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THE JACOBI-TODA SYSTEM AND FOLIATED INTERFACES

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Dedicated to Professor L. Nirenberg on the occasion of his 85th birthday, with deep admiration

Abstract. Let $(\mathcal{M}, \tilde{g})$ be an N-dimensional smooth (compact or noncompact) Riemannian manifold. We introduce the elliptic *Jacobi-Toda system* on $(\mathcal{M}, \tilde{g})$. We review various recent results on its role in the construction of solutions with multiple interfaces of the Allen-Cahn equation on compact manifolds and entire space, as well as multiple-front traveling waves for its parabolic counterpart.

1. **Introduction.** In the gradient theory of phase transitions by Allen-Cahn [1], two phases of a material, $+1$ and -1 coexist in a region $\Omega \subset \mathbb{R}^N$ separated by an $(N-1)$-dimensional interface. The phase is idealized as a smooth ε-regularization of the discrete function, which is selected as a critical point of the energy

$$J_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{4\varepsilon}(1-u^2)^2,$$

where $\varepsilon > 0$ is a small parameter. While any function with values ± 1 minimizes exactly the second term, the presence of the gradient term conveys a balance in which the interface is selected asymptotically as stationary for perimeter (namely a generalized minimal surface). The mathematical problem is that of finding critical points of J_ε in $H^1(\Omega)$, namely solutions u_ε of the *Allen-Cahn equation*

$$\varepsilon^2 \Delta u + u - u^3 = 0, \quad \text{in } \Omega, \quad \partial_\nu u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

with the property that

$$u_\varepsilon \to \chi_\Lambda - \chi_{\Omega\setminus\Lambda} \quad \text{in } L^1_{loc}(\Omega). \tag{1.2}$$

where Λ is an open subset of Ω with $\Gamma = \partial\Lambda \cap \Omega$ having *minimal perimeter*. Modica and Mortola proved that this is precisely the situation (after passing to a subsequence) for a family of local minimizers u_ε with $\sup_{\varepsilon>0} J_\varepsilon(u_\varepsilon) < +\infty$

Moreover,

$$J_\varepsilon(u_\varepsilon) \to \frac{2}{3}\sqrt{2}\mathcal{H}^{N-1}(\Gamma). \tag{1.3}$$

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The formal reason why this asymptotic behavior for the energy holds, is that at main order the asymptotic behavior of u_ε is governed near the interface Γ by

$$u_\varepsilon(x) \approx w(z/\varepsilon) \tag{1.4}$$

where z is a choice of normal coordinate to Γ and w is the unique solution of the ordinary differential equation

$$w'' + w - w^3 = 0, \quad w(0) = 0, \quad w(\pm\infty) = \pm 1, \tag{1.5}$$

namely

$$w(\zeta) := \tanh(\zeta/\sqrt{2}). \tag{1.6}$$

Kohn and Sternberg [29] built local minimizers with this property in the two-dimensional case: Associated to a straight line segment Γ_0 contained in Ω which locally minimizes length among all curves nearby with endpoints lying on $\partial\Omega$, they find a local minimizer u_ε of J_ε with asymptotic interface given by this segment and property (1.3). Kowalczyk [31] considered a segment perpendicular to the boundary, not necessarily minimizing, but non-degenerate and found a solution with the above features. In [49] Pacard and Ritoré considered the Allen-Cahn equation on a compact Riemannian manifold and established that, associated to a minimal hypersurface, non-degenerate in the sense that its *Jacobi operator* is non-singular, a solution with a single interface exists.

In this paper we review some recent results on solutions to the Allen-Cahn equation that exhibit *multiple interfaces*. The prototype of the phenomena we deal with is the existence of a solution u_ε of (1.1) for $N = 2$, such that

$$J_\varepsilon(u_\varepsilon) \to m\,\frac{2}{3}\sqrt{2}\mathcal{H}^1(\Gamma). \tag{1.7}$$

for each given $m \geq 1$, which exhibit the transition behavior (1.4) along m close, nearly parallel copies of a critical segment. As proven in [15], such a solution indeed exists for instance associated to the shorter axis of an ellipse. A notable feature of these solutions is that their equilibrium location is governed by the well-known integrable *Toda System*.

On a compact manifold equilibrium of multiple interfaces is governed by the *Jacobi-Toda system* as found in [21]. This type of system also arises in building entire solutions to the Allen-Cahn equation in Euclidean space, and in the construction of traveling waves to the associated parabolic flow. Next we will introduce this system, and then discuss various results of this type.

2. The Toda and Jacobi-Toda systems.

2.1. The Jacobi operator. Let $(\mathcal{M}, \tilde g)$ be a compact or noncompact N-dimensional Riemannian manifold and $\mathcal{K}$ be a minimal $(N-1)$-dimensional embedded submanifold of $\mathcal{M}$. The *Jacobi operator* $\mathcal{J}$ of $\mathcal{K}$, corresponds to the second variation of N-volume along normal perturbations of $\mathcal{K}$ inside $\mathcal{M}$: given any smooth small function v on $\mathcal{K}$, let us consider the manifold $\mathcal{K}(v)$, the normal graph on $\mathcal{K}$ of the function v, namely the image of $\mathcal{K}$ by the map $p \in \mathcal{K} \mapsto \exp_p\big(v(p)\nu_{\mathcal{K}}(p)\big)$. If $H(v)$ denotes the mean curvature of $\mathcal{K}(v)$, defined as the arithmetic mean of the principal curvatures, then the linear operator $\mathcal{J}$ is the differential of the map $v \mapsto nH(v)$ at $v = 0$. More explicitly, it can be shown that

$$\mathcal{J}\psi \;=\; \Delta_{\mathcal{K}}\psi + |A_{\mathcal{K}}|^2\psi + \mathrm{Ric}_{\tilde g}(\nu_{\mathcal{K}}, \nu_{\mathcal{K}})\psi, \tag{2.1}$$

where $\Delta_{\mathcal{K}}$ is the Laplace-Beltrami operator on $\mathcal{K}$, $|A_{\mathcal{K}}|^2$ denotes the norm of the second fundamental form of $\mathcal{K}$, $\mathrm{Ric}_{\tilde{g}}$ is the Ricci tensor of $\mathcal{M}$ and $\nu_{\mathcal{K}}$ is a unit normal to $\mathcal{K}$.

The minimal submanifold $\mathcal{K}$ is said to be *nondegenerate* if the are no nontrivial smooth solutions to the homogeneous problem

$$\mathcal{J}\psi = 0 \quad \text{in } \mathcal{K}. \tag{2.2}$$

This condition implies that $\mathcal{K}$ is isolated as a minimal submanifold of $\mathcal{M}$.

In the noncompact case $\mathcal{M} = \mathbb{R}^N$, we have $\mathrm{Ric}_{\tilde{g}}(\nu_{\mathcal{K}}, \nu_{\mathcal{K}}) = 0$. If $\mathcal{K} = \mathbb{R}^{N-1}$, $\mathcal{J} = \Delta_{\mathbb{R}^{N-1}}$.

2.2. **The Toda system.** Next we introduce the integrable Toda system in $\mathbb{R}^N$. First we have the one-dimensional *Toda system*, given by

$$c_p^2 f_j'' = e^{f_{j-1}-f_j} - e^{f_j - f_{j+1}} \quad \text{in } \mathbb{R}, \quad j = 1, \dots, m, \tag{2.3}$$

with the conventions $f_0 = -\infty$, $f_{m+1} = +\infty$, where c_p is an explicit positive constant. Without loss of generality we may take $c_p = 1$. Equation (2.3) models m particles on the whole line interacting with the neighbors exponentially.

We introduce variables

$$u_j = q_{j+1} - q_j. \tag{2.4}$$

In terms of $\mathbf{u} = (u_1, \dots, u_{k-1})$ system (2.3) becomes

$$\mathbf{u}'' + M e^{\mathbf{u}} = 0 \tag{2.5}$$

where

$$M = \begin{pmatrix} 2 & -1 & 0\cdots & 0 \\ -1 & 2 & -1\cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 2 & -1 \\ 0 & \cdots & -1 & 2 \end{pmatrix}, \quad e^{\mathbf{u}} = \begin{pmatrix} e^{u_1} \\ \vdots \\ e^{u_{m-1}} \end{pmatrix}.$$

According to classical results of Kostant [30] and Moser [45], all solutions to (2.5) can be written explicitly (see formula (7.7.10) in [30]). They depend on $2(k-1)$ parameters. Given numbers $w_1, \dots, w_k \in \mathbb{R}$ such that

$$\sum_{j=1}^m w_j = 0, \quad \text{and } w_j > w_{j+1}, \quad j = 1, \dots, m, \tag{2.6}$$

we define the matrix

$$\mathbf{w}_0 = \mathrm{diag}\,(w_1, \dots, w_m).$$

Next, given numbers $g_1, \dots, g_m \in \mathbb{R}$ such that

$$\prod_{j=1}^m g_j = 1, \quad \text{and } g_j > 0, \quad j = 1, \dots, m, \tag{2.7}$$

we define the matrix

$$\mathbf{g}_0 = \mathrm{diag}\,(g_1, \dots, g_m).$$

Furthermore, we define functions $\Phi_j(\mathbf{g}_0, \mathbf{w}_0; z)$, $z \in \mathbb{R}$, $j = 0, \dots, m$, by

$$\begin{aligned} &\Phi_0 = \Phi_k \equiv 1 \\ &\Phi_j(\mathbf{g}_0, \mathbf{w}_0; z) = \\ &(-1)^{j(m-j)} \sum_{1 \le i_i < \cdots < i_j \le m} r_{i_1 \dots i_j}(\mathbf{w}_0) g_{i_1} \dots g_{i_j} \exp[-z(w_{i_1} + \cdots + w_{i_j})], \end{aligned} \tag{2.8}$$

where $r_{i_1\ldots i_j}(\mathbf{w}_0)$ are rational functions of the entries of the matrix $\mathbf{w}_0$.

According to [30] all solutions to (2.5) are given by

$$u_j(z) = 2\log\Phi_j(\mathbf{g}_0, \mathbf{w}_0; z) - \log\Phi_{j-1}(\mathbf{g}_0, \mathbf{w}_0; z) - \log\Phi_{j+1}(\mathbf{g}_0, \mathbf{w}_0; z). \tag{2.9}$$

Analogously, there is the two-dimensional Toda system

$$\begin{cases} \Delta\mathbf{u} + Me^{\mathbf{u}} = 0, \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{u_j} < +\infty, j = 1, ..., m-1. \end{cases} \tag{2.10}$$

System (2.10) is a natural generalization of the Liouville equation

$$\Delta u + e^u = 0 \text{ in } \mathbb{R}^2, \int_{\mathbb{R}^2} e^u < \infty. \tag{2.11}$$

The Liouville equation (2.11) and the Toda system (2.10) arise in many physical models. In Chern-Simons theories, the Liouville equation is related to abelian models, while the Toda system is related to nonabelian models. We refer to the books by Dunne [22] and Yang [55] for physical backgrounds. The $SU(3)$ Chern-Simons model has been studed in many papers. We refer to Jost-Wang [26], Jost-Lin-Wang [25], Malchiodi-Ndiaye [39], Ohtsuka-Suzuki [48] and the references therein.

Using algebraic geometry results, Jost and Wang ([27]) classified all solutions to (2.10). When $m = 2$, (2.10) is reduced to Liouville equation and the solutions are given

$$u = \log\frac{8a^2}{(a^2 + |x - x_0|^2)^2} \tag{2.12}$$

which is nondegenerate ([11]). When $m = 3$, all solutions to (2.10) can be written as follows:

$$u(z) = \log\frac{4\left(a_1^2a_2^2 + a_1^2|2z + c|^2 + a_2^2|z^2 + 2bz + bc - d|^2\right)}{\left(a_1^2 + a_2^2|z + b|^2 + |z^2 + cz + d|^2\right)^2}, \tag{2.13}$$

$$v(z) = \log\frac{16a_1^2a_2^2\left(a_1^2 + a_2^2|z + b|^2 + |z^2 + cz + d|^2\right)}{\left(a_1^2a_2^2 + a_1^2|2z + c|^2 + a_2^2|z^2 + 2bz + bc - d|^2\right)^2}, \tag{2.14}$$

where $z = x_1 + ix_2 \in \mathbb{C}$, $a_1 > 0$, $a_2 > 0$ are real numbers and $b = b_1 + ib_2 \in \mathbb{C}$, $c = c_1 + ic_2 \in \mathbb{C}$, $d = d_1 + id_2 \in \mathbb{C}$. Note that there are eight parameters $(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2) \in \mathbb{R}^8$. The nondegeneracy of the solutions has only been proved in the case of $m = 3$ ([54]).

2.3. **The Jacobi-Toda system.** Now we introduce the following the *Jacobi-Toda system*

$$\sigma\Big(\Delta_{\mathcal{K}} f_j + \big(|A_{\mathcal{K}}|^2 + \mathrm{Ric}_{\tilde{g}}(\nu_{\mathcal{K}}, \nu_{\mathcal{K}})\big) f_j\Big) - \left[e^{-(f_j - f_{j-1})} - e^{-(f_{j+1} - f_j)}\right] = 0 \tag{2.15}$$

on $\mathcal{K}$, $j = 1, \ldots, k$, with the convention $f_0 = -\infty$, $f_{k+1} = +\infty$.

Here $\sigma > 0$ is a positive constant (in many cases we also assume that σ is small).

Observe that if $\mathcal{K} = \mathbb{R}^N, N = 1, 2$, then (2.15) becomes the Toda system (2.3), (2.10). On the other hand, if we sum up (2.15) and let $\bar{f} = \sum_{j=1}^m f_j$, then we obtain the Jacobi operator $\mathcal{J}(\bar{f}) = 0$. Thus *both* the Jacobi operator and the Toda system are embedded in the system (2.15).

As we have mentioned in the introduction, the Toda system was used first in [15] to construct clustered interfaces in a bounded domain in $\mathbb{R}^2$ for the Allen-Cahn equation (1.1). Later it was used to construct multiple end solutions to the Allen-Cahn equation and nonlinear Schrodinger equation in entire $\mathbb{R}^2$ in [12] and [13].

The role of the Jacobi operator was found in [49] in constructing solutions to Allen-Cahn equation. The Jacobi-Toda system was discovered in [21]. Later different Jacobi-Toda systems have been derived in ([18], [19]).

In the following, we explain how we (2.15) arises in *various settings* and how we solve (2.15) in compact or noncompact cases: In Section 2, we consider Allen-Cahn equation on a compact manifold. In Secton 3, we derive (2.15) formally on a compact manifold. In Section 4, we explain how to solve (2.15) on a compact manifold. In Section 5, we derive and solve the Jacobi-Toda system on catenoid in $\mathbb{R}^3$. In Section 6, we show that how Jacobi-Toda system arises in traveling waves with multiple fronts in $\mathbb{R}^N$.

3. **The Jacobi-Toda system I: Foliated interfaces of Allen-Cahn equation on compact manifolds.**

3.1. **Allen-Cahn equation on manifolds.**

It is natural to consider situations in which phase transitions take place in a manifold rather than in a subset of Euclidean space. Namely, we consider the Allen-Cahn equation on a compact N-dimensional Riemannian manifold $(\mathcal{M}, \tilde{g})$

$$\varepsilon^2 \Delta_{\tilde{g}} u \,+\, (1-u^2)u \,=\, 0 \quad \text{in } \mathcal{M}, \tag{3.1}$$

where $\Delta_{\tilde{g}}$ is the Laplace-Beltrami operator on $\mathcal{M}$.

In [49], Pacard and Ritoré proved the following result: *assume that $\mathcal{K}$ is non-degenerate minimal $(N-1)$-dimensional submanifold which divides $\mathcal{M}$ into two connected components. Then there exists a solution u_ε to equation (3.1) with values close to ± 1 inside $\mathcal{M}_\pm$, whose (sharp) 0-level set is a smooth manifold which lies ε-close to $\mathcal{K}$.* More precisely, let $w(z) := \tanh\left(\frac{z}{\sqrt{2}}\right)$ be the unique solution of Problem (1.5). Then the solution u_ε in [49] resembles near $\mathcal{K}$ the function $w(z/\varepsilon)$, where z is a choice of signed geodesic distance to Γ.

3.2. **Multple interfaces.**

In [21], we found a a new phenomenon induced by the presence of positive curvature in the ambient manifold $\mathcal{M}$: in addition to non-degeneracy of $\mathcal{K}$, let us assume that

$$K \,:=\, |A_{\mathcal{K}}|^2 + \mathrm{Ric}_{\tilde{g}}(\nu_{\mathcal{K}}, \nu_{\mathcal{K}}) > 0 \quad \text{on } \mathcal{K}. \tag{3.2}$$

Then, besides the solution by Pacard and Ritoré, there are solutions with *multiple interfaces* collapsing onto $\mathcal{K}$. In fact, given any integer $m \geq 2$, we find a solution u_ε such that $u_\varepsilon^2 - 1$ approaches 0 in $\mathcal{M}_\pm$ as $\varepsilon \to 0$, with zero level set constituted by m smooth components $O(\varepsilon|\log\varepsilon|)$ distant one to each other and to $\mathcal{K}$.

Condition (3.2) is satisfied automatically if the manifold $\mathcal{M}$ has non-negative Ricci curvature. If $N = 2$, K corresponds simply to Gauss curvature of $\mathcal{M}$ measured along the geodesic $\mathcal{K}$.

The nature of these solutions is drastically different from the single-interface solution by Pacard and Ritoré. They are actually defined only if ε satisfies a *nonresonance condition* in ε. In fact, in the construction ε must remain suitably away from certain values where a shift in Morse index occurs. We expect that the solutions we find have a Morse index $O(|\log\varepsilon|^a)$ for some $a > 0$ as critical points of J_ε, while the single interface solution is likely to have its Morse index uniformly bounded by the index of $\mathcal{K}$ (namely the number of negative eigenvalues of the operator $\mathcal{J}$.

Theorem 1 ([21]). *Assume that $\mathcal{K}$ is nondegenerate, and that condition (3.2) is satisfied. Then, for each $m \geq 2$, there exists a sequence of values $\varepsilon = \varepsilon_j \to 0$ such that problem (3.1) has a solution u_ε such that $u_\varepsilon^2 - 1 \to 0$ uniformly on compact subsets of $\mathcal{M}_\pm$, while near $\mathcal{K}$, we have*

$$u_\varepsilon(x) = \sum_{\ell=1}^{m} w\left(\frac{z - \varepsilon f_\ell(y)}{\varepsilon}\right) + \frac{1}{2}\left((-1)^{m-1} - 1\right) + o(1),$$

where (z, y) are the Fermi coordinates defined near $\mathcal{K}$ through the exponential map, and the functions f_ℓ satisfy

$$f_\ell(y) = \left(\ell - \frac{m+1}{2}\right)\left[\sqrt{2}\log\frac{1}{\varepsilon} - \frac{1}{\sqrt{2}}\log\log\frac{1}{\varepsilon}\right] + O(1). \tag{3.3}$$

Moreover, when $N = 2$, there exist positive numbers $\nu_1, \ldots, \nu_{m-1}$ such that gi ven $c > 0$ and all sufficiently small $\varepsilon > 0$ satisfying

$$\left|\frac{1}{\log\frac{1}{\varepsilon}} - \frac{\nu_i}{j^2}\right| > \frac{c}{j^3}, \qquad \text{for all} \quad i = 1, \ldots, m-1, \quad j = 1, 2, \ldots. \tag{3.4}$$

a solution u_ε with the above properties exists.

We observe that the same result holds if m is *even* and $\mathcal{M} \setminus \mathcal{K}$ consists of just one component. Thus the condition that $\mathcal{K}$ divides $\mathcal{M}$ into two connected components is not essential in general.

As we shall derive formally, the equilibrium location of the interfaces is asymptotically governed by a small perturbation of the *Jacobi-Toda system*

$$\varepsilon^2\Big(\Delta_{\mathcal{K}} f_j + \left(|A_{\mathcal{K}}|^2 + \mathrm{Ric}_{\tilde{g}}(\nu_{\mathcal{K}}, \nu_{\mathcal{K}})\right) f_j\Big) - a_0\left[e^{-(f_j - f_{j-1})} - e^{-(f_{j+1} - f_j)}\right] = 0 \tag{3.5}$$

on $\mathcal{K}$, $j = 1, \ldots, m$, with the convention $f_0 = -\infty$, $f_{m+1} = +\infty$.

Heuristically, the interface foliation near $\mathcal{K}$ is possible due to a balance between the interfacial energy, which decreases as interfaces app roach each other, and the fact that the length or area of each individual interface increases as the interface is closer to $\mathcal{K}$ since $\mathcal{M}$ is positively curved near $\mathcal{K}$. What is unexpected, is the need of a *nonresonance condition* in order to solve the Jacobi-Toda system.

Similar resonance has been observed in (simple) concentration phenomena for various problems, see [14, 34, 37, 38]. The phenomenon of clustering of interfaces here discovered has an interesting resemblance with the problem of foliations of a neighborhood of a geodesic by CMC tubes considered in [35, 41].

Our result deals with situations in which the minimal submanifold is local but not globally area minimizing. In fact, since condition (3.2) holds, the Jacobi opeator has at least one negative eigenvalue, and near $\mathcal{K}$, $\mathcal{M}$ cannot have parabolic points. In the case of a bounded domain Ω of $\mathbb{R}^2$ under Neumann boundary conditions, a multiple-layer solution near a non -minimizing straight segment orthogonal to the boundary was built in [15]. In ODE cases for the Allen-Cahn equation, clustering interfaces had been previously observed in [9, 46, 47]. No resonance phenomenon is present in those situations, constituting a major qualitative difference with the current setting.

We do not expect that interface foliation occurs if the limiting interface is a minimizer of the perimeter. On the other hand, negative Gauss curvature seems also prevent interface foliation. This is suggested by a version of De Giorgi- Gibbons conjecture for problem (3.1) with $\mathcal{M}$ the hyperbolic space, established in [4].

4. **Formal derivation of Jacobi-Toda system.** In this section, we derive formally the Jacobi-Toda system on a compact manifold $\mathcal{M}$ on which we consider singularly perturbed Allen-Cahn equation (3.1).

4.1. **The Laplace-Beltrami and Jacobi operators.** If $(\mathcal{M}, \tilde{g})$ is an N-dimensional Riemannian manifold and $\mathcal{K} \subset \mathcal{M}$ be an $(N-1)$-dimensional closed smooth embedded submanifold associated with the metric $\tilde{g}_0$ induced from $(\mathcal{M}, \tilde{g})$. Let $\Delta_{\mathcal{K}}$ be the Laplace-Beltrami operator defined on $\mathcal{K}$.

Let us consider the space $C^\infty(N\mathcal{K})$ of all smooth normal vector fields on $\mathcal{K}$. Since $\mathcal{K}$ is a submanifold of codimension 1, then given a choice of orientation and unit normal vector field along $\mathcal{K}$, denoted by $\nu_{\mathcal{K}} \in N\mathcal{K}$, we can write $\Phi \in C^\infty(N\mathcal{K})$ as $\Phi = \phi\nu_{\mathcal{K}}$, where $\phi \in C^\infty(\mathcal{K})$.

For $\Phi \in C^\infty(N\mathcal{K})$, we consider the one-parameter family of submanifolds $t \to \mathcal{K}_{t,\Phi}$ given by

$$\mathcal{K}_{t,\Phi} \equiv \Big\{ \exp_{f(\tilde{y})}\big(t\Phi(f(\tilde{y}))\big) : f(\tilde{y}) \in \mathcal{K} \Big\}. \tag{4.1}$$

The first variation formula of the volume functional is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Vol}(\mathcal{K}_{t,\Phi}) = \int_{\mathcal{K}} < \Phi, \mathbf{h} >_N \mathrm{d}V_{\mathcal{K}}, \tag{4.2}$$

where $\mathbf{h}$ is the mean curvature vector of $\mathcal{K}$ in $\mathcal{M}$, $< \cdot, \cdot >_N$ denotes the restriction of $\tilde{g}$ to $N\mathcal{K}$, and $\mathrm{d}V_{\mathcal{K}}$ the volume element of $\mathcal{K}$.

The submanifold $\mathcal{K}$ is said to be *minimal* if it is stationary point for the volume functional, namely if

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Vol}(\mathcal{K}_{t,\Phi}) = 0 \quad \text{for any } \Phi \in C^\infty(N\mathcal{K}), \tag{4.3}$$

or equivalently by (4.2), if the mean curvature $\mathbf{h}$ is identically zero on $\mathcal{K}$.

The *Jacobi operator* $\mathcal{J}$ appears in the expression of the second variation of the volume functional for a minimal submanifold $\mathcal{K}$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} \mathrm{Vol}(\mathcal{K}_{t,\Phi}) = -\int_{\Gamma} < \mathcal{J}\Phi, \Phi >_N \mathrm{d}V_{\mathcal{K}} \quad \text{for any } \Phi \in C^\infty(N\mathcal{K}) \tag{4.4}$$

and is given by

$$\mathcal{J}\phi = -\Delta_{\mathcal{K}}\phi - \mathrm{Ric}_{\tilde{g}}(\nu_{\mathcal{K}}, \nu_{\mathcal{K}})\phi - |A_{\mathcal{K}}|^2\phi, \tag{4.5}$$

where $\Phi = \phi\nu_{\mathcal{K}}$, as has been explained above.

The submanifold $\mathcal{K}$ is said to be *non-degenerate* if the Jacobi operator $\mathcal{J}$ is invertible, or equivalently if the equation $\mathcal{J}\phi = 0$ has only the trivial solution in $C^\infty(\mathcal{K})$.

4.2. **Stretched Fermi coordinates and expansion of the Laplace-Beltrami operator.** To construct the approximation to a solution of (3.1), which concentrates near $\mathcal{K}$, after rescaling, in $\mathcal{M}/\varepsilon$. We introduce stretched Fermi coordinates in the neighborhood $\varepsilon^{-1}\mathcal{V}$ of the point $\varepsilon^{-1}p \in \varepsilon^{-1}\mathcal{K}$ by

$$\Phi_\varepsilon(\mathtt{y}, z) = \frac{1}{\varepsilon}\Phi^0(\varepsilon\mathtt{y}, \varepsilon z), \quad (\mathtt{y}, z) = (\mathtt{y}_1, \cdots, \mathtt{y}_{N-1}, z) \in \varepsilon^{-1}\mathcal{V} \times (-\frac{\delta_0}{\varepsilon}, \frac{\delta_0}{\varepsilon}). \tag{4.6}$$

Obviously, in $\mathcal{M}_\varepsilon = \varepsilon^{-1}\mathcal{M}$ the new coefficients g_{ab}'s of the Riemannian metric, after rescaling, can be written as

$$g_{ab}(\mathtt{y}, z) = \tilde{g}_{ab}(\varepsilon\mathtt{y}, \varepsilon z,), \quad a, b = 1, 2, \cdots, N.$$

Lemma 4.1. *In the above coordinates* (y, z), *for any* $i, j = 1, 2, \cdots, N-1$, *we have*

$$g_{ij}(\mathrm{y}, z) \;=\; \delta_i^j - 2\varepsilon\Gamma_i^j(E_N)z + \varepsilon^2\Big[-R(X_j, X_N, X_N, X_i) + \sum_{k=1}^{N-1}\Gamma_i^k(E_N)\Gamma_k^j(E_N)\Big]z^2 + O(|\varepsilon z|^3), \tag{4.7}$$

$$g_{iN}(\mathrm{y}, z) \;=\; O((\varepsilon z)^2), \tag{4.8}$$

$$g_{NN}(\mathrm{y}, z) \;=\; 1 + O(|\varepsilon z|^3). \tag{4.9}$$

Here $R(\cdot)$ *and* Γ_a^b *are the Ricci curvature tensor and Christoffel symbols repectively. They are computed at the point* $p \in \mathcal{K}$ *parameterized by* $(0, 0)$.

Now we will focus on the expansion of the Laplace-Beltrami operator defined by

$$\begin{aligned}\Delta_{\mathcal{M}_\varepsilon} &= \frac{1}{\sqrt{\det g}}\,\partial_a\Big(\,g^{ab}\,\sqrt{\det g}\,\partial_b\,\Big)\\ &= g^{ab}\,\partial_a\partial_b \,+\, (\partial_a g^{ab})\,\partial_b \,+\, \frac{1}{2}\,\partial_a\big(\log(\det g)\,\big)\,g^{ab}\,\partial_b.\end{aligned} \tag{4.10}$$

Using the assumption that the submanifold $\mathcal{K}$ is minimal, direct computation gives that

$$\det g = 1 - \varepsilon^2 K(\varepsilon y) z^2 + O(\varepsilon^3 |z|^3),$$

where we have denoted

$$K = \mathrm{Ric}_{\tilde{g}}(\nu_{\mathcal{K}}, \nu_{\mathcal{K}}) + |A_{\mathcal{K}}|^2. \tag{4.11}$$

This gives

$$\log(\det g) = -\varepsilon^2 K(\varepsilon y) z^2 + O(\varepsilon^3 |z|^3).$$

Hence, we have the expansion

$$\Delta_{\mathcal{M}_\varepsilon} = \partial_{zz} \,+\, \Delta_{\mathcal{K}_\varepsilon} \,+\, \varepsilon^2 z K(\varepsilon y)\,\partial_z \,+\, B \tag{4.12}$$

where the operator B has the form

$$B \;=\; \varepsilon z\, a_{ij}^1\,\partial_{ij} \,+\, \varepsilon^2 z^2\, a_{iN}^2\,\partial_{iz} \,+\, \varepsilon^3 z^3\, a_{NN}^3\,\partial_{zz} \,+\, \varepsilon^2 z\, b_i^1 \partial_i \,+\, \varepsilon^3 z^2\, b_N^2\,\partial_z\,. \tag{4.13}$$

and all the coefficients are smooth functions defined on a neighborhood of $\mathcal{K}$ in $\mathcal{M}$, evaluated at $(\varepsilon y, \varepsilon z)$.

4.3. **The approximate solution.** If we set $u(x) := \tilde{u}(\varepsilon x)$, then problem (3.1) is thus equivalent to

$$\Delta_{\mathcal{M}_\varepsilon} u \,+\, F(u) = 0 \quad \text{ in } \mathcal{M}_\varepsilon, \tag{4.14}$$

where $F(u) \equiv u - u^3$. In the sequel, we denote by $\mathcal{M}_\varepsilon$ and $\mathcal{K}_\varepsilon$ the ε^{-1}-dilations of $\mathcal{M}$ and $\mathcal{K}$.

To define the approximate solution we observe the heteroclinic solution to (1.5) has the asymptotic properties

$$\begin{aligned} w(z) \,-\, 1 &= -2\,e^{-\sqrt{2}\,|z|} \,+\, O\big(e^{-2\sqrt{2}\,|z|}\big), \quad z > 1,\\ w(z) \,+\, 1 &= 2\,e^{-\sqrt{2}\,|z|} \,+\, O\big(e^{-2\sqrt{2}\,|z|}\big), \quad z < -1,\\ w'(z) &= 2\sqrt{2}\,e^{-\sqrt{2}\,|z|} \,+\, O\big(e^{-2\sqrt{2}\,|z|}\big), \quad |z| > 1,\end{aligned} \tag{4.15}$$

where A_0 is a universal constant. For a fixed integer $m \geq 2$, we assume that the location of the m phase transition layers are characterized in the coordinate (y, z) defined in (4.6) by the functions $z = f_j(\varepsilon y)$, $1 \leq j \leq m$ with

$$f_1(\varepsilon y) < f_2(\varepsilon y) < \cdots < f_m(\varepsilon y),$$

separated one to each other by large distances $O(\log \varepsilon)$, and define in coordinates (y, z) the first approximation

$$u_0(y, z) := \sum_{j=1}^{m} w_j(z - f_j(\varepsilon y)) + \frac{(-1)^{m-1} - 1}{2}, \quad w_j(t) := (-1)^{j-1} w(t). \tag{4.16}$$

With this definition we have that $u_0(y, z) \approx w_j(z - f_j(\varepsilon y))$ for values of z close to $f_j(\varepsilon y)$.

The functions $f_j : \mathcal{K} \to \mathbb{R}$ will be left as parameters, on which we will assume a set of constraints that we describe next.

Let us fix numbers $p > N$, $M > 0$, and consider functions $h_j \in W^{2,p}(\mathcal{K})$, $j = 1, \dots, m$ such that

$$\begin{aligned} &\|h_j\|_{W^{2,p}(\mathcal{K})} := \|D^2_{\mathcal{K}} h_j\|_{L^p(\mathcal{K})} + \|D_{\mathcal{K}} h_j\|_{L^p(\mathcal{K})} + \|h_j\|_{L^\infty(\mathcal{K})} \\ \leq &M, \quad \text{for all} \quad j = 1, \dots, m. \end{aligned} \tag{4.17}$$

For a small $\varepsilon > 0$, we consider the unique number $\rho = \rho_\varepsilon$ with

$$e^{-\sqrt{2}\,\rho} = \varepsilon^2 \rho. \tag{4.18}$$

We observe that ρ_ε is a large number that can be expanded in ε as

$$\rho_\varepsilon = \sqrt{2} \log \frac{1}{\varepsilon} - \frac{1}{\sqrt{2}} \log\Big(\sqrt{2} \log \frac{1}{\varepsilon}\Big) + O\Big(\frac{\log \log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon}}\Big).$$

Then we assume that the m functions $f_j : \mathcal{K} \to \mathbb{R}$ are given by the relations

$$f_k(y) = \left(k - \frac{m+1}{2}\right) \rho_\varepsilon + h_k(y), \quad k = 1, \dots, m. \tag{4.19}$$

so that

$$f_{k+1}(y) - f_k(y) = \rho_\varepsilon + h_{k+1}(y) - h_k(y)\,, \quad k = 1, 2, \dots, m-1. \tag{4.20}$$

We we will use in addition the conventions $h_0 \equiv -\infty$, $h_{m+1} \equiv +\infty$.

Our first goal is to compute the error of approximation in a δ_0/ε neighborhood of $\mathcal{K}_\varepsilon$, namely the quantity:

$$S(u_0) \equiv \Delta_{\mathcal{M}_\varepsilon} u_0 + F(u_0). \tag{4.21}$$

For each fixed ℓ, $1 \leq \ell \leq m$, this error reproduces a similar pattern on each set of the form

$$A_\ell = \left\{ (y, z) \in \big(-\frac{\delta_0}{\varepsilon}, \frac{\delta_0}{\varepsilon}\big) \times \mathcal{K}_\varepsilon \ / \ |z - f_\ell(\varepsilon y)| \leq \frac{1}{2}\rho_\varepsilon + M \,\right\}. \tag{4.22}$$

For $(y, z) \in A_\ell$, we write $t = z - f_\ell(\varepsilon y)$ and estimate in this range the quantity $S(u_0)(t + f_\ell(\varepsilon y), y)$. We have the validity of the following expression.

Lemma 4.2. *For $\ell \in \{1,\dots,m\}$ and $(y,z)\in A_\ell$ we have*

$$
\begin{aligned}
&(-1)^{\ell-1}S(u_0)(t+f_\ell,y)\\
=&6(1-w^2(t))\varepsilon^2\rho_\varepsilon\left[e^{-\sqrt{2}(h_\ell-h_{\ell-1})}e^{\sqrt{2}t}-e^{-\sqrt{2}(h_{\ell+1}-h_\ell)}e^{-\sqrt{2}t}\right]\\
&-\varepsilon^2\Big(\Delta_{\mathcal{K}}h_\ell+(t+f_\ell)K\Big)w'(t)+\varepsilon^2|\nabla_{\mathcal{K}}h_\ell|^2w''(t)+(-1)^{\ell-1}\Theta_\ell(\varepsilon y,t)\,. \quad (4.23)
\end{aligned}
$$

where for some $\tau,\sigma>0$ we have

$$\|\Theta_\ell(\cdot,t)\|_{L^p(\mathcal{K})}\le C\varepsilon^{2+\tau}e^{-\sigma|t|}.$$

The proof of this lemma is quite lengthy and can be found in Lemma 2.4 of [21].

4.4. **The Jacobi-Toda system as a solvability condition.** We want to find a solution to

$$S(u)=\Delta_{\mathcal{M}_\varepsilon}u+F(u)$$

in the form

$$u=u_0+\phi.$$

Expanding the operator S

$$S(u_0+\phi)=S(u_0)+S^{'}(u_0)[\phi]+N[\phi] \quad (4.24)$$

where $N[\phi]=O(\phi^2)$ and $S^{'}(u_0)[\phi]$ is linearized operator at u_0 which can be formally approximated by

$$S^{'}(u_0)[\phi]\sim\phi_{tt}+F^{'}(w)\phi+\Delta_{\mathcal{K}_\epsilon}\phi.$$

The leading order operator $\phi_{tt}+F^{'}(w)\phi$ has a one-dimensional kernel $w^{'}$.

The infinite-dimensional reduction method, developed in [14], allows us to reduce the problem (4.24) to finding an $\mathtt{h}$ such that for all $\ell=1,\dots,m$, we have

$$I_\ell(y)=\int_{\mathbb{R}}(S_\ell(\mathtt{h}))\,w'\,dt\;=0\quad\text{for all}\quad y\in\mathcal{K}_\varepsilon. \quad (4.25)$$

Let us compute this function more explicitly.

Using the definition of S_ℓ in (4.23), we get

$$
\begin{aligned}
&\varepsilon^{-2}I_\ell(\varepsilon^{-1}y)\\
=&b_1\Big(\Delta_{\mathcal{K}}h_\ell+K(y)b_\ell,\Big)-b_2\rho_\varepsilon\left[e^{-\sqrt{2}(h_\ell-h_{\ell-1})}-e^{-\sqrt{2}(h_{\ell+1}-h_\ell)}\right]+\theta_\ell(\mathtt{h}), \quad (4.26)
\end{aligned}
$$

where θ_ℓ is a small operator:

$$\|\theta_\ell(\mathtt{h})\|_{L^p(\mathcal{K})}\;=\;O(\varepsilon^{1-\tau})$$

for any $\tau>\frac{N-1}{p}$, uniformly on $\mathtt{h}$.

The constants b_1,b_2 are given by

$$b_1=\int_{\mathbb{R}}w'(t)^2\,dt,\quad b_2=\int_{\mathbb{R}}6(1-w^2(t))e^{\sqrt{2}t}w'(t)dt=\int_{\mathbb{R}}6(1-w^2(t))e^{-\sqrt{2}t}w'(t)dt.$$

We recall that $f_\ell(y)=(\ell-\frac{m+1}{2})\rho_\varepsilon+h_\ell(y)$. Since we want that the functions h_ℓ make the quantities I_ℓ as small as possible, it is reasonable to find first an $\mathtt{h}$ such that the equations

$$b_1\left(\Delta_{\mathcal{K}} h_\ell + K(y)h_\ell,\right) - b_2\rho_\varepsilon\left[e^{-\sqrt{2}(h_\ell - h_{\ell-1})} - e^{-\sqrt{2}(h_{\ell+1}-h_\ell)}\right] = 0, \quad \ell = 1,\dots m, \tag{4.27}$$

be approximately satisfied. Thus we have derived the location of the interfaces *formally* satisfy (4.27) which is the Jacobi-Toda system (2.15). In fact, we set

$$R_\ell(\mathtt{h}) := \sigma\Big(\Delta_{\mathcal{K}} h_\ell + K(y)h_\ell\Big) - \left[e^{-\sqrt{2}(h_\ell - h_{\ell-1})} - e^{-\sqrt{2}(h_{\ell+1}-h_\ell)}\right], \tag{4.28}$$

where

$$\sigma := \sigma_\varepsilon = \rho_\varepsilon^{-1} b_1 b_2^{-1} \sim (\log\frac{1}{\varepsilon})^{-1}\ .$$

Then the Jacobi-Toda system becomes

$$\mathbf{R}(\mathtt{h}) := \begin{bmatrix} R_1(\mathtt{h}) \\ \vdots \\ R_m(\mathtt{h}) \end{bmatrix}. \tag{4.29}$$

We shall solve this system in the next section.

5. **Solvability of Jacobi-Toda system.** In this section, we proceed to find a solution $\mathtt{h}$ to the system $\mathbf{R}(\mathtt{h}) = 0$.

Before we solve the most general case, let us consider the simple case $m = 2$ first. Our aim is to find what the difficulties are. If we set

$$\mathtt{v}_1 = h_1 - h_2, \mathtt{v}_2 = h_1 + h_2$$

then $\mathtt{v}_2$ satisfies

$$\mathcal{J}(\mathtt{v}_2) = 0 \tag{5.1}$$

which implies that $\mathtt{v}_2 = 0$ under the nondegeneracy condition of $\mathcal{K}$. Now the equation for $\mathtt{v}_1$ becomes

$$\sigma(\Delta_{\mathcal{K}}\mathtt{v}_1 + K(y)\mathtt{v}_1) + e^{\sqrt{2}\mathtt{v}_1} = 0 \text{ on } \mathcal{K}. \tag{5.2}$$

Equation (5.2) becomes *supercritical* as long as $N \geq 3$. Even in one or two-dimensional case, it is unclear how to obtain a solution and furthermore we have to perturb such solution (because in reality a small right hand side is also prsent).

Let us assume that $K(y) \equiv 1$. Then (5.3) does have a solution satisfies the *algebraic equation*

$$\sigma(\mathtt{v}_1^0) + e^{\sqrt{2}\mathtt{v}_1^0} = 0 \text{ on } \mathcal{K}. \tag{5.3}$$

But if we perturb this solution $\mathtt{v}_1^0$, the linearized operator becomes

$$\Delta_{\mathcal{K}} h + K(y)h - \mathtt{v}_1^0 h \tag{5.4}$$

which has *resonance* on $\mathcal{K}$ since $\mathtt{v}_1^0 \sim \log\sigma$.

In conclusion, we see that even in the constant $K(y)$ case we should expect *resonance*. When $m > 2$, the matter becomes worse because we have coupled systems.

However, the above procedure also suggests a general scheme of solving $\mathbf{R}(h) = 0$ we shall now describe next.

We find first a convenient representation of the operator $\mathbf{R}(\mathtt{h})$. Let us consider the auxiliary variables

$$\mathtt{v} := \begin{bmatrix} \bar{\mathtt{v}} \\ v_m \end{bmatrix}, \quad \bar{\mathtt{v}} = \begin{bmatrix} v_1 \\ \vdots \\ v_{m-1} \end{bmatrix},$$

defined in terms of $\mathtt{h}$ as

$$v_\ell = h_{\ell+1} - h_\ell, \quad \ell = 1, \ldots, m-1, \quad v_m = \sum_{\ell=1}^{m} h_\ell,$$

and define the operator

$$\mathbf{S}(\mathtt{v}) := \begin{bmatrix} \bar{\mathbf{S}}(\bar{\mathtt{v}}) \\ S_m(v_m) \end{bmatrix}, \quad \bar{\mathbf{S}}(\bar{\mathtt{v}}) = \begin{bmatrix} S_1(\bar{\mathtt{v}}) \\ \vdots \\ S_{m-1}(\bar{\mathtt{v}}) \end{bmatrix},$$

where for $l = 1, \ldots, m-1$, and with the conventions $v_0 = v_{m+1} = +\infty$ we set

$$S_\ell(\mathtt{v}) := R_{\ell+1}(\mathtt{h}) - R_\ell(\mathtt{h}) \ =$$

$$\sigma\Big(\Delta_{\mathcal{K}} v_\ell + K(y)(\rho_\varepsilon + v_\ell),\Big) + \begin{cases} e^{-\sqrt{2}v_2} - 2e^{-\sqrt{2}v_1} & \text{if} \quad \ell = 1 \\ e^{-\sqrt{2}v_{\ell+1}} - 2e^{-\sqrt{2}v_\ell} + e^{-\sqrt{2}v_{\ell-1}} & \text{if} \quad 1 < \ell < m-1 \\ -2e^{-\sqrt{2}v_{m-1}} + e^{-\sqrt{2}v_{m-2}} & \text{if} \quad \ell = m-1 \end{cases}$$

and

$$S_m(\mathtt{v}) := \sum_{\ell=1}^{m} R_\ell(\mathtt{h}) = \sigma\Big(\Delta_{\mathcal{K}} v_m + K(y) v_m\Big).$$

Then the operators $\mathbf{R}$ and $\mathbf{S}$ are in correspondence through the formula

$$\mathbf{S}(\mathtt{v}) \ = \ \mathbf{B}\,\mathbf{R}\left(\mathbf{B}^{-1}\mathtt{v}\right), \tag{5.5}$$

where $\mathbf{B}$ is the constant, invertible $N \times N$ matrix

$$\mathbf{B} \ = \ \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \\ 1 & \cdots & 1 & 1 & 1 \end{bmatrix}, \tag{5.6}$$

and then the system $\mathbf{R}(\mathtt{h}) = 0$ is equivalent to $\mathbf{S}(\mathtt{v}) = 0$, which setting $\beta = b_2 b_1^{-1}$ decouples into

$$\bar{\mathbf{S}}(\bar{\mathtt{v}}) \ = \ \sigma\Big[\Delta_{\mathcal{K}}\bar{\mathtt{v}} + K(y)\bar{\mathtt{v}}\Big] \ + \ \beta K(y) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \ + \ \bar{\mathbf{S}}_0(\bar{\mathtt{v}}) \ = \ 0, \tag{5.7}$$

$$S_m(v_m) \ = \ \sigma\left(\Delta_{\mathcal{K}} v_m + K(y) v_m\right) \ = \ 0, \tag{5.8}$$

where

$$\bar{\mathbf{S}}_0(\bar{\mathtt{v}}) := -\mathbf{C} \begin{bmatrix} e^{-\sqrt{2}v_1} \\ \vdots \\ e^{-\sqrt{2}v_{m-1}} \end{bmatrix}, \quad \mathbf{C} \ = \ \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & & -1 & 2 \end{bmatrix}. \tag{5.9}$$

In system (5.7)-(5.8), the second relation and our non-degeneracy assumption forces $v_m = 0$. Thus we look for a solution $\mathtt{v} = (\bar{\mathtt{v}}, 0)$ of the system, where $\bar{\mathtt{v}}$ satisfies (5.7).

To solve (5.7), we proceed in two steps:

Step I: Good approximate solutions

Rather than finding an exact solution $\bar{\mathtt{v}}$ of $\bar{\mathbf{S}}(\bar{\mathtt{v}}) = 0$ we will find a good approximation. More precisely, by means of a simple iterative procedure, we will find for each $k \geq 1$ a function $\bar{\mathtt{v}}^k$ with the property that

$$\bar{\mathbf{S}}(\bar{\mathtt{v}}^k) = O(\sigma^k). \tag{5.10}$$

Step II: Inverting the linearized Jacobi-Toda operator

Our next task is to prove the existence of of a true solution to (5.7)-(5.8) in the form

$$\bar{\mathtt{v}} = \bar{\mathtt{v}}^k + \omega \tag{5.11}$$

The problem is then to invert the linearized operator at $\mathtt{v}^k$.

5.1. **Iteration: Good approximate solutions.** Let us find a function $\bar{\mathtt{v}}^1$ with the desired property (5.10) for $k = 1$. We consider the vector $\bar{\mathtt{v}}^1(y)$ defined by the relations

$$\bar{\mathbf{S}}_0(\bar{\mathtt{v}}^1) \;=\; -\mathbf{C}\begin{bmatrix} e^{-\sqrt{2}v_1^1} \\ \vdots \\ e^{-\sqrt{2}v_{m-1}^1} \end{bmatrix} = -\beta K(y)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

We compute explicitly

$$v_\ell^1(y) = \frac{1}{\sqrt{2}}\log\left[\frac{\beta}{2}K(y)\,(m-\ell)\,\ell\right], \ell = 1, \dots, m-1, \tag{5.12}$$

and get from (5.7)

$$\bar{\mathbf{S}}(\bar{\mathtt{v}}^1) \;=\; \sigma\left[\Delta_{\mathcal{K}}\bar{\mathtt{v}}^1 + K(y)\bar{\mathtt{v}}^1\right] \;=\; O(\sigma).$$

This approximation can be improved to any order in powers of σ, as the following lemma states.

Lemma 5.1. *Given $k \geq 1$, there exists a function of the form*

$$\bar{\mathtt{v}}^k(y,\sigma) = \bar{\mathtt{v}}^1(y) + \sigma\xi_k(y,\sigma),$$

where $\bar{\mathtt{v}}^1(y)$ is defined by (5.12), $\xi_1 \equiv 0$, and ξ_k is smooth on $[0,\infty)\times\mathcal{K}$, such that

$$\bar{\mathbf{S}}(\bar{\mathtt{v}}^k) = O(\sigma^k)$$

as $\sigma \to 0$, uniformly on $\mathcal{K}$. In particular,

$$\mathtt{h}^k := \mathbf{B}^{-1}\begin{bmatrix} \bar{\mathtt{v}}^k \\ 0 \end{bmatrix},$$

with $\mathbf{B}$ is given by (5.6), solves approximately system (4.27)in the sense that

$$\mathbf{R}(\mathtt{h}^k) = O(\sigma^k).$$

Proof. In order to find a subsequent improvement of approximation beyond $\mathtt{v}^1$, we set $\bar{\mathtt{v}}^2 = \bar{\mathtt{v}}^1 + \omega_1$. Let us expand

$$\bar{\mathbf{S}}(\bar{\mathtt{v}}^1 + \omega) \;=\; \sigma\Big[\Delta_{\mathcal{K}}\omega + K(y)\omega\Big] + \sigma(\Delta_{\mathcal{K}}\mathtt{v}^1 + K(y)\mathtt{v}^1) + D\bar{\mathbf{S}}_0(\bar{\mathtt{v}}^1)\omega + \mathbf{N}(\omega), \tag{5.13}$$

where

$$D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^1) \;=\; \sqrt{2}\mathbf{C}\begin{bmatrix} e^{-\sqrt{2}v_1^1} & 0 & \cdots & 0 \\ 0 & e^{-\sqrt{2}v_2^1} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\sqrt{2}v_{m-1}^1} \end{bmatrix}$$

$$= \frac{\beta}{\sqrt{2}}\,K(y)\begin{bmatrix} 2a_1 & -a_2 & 0 & & \cdots & 0 \\ -a_1 & 2a_2 & -a_3 & & \cdots & 0 \\ 0 & -a_2 & 2a_3 & & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & \cdots & -a_{m-3} & 2a_{m-2} & -a_{m-1} \\ 0 & & \cdots & & -a_{m-2} & 2a_{m-1} \end{bmatrix}, \tag{5.14}$$

with

$$a_\ell \;=\; (m-\ell)\,\ell \quad \ell = 1,\dots,m-1, \tag{5.15}$$

and

$$\mathbf{N}(\bar{\mathrm{v}}) \;=\; \frac{\beta}{2}\,\mathbf{C}\begin{bmatrix} a_2(\,e^{-\sqrt{2}v_1^0} - 1 + \sqrt{2}\,v_1^0\,) \\ \vdots \\ a_m(\,e^{-\sqrt{2}v_m^0} - 1 + \sqrt{2}\,v_m^0\,) \end{bmatrix}.$$

The matrix $D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^1)$ is clearly invertible. Let us consider the unique solution $\omega_1 = O(\sigma)$ of the linear system

$$D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^1)\omega_1 \;=\; -\sigma(\Delta_{\mathcal{K}}\bar{\mathrm{v}}^1 + K(y)\bar{\mathrm{v}}^1) \;=\; O(\sigma) \tag{5.16}$$

and define $\bar{\mathrm{v}}^2 = \bar{\mathrm{v}}^1 + \omega_1$. Then from (5.13) we have

$$\bar{\mathbf{S}}(\bar{\mathrm{v}}^2) \;=\; \sigma(\Delta_{\mathcal{K}}\omega_1 + K(y)\omega_1) + \mathbf{N}(\omega_1) \;=\; O(\sigma^2). \tag{5.17}$$

Next we define $\bar{\mathrm{v}}^3 = \bar{\mathrm{v}}^2 + \omega_2$ where $\omega_2 = O(\sigma^2)$ is the unique solution of

$$-\,D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^1)\omega_2 \;=\; \sigma(\Delta_{\mathcal{K}}\omega_1 + K(y)\omega_1) + \mathbf{N}(\omega_1). \tag{5.18}$$

Then from (5.13) we get

$$\bar{\mathbf{S}}(\bar{\mathrm{v}}^3) \;=\; \sigma(\Delta_{\mathcal{K}}\bar{\omega}_2 + K(y)\bar{\omega}_2) + \mathbf{N}(\omega_1+\omega_2) - \mathbf{N}(\omega_1) \;=\; O(\sigma^2). \tag{5.19}$$

In general, we define inductively, for $k \geq 3$, $\bar{\mathrm{v}}^{k+1} = \bar{\mathrm{v}}^k + \omega_k$ where ω_k is the unique solution of the linear system

$$-\,D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^1)\,\omega_k = \sigma(\Delta_{\mathcal{K}}\omega_{k-1} + K(y)\omega_{k-1}) + \mathbf{N}(\omega_1+\cdots+\omega_{k-1}) - \mathbf{N}(\omega_1+\cdots+\omega_{k-2}). \tag{5.20}$$

Then clearly $\omega_k = O(\sigma^k)$. Let us estimate the size of $\bar{\mathbf{S}}(\bar{\mathrm{v}}^{k+1})$ From (5.13) we have

$$\bar{\mathbf{S}}(\bar{\mathrm{v}}^{k+1}) \;=\; \sigma(\Delta_{\mathcal{K}}\bar{\mathrm{v}}^1 + K(y)\bar{\mathrm{v}}^1) + \big[\sigma(\Delta_{\mathcal{K}} + K) + D\mathbf{S}_0(\bar{\mathrm{v}}^1) + \mathbf{N}\,\big]\,(\sum_{i=1}^{k}\omega_i\,).$$

Now, using (5.16), (5.18) and (5.20) we get

$$[\sigma(\Delta_{\mathcal K}+K)+D\mathbf{S}_0(\bar{\mathrm v}^1)]\,(\sum_{i=1}^{k}\omega^i\,)\;=\;\sigma(\Delta_{\mathcal K}\bar{\mathrm v}^1+K\bar{\mathrm v}^1)\;+\;D\bar{\mathbf S}_0(\bar{\mathrm v}^1)\omega^1\;+$$

$$\sigma(\Delta_{\mathcal K}\omega_k+K\omega_k)\;+\;\sum_{i=2}^{k}\big[\sigma(\Delta_{\mathcal K}\omega_{i-1}+K\omega_{i-1})+D\bar{\mathbf S}_0(\bar{\mathrm v}^1)\omega_i\big]\;=$$

$$0-\mathbf{N}(\omega^1)-\sum_{i=3}^{k}[\mathbf{N}(\omega_1+\cdots+\omega_{i-1})-\mathbf{N}(\omega_1+\cdots+\omega_{i-2})]\;=-\mathbf{N}(\omega_1+\cdots+\omega_k)\;.$$

Hence,

$$\bar{\mathbf S}(\bar{\mathrm v}^{k+1})\;=\;\sigma(\Delta_{\mathcal K}\omega_k+K\omega_k)+\mathbf{N}(\omega_1+\cdots+\omega_{k-1}+\omega_k)-\mathbf{N}(\omega_1+\cdots+\omega_{k-1})\;=\;O(\sigma^{k+1}),\tag{5.21}$$

Finally, the functions $\xi_1\equiv 0$ and

$$\xi_k:=\sigma^{-1}(\omega_1+\cdots+\omega_{k-1}),\quad k\ge 2,$$

clearly satisfy the conclusions of the lemma, and the proof is concluded. □

The question now, is how to use the approximation $\mathtt{h}^k$ just constructed to find an exact $\mathtt{h}$ solution to system (4.25). This system takes the form

$$\mathbf{R}(\mathtt{h})=g\tag{5.22}$$

where g is a small function, actually a small nonlinear operator in $\mathtt{h}$. For the moment we will think of g as a fixed function. Since the operator $\mathbf{R}$ decouples as in (5.5) when expressed in terms of $\mathbf{S}$, it is more convenient to consider the equivalent problem

$$\mathbf{S}(\mathrm{v})=g\tag{5.23}$$

which, according to expressions (5.7) and (5.8), decouples as

$$\bar{\mathbf S}(\bar{\mathrm v})\;=\;\sigma\Big[\Delta_{\mathcal K}\bar{\mathrm v}+K(y)\bar{\mathrm v}\Big]\;+\;\beta K(y)\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}\;+\;\bar{\mathbf S}_0(\bar{\mathrm v})\;=\;\bar g,\tag{5.24}$$

$$\mathbf{S}_m(v_m)\;=\;\sigma\left(\Delta_{\mathcal K}v_m+K(y)v_m\right)\;=\;g_m,\tag{5.25}$$

Equation (5.25) has a unique solution v_m for any given function g_m, thanks to the nondegeneracy assumption. Therefore we will concentrate in solving Problem (5.24), for a small given $\bar g$. Let us write

$$\bar{\mathrm v}=\bar{\mathrm v}^k+\omega$$

where $\bar{\mathrm v}^k$ is the approximation in Lemma 5.1. We express (5.24) in the form

$$\tilde L_\sigma(\omega)\;:=\;-\sigma\Big[\Delta_{\mathcal K}\omega+K(y)\omega\Big]-D\bar{\mathbf S}_0(\bar{\mathrm v}^k)\omega\;=\;\bar{\mathbf S}(\bar{\mathrm v}^k)+\mathbf{N}_1(\omega)+\bar g\tag{5.26}$$

where

$$\mathbf{N}_1(\omega):=\bar{\mathbf S}_0(\bar{\mathrm v}^k+\omega)-\bar{\mathbf S}_0(\bar{\mathrm v}^k)-D\bar{\mathbf S}_0(\bar{\mathrm v}^k)\omega\tag{5.27}$$

and $\mathbf{S}_0$ is the operator in (5.9).

The desired solvability theory will be a consequence of a suitable invertibility statement for the linear operator $\tilde{L}_\sigma$. Thus we consider the equation

$$\tilde{L}_\sigma(\omega) = \tilde{g} \quad \text{in } \mathcal{K}. \tag{5.28}$$

This operator is vector valued. It is convenient to express it in self-adjoint form by replacing the matrix $D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^k)$ with a symmetric one. We recall that we have

$$D\bar{\mathbf{S}}_0(\bar{\mathrm{v}}^k) \;=\; \sqrt{2}\mathbf{C} \begin{bmatrix} e^{-\sqrt{2}v_1^k} & 0 & \cdots & 0 \\ 0 & e^{-\sqrt{2}v_2^k} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\sqrt{2}v_{m-1}^k} \end{bmatrix}$$

where the matrix $\mathbf{C}$ is given in (5.9). $\mathbf{C}$ is symmetric and positive definite. Indeed, its eigenvalues are explicitly computed as

$$1, \frac{1}{2}, \ldots, \frac{m-1}{m}.$$

We consider the symmetric, positive definite square root matrix of $\mathbf{C}$ and denote it by $\mathbf{C}^{\frac{1}{2}}$. Then setting

$$\omega := \mathbf{C}^{\frac{1}{2}}\psi, \quad g := \mathbf{C}^{-\frac{1}{2}}\tilde{g},$$

we see that equation (5.28) becomes

$$L_\sigma(\psi) := -\sigma\Delta_{\mathcal{K}}\psi - \mathbf{A}(y,\sigma)\psi \;=\; g \quad \text{in } \mathcal{K} \tag{5.29}$$

where $\mathbf{A}$ is the symmetric matrix

$$\mathbf{A}(y,\sigma) \;=\; \sigma K(y)\,\mathbf{I}_{m-1} + \sqrt{2}\,\mathbf{C}^{\frac{1}{2}} \begin{bmatrix} e^{-\sqrt{2}v_1^k} & 0 & \cdots & 0 \\ 0 & e^{-\sqrt{2}v_2^k} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\sqrt{2}v_{m-1}^k} \end{bmatrix} \mathbf{C}^{\frac{1}{2}}. \tag{5.30}$$

Since

$$\mathrm{v}^k = \mathrm{v}^1(y) + \sigma\xi^k(y,\sigma)$$

we have that A is smooth in its variables and

$$\mathbf{A}(y,0) \;=\; \frac{\beta}{\sqrt{2}}\,K(y)\,\mathbf{C}^{\frac{1}{2}} \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m-1} \end{bmatrix} \mathbf{C}^{\frac{1}{2}}. \tag{5.31}$$

where $a_\ell = \ell(m-\ell)$. In particular, $\mathbf{A}(y,\sigma)$ has uniformly positive eigenvalues whenever σ is sufficiently small.

5.2. **Inverting the linearized Jacobi-Toda operator.** Our main result concerning uniform solvability of Problem (5.29) is the following.

Proposition 5.1. *There exists a sequence of values $\sigma = \sigma_\ell \to 0$ such that L_σ is invertible. More precisely, for any $g \in L^2(\mathcal{K})$ there exists a unique solution $\psi = L_\sigma^{-1} g \in H^1(\mathcal{K})$ to equation (5.29). This solution satisfies*

$$\sigma\|D^2_{\mathcal{K}}\psi\|_{L^2(\mathcal{K})} + \|\psi\|_{L^2(\mathcal{K})} \;\leq\; C\sigma^{-\frac{N-1}{2}}\|g\|_{L^2(\mathcal{K})}. \tag{5.32}$$

Moreover, if $p > N-1$, there exist $C, \nu > 0$ such that the solution satisfies besides the estimate

$$\|D^2_{\mathcal{K}}\psi\|_{L^p(\mathcal{K})} + \|D_{\mathcal{K}}\psi\|_{L^\infty(\mathcal{K})} + \|\psi\|_{L^\infty(\mathcal{K})} \le C\sigma^{-\frac{N-1}{2}-\nu}\,\|g\|_{L^p(\mathcal{K})}.$$

In addition, for $N = 2$, we have the existence of positive numbers $\nu_1, \nu_2, \ldots, \nu_{m-1}$ such that for all small σ with

$$|\nu_i\sigma - j^2| > c\sigma^{-\frac{1}{2}} \quad \textit{for all} \quad j \ge 1, \quad i = 1, \ldots, m-1,$$

for some $c > 0$, then $\psi = L_\sigma^{-1}g$ exists and estimate (5.32) holds.

The first part of the result holds in larger generality. Actually the properties we will use in the matrix function $\mathbf{A}(y,\sigma)$ are its symmetry, its smooth dependence in its variables on $\mathcal{K}\times[0,\sigma_0)$ and the fact for certain numbers $\gamma_\pm > 0$, we have

$$\gamma_-|\xi|^2 \le \xi^T\mathbf{A}(y,\sigma)\xi \le \gamma_+|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^{m-1},\ (y,\sigma) \in \mathcal{K}\times[0,\sigma_0). \tag{5.33}$$

Most of the work in the proof consists in finding the sequence σ_ℓ such that 0 lies suitably away from the spectrum of L_{σ_ℓ}, when this operator is regarded as self-adjoint in $L^2(\mathcal{K})$. The result will be a consequence of various considerations on the asymptotic behavior of the small eigenvalues of L_σ as $\sigma \to 0$. The general scheme below has already been used in related settings, see [37, 38, 40, 34, 35], using the theory of smooth and analytic dependence of eigenvalues of families of Fredholm operators due to T. Kato. Our proof relies only on elementary considerations on the variational characterization of the eigenvalues of L_σ and Weyl's asymptotic formula.

Thus, we consider the eigenvalue problem

$$L_\sigma\phi = \lambda\phi \quad \text{in } \mathcal{K}. \tag{5.34}$$

For each $\sigma > 0$ the eigenvalues are given by a sequence $\lambda_j(\sigma)$, characterized by the Courant-Fisher formulas

$$\lambda_j(\sigma) = \sup_{dim(M)=j-1}\ \inf_{\phi\in M^\perp\setminus\{0\}} Q_\sigma(\phi,\phi) = \inf_{dim(M)=j}\ \sup_{\phi\in M\setminus\{0\}} Q_\sigma(\phi,\phi) \tag{5.35}$$

where

$$Q_\sigma(\phi,\phi) = \frac{\int_{\mathcal{K}} L_\sigma\phi\cdot\phi}{\int_{\mathcal{K}}|\phi|^2} = \frac{\int_{\mathcal{K}} \sigma|\nabla\phi|^2 - \phi^T\mathbf{A}(y,\sigma)\phi}{\int_{\mathcal{K}}|\phi|^2}.$$

We have the validity of the following result.

Lemma 5.2. *There is a number $\sigma_* > 0$ such that for all $0 < \sigma_1 < \sigma_2 < \sigma_*$ and all $j \ge 1$ the following inequalities hold.*

$$(\sigma_2-\sigma_1)\frac{\gamma_-}{2\sigma_2^2} \le \sigma_2^{-1}\lambda_j(\sigma_2) - \sigma_1^{-1}\lambda_j(\sigma_1) \le 2(\sigma_2-\sigma_1)\frac{\gamma_+}{\sigma_1^2}. \tag{5.36}$$

In particular, the functions $\sigma \in (0,\sigma_*) \mapsto \lambda_j(\sigma)$ are continuous.

Proof. Let us consider small numbers $0 < \sigma_1 < \sigma_2$. We observe that for any ϕ with $\int_{\mathcal{K}}|\phi|^2 = 1$ we have

$$\sigma_1^{-1}Q_{\sigma_1}(\phi,\phi) - \sigma_2^{-1}Q_{\sigma_2}(\phi,\phi) = -\int_{\mathcal{K}}\phi^T(\sigma_1^{-1}\mathbf{A}(y,\sigma_1) - \sigma_2^{-1}\mathbf{A}(y,\sigma_2))\phi =$$

$$(\sigma_1-\sigma_2)\int_{\mathcal{K}}\phi^T(\sigma^{-2}\mathbf{A}(y,\sigma) - \sigma^{-1}\partial_\sigma\mathbf{A}(y,\sigma))\phi \tag{5.37}$$

for some $\sigma \in (\sigma_1,\sigma_2)$. From the assumption (5.33) on the matrix A we then find that

$$\sigma_1^{-1}Q_{\sigma_1}(\phi,\phi)+(\sigma_2-\sigma_1)\frac{\gamma_-}{2\sigma_2^2}\le\sigma_2^{-1}Q_{\sigma_2}(\phi,\phi)\le\sigma_1^{-1}Q_{\sigma_1}(\phi,\phi)+2(\sigma_2-\sigma_1)\frac{\gamma_+}{\sigma_1^2}.$$

From here, and formulas (5.35), inequality (5.36) follows. □

Corollary 5.1. *There exists a number $\delta>0$ such that for any $\sigma_2>0$ and j such that*

$$\sigma_2+|\lambda_j(\sigma_2)|<\delta,$$

and any σ_1 with $\frac{1}{2}\sigma_2\le\sigma_1<\sigma_2$, we have that

$$\lambda_j(\sigma_1)\;<\;\lambda_j(\sigma_2).$$

Proof. Let us consider small numbers $0<\sigma_1<\sigma_2$ such that $\sigma_1\ge\frac{\sigma_2}{2}$. Then from (5.36) we find that

$$\lambda_j(\sigma_1)\le\lambda_j(\sigma_2)+(\sigma_2-\sigma_1)\frac{1}{\sigma_2}[\lambda_j(\sigma_2)-\gamma\frac{\sigma_1}{\sigma_2}],$$

for some $\gamma>0$. From here the desired result immediately follows. □

5.3. **Proof of Proposition 5.1, general N.** Let us consider the numbers $\bar\sigma_\ell:=2^{-\ell}$ for large $\ell\ge1$. We will find a sequence of values $\sigma_\ell\in(\bar\sigma_{\ell+1},\bar\sigma_\ell)$ as in the statement of the lemma.

We define

$$\Gamma_\ell^1=\{\sigma\in(\bar\sigma_{\ell+1},\bar\sigma_\ell):\ \ker L_\sigma\ne\{0\}\}. \tag{5.38}$$

If $\sigma\in\Gamma_l^1$ then for some j we have that $\lambda_j(\sigma)=0$. It follows that $\lambda_j(\bar\sigma_{l+1})<0$. Indeed, let us assume the opposite. Then, given $\delta>0$, the continuity of λ_j implies the existence of $\tilde\sigma$ with $\frac{1}{2}\sigma\le\tilde\sigma<\sigma$ and $0\le\lambda_j(\tilde\sigma)<\delta$. If δ is chosen as in Corollary 5.1, and ℓ is so large that $2^{-\ell}<\delta$, we obtain a contradiction.

As a conclusion, we find that for all large ℓ

$$\mathrm{card}\,(\Gamma_\ell^1)\le N(\bar\sigma_{\ell+1}), \tag{5.39}$$

where $N(\sigma)$ denotes the number of negative eigenvalues of problem (5.34). We estimate next this number for small σ. Let us consider $a_+>0$ such that

$$\xi^T\mathbf{A}(y,\sigma)\xi\le a_+|\xi|^2\quad\text{for all}\quad\xi\in\mathbb{R}^{m-1},\ (y,\sigma)\in\mathcal{K}\times[0,\sigma_0),$$

and the operator

$$L_\sigma^+=-\Delta_{\mathcal{K}}-\frac{a^+}{\sigma}. \tag{5.40}$$

Let $\lambda_j^+(\sigma)$ denote its eigenvalues. From the Courant-Fisher characterization we see that $\lambda_j^+(\sigma)\le\lambda_j(\sigma)$. Hence $N(\sigma)\le N_+(\sigma)$, where the latter quantity designates the number of negative eigenvalues of L_σ^+.

Let us denote by μ_j the eigenvalues of $-\Delta_{\mathcal{K}}$. Then Weyl's asymptotic formula for eigenvalues of the Laplace-Beltrami operator, see for instance [33, 42], asserts that for a certain constant $C_{\mathcal{K}}>0$,

$$\mu_j=C_{\mathcal{K}}j^{\frac{2}{N-1}}\;+\;o(j^{\frac{2}{N-1}})\quad\text{as } j\to+\infty. \tag{5.41}$$

Using the fact that $\lambda_j^+(\sigma)=\mu_j-\frac{a^+}{\sigma}$ and (5.41) we then find that

$$N_+(\sigma)=C\sigma^{-\frac{N-1}{2}}+o(\sigma^{-\frac{N-1}{2}})\quad\text{as }\sigma\to0. \tag{5.42}$$

As a conclusion, using (5.39) we find

$$\operatorname{card}(\Gamma_\ell^1) \le N(\bar\sigma_{\ell+1}) \le C\bar\sigma_{\ell+1}^{-\frac{N-1}{2}} \le C2^{\ell\frac{N-1}{2}}. \tag{5.43}$$

Hence there exists an interval $(a_\ell, b_\ell) \subset (\bar\sigma_{\ell+1}, \bar\sigma_\ell)$ such that $a_\ell, b_\ell \in A_\ell$, $\operatorname{Ker}(L_\sigma) = \{0\}$, $\sigma \in (a_\ell, b_\ell)$ and

$$b_\ell - a_\ell \ge \frac{\bar\sigma_\ell - \bar\sigma_{\ell+1}}{\operatorname{card}(A_\ell)} \ge C\bar\sigma_\ell^{1+\frac{N-1}{2}}. \tag{5.44}$$

Let

$$\sigma_\ell := \frac{1}{2}(b_\ell + a_\ell).$$

We will analyze the spectrum of L_{σ_ℓ}. If some $c > 0$, and all j we have

$$|\mu_j(\sigma_\ell)| \ge c\bar\sigma_\ell^{\frac{N-1}{2}},$$

then we have the validity of the existence assertion and estimate (5.32). Assume the opposite, namely that for some j we have $|\mu_j(\sigma_\ell)| \le \delta\sigma_\ell^{\frac{N-1}{2}}$, with δ arbitrarily small. Let us assume first that $0 < \lambda_j(\sigma_\ell) < \delta\sigma_\ell^{\frac{N-1}{2}}$. Then we have from Lemma 5.2,

$$\lambda_j(a_\ell) \le \lambda_j(\sigma_\ell) + (\sigma_\ell - a_\ell)\frac{1}{a_\ell}[\lambda_j(\sigma_\ell) - \gamma\frac{a_\ell}{\sigma_\ell}].$$

Hence,

$$\lambda_j(a_\ell) \le \delta\sigma_\ell^{\frac{N-1}{2}} + C\sigma_\ell^{\frac{N-1}{2}}(\sigma_\ell^{1+\frac{N-1}{2}} - \gamma) < 0$$

if δ was chosen a priori sufficiently small. It follows that $\lambda_j(\sigma)$ must vanish at some $\sigma \in (\sigma_\ell, b_\ell)$, and we have thus reached a contradiction with the choice of the interval (a_ℓ, b_ℓ).

The case $-\delta\sigma_\ell^{\frac{N-1}{2}} < \lambda_j(\sigma_\ell) < 0$ is handled similarly. In that case we get $\lambda_j(b_\ell) > 0$. The proof of existence and estimate (5.32) is thus complete.

Let us consider now a number $p > N-1$. Now we want to estimate the inverse of L_{σ_ℓ} in Sobolev norms. The equation satisfied by $\psi = L_{\sigma_\ell}^{-1}g$ has the form

$$\Delta_{\mathcal K}\psi = O(\sigma^{-1})[\psi + g]$$

for $\sigma = \sigma_\ell$. Then from elliptic estimates we get

$$\|\psi\|_{W^{2,q}(\mathcal K)} \le C\sigma^{-1}[\|\psi\|_{L^q(\mathcal K)} + \|g\|_{L^q(\mathcal K)}] \tag{5.45}$$

Using this for $q = 2$ and estimate (5.32) we get

$$\|\psi\|_{W^{2,2}(\mathcal K)} \le C\sigma^{-1}[\|\psi\|_{L^2(\mathcal K)} + \|g\|_{L^2(\mathcal K)}] \le C\sigma^{-\frac{N}{2}-1}\|g\|_{L^p(\mathcal K)}.$$

From Sobolev's embedding we then get

$$\|\psi\|_{L^q(\mathcal K)} \le C\sigma^{-\frac{N-1}{2}-1}\|g\|_{L^p(\mathcal K)}.$$

for any $1 < q \le \frac{2(N-1)}{N-5}$ if $N > 5$, and any $q > 1$ if $N \le 5$. If $q = p$ is admissible in this range, the estimate follows from (5.45). If not, we apply it for $q = \frac{2(N-1)}{N-5}$, and then Sobolev's embedding yields

$$\|\psi\|_{L^s(\mathcal K)} \le C\sigma^{-\frac{N-1}{2}-2}\|g\|_{L^p(\mathcal K)}.$$

for any $1 < s \le \frac{2(N-1)}{N-6}$ if $N > 6$, and any $s > 1$ if $N \le 6$. Iterating this argument, we obtain the desired estimate after a finite number of steps. The proof of the first part of the proposition is concluded. $\square$

5.4. **The case $N = 2$. Conclusion of the proof.** We consider now the problem of solving system (5.29) when $N = 2$. We consider first the problem of solving

$$-\sigma\Delta_{\mathcal{K}}\psi - \mathbf{A}(y,0)\psi \;=\; g \quad \text{in } \mathcal{K} \tag{5.46}$$

A main observation is the following: the linear system (5.46) can be decoupled: If $\Lambda_1, \dots, \Lambda_{m-1}$ denote the eigenvalues of the matrix

$$\mathbf{Q} := \mathbf{C}^{\frac{1}{2}} \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m-1} \end{bmatrix} \mathbf{C}^{\frac{1}{2}},$$

which coincide with those of

$$\begin{bmatrix} 2a_1 & -a_2 & 0 & & \cdots & 0 \\ -a_1 & 2a_2 & -a_3 & & \cdots & 0 \\ 0 & -a_2 & 2a_3 & & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & \cdots & -a_{m-3} & 2a_{m-2} & -a_{m-1} \\ 0 & & \cdots & & -a_{m-2} & 2a_{m-1} \end{bmatrix},$$

then system (5.46) expressed in coordinates associated to eigenfunctions of $\mathbf{Q}$ decouples into $m-1$ equations of the form

$$-\sigma\Delta_{\mathcal{K}}\psi_j - \frac{\beta}{\sqrt{2}}\Lambda_j\, K(y)\,\psi_j \;=\; g_j \quad \text{in } \mathcal{K}, \quad j = 1, \dots, m-1 \tag{5.47}$$

When $N = 2$ this problem reduces to an ODE. $\mathcal{K}$ is then a geodesic of $\mathcal{M}$ and $K(y)$ will simply be Gauss curvature measured along $\mathcal{K}$. Using y as arclength coordinate, and dropping the index j, Equations (5.47) take the generic form

$$\begin{aligned} -\sigma\psi'' - \mu\, K(y)\,\psi = g \quad \text{in } (0,\ell), \\ \psi(0) \;=\; \psi(\ell), \quad \psi'(0) \;=\; \psi'(\ell). \end{aligned} \tag{5.48}$$

where μ is given and fixed, and ℓ is the total length of $\mathcal{K}$.

For this problem to be uniquely solvable, we need that $\mu\sigma^{-1}$ differs from the eigenvalues $\lambda = \lambda_j$ of the problem

$$\begin{aligned} -\varphi'' = \lambda\, K(y)\,\varphi \quad \text{in } (0,\ell), \\ \varphi(0) \;=\; \varphi(\ell), \quad \varphi'(0) \;=\; \varphi'(\ell). \end{aligned} \tag{5.49}$$

More precisely, in such a case we have that the solution of (5.48) satisfies

$$\|\psi\|_{L^2(\mathcal{K})} \le \frac{\sigma^{-1}}{\min_j |\lambda_j - \sigma^{-1}\mu|}\|g\|_{L^2(\mathcal{K})} \tag{5.50}$$

Now, we restate Problem (5.49) using the following Liouville transformation:

$$\ell_0 = \int_0^\ell \sqrt{K(y)}\,\mathrm{d}\,y, \qquad t = \frac{\pi}{\ell_0}\int_0^y \sqrt{K(\theta)}\,\mathrm{d}\,\theta,\ t \in [0,\pi),$$

$$\Psi(y) = K(y)^{-\frac{1}{4}}, \qquad e(t) = \varphi(y)/\,\Psi(y), \qquad q(t) = \frac{\ell_0^2\,\Psi''(y)}{\pi^2\,\Psi^2(y)\,K(y)}.$$

Equation (5.49) then becomes

$$-e'' - q(t)\,e = \frac{\ell_0^2}{\pi^2}\,\lambda\,e \quad \text{ in } (0,\pi),\ e(0) = e(\pi),\ e'(0) = e'(\pi).$$

A result in [32] shows that, as $j \to \infty$ we have

$$\lambda_j = \frac{4\pi^2 j^2}{\ell_0^2} + O(j^{-2}).$$

Hence, if for some $c > 0$ we have that

$$|\sigma^{-1}\mu - \frac{4\pi^2 j^2}{\ell_0^2}| > c\sigma^{-\frac{1}{2}} \quad \text{for all} \quad j \geq 1,$$

and σ is sufficiently small, then the problem will be solvable, and thanks to (5.50), we will have the estimate

$$\|\psi\|_{L^2(\mathcal{K})} \leq C\sigma^{-\frac{1}{2}}\|g\|_{L^2(\mathcal{K})} \tag{5.51}$$

for the unique solution of Problem (5.48). It follows that, under these conditions System (5.47) is uniquely solvable and its solution $\psi = -(\sigma\Delta_{\mathcal{K}}\psi + \mathbf{A}(y,0))^{-1}g$ satisfies estimate (5.51).

Now, for σ as above, we can write system (5.29) in the fixed point form in $L^2(\mathcal{K})$,

$$\psi + T(\psi) = -(\sigma\Delta_{\mathcal{K}}\psi + \mathbf{A}(y,0))^{-1}g, \quad \psi \in L^2(\mathcal{K}), \tag{5.52}$$

where

$$T(\psi) := (\sigma\Delta_{\mathcal{K}}\psi + \mathbf{A}(y,0))^{-1}[(\mathbf{A}(y,\sigma) - \mathbf{A}(y,0))\psi]\,.$$

We observe that, as an operator in $L^2(\mathcal{K})$, $\|T\| = O(\sigma^{\frac{1}{2}})$. Thus, for small σ, Problem (5.52) is uniquely solvable, and satisfies (5.51). Finally, for the L^p case, we argue with the same bootstrap procedure of §5.3.

The proof of the proposition is complete. □

6. **The Jacobi-Toda system II: nested catenoidal interfaces in $\mathbb{R}^3$.** In this section, we consider a simple noncompact case, i.e., $\mathcal{M} = \mathbb{R}^3$ and $\mathcal{K}$ is a complete embedded minimal surfaces in $\mathbb{R}^3$. First let us review the results on Allen-Cahn equation in $\mathbb{R}^N$

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N \tag{6.1}$$

6.1. **De Giorgi conjecture.** E. De Giorgi [10] formulated in 1978 a celebrated conjecture on the Allen-Cahn equation (6.1), parallel to Bernstein's theorem for minimal surfaces: The level sets $[u = \lambda]$ of a bounded entire solution u to (6.1), which is also monoton e in one direction, must be hyperplanes, at least for dimension $N \leq 8$. Equivalently, up to a translation and a rotation, $u = w(x_1)$. This conjecture has been proven in dimensions $N = 2$ by Ghoussoub and Gui [23], $N = 3$ by Ambrosio and Cabré [2], and under a mild additional assumption by Savin [52] for $4 \leq N \leq 8$. A counterexample was recently built for $N \geq 9$ in [16]. See [20] for a recent survey on the state of the art of this question.

6.2. **Finite Morse index solutions to Allen-Cahn equation equation in $\mathbb{R}^3$ and embedded minimal surfaces of finite total curvature.** In [17], we constructed a new class of entire solutions to the Allen-Cahn equation in $\mathbb{R}^3$, and also finite Morse index, whose level sets resemble a large dilation of a given complete, embedded minimal surface $\mathcal{K}$ of $\mathbb{R}^3$, asymptotically flat in the sense that it has *finite total curvature*, namely

$$\int_{\mathcal{K}} |K|\, dV \ < \ +\infty$$

where K denotes Gauss curvature of the manifold, which is also *non-degenerate*. Furtheremore, we also proved that the Morse index of the solutions to (6.1) equals the index $i(\mathcal{K})$ of the minimal surface.

Theorem 2. [17] *Let $N = 3$ and $\mathcal{K}$ be a minimal surface embedded, complete with finite total curvature which is nondegenerate. Then, there exists a bounded solution u_α of equation (6.1), defined for all sufficiently small α, such that*

$$u_\alpha(x) = w(z - q(y)) + O(\alpha) \quad \textit{for all} \quad x = y + z\nu(\alpha y), \quad |z - q(y)| < \frac{\delta}{\alpha}, \tag{6.2}$$

where the function q satisfies

$$q(y) \ = \ (-1)^k \beta_k \, \log |\alpha y'| + O(1) \quad y \in M_{k,\alpha}, \quad k = 1, \dots, m,$$

for some numbers β_k.

Furthermore, for all sufficiently small α we have

$$m(u_\alpha) \ = \ i(\mathcal{K}).$$

6.3. **Nested catenoids.** We discuss another new phenomena for the Allen-Cahn equation in $\mathbb{R}^3$

$$\Delta u \, + \, u - u^3 \, = \, 0 \quad \text{in } \mathbb{R}^3\, . \tag{6.3}$$

Theorem 2 establishes an almost one-to-one correspondence between finite Morse index solutions to (6.3) and the embedded complete minimal surfaces with finite total curvature in $\mathbb{R}^3$.

In the following, we present a new phenomena associated with (6.3) which has not found any analog in the theory of minimal surfaces. Namely we find a solution to (6.3) with *large number* of Morse index and *large number* of nested catenoids.

We consider the simpest minimal surfaces $\mathcal{K}$ =catenoid. Its parametrization is given by

$$X(u, v) = (\cosh(u)v, u) : \mathbb{R} \times S^1 \tag{6.4}$$

where $u \in \mathbb{R}, v \in S^1$. The Jacobi operator can be computed explicitly

$$\mathcal{J} := \frac{1}{(\cosh u)^2} (\partial_{uu} + \partial_{vv} + 2\text{sech}^2 u) \tag{6.5}$$

Let

$$X_w = X + t\mathbf{n}$$

be the Fermi coordinate. Here $\mathbf{n}$ is the unit normal vector and t is the signed distance to the catenoid.

We state the following theorem on the existence of *nested catenoids*:

Theorem 1. [18] *Let $N = 3$ and $\mathcal{K}$ be the catenoid. Let $m \geq 2$ be any fixed positive integer. Then for α small, there exists a bounded solution u_α of equation (6.3), defined for all sufficiently small α, such that*

$$u_\alpha(x) = \sum_{j=1}^{m} (-1)^j w(t - q_j(y)) + O(\alpha) \quad \text{for all} \quad x = y + z\nu(\alpha y), \quad |z - q(y)| < \frac{\delta}{\alpha}, \tag{6.6}$$

where the function q_j satisfies the Jacobi-Toda system

$$\varepsilon^2 \Big(\Delta_{\mathcal{K}} q_j + |A_{\mathcal{K}}|^2 q_j \Big) \, - \, a_0 \left[e^{-\sqrt{2}(q_j - q_{j-1})} \, - \, e^{-\sqrt{2}(q_{j+1} - q_j)} \right] \, = \, 0 \tag{6.7}$$

on $\mathcal{K}$, $j = 1, \ldots, m$, with the convention $q_0 = -\infty$, $q_{m+1} = +\infty$. The Morse index of u_α approaches $+\infty$ as $\alpha \to 0$.

6.4. **Solvability of the Jacobi-Toda system (6.7).** In the following, we discuss how to solve the new Jacobi-Toda system (6.7). Of course, the main problem is that $\mathcal{K}$ is noncompact. We consider the simplest case $m = 2$ and furthermore we assume that q depends on u only. Similar to Section 4, we can only need to solve the following scalar equation

$$\epsilon^2 \mathcal{J}[h] = e^{-\sqrt{2}h}, \text{where} \;\; h = \sqrt{2}(q_1 - q_2). \tag{6.8}$$

If we use the arclength as paramter

$$s = \sinh u$$

then the Jacobi operator becomes

$$\mathcal{J}[h] = h_{ss} + h_{vv} \frac{s}{1 + s^2} h_s + \frac{2h}{(1 + s^2)^2} h \tag{6.9}$$

As in Section 4, we proceed in two steps: first we solve an algebraic equation and improve the error, secondly we solve the linearized problem.

Let h_0 solve the algebraic equation

$$\frac{2\epsilon^2 h_0}{(1 + s^2)^2} = e^{-\sqrt{2}h_0} \tag{6.10}$$

Then it is easy to see that

$$h_0 = \sqrt{2} \log \frac{1}{\epsilon} + \log \log \frac{1}{\epsilon} + \log(1 + s^2)^2 \tag{6.11}$$

Let

$$\sigma = \frac{1}{\log \frac{1}{\epsilon}}$$

Then we need to analyze the following linear problem

$$\sigma \mathcal{J}[\phi] + (1 + \sigma \log(1 + s^2)^2)[\phi] = g \tag{6.12}$$

To overcome the non-compact case, we use stereographic projection

$$u = \tan \frac{\theta}{2}$$

so that (6.12) can be reduced to a problem on the compact manifold $\mathbb{S}^2$

$$\sigma[\Delta_{\mathbb{S}^2}[\phi] + a(\theta)\phi_\theta + 2\phi] + (1 + \sigma V(\theta))[\phi] = g \text{ on } \mathbb{S}^2 \tag{6.13}$$

where $V(\theta) = \log\cos\theta \in L^p$ for any $p > 1$.

If $V(\theta)$ is smooth, then we can solve (6.13) using gap condition, as in Section 4.3 or in [14]. But the main problem is V has a singularity like $\log|\theta|$. This can be overcome by the following simple lemma:

Lemma 6.1. *There exists $\sigma_i \to 0$ such that*

$$\|\phi\|_{H^2(\mathbb{S}^2)} \leq \frac{1}{\sigma}\|g\|_{L^2(\mathbb{S}^2)} \tag{6.14}$$

Proof. Since we are in symmetric class, we may find gaps $\sigma_i \to 0$ (as in [14]) such that

$$\|\phi\|_{L^2} \leq \frac{1}{\sqrt{\sigma_i}}\|g\|_{L^2}. \tag{6.15}$$

Combining (6.14) and (6.15), we see that

$$\|\phi\|_{L^p} \leq \frac{1}{\sigma^{(1+\tau)/2}}\|g\|_{L^2} \tag{6.16}$$

for some $p > 2$ and τ is small.

Hence

$$\begin{aligned}\|\phi\|_{H^2} &\leq C\|V(\theta)\phi\|_{L^2} + \frac{1}{\sigma}\|g\|_{L^2}\\ &\leq C\frac{1}{\sigma^{(1+\sigma)/2}}\|\phi\|_{L^p} + \frac{1}{\sigma}\|g\|_{L^2}\\ &\leq C\frac{1}{\sigma}\|g\|_{L^2}.\end{aligned}$$

□

7. The Jacobi-Toda System III: Nested traveling wave solutions for Allen-Cahn equation.

In this section, we show that another kind of Jacobi-Toda system arises in the study of nested traveling waves to Allen-Cahn equation. Interestingly enough, we can solve the corresponding Jacobi-Toda system *without the resonance condition*.

We consider traveling wave solutions to the following parabolic Allen-Cahn equation

$$u_t = \Delta u + u - u^3, \quad \text{in } \mathbb{R}^{N+1} \times (-\infty, \infty), \quad N > 1, \tag{7.1}$$

in the following form:

$$u(x,t) = U(x', x_{N+1} - ct), \quad x = (x', x_{N+1}). \tag{7.2}$$

Then U will satisfy the following traveling wave Allen-Cahn equation

$$c\partial_{x_{N+1}}U + \Delta U + U - U^3 = 0 \text{ in } \mathbb{R}^{N+1}. \tag{7.3}$$

In [7], the existence of a traveling wave in the form $U(r, x_{N+1})$, $|x'| = r$, is obtained for any speed $c > 0$. Furthermore, it is shown that, asymptotically, the $0-$level set of U—called Γ, is paraboloid- like

$$\lim_{x_{N+1}\to+\infty,(x',x_{N+1})\in\Gamma} \frac{r^2}{2x_{N+1}} = \frac{N-1}{c}, \quad \text{if } N > 1.$$

Let us assume $c = \epsilon \ll 1, N > 1$. In [19], we established the existence of traveling wave solution to (7.3) whose zero level set consists on k disjoint components which foliate the neighborhood of the rotationally symmetric eternal solution to the mean curvature flow. To explain this we will recall that (7.3) is related with the problem of eternal solutions to the mean curvature flow.

Here we will discuss a special eternal solution of the mean curvature flow for which $\Sigma(t) = \{F(x') + t\}$, where $F\colon \mathbb{R}^N \to \mathbb{R}$, is a graph of a smooth function satisfying

$$\nabla\left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\right) = \frac{1}{\sqrt{1+|\nabla F|^2}}. \tag{7.4}$$

It is known from [8] that there exists a unique radially symmetric solution F of (7.4) with the following asymptotic behavior

$$F(r) = \frac{r^2}{2(N-1)} - \log r + 1 + O(r^{-1}), \quad r \gg 1. \tag{7.5}$$

Notice that this asymptotic behavior corresponds (to the leading order) to the asympto tic behavior of the nodal set Γ of solutions to (6.1) found in [7].

Theorem 3 ([19])**.** *Let $c = \varepsilon > 0$ be small. Then (7.3) has a solution u_ϵ whose 0-level set consists of k interfaces given as graphs of smooth functions $\{x_{N+1} = F_{\epsilon,j}(r)\}$ where*

$$F_{\varepsilon,j}(r) = \varepsilon^{-1}F(\varepsilon r) + 2\left(j - \frac{k-2}{2}\right)\varepsilon \log\frac{(1+r)}{\varepsilon} + O\left(\varepsilon \log\log\frac{1}{\varepsilon}\right), \tag{7.6}$$

and $r = |x'|$.

Functions $F_{\varepsilon,j}(r)$ can also be seen as normal graphs over the surface $\Gamma_\varepsilon = \{x_{N+1} = \varepsilon^{-1}F(\varepsilon r)\}$. Although these functions diverge logarithmically along the ends of Γ_ε but since this growth is small relative to the asymptotic behavior of Γ_ε at ∞ (which is quadratic in r) and their graphs do not intersect we can speak of a foliation of a neighborhood of Γ_ε by the nodal sets of u_ε.

The relation between the eternal solution Γ_ε and the traveling wave solution u_ε is well known and in fact easy to explain, at least formally. On the other hand it is not at all obvious what kind of geometric conditions should functions $F_{\varepsilon,j}$ satisfy. The reason for this is that in the geometric context of the MC flow the phenomenon of foliations does not have an analog. Indeed as we will see the way the foliation by the traveling wave is determined depends on a delicate mechanism which mediates between the local geometry effect due to the curvature of the nodal sets and the long range interactions between them. This mechanism has no analogue in the geometric context.

It turns out that as before, a new system of nonlinear PDEs, which we call *the Jacobi-Toda system*, gives a sufficient condition (at least formally) for the eternal solution of the MC flow to generate a foliation by traveling waves. To this end we will fix an eternal solution Σ and consider k functions $f_{\varepsilon,1}, \dots, f_{\varepsilon,k} \in \mathcal{C}^2(\Sigma)$ such that $f_{\varepsilon,1} < \cdots < f_{\varepsilon,k}$. We will also denote the Laplace-Beltrami operator, the second fundamental form, and the gradient on Σ, respectively, by Δ_Σ A_Σ, and ∇_Σ. Then functions $f_{\varepsilon,j}$, $j = 1, \dots, k$ solve the following *new* Jacobi-Toda system

$$\begin{aligned} &\varepsilon^2\alpha_0\big[(\Delta_\Sigma + |A_\Sigma|^2)f_{\varepsilon,j} + \nabla_\Sigma f_{\varepsilon,j}\cdot\nabla_\Sigma(x_{N+1})\big] \\ &\qquad - e^{\sqrt{2}(f_{\varepsilon,j-1}-f_{\varepsilon,j})} + e^{\sqrt{2}(f_{\varepsilon,j}-f_{\varepsilon,j+1})} = 0, \end{aligned} \tag{7.7}$$

where we always agree that $f_{\varepsilon,0} = -\infty$, $f_{\varepsilon,m+1} = \infty$. Here $\alpha_0 > 0$ is a universal constant. The normal graphs of functions $f_{\varepsilon,j}(\varepsilon\cdot)$ over Σ_ε represent the foliations by trave lling waves. The relation between functions $f_{\varepsilon,j}$ and $F_{\varepsilon,j}$ introduced in Theorem 3 is easy to obtain.

Considering this new Jacobi-Toda system (7.7) we have:

Theorem 4. *Consider (7.7) with Σ replaced by Γ. For each $m > 1$ there exists a solution to the Jacobi-Toda system.*

A remarkable fact in Theorem 4 is that unlike Theorem 1 and Theorem 1, we do not need *nonresonance condition* to solve the Jacobi-Toda system (7.7). This may be due to the effect of the first order term. In the following we sketch the main steps in proving Theorem 4.

7.1. **Rotationally symmetric eternal solutions.** For what follows it will be convenient to denote:

$$L_0[v] = \Delta_\Gamma v + |A_\Gamma|^2 v + \nabla_\Gamma v \cdot \nabla_\Gamma F. \tag{7.8}$$

Our theory of solvability of the Jacobi-Toda system will be valid for functions of the radial variable r only and so we need to express the operator L_0 in terms of the radial variable r:

$$L_0[v] = \frac{v_{rr}}{1+F_r^2} + \frac{(N-1)v_r}{r} + \Big(\frac{(N-1)F_r^2}{r^2(1+F_r^2)} + \frac{F_{rr}^2}{(1+F_r^2)^3}\Big)v. \tag{7.9}$$

Let us change the independent variable

$$s = \int_0^r \sqrt{1+F_r^2}\,dr. \tag{7.10}$$

The new variable s is nothing else but the arc length of the curve $\gamma = \{(r, F(r)), r > 0\}$ in $\mathbb{R}^2$. Using the asymptotic formula (7.5) for F we get that

$$s \sim r, \quad r \ll 1, \qquad s = \frac{r^2}{2(N-1)} + \mathcal{O}(\log r), \quad r \gg 1. \tag{7.11}$$

By a straightforward computation we obtain the following expression for the operator L_0:

$$L_0[v] = v_{ss} + a(s)v_s + b(s)v \tag{7.12}$$

where

$$a(s) = \frac{F_r + \frac{N-1}{r}}{\sqrt{1+F_r^2}}, \quad b(s) = |A_\Gamma(r)|^2, \quad r = r(s). \tag{7.13}$$

Note that

$$\begin{aligned} a(s) &= \frac{N-1}{s}\big(1+\mathcal{O}(s^2)\big), \quad s \ll 1, \qquad a(s) = 1 + \mathcal{O}(s^{-1}), \quad s \gg 1, \\ b(s) &= \frac{(N-1)}{r^2} + \mathcal{O}(r^{-4}) = \frac{1}{2s} + \mathcal{O}(s^{-2}\log s), \quad s \gg 1, \end{aligned} \tag{7.14}$$

and that in general $a(s), b(s) > 0$ since Γ is convex and $F_r(0) = 0$. We also have $b(0) = 1$ and $b'(0) = 0$. Another important fact is that

$$b''(0) = -\frac{N^2+4N+2}{N^4(N+2)} < 0, \quad N = 2, \dots. \tag{7.15}$$

This last identity follows by a direct computation. Setting $\mathfrak{b}_N = \frac{N^2+4N+2}{2N^4(N+2)}$ we have

$$b(s) = 1 - \mathfrak{b}_N s^2 + \mathcal{O}(s^4), \quad s \to 0. \tag{7.16}$$

7.2. **A non-homogeneous Jacobi-Toda system.** In reality we have to deal in general with the non-homogeneous Jacobi-Toda system. Thus we will consider the following problem:

$$\varepsilon^2\alpha_0 L_0[f_{\varepsilon,j}] - e^{\sqrt{2}(f_{\varepsilon,j-1}-f_{\varepsilon,j})} + e^{\sqrt{2}(f_{\varepsilon,j}-f_{\varepsilon,j+1})} = \varepsilon^2 h_{\varepsilon,j}, \tag{7.17}$$

where $f_{\varepsilon,j} = f_{\varepsilon,j}(r)$, $h_{\varepsilon,j} = h_{\varepsilon,j}(r)$. The above problem can also be seen in terms of the arc length variable s.

Thus we state the following:

Proposition 7.1. *Let $m \geq 2$ and consider the following problem:*

$$\varepsilon^2\alpha_0 L_0[f_{\varepsilon,j}] - e^{\sqrt{2}(f_{\varepsilon,j-1}-f_{\varepsilon,j})} + e^{\sqrt{2}(f_{\varepsilon,j}-f_{\varepsilon,j+1})} = \varepsilon^2 h_{\varepsilon,j}, \quad j = 1, \ldots, m \tag{7.18}$$

where $h_{\varepsilon,j}$ are smooth functions such that it holds:

$$|h_{\varepsilon,j}(s)| \leq C\varepsilon^\tau (2+s)^{-\frac{3}{2}}, \quad j = 1, \ldots, m, \tag{7.19}$$

with similar estimates for the derivatives. There exists a solution of this problem such that if we denote

$$u_{\varepsilon,j} = f_{\varepsilon,j} - f_{\varepsilon,j-1}, \quad j = 2, \ldots, m, \quad v_\varepsilon = \sum_{j=1}^{k} f_{\varepsilon,j},$$

then functions $u_{\varepsilon,j}, v_\varepsilon$ satisfy

$$\begin{aligned} u_{\varepsilon,j}(s) &= \log \frac{2\sqrt{2}}{\varepsilon^2\alpha_0 b(s)} + \mathcal{O}(\log\log \frac{1}{\varepsilon^2 b(s)}), \quad s \to \infty \\ |v_\varepsilon(s)| &\leq C\varepsilon^\tau (2+s)^{-\frac{1}{2}} \log(s+2). \end{aligned} \tag{7.20}$$

To describe the strategy we use we will assume for simplification that $m = 2$, and denote $u_\varepsilon = \sqrt{2}(f_{\varepsilon,2} - f_{\varepsilon,1})$ and $v_\varepsilon = \sqrt{2}(f_{\varepsilon,1} + f_{\varepsilon,2})$, and respectively $h_\varepsilon = \frac{\sqrt{2}}{\alpha_0}(h_{\varepsilon,2} - h_{\varepsilon,1})$ and $g_\varepsilon = \frac{\sqrt{2}}{\alpha_0}(h_{\varepsilon,2} + h_{\varepsilon,1})$. Then we get the following decoupled system:

$$L_0[u_\varepsilon] - \frac{2\sqrt{2}}{\varepsilon^2\alpha_0} e^{-u_\varepsilon} = h_\varepsilon \tag{7.21}$$

$$L_0[v_\varepsilon] = g_\varepsilon \tag{7.22}$$

Let us discuss briefly the second of the above equations. We will see that the right hand side of this equation satisfies:

$$g_\varepsilon \sim \varepsilon^\tau (1+s)^{-\frac{3}{2}-\beta}, \quad \tau > 0, \beta \geq 0. \tag{7.23}$$

When the right hand side of (7.22) has this behavior then the equation can be solved by using the nondegeneracy of the traveling graph. The key observation is that the operator L_0 has a decaying, positive element in its kernel

$$\phi_0 = \frac{1}{\sqrt{1+F_r^2}} \sim \frac{1}{r} \quad (\sim \frac{1}{\sqrt{s}}), \tag{7.24}$$

from which (7.22) by a standard ODE method.

The solvability theory for the nonlinear equation (7.21) is another story. Even when the right hand side decays at the same rate as in (7.23) we still have the nonlinear term to deal with. In general the decay rate of this term will be actually slower and in addition it is a term of order ε^{-2}. In other words the real difficulty is in solving the non-homogeneous nonlinear problem (7.21). To do this we will first use an approximation scheme to find a suitable asymptotic approximation of the solution of (7.21) and after this we will be in the position to use a fixed point argument to solve the non-homogeneous problem with the right hand side of the type (7.23).

7.3. **Solving for u_ε: Approximations.** Our goal in this and the following section is to solve (7.21). Of course once this is done the Proposition 7.1 will be proven. We begin by finding an approximate solution of (7.21) assuming that $h_{\varepsilon,j}$ satisfies (7.19), which is equivalent to solving:

$$\mathfrak{S}_\delta[u] = h_\delta, \quad u > 0, \quad |h_\delta(s)| \le C\delta^\tau(2+s)^{-\frac{3}{2}}, \tag{7.25}$$

where

$$\mathfrak{S}_\delta[u] \equiv L_0[u] - \delta^{-2}e^{-u}, \quad \delta = \frac{\varepsilon\sqrt{\alpha_0}}{2^{3/4}}. \tag{7.26}$$

Once a good approximation is found the nonlinear problem (7.21) can be reduced to a fixed point problem. This step involves inverting the linear operator obtained by linearization of the nonlinear operator $\mathfrak{S}_\delta$ around the approximate solution. This problem will be dealt with in the next section.

The nonlinear operator $\mathfrak{S}_\delta$ can be written explicitly in terms of the arc length variable s:

$$\mathfrak{S}_\delta[u] = u_{ss} + a(s)u_s + b(s)u - \delta^{-2}e^{-u}.$$

In order to find a solution of (7.25) we will first build an approximate solution. The leading order term of this approximation is found by solving for $\mathtt{u}_0$ the following equation:

$$b(s)\mathtt{u}_0 = \frac{1}{\delta^2}e^{-\mathtt{u}_0} \Longrightarrow \mathtt{u}_0 e^{\mathtt{u}_0} = \frac{1}{\delta^2 b(s)}, \tag{7.27}$$

which implies that

$$\mathtt{u}_0(s) = \log\frac{1}{\delta^2 b(s)} - \log\log\frac{1}{\delta^2 b(s)} + \mathcal{O}(\log\log|\log\delta^2 b(s)|). \tag{7.28}$$

Let us also observe the following relations:

$$\mathtt{u}_0' = \frac{b'}{b}\frac{\mathtt{u}_0}{1+u_0}, \quad \mathtt{u}_0'' = \left(\frac{b'}{b}\right)'\frac{\mathtt{u}_0}{1+\mathtt{u}_0} + \left(\frac{b'}{b}\right)^2\frac{\mathtt{u}_0}{(1+\mathtt{u}_0)^3}. \tag{7.29}$$

Accepting the function $\mathtt{u}_0$ as the leading order approximation, and assuming that the approximate solution is of the form $u+\mathtt{u}_0$, we are left with the following problem:

$$u'' + a(s)u' + b(s)u = b(s)\mathtt{u}_0(e^{-u} - 1) - a(s)\mathtt{u}_0' - \mathtt{u}_0'' + h_\delta. \tag{7.30}$$

Next terms in the approximate solutions will be determined considering two regions depending on the behavior of the functions $a(s)$, $b(s)$. It is important to keep in mind the the approximations we want to construct must be decaying functions of s while at the same time there size must become small in terms of powers of $\frac{1}{\log\frac{1}{\delta^2}}$. This is the reason why we need to consider the two regimes for s. Our goal is to find, for any $k > 0$, an approximation to the above problem, denoted by $\mathtt{u}_k$ such that

$$|\mathfrak{S}_\delta[\mathtt{u}_k](s)| \le \frac{C(2+s)^{-k-1}}{\left[\log\frac{1}{\delta^2}\right]^k}. \tag{7.31}$$

We introduce the following weighted Hölder norms for functions $u\colon \mathbb{R}_+ \to \mathbb{R}$:

$$\|u\|_{\mathcal{C}^{\ell}_{\beta,\mu}(\mathbb{R}_+)} := \sum_{j=0}^{\ell} \sup_{s>1} \big\{(2+s)^{\beta+j}\big[\log\big(\frac{2+s}{\delta^2}\big)\big]^{\mu}\|u\|_{\mathcal{C}^j((s-1,s+1))}\big\},$$

where β, and $\mu \geq 0$.

We summarize the approximation in the following:

Lemma 7.1. *For each $k \geq 1$ the function*

$$\mathtt{u}_k = \mathtt{u}_0 + \sum_{j=1}^{k}[\chi u_{i,j} + (1-\chi)u_{o,j}],$$

satisfies, with some constant C (that may depend on k):

$$\|\chi u_{i,j} + (1-\chi)u_{o,j}\|_{\mathcal{C}^2_{j-1,j}(\mathbb{R}_+)} \leq C, \quad j \geq 1. \tag{7.32}$$

The error of the approximation can be estimated by:

$$\|\mathfrak{S}_\delta[\mathtt{u}_k]\|_{\mathcal{C}^0_{k+1,k}(\mathbb{R}_+)} \leq C. \tag{7.33}$$

The proof of Lemma 7.1 is as follows: we will consider the inner and the outer approximation separately, construct suitable approximations and then "glue" the approximate solutions. We refer to [19] for detailed proofs.

7.4. **Definition of the linearized operator $\mathfrak{L}_\delta$.** From the above considerations we see that linearization of $\mathfrak{S}_\delta$ around the approximate solution $\mathtt{u}_k$ is the following operator

$$\mathfrak{L}_\delta[h] = h_{ss} + a(s)h_s + p_\delta(s)h, \quad p_\delta(s) = b(s)(1 + \mathtt{u}_0 e^{-\mathtt{u}_k + \mathtt{u}_0}). \tag{7.34}$$

For future reference we observe that from (7.27)–(7.28) and Lemma 7.1 it follows that

$$0 < p_\delta(s) \leq C \log\frac{1}{\delta^2}, \tag{7.35}$$

while when $s \geq \bar{s}$ then $p_\delta(s)$ satisfies

$$\frac{1}{C(2+s)}\log\big(\frac{2+s}{\delta^2}\big) < p_\delta(s) \leq \frac{C}{2+s}\log\big(\frac{2+s}{\delta^2}\big), \quad s \geq 0. \tag{7.36}$$

7.5. **The right bounded inverse of $\mathfrak{L}_\delta$.** The key is to solve the following problem

$$\mathfrak{L}_\delta[h] = g(s). \tag{7.37}$$

The point is to construct a right inverse of $\mathfrak{L}_\delta$ which is bounded in the weighted norm defined above. More precisely we will show:

Lemma 7.2. *Suppose that $\beta > 1$. Then there exists a constant $C > 0$ and a solution to (7.37) such that*

$$\|h\|_{\mathcal{C}^{2,\mu}_{\beta}(\mathbb{R}_+)} \leq C \log\frac{1}{\delta^2}\|g\|_{\mathcal{C}^{0,\mu}_{\beta+1}(\mathbb{R}_+)}. \tag{7.38}$$

To begin we make the following transformation:

$$\hat{h} = \exp\Big(\frac{1}{2}\int_1^s a(\tau)\,d\tau\Big)h. \tag{7.39}$$

Then near $s \sim 0$, $\hat{h} = s^{(N-1)/2} h$ and near $s \to +\infty$, $\hat{h} \sim e^{s/2} h$, by (7.14). Equation (7.37) is transformed to

$$\hat{h}'' + (p_\delta(s) - \hat{a}(s))\hat{h}(s) = \hat{g}, \tag{7.40}$$

where

$$\hat{a} = \frac{1}{2}a' + \frac{1}{4}a^2, \quad \hat{g} = \exp\left(\frac{1}{2}\int_1^s a(\tau)\, d\tau\right) g.$$

We mainly work with the transformed equation (7.40). The idea of the proof of the Lemma is the following: we will consider the inner and the outer problem separately, construct suitable inverses of $\mathfrak{L}_\delta$ for these problems and then "glue" the solutions. The situation now is more complicated since we have to consider the full second order problem. It is at this level that we take full advantage of some special properties of the eternal solution to the mean curvature flow. We refer to [19] for full details.

8. **Concluding remarks.** We have derived the *Jacobi-Toda system* for the Allen-Cahn equation in three different settings: the compact case, the non-compact $\mathbb{R}^3$ case and non-compact traveling wave case. We believe that in general the Jacobi-Toda system provides a natural theory for the interaction of interfaces.

We note that Jacobi-Toda system has no analog in the theory of minimal surfaces or the theory of CMC surfaces: the reason is clear. In CMC surfaces, the interaction of the surfaces is Dirac, i.e., either the surfaces touch each other or have no interaction at all. The Toda system appears only for semilinear PDE because of the weak interactions of the solutions.

Several important questions appear:

1. Under the condition $K(y) > 0$, it is possible to solve the Jacobi-Toda system. What if $K(y) \geq 0$? A natural condition for the solvability of the Jacobi-Toda system seems to be that the Morse index of the Jacobi operator is positive.

2. In the non-compact setting, whether or not resonance is needed depends very much on the decay of the potential K. In the catenoid case, the potential K decays like $\frac{1}{r^4}$. In the traveling wave case, the potential decays only like $\frac{1}{r^2}$. It will be interesting to know for what kind of potential the resonance condition is needed.

3. In [16], we constructed a single interface solution to the Allen-Cahn equation in $\mathbb{R}^9$ using the Bomberie-De Giorgi-Giusti graph. In view of the nested catenoid, a natural question is: is there a nested Bomberie-De Giorgi-Giusti graph?

4. Interfaces solutions are also constructed for the singularly perturbed Schrodinger equation

$$\epsilon^2 \Delta u - V(x)u + u^p = 0 \text{ in } \mathbb{R}^N \tag{8.41}$$

Even for single interfaces, resonance conditions are needed [14]. If we consider interface foliations, further difficulties arise for multiple non-resonance conditions.

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REFERENCES

[1] S. Allen and J. W. Cahn, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening,* Acta. Metall., **27** (1979), 1084–1095.

[2] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in R3 and a conjecture of De Giorgi,* Journal Amer. Math. Soc., **13** (2000), 725–739.

[3] M. Artin, "Algebra," Upper Saddle River, N.J. : Prentice Hall, 1991.

[4] I. Birindelli and R. Mazzeo, *Symmetry of solution of two-phase semilinear elliptic equations on hyperbolic space,* arXiv:0806.2952v1.

[5] L. Bronsard and B. Stoth, *On the existence of high multiplicity interfaces,* Math. Res. Lett., **3** (1996), 117–131.

[6] L. Caffarelli and A. Córdoba, *Uniform convergence of a singular perturbation problem,* Comm. Pure Appl. Math., **XLVII** (1995), 1–12.

[7] X. Chen, J. Guo, F. Hamel, H. Ninomiya and J.-M. Roquejoffre, *Traveling waves with paraboloid like interfaces for balanced bistable dynamics,* Ann. I. H. Poincare-AN, **24** (2007), 369–393.

[8] J. Clutterbuck, O. C. Schnürer and F. Schulze, *Stability of transalting solutions to mean curvature flow,* Cal. Var., **29** (2007), 281–293.

[9] E. N. Dancer and S. Yan, *Multi-layer solutions for an elliptic problem,* J. Diff. Eqns., **194** (2003), 382–405.

[10] E. De Giorgi, *Convergence problems for functionals and operators,* Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131–188, Pitagora, Bologna(1979).

[11] M. del Pino, M. Kowalczyk and M. Musso, *Singular limits in Liouville-type equations,* Calc. Var. Partial Differential Equations, **24** (2005), 47–81.

[12] M. del Pino, M. Kowalczyk, F. Pacard and J. Wei, *Multiple-end solutions to the Allen-Cahn equation in* $\mathbb{R}^2$*,* J. Functional Analysis, **258** (2010), 458–503.

[13] M. del Pino, M. Kowalczyk, F. Pacard and J. Wei, *The Toda system and multiple-end solutions of autonomous planar elliptic problems,* Advances in Math, to appear.

[14] M. del Pino, M. Kowalczyk and J. Wei, *Concentration on curves for nonlinear Schrödinger equations,* Comm. Pure Appl. Math., **70** (2007), 113–146.

[15] M. del Pino, M. Kowalczyk and J. Wei, *The Toda system and clustering interface in the Allen-Cahn equation,* Archive Rational Mechanical Analysis, **190** (2008), 141–187.

[16] M. del Pino, M. Kowalczyk and J. Wei, *On De Giorgi's Conjecture in dimensions* $N \geq 9$*,* preprint 2008, arXiv:0806.3141.

[17] M. del Pino, M. Kowalczyk and J. Wei, *Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature in* $\mathbb{R}^3$*,* preprint 2009, arXiv.org.

[18] M. del Pino, M. Kowalczyk and J. Wei, *Nested catenoids for Allen-Cahn equation in* $\mathbb{R}^3$*,* preprint 2010.

[19] M. del Pino, M. Kowalczyk and J. Wei, *Traveling wave foliations near eternal solutions of the mean curvature flow and the Jacobi-Toda system,* preprint 2010.

[20] M. del Pino, M. Kowalczyk and J. Wei, *On De Giorgi Conjecture and Beyond,* preprint 2010.

[21] M. del Pino, M. Kowalczyk, J. Wei and J. Yang, *Interface foliation near minimal submanifoldsin Riemannian manifolds with positive Ricci curvature,* preprint 2009.

[22] G. Dunne, "Self-dual Chern-Simons Theories," Lecture Notes in Physics, No. **36**, Springer, Berlin, 1995.

[23] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems,* Math. Ann., **311** (1998), 481–491.

[24] C. E. Garza-Hume and P. Padilla, *Closed geodesic on oval surfaces and pattern formation,* Comm. Anal. Geom., **11** (2003), 223–233.

[25] J. Jost, CS Lin and G. Wang, *Analytic aspects of Toda system. II. Bubbling behavior and existence of solutions,* Comm. Pure Appl. Math., **59** (2006), 526–558.

[26] J. Jost and G. Wang, *Analytic aspects of the Toda system: I. A Moser-Trudinger inequality,* Comm. Pure Appl. Math., **54** (2001), 1289–1319.

[27] Jürgen Jost and Guofang Wang, *Classification of solutions of a Toda system in* $\mathbb{R}^2$*,* Int. Math. Res. Not., (2002), 277–290.

[28] T. Kato, "Perturbation Theory for Linear Operators," Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[29] R. V. Kohn and P. Sternberg, *Local minimizers and singular perturbations,* Proc. Royal Soc. Edinburgh A, **11** (1989), 69–84.

[30] B. Kostant, *The solution to a generalized Toda lattice and representation theory,* Adv. Math., **34** (1979), 195–338.

[31] M. Kowalczyk, *On the existence and Morse index of solutions to the Allen-Cahn equation in two dimensions*, Ann. Mat. Pura Appl., **184** (2005), 17–52.

[32] B. M. Levitan and I. S. Sargsjan, "Sturm-Liouville and Dirac Operator," Mathematics and its application (Soviet Series), **59**, Kluwer Acadamic Publishers Group, Dordrecht, 1991.

[33] Peter Li and Shing-Tung Yau, *On the Schrödinger equation and the eigenvalue problem,* Commun. Math. Phys., **88** (1983), 309–318.

[34] F. Mahmoudi and A. Malchiodi, *Concentration on minimal submanifolds for a singularly perturbed Neumann problem,* Adv. Math., **209** (2007), 460–525.

[35] F. Mahmoudi, R. Mazzeo and F. Pacard, *Constant mean curvature hypersurfaces condensing on a submanifold,* Geom. Funct. Anal., **16** (2006), 924–958

[36] A. Malchiodi, *Concentration at curves for a singularly perturbed Neumann problem in three-dimensional domains,* Geom. Funct. Anal., **15** (2005), 1162–1222.

[37] A. Malchiodi and M. Montenegro, *Boundary concentration phenomena for a singularly perturbed elliptic problem,* Commun. Pure Appl. Math., **55** (2002), 1507–1568.

[38] A. Malchiodi and M. Montenegro, *Multidimensional boundary layers for a singularly perturbed Neumann problem,* Duke Math. J., **124** (2004), 105–143.

[39] A. Malchiodi and C. B. Ndiaye, *Some existence results for the Toda system on closed surfaces,* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Math. Appl., **18** (2007), 391–412.

[40] A. Malchiodi and J. Wei, *Boundary interface for the Allen-Cahn equation,* J. Fixed Point Theory Appl., **1** (2007), 305–336.

[41] R. Mazzeo and F. Pacard, *Foliations by constant mean curvature tubes,* Comm. Anal. Geom., **13** (2005), 633–670.

[42] S. Minakshisundaram and A. Pleijel, *Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds,* Canad. J. Math., **1** (1949), 242–256.

[43] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion,* Arch. Rat. Mech. Anal., **98** (1987), 357–383.

[44] L. Modica, *Convergence to minimal surfaces problem and global solutions of* $\Delta u = 2(u^3 - u)$, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), pp. 223–244, Pitagora, Bologna, (1979).

[45] J. Moser, *Finitely many mass points on the line under the influence of an exponential potential–an integrable system,* Dynamical systems, theory and applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974), pp. 467–497. Lecture Notes in Phys., Vol. **38**, Springer, Berlin, 1975.

[46] K. Nakashima, *Multi-layered stationary solutions for a spatially inhomogeneous Allen-Cahn equation,* J. Diff. Eqns., **191** (2003), 234–276.

[47] K. Nakashima and K. Tanaka, *Clustering layers and boundary layers in spatially inhomogeneous phase transition problems,* Ann. Inst. H. Poincaré Anal. Non Linéaire, **20** (2003), 107–143.

[48] H. Ohtsuka and T. Suzuki, *Blow-up analysis for* $SU(3)$ *Toda system,* J. Diff. Eqns., **232** (2007), 419–440.

[49] F. Pacard and M. Ritoré, *From constant mean curvature hypersurfaces to the gradient theory of phase transitions,* J. Diff. Geom., **64** (2003), 359–423.

[50] P. Padilla and Y. Tonegawa, *On the convergence of stable phase transitions,* Comm. Pure Appl. Math., **51** (1998), 551–579.

[51] M. Röger and Y. Tonegawa, *Convergence of phase-field approximations to the Gibbs-Thomson law,* Cal. Var. PDE, **32** (2008), 111–136.

[52] O. Savin, *Regularity of flat level sets in phase transitions,* To appear in Ann. of Math..

[53] Y. Tonegawa, *Phase field model with a variable chemical potential,* Proc. Roy. Soc. Edinburgh Sect. A, **132** (2002), 993–1019.

[54] J. Wei, CY Zhao and F. Zhou, *On non-degeneracy of solutions to SU(*3*) Toda system,* preprint 2010.

[55] Y. Yang, "Solitons in Field Theory and Nonlinear Analysis," Springer Monographs in Mathematics. Springer, New York, 2001.

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