilon$$

and by Fatou's lemma we also have

$$\int_U u \,\mathrm{d}x \leqslant C\varepsilon.$$

Hence

$$\|u_i^* - u\|_1 = \int_{\Omega\setminus U} |u_i^* - u| \,\mathrm{d}x + \int_U |u_i^* - u| \,\mathrm{d}x \leqslant \int_{\Omega\setminus U} |u_i^* - u| \,\mathrm{d}x + 2C\varepsilon$$

and therefore

$$\limsup_i \|u_i^* - u\|_1 \leqslant 2C\varepsilon.$$

Since ε was arbitrary we conclude that $u_i^* \to u$ in $L^1(\Omega)$. Finally, from this convergence in $L^1(\Omega)$ and from (57) we conclude that $u_i^* \to u$ in $L^p(\Omega)$ for any $1 \leqslant p < n/(n-1)$. □

Proof of (9) in Theorem 1.7. By Lemma 5.1, we can extract a subsequence (which we denote the same) such that $u_i^* \to u$ in $L^p(\Omega)$ and a.e., where we fix some $1 \leqslant p < n/(n-1)$. Let $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$ and let ζ_i be the solution of

$$\begin{cases} -\Delta\zeta_i = \varphi & \text{in } \Omega, \\ \dfrac{\partial\zeta_i}{\partial\nu} + \sigma_i\zeta_i = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 3.16 we have that $\zeta_i \rightharpoonup \zeta$ in $H^1(\Omega)$ weakly, where ζ is the solution of

$$\begin{cases} -\Delta\zeta = \varphi & \text{in } \Omega, \\ \dfrac{\partial\zeta}{\partial\nu} + \sigma_\infty\zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that since φ is smooth, we have that ζ_i is bounded in $C^k_{\mathrm{loc}}(\Omega)$ for any $k \geqslant 0$, and therefore $\zeta_i \to \zeta$ in $C^k_{\mathrm{loc}}(\Omega)$ for any $k \geqslant 0$. In particular we have a.e. convergence. Taking ζ_i as a test function in the weak formulation of

$$\begin{cases} -\Delta u_i^* = \lambda_i^* f(u_i^*) & \text{in } \Omega, \\ \dfrac{\partial u_i^*}{\partial\nu} + \sigma_i u_i^* = 0 & \text{on } \partial\Omega \end{cases}$$

we find

$$\int_\Omega u_i^*\varphi\,\mathrm{d}x = \lambda_i^* \int_\Omega f(u_i^*)\zeta_i\,\mathrm{d}x.$$

By passing to the limit as $i \to \infty$ and using Fatou's lemma on the right-hand side we find

$$\int_\Omega u\varphi\,\mathrm{d}x \geqslant \lambda_\infty^* \int_\Omega f(u)\zeta\,\mathrm{d}x.$$

This shows that u is a weak supersolution of

$$\begin{cases} -\Delta u = \lambda_\infty^* f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu} + \sigma_\infty u = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 3.14 property (v), we conclude that $u = u_\infty^*$ and this finishes the proof of (9) in Theorem 1.7. □

5.2. Asymptotic behavior of $\sup_\Omega u^*(\lambda_i)$

In this section we prove the second part of Theorem 1.7, which we recall now: if u_∞^* is unbounded then

$$\|u_i^*\|_\infty \to \infty$$

and if $u_\infty^* \in L^\infty(\Omega)$ then

$$\limsup \|u^*(\sigma_i)\|_\infty < \infty.$$

Step 1. *If u^* is unbounded then*

$$\|u_i^*\|_\infty \to \infty.$$

Proof. This is a consequence of the fact that

$$u_i^* \to u_\infty^* \quad \text{in } L^p(\Omega), \quad 1 \leqslant p < \frac{n}{n-1}.$$

Step 2. *If* $u_\infty^* \in L^\infty(\Omega)$ *then*

$$\limsup \left\| u^*(\sigma_i) \right\|_\infty < \infty.$$

Proof. Suppose not and consider a subsequence (denoted the same) such that $\sup_\Omega u_i^* \nearrow \infty$. We fix now $M = C_1 + 2 < \infty$, where C_1 is to be chosen later. Now, for each fixed i because of property (vii) in Theorem 3.14 we can select $0 < \lambda_i \leqslant \lambda_i^*$ such that the minimal solution u_i of the problem

$$\begin{cases} -\Delta u_i = \lambda_i f(u_i) & \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial \nu} + \sigma_i u_i = 0 & \text{on } \partial\Omega \end{cases} \tag{60}$$

satisfies

$$\sup_\Omega u_i = M. \tag{61}$$

Note that the sequence λ_i is bounded, so up to a new subsequence $\lambda_i \to \widetilde{\lambda}$.

Claim.

$$u_i \rightharpoonup \widetilde{u} \quad \textit{in } H^1(\Omega) \textit{ weakly}, \tag{62}$$

where $\widetilde{u}$ *is the minimal solution of*

$$\begin{cases} -\Delta \widetilde{u} = \widetilde{\lambda} f(\widetilde{u}) & \textit{in } \Omega, \\ \dfrac{\partial \widetilde{u}}{\partial \nu} + \sigma_\infty \widetilde{u} = 0 & \textit{on } \partial\Omega. \end{cases} \tag{63}$$

In particular $\widetilde{\lambda} \leqslant \lambda_\infty^*$ *and* $\widetilde{u} \leqslant u_\infty^*$.

Proof of Step 2 completed. Let v_i be the solution of

$$\begin{cases} -\Delta v_i = \lambda_\infty^* f(u_\infty^*) & \text{in } \Omega, \\ \dfrac{\partial v_i}{\partial \nu} + \sigma_i v_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{64}$$

We note here that by Proposition 3.1 we have

$$v_i \leqslant C_1 \quad \text{in } \Omega,$$

where C_1 depends on λ^*_∞, u^*_∞, Ω, n and $\lambda_1(\sigma_i)$, which is bounded away from zero. At this point we make the choice of C_1.

Recall that we assume $u^*_\infty \in L^\infty(\Omega)$, hence by Lemma 3.16 we have $v_i \rightharpoonup u^*_\infty$ in $H^1(\Omega)$ weakly. But subtracting (64) from (60) and using Proposition 3.1 we have

$$\sup_\Omega u_i - v_i \leqslant C\big\|\big(\lambda_i f(u_i) - \lambda^*_\infty f\big(u^*_\infty\big)\big)^+\big\|_p,$$

where we fix some $n/2 < p < \infty$, and C is independent of i. But $\lambda_i f(u_i)$ is bounded in $L^\infty(\Omega)$ and converges pointwise to $\widetilde{\lambda} f(\widetilde{u}) \leqslant \lambda^*_\infty f(u^*_\infty)$. Therefore

$$\big\|\big(\lambda_i f(u_i) - \lambda^*_\infty f\big(u^*_\infty\big)\big)^+\big\|_p \to 0 \quad \text{as } i \to \infty.$$

Hence, for i large we have

$$M = \sup_\Omega u_i \leqslant 1 + \sup_\Omega v_i \leqslant 1 + C_1$$

which is impossible.

Proof of (62). From (60), (61) and the fact that $\lambda_1(\sigma_i)$ stays away from zero, we have that u_i is bounded in $H^1(\Omega)$ and $L^\infty(\Omega)$. Hence by taking a subsequence we can assume that $u_i \rightharpoonup \widetilde{u}$ in $H^1(\Omega)$ weakly, a.e. and in $L^p(\Omega)$ strongly for $1 \leqslant p < \infty$. We also can assume that $\lambda_i \to \widetilde{\lambda}$. Note that $\widetilde{u}$ satisfies (63). Indeed, take $\varphi \in C_0^\infty(\Omega)$ and ζ_i the solution of

$$\begin{cases} -\Delta \zeta_i = \varphi & \text{in } \Omega, \\ \dfrac{\partial \zeta_i}{\partial \nu} + \sigma_i \zeta_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{65}$$

Then by Lemma 3.16 we have that $\zeta_i \rightharpoonup \zeta$ which is the solution

$$\begin{cases} -\Delta \zeta = \varphi & \text{in } \Omega, \\ \dfrac{\partial \zeta}{\partial \nu} + \sigma_\infty \zeta = 0 & \text{on } \partial\Omega. \end{cases} \tag{66}$$

Hence, we can take the limit as $i \to \infty$ in

$$\int_\Omega u_i \varphi = \lambda_i \int_\Omega f(u_i)\zeta_i.$$

We also have

$$\int_\Omega |\nabla \zeta|^2 + \int_{\overline{\Omega}} \zeta^2 \, \mathrm{d}\sigma_\infty \geqslant \widetilde{\lambda} \int_\Omega f'(\widetilde{u})\zeta^2 \quad \text{for all } \zeta \in H_{\sigma_\infty} \tag{67}$$

which is obtained from the corresponding stability inequality for u_i as follows: take $\varphi \in C_0^\infty(\Omega)$, ζ_i the solution of (65) and ζ the solution of (66). We have $\zeta_i \in H_{\sigma_i}$ and $\zeta_i \rightharpoonup \zeta$ in $H^1(\Omega)$ weakly. Therefore, by property (iii) in Theorem 3.14 we have

$$\int_\Omega |\nabla \zeta_i|^2 + \int_{\overline{\Omega}} \zeta_i^2 \, \mathrm{d}\sigma_i \geqslant \lambda_i \int_\Omega f'(u_i)\zeta_i^2. \tag{68}$$

Now, multiplying (65) by ζ_i and integrating by parts we get

$$\int_\Omega |\nabla\zeta_i|^2 + \int_{\overline{\Omega}} \zeta_i^2 \,\mathrm{d}\sigma_i = \int_\Omega \varphi\zeta_i.$$

Since $\zeta_i \rightharpoonup \zeta$ in $H^1(\Omega)$ weakly, this equality shows that

$$\int_\Omega |\nabla\zeta_i|^2 + \int_{\overline{\Omega}} \zeta_i^2 \,\mathrm{d}\sigma_i \to \int_\Omega |\nabla\zeta|^2 + \int_{\overline{\Omega}} \zeta^2 \,\mathrm{d}\sigma_\infty.$$

Taking $i \to \infty$ in (68) and using Fatou's lemma on the right-hand side, we obtain (67) for ζ in a subset of H_{σ_∞}, namely the ones that are solutions of (66) for some $\varphi \in C_0^\infty(\Omega)$. But this subset is dense in H_{σ_∞} and (67) follows.

By Theorem 3.14 property (i) we must have $\widetilde{\lambda} \leqslant \lambda_\infty^*$, and by property (ix) of the same theorem $\widetilde{u}$ is the minimal solution of (63). $\square$

Appendix

Proof of Lemma 3.11. Recall that we assume that u is a weak solution of

$$\begin{cases} -\Delta u + \sigma u = h & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\sigma \in \mathcal{M}$ and $h \in L^1_\chi$. For $m > 0$ let $h_m = h$ if $|h| \leqslant m$, $h_m = -m$ if $h < -m$ and $h_m = m$ if $h > m$, and let u_m denote the H^1-solution of

$$\begin{cases} -\Delta u_m + \sigma u_m = h_m & \text{in } \Omega, \\ \dfrac{\partial u_m}{\partial \nu} + \sigma u_m = 0 & \text{on } \partial\Omega. \end{cases} \tag{69}$$

Note that $u_m \to u$ in $L^1(\Omega)$. Let $\varphi \in C_0^\infty(\Omega)$ and suppose that the solution ζ of

$$\begin{cases} -\Delta \zeta + \sigma \zeta = \varphi & \text{in } \Omega, \\ \dfrac{\partial \zeta}{\partial \nu} + \sigma \zeta = 0 & \text{on } \partial\Omega \end{cases} \tag{70}$$

is nonnegative.

Note that $\Phi'(u_m)\zeta \in H_\sigma$ because $\Phi' \in L^\infty$, $\zeta \in H_\sigma$ and $\nabla(\Phi'(u_m)\zeta) \in L^2(\Omega)$. Using $\Phi'(u_m)\zeta$ as a test function in (69) we find that

$$\int_\Omega \nabla u_m\big(\Phi''(u_m)\nabla u_m \zeta + \Phi'(u_m)\nabla\zeta\big)\,\mathrm{d}x + \int_{\overline{\Omega}} \Phi'(u_m)u_m\zeta \,\mathrm{d}\sigma = \int_\Omega h_m \Phi'(u_m)\zeta \,\mathrm{d}x.$$

But $\Phi'' \leqslant 0$ because Φ is concave, and $\Phi'(u)u \leqslant \Phi(u)$ (this follows from the concavity of Φ and $\Phi(0) = 0$). Hence

$$\int_\Omega \nabla\big(\Phi(u_m)\big)\nabla\zeta \,\mathrm{d}x + \int_{\overline{\Omega}} \Phi(u_m)\zeta \,\mathrm{d}\sigma \geqslant \int_\Omega h_m\Phi'(u_m)\zeta \,\mathrm{d}x. \tag{71}$$

Note that $\Phi(u_m) \in H_\sigma$ because $\Phi(u) \leqslant \|\Phi'\|_\infty |u| \in L^2(\overline{\Omega}, \sigma)$. Using $\Phi(u_m)$ in (70) we obtain

$$\int_\Omega \nabla\big(\Phi(u_m)\big)\nabla\zeta \,\mathrm{d}x + \int_{\overline{\Omega}} \Phi(u_m)\zeta \,\mathrm{d}\sigma = \int_\Omega \Phi(u_m)\varphi \,\mathrm{d}x. \tag{72}$$

Combining (71) and (72) we get

$$\int_\Omega \Phi(u_m)\varphi \,\mathrm{d}x \geqslant \int_\Omega h_m \Phi'(u_m)\zeta \,\mathrm{d}x.$$

Now we let $m \to \infty$:

$$\int_\Omega \big|\Phi(u_m) - \Phi(u)\big||\varphi| \,\mathrm{d}x \leqslant \|\varphi\|_\infty \|\Phi'\|_\infty \int_\Omega |u_m - u| \,\mathrm{d}x \to 0$$

and

$$\int_\Omega h_m \Phi'(u_m)\zeta \,\mathrm{d}x \to \int_\Omega h\Phi'(u)\zeta \,\mathrm{d}x$$

since we have convergence a.e. (at least for a subsequence) and

$$\big|h_m \Phi'(u_m)\zeta\big| \leqslant \|\Phi'\|_\infty |h|\zeta \in L^1(\Omega)$$

by the assumption $h \in L^1_\chi$. $\square$

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