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THE EXTREMAL SOLUTION OF A BOUNDARY REACTION PROBLEM

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ABSTRACT. We consider

$$\Delta u = 0 \ \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = \lambda f(u) \ \text{ on } \Gamma_1, \qquad u = 0 \ \text{ on } \Gamma_2$$

where $\lambda > 0$, $f(u) = e^u$ or $f(u) = (1+u)^p$ and Γ_1, Γ_2 is a partition of $\partial\Omega$ and $\Omega \subset \mathbb{R}^N$. We determine sharp conditions on the dimension N and $p > 1$ such that the extremal solution is bounded, where the extremal solution refers to the one associated to the largest λ for which a solution exists.

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1. Introduction

We study the semilinear boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda f(u) & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2 \end{cases} \tag{1}$$

where $\lambda > 0$ is a parameter, $f(u)$ is a nonlinear smooth function of u, $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain and Γ_1, Γ_2 is a partition of $\partial\Omega$ into surfaces separated by a smooth interface. We will assume that

$$f \text{ is smooth, nondecreasing, convex and } f(0) > 0, \tag{2}$$

$$\lim_{u\to\infty} \frac{f(u)}{u} = \infty, \quad \text{and} \tag{3}$$

$$\liminf_{t\to+\infty} \frac{f'(t)t}{f(t)} > 1. \tag{4}$$

Assumption (4) is not essential, but it simplifies some of the arguments and holds for the examples $f(u) = e^u$, $f(u) = (1+u)^p$, $p > 1$.

We say that u is a weak solution of (1) if $u \in W^{1,1}(\Omega)$, $f(u) \in L^1(\Gamma_1)$ and

$$\int_\Omega u(-\Delta\varphi) = \int_{\Gamma_1} \lambda f(u)\varphi \qquad \text{for all } \varphi \in C^2(\bar\Omega) \text{ such that } \varphi\Big|_{\Gamma_2} \equiv 0 \text{ and } \frac{\partial\varphi}{\partial\nu}\Big|_{\Gamma_1} \equiv 0.$$

Problem (1) shares many properties with the following generalization of the so-called Gelfand's problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

which has been widely considered [3, 4, 10, 11, 19, 20]. In particular, the following result can be proved as in [3].

Proposition 1.1. *There exists $\lambda^* \in (0,\infty)$ such that*

- *(1) has a smooth solution for $0 \le \lambda < \lambda^*$,*
- *(1) has a weak solution for $\lambda = \lambda^*$,*
- *(1) has no solution for $\lambda > \lambda^*$ (even in the weak sense) .*

Moreover, for $0 \le \lambda < \lambda^*$ there exists a minimal solution u_λ which is bounded, positive and stable, in the sense that the linearized operator at u_λ is positive, i.e.

$$\inf_{\varphi\in C^1(\overline\Omega),\varphi=0 \text{ on } \Gamma_2} \frac{\int_\Omega |\nabla\varphi|^2\,dx - \lambda\int_{\Gamma_1} f'(u_\lambda)\varphi^2\,ds}{\int_{\Gamma_1}\varphi^2\,ds} > 0. \tag{6}$$

The monotone limit $u^* := \lim_{\lambda\nearrow\lambda^*} u_\lambda$ is a weak solution for $\lambda = \lambda^*$ and satisfies

$$\lambda^*\int_{\Gamma_1} f'(u^*)\varphi^2 \le \int_\Omega |\nabla\varphi|^2\,dx, \quad \forall\varphi \in C^1(\overline\Omega), \varphi = 0 \text{ on } \Gamma_2. \tag{7}$$

We call u^* the extremal solution of (1).

Remark 1.2. *Under assumption (4) we have $u^* \in H^1(\Omega)$. The proof is analogous to the argument for (5) in [4], so we skip it.*

Proposition 1.1 suggests the following natural question : is u^* a bounded solution?

In the context of (5), no complete answer has been given yet. For the case $f(u) = e^u$, that is the original Gelfand problem, it was shown by Joseph and Lundgren [19] that if Ω is a ball, then u^* is bounded if and only if $N < 10$. Crandall and Rabinowitz [11] showed that if $f(u) = e^u$ and $N < 10$ then for *any* smooth and bounded domain, u^* is bounded. Brezis and Vázquez [4] provided a different proof of the unboundedness of u^* in the case $\Omega = B_1$ and $N \ge 10$: they established in particular that a singular solution which is stable must be the extremal one. In applying this criterion in dimension $N \ge 10$ they use Hardy's inequality valid for $N \ge 3$:

$$\frac{(N-2)^2}{4}\int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \le \int_{\mathbb{R}^N} |\nabla\varphi|^2, \qquad \forall\varphi \in C_0^\infty(\mathbb{R}^N) \tag{8}$$

Other explicit nonlinearities, for instance $f(u) = (1+u)^p$ with $p > 1$, are considered in these references, but in the general case, little is known. In this direction, we mention the result of Nedev [21], which asserts that for any function f satisfying (2) and (3), and any smooth bounded domain in $\mathbb{R}^N$, $N \le 3$, u^* is bounded. This result has been extended by Cabré to the case $N = 4$ and Ω strictly convex [5]. Finally, Cabré and Capella [6] showed that if Ω is a ball and $N \leq 9$ then for any nonlinearity f satisfying (2),(3) the extremal solution is bounded.

Proving that u^* is unbounded seems to be much more difficult. Besides the radial case Dávila and Dupaigne [13] have shown that in domains that are small perturbations of a ball and for the nonlinearities e^u and $(1+u)^p$ the extremal solution is singular for large dimensions ($N \geq 11$ and $N > 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$ respectively).

Returning to (1), we are interested in determining whether the extremal solution u^* is bounded or singular in the cases $f(u) = e^u$ and $f(u) = (1+u)^p$, $p > 1$.

Theorem 1.3. *Let $f(u) = e^u$. In any dimension $N \geq 10$ there exists a domain $\Omega \subset \mathbb{R}^N$ and a partition in smooth sets Γ_1, Γ_2 of $\partial\Omega$ such that $u^* \notin L^\infty(\Omega)$.*

The proof is an adaptation of the argument of Brezis and Vázquez [4], using a stability criterion. In our case the singular solution has the form $u_0(x) = -\log|x|$ for $x \in \partial\mathbb{R}^N_+$ and its linearized stability in dimension $N \geq 10$ is obtained thanks to :

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 \geq H_N \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|}, \quad \forall\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+}), \tag{9}$$

which holds for $N \geq 3$ and where the best constant

$$H_N := \inf\left\{ \frac{\int_{\mathbb{R}^N_+} |\nabla\varphi|^2}{\int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|}} \, : \, \varphi \in H^1(\mathbb{R}^N_+), \varphi\Big|_{\partial\mathbb{R}^N_+} \not\equiv 0 \right\} \tag{10}$$

is given by

$$H_N = 2\frac{\Gamma(\frac{N}{4})^2}{\Gamma(\frac{N-2}{4})^2} \qquad \forall N \geq 3, \tag{11}$$

where Γ is the Gamma function. Inequality (9) is known as Kato's inequality and a proof of it was given by Herbst [18].

We will give here a different proof of this result which offers a sharper version, analogous to improvements of (8) found by Brezis and Vázquez [4] or Vázquez and Zuazua [22] (see also [2, 4, 12, 17, 22] for other improved versions of the Hardy inequality (8)) :

Theorem 1.4. *Let $B = B_1(0)$ be the unit ball in $\mathbb{R}^N$, $N \geq 3$. Then for any $1 \leq q < 2$ there exists $c = c(N,q) > 0$ such that*

$$\int_{\mathbb{R}^N_+ \cap B} |\nabla\varphi|^2 \geq H_N \int_{\partial\mathbb{R}^N_+ \cap B} \frac{\varphi^2}{|x|} + c\|\varphi\|^2_{W^{1,q}(\mathbb{R}^N_+ \cap B)}, \quad \forall\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+} \cap B),$$

As a converse to Theorem 1.3 we prove :

Theorem 1.5. *Let $f(u) = e^u$, $N \leq 9$ and suppose $\Omega \subset \mathbb{R}^N_+$ is open, bounded and satisfies:*

- *$\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}^N_+$ and $\Gamma_2 \subset \mathbb{R}^N_+$*
- *Ω is symmetric with respect to the hyperplanes $x_1 = 0, \ldots, x_{N-1} = 0$, and*
- *Ω is convex with respect to all directions $x_1, \ldots, x_{N-1}$.*

Then the extremal solution u^* of (1) belongs to $L^\infty(\Omega)$.

Remark 1.6. *In order to prove Theorem 1.5, one is at first tempted to imitate the classical argument of Crandall and Rabinowitz* [11]*: roughly speaking, one uses the stability inequality* (7) *and the equation* (1) *with test functions of the form* $\varphi = e^{ju}$*,* $j \geq 1$*. This does not lead to the optimal dimension* $N = 9$ *but applies to general domains (see Proposition 1.7 below). We use instead test functions* φ*, which are not functions of* u*, but which have the expected behavior of* e^{ju} *near a singular point, assuming it exists.*

Proposition 1.7. *Let* $f(u) = e^u$ *and assume* $\Omega \subset \mathbb{R}^N$ *is a smooth bounded domain such that* $\partial\Omega = \Gamma_1 \cup \Gamma_2$*, where* $\Gamma_1 \subset \partial\mathbb{R}^N_+$ *and* $\Gamma_2 \subset \mathbb{R}^N_+$*. Assume further that* $N < 6$*. Then the extremal solution* u^* *of* (1) *belongs to* $L^\infty(\Omega)$*.*

This raises the following question

Open Problem 1. *Does Theorem 1.5 hold in any smooth bounded domain* $\Omega \subset \mathbb{R}^N_+$ *such that* $\partial\Omega = \Gamma_1 \cup \Gamma_2$*, where* $\Gamma_1 \subset \partial\mathbb{R}^N_+$ *and* $\Gamma_2 \subset \mathbb{R}^N_+$ *?*

Next we look at (1) in the case $f(u) = (1+u)^p$, $p > 1$. Given $0 < \alpha < N-1$ define

$$w_\alpha(x) = \int_{\partial\mathbb{R}^N_+} K(x,y)|y|^{-\alpha} dy \qquad \text{for } x \in \mathbb{R}^N_+, \tag{12}$$

where $K(x,y) = \frac{2x_N}{N\omega_N}|x-y|^{-N}$ is the Green's function for the Dirichlet problem in $\mathbb{R}^N_+$ (see e.g. [16]). Clearly, $w_\alpha > 0$ in $\mathbb{R}^N_+$. Moreover w_α is harmonic in $\mathbb{R}^N_+$ and w_α extends to a function belonging to $C^\infty(\overline{\mathbb{R}^N_+} \setminus \{0\})$ with

$$w_\alpha(x) = |x|^{-\alpha} \quad \text{for all } x \in \partial\mathbb{R}^N_+ \setminus \{0\}. \tag{13}$$

It is not difficult to verify that for some constant $C(N,\alpha)$ we have

$$\frac{\partial w_\alpha}{\partial\nu}(x) = C(N,\alpha)|x|^{-\alpha-1} \qquad \forall x \in \partial\mathbb{R}^N_+ \setminus \{0\}.$$

In Section 2 we shall show that

$$C(N,\alpha) = 2\frac{\Gamma(\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{N-1}{2}-\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{N-2}{2}-\frac{\alpha}{2})}. \tag{14}$$

A heuristic calculation shows that for (1) with nonlinearity $f(u) = (1+u)^p$, the expected behavior of a solution u which is singular at $0 \in \partial\Omega$ should be $u(x) \sim |x|^{\frac{1}{p-1}}$. The boundedness of u^* is then related to the value of $C(N, \frac{1}{p-1})$. Observe that $C(N, \frac{1}{p-1})$ is defined for $p > \frac{N}{N-1}$. In the sequel, when writing $C(N, \frac{1}{p-1})$ we will implicitly assume that this condition holds.

Theorem 1.8. *Consider* (1) *with* $f(u) = (1+u)^p$*. Assume* $\Omega \subset \mathbb{R}^N_+$ *is a bounded domain such that* $\partial\Omega = \Gamma_1 \cup \Gamma_2$*, where* $\Gamma_1 \subset \partial\mathbb{R}^N_+$ *and* $\Gamma_2 \subset \mathbb{R}^N_+$ *and such that the following condition holds*

- Ω *is convex with respect to* x' *and*
- $\Pi_N(\Omega) = \Gamma_1$*, where* Π_N *is the projection on* $\partial\mathbb{R}^N_+$*.*

If $p\,C(N, \frac{1}{p-1}) > H_N$ *or* $1 < p < \frac{N}{N-2}$ *then* u^* *is bounded.*

In the above, Ω is said to be convex with respect to x' if $(tx', x_N) + ((1-t)y', x_N) \in \Omega$ whenever $t \in [0,1]$, $x = (x', x_N) \in \Omega$ and $y = (y', x_N) \in \Omega$. Π_N is defined by $\Pi_N(x', x_N) = x_N$ for all $x = (x', x_N) \in \mathbb{R}^N_+$.

Remark 1.9. *It is not difficult to verify that the same result holds if*

- *Ω is convex with respect to all directions $x_1, \dots, x_{N-1}$ and*
- *Ω is symmetric with respect to the hyperplanes $x_1 = 0, \dots, x_{N-1} = 0$.*

Theorem 1.10. *Consider* (1) *with $f(u) = (1+u)^p$. If $p\,C(N, \frac{1}{p-1}) \le H_N$ and $p \ge \frac{N}{N-2}$ there exists a domain Ω such that u^* is singular.*

Remark 1.11. *The condition $p\,C(N, \frac{1}{p-1}) \le H_N$ is not enough to guarantee that the extremal solution is singular for some domain. Actually this condition can hold for some values of p in the range $\frac{N}{N-1} < p < \frac{N}{N-2}$. In this case a singular solution exists in some domains, but it does not correspond to the extremal one. See Theorem 6.2 in* [4] *for a similar phenomenon.*

The organization of the paper is as follows. In Section 2 we derive formula (14) and we prove Theorem 1.4 in Section 3. In Section 4 we analyze the exponential case and give a proof of Theorems 1.3 and 1.5. The proofs of Theorems 1.8 and 1.10 are given in Section 5.

2. Computation of $C(N, \alpha)$

We write $x = (x', x_N) \in \mathbb{R}^N_+$ with $x' \in \mathbb{R}^{N-1}$, $x_N > 0$. It follows from (12) and a simple change of variables that

$$w_\alpha(x', x_N) = w_\alpha(e(x'), x_N) \qquad \text{for all rotations } e \in \mathcal{O}(N-1).$$

and similarly

$$w_\alpha(Rx', Rx_N) = R^{-\alpha} w_\alpha(x', x_N). \tag{15}$$

Differentiating with respect to x_N yields

$$\frac{\partial w_\alpha}{\partial x_N}(Rx', Rx_N) = R^{-\alpha-1}\frac{\partial w_\alpha}{\partial x_N}(x', x_N).$$

Let $x \in \partial\mathbb{R}^N_+$, $x = (x', 0)$ and plug $R = \frac{1}{|x|} = \frac{1}{|x'|}$ in the previous formula to find

$$\frac{\partial w_\alpha}{\partial \nu}(x) = -\frac{\partial w_\alpha}{\partial x_N}(x', 0) = |x|^{-\alpha-1}\left(-\frac{\partial w_\alpha}{\partial x_N}\Big(\frac{x'}{|x'|}, 0\Big)\right).$$

Define

$$C(N,\alpha) = -\frac{\partial w_\alpha}{\partial x_N}\Big(\frac{x'}{|x'|}, 0\Big) \tag{16}$$

and observe that it is independent of $x' \in \mathbb{R}^{N-1}$.

Using (15) and the radial symmetry of w in the variables x', there exists a function $v : [0, \infty) \to \mathbb{R}$ such that

$$w_\alpha(x', x_N) = |x'|^{-\alpha} w_\alpha\Big(\frac{x'}{|x'|}, \frac{x_N}{|x'|}\Big) = |x'|^{-\alpha} v\Big(\frac{x_N}{|x'|}\Big). \tag{17}$$

Writing $r = |x'|$, $t = \frac{x_N}{|x'|}$, we have

$$r^{-\alpha} v(t) = w_\alpha(x', rt), \quad \forall x' \in \mathbb{R}^{N-1},\ |x'| = r.$$

The equation $\Delta w = 0$ is equivalent to

$$(1+t^2)v''(t) + (2\alpha + 4 - N)tv'(t) + \alpha(\alpha - N + 3)v(t) = 0, \qquad t > 0, \tag{18}$$

while (13) implies

$$v(0) = 1.$$

The initial condition for v' is related to (16)

$$v'(0) = -C(N, \alpha).$$

In addition to these initial conditions we remark that w_α is a smooth function in $\mathbb{R}^N_+$ and this together with (17) implies that

$$\lim_{t\to\infty} v(t)t^\alpha \text{ exists.} \tag{19}$$

Using the change of variables $z = it$ with i the imaginary unit and defining the new unknown $h(z) := v(-iz)$ equation (18) becomes

$$(1 - z^2)h''(z) - (2\alpha + 4 - N)zh'(z) - \alpha(\alpha - N + 3)h(z) = 0, \tag{20}$$

with initial conditions

$$\lim_{t>0,\ t\to 0} h(it) = 1, \qquad \lim_{t>0,\ t\to 0} h'(it) = iC(N, \alpha). \tag{21}$$

On the other hand (19) implies

$$\lim_{t\in\mathbb{R},\ t\to\infty} h(it)t^\alpha \text{ exists.} \tag{22}$$

The substitution

$$g(z) = (1 - z^2)^{\frac{\alpha}{2}+\frac{1}{2}-\frac{N}{4}} h(z) \tag{23}$$

transforms equation (20) into

$$(1 - z^2)g''(z) - 2zg'(z) + \big(\nu(\nu + 1) - \frac{\mu^2}{1 - z^2}\big)g(z) = 0, \tag{24}$$

with

$$\mu = \alpha + \frac{2 - N}{2}, \quad \nu = \frac{N - 4}{2}. \tag{25}$$

The general solution to (24) is well known. Indeed, equation (24) belongs to the class of Legendre's equations. Following [1], two linearly independent solutions of (24) are given by the Legendre functions $P^\mu_\nu(z)$, $Q^\mu_\nu(z)$, which are defined in $\mathbb{C} \setminus \{-1, 1\}$ and analytic in $\mathbb{C} \setminus (-\infty, 1]$ (see [1, Formulas 8.1.2 – 8.1.6]). Moreover the limits of $P^\mu_\nu(z)$, $Q^\mu_\nu(z)$ on both sides of $(-1, 1)$ exist and we shall use the notation

$$\begin{aligned} P^\mu_\nu(x + i0) &= \lim_{z\to x, \mathrm{Re}(z)>0} P^\mu_\nu(z), \quad -1 < x < 1, \\ P^\mu_\nu(x - i0) &= \lim_{z\to x, \mathrm{Re}(z)<0} P^\mu_\nu(z), \quad -1 < x < 1, \end{aligned} \tag{26}$$

and a similar notation for Q^μ_ν.

The solution g of (24) is therefore given by

$$g(z) = c_1 P^\mu_\nu(z) + c_2 Q^\mu_\nu(z),$$

for appropriate constants c_1, c_2. These constants are determined by the initial conditions (21), which imply:

$$c_1 P^\mu_\nu(0 + i0) + c_2 Q^\mu_\nu(0 + i0) = 1, \tag{27}$$

$$c_1 \frac{d}{dz} P^\mu_\nu(0 + i0) + c_2 \frac{d}{dz} Q^\mu_\nu(0 + i0) = iC(N, \alpha). \tag{28}$$

In order to evaluate $C(N,\alpha)$, we use also condition (22), which is equivalent to

$$\lim_{t\to\infty,\, t\in\mathbb{R}} (c_1 P_\nu^\mu(it) + c_2 Q_\nu^\mu(it)) t^{\frac{N}{2}-1} \text{ exists.} \tag{29}$$

But according to [1, Formulas 8.1.3, 8.1.5]

$$P_\nu^\mu(z) \sim z^\nu \text{ as } |z| \to \infty$$
$$Q_\nu^\mu(z) \sim z^{-\nu-1} \text{ as } |z| \to \infty$$

This and (23),(29) imply that $c_1 = 0$ and we obtain from (27),(28)

$$C(N,\alpha) = -i \frac{\frac{d}{dz} Q_\nu^\mu(0+i0)}{Q_\nu^\mu(0+i0)} \tag{30}$$

From the properties and formulas in [1] the following values can be deduced:

$$Q_\nu^0(0+i0) = -i 2^{\mu-1} \pi^{\frac{1}{2}} e^{i\mu\pi - i\nu\frac{\pi}{2}} \frac{\Gamma(\frac{\nu}{2}+\frac{\mu}{2}+\frac{1}{2})}{\Gamma(\frac{\nu}{2}-\frac{\mu}{2}+1)} \tag{31}$$

$$\frac{d}{dz} Q_\nu^0(0+i0) = 2^\mu \pi^{\frac{1}{2}} e^{i\mu\pi - i\nu\frac{\pi}{2}} \frac{\Gamma(\frac{\nu}{2}+\frac{\mu}{2}+1)}{\Gamma(\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{2})} \tag{32}$$

The relations (30),(31),(32) and the values (25) yield formula (14).

□

3. Improved Kato inequality

We begin with some remarks on (9).

Remark 3.1. *a) The singular weight* $\frac{1}{|x|}$ *in the right-hand side of* (9) *is optimal, in the sense that it may not be replaced by* $\frac{1}{|x|^\alpha}$ *with* $\alpha > 1$. *This can be easily seen by choosing* $\varphi \in H^1(\mathbb{R}_+^N)$ *such that* $\varphi(x) = |x|^{-\frac{N-2}{2}+\frac{\alpha-1}{2}}$ *in a neighborhood of the origin.*

Moreover, the infimum in (10) *is not achieved.*

b) In dimension $N = 2$ *the infimum* (10) *is zero, see* [14]. *Nonetheless, if the test-functions* φ *are required to vanish on the half line* $x_1 > 0$ *then the infimum has been computed in* [14] *:*

$$\inf \left\{ \frac{\int_{\mathbb{R}_+^2} |\nabla\varphi|^2}{\int_{\partial\mathbb{R}_+^2} \frac{\varphi^2}{|x|}} : \varphi \in H^1(\mathbb{R}_+^2), \varphi(x_1,0) = 0 \quad \textit{if} \quad x_1 > 0, \varphi\Big|_{\partial\mathbb{R}_+^2} \not\equiv 0 \right\} = \frac{1}{\pi} \tag{33}$$

c) Using Stirling's formula it is easy to see that

$$H_N = \frac{N-3}{2} + O(\frac{1}{N}) \qquad \textit{as } N \to \infty. \tag{34}$$

d) The estimates

$$\frac{N-3}{2} \le H_N \le \frac{\sqrt{(N-3)^2+1}}{2} \tag{35}$$

can be obtained in a more straightforward way using particular test functions. We give a proof of this at the end of Section 3. Also observe that (34) *could be deduced from* (35).

Let us explain first informally the idea behind the proof of Theorem 1.4, assuming for a moment that a minimizer $\overline{w} \in H^1(\mathbb{R}^N_+)$ of (10) exists. $\overline{w}$ then satisfies the associated Euler-Lagrange equation:

$$\begin{cases} \Delta \overline{w} = 0 & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial \overline{w}}{\partial \nu} = H_N \dfrac{\overline{w}}{|x|} & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{36}$$

Elementary changes of variables show that given $R > 0$ and a rotation $e \in \mathcal{O}(N-1)$, $\overline{w_R} := R^{\frac{2-N}{2}} \overline{w}(Rx)$ and $\overline{w_e} := \bar{w}(e(x'), x_N)$ are also minimizers of (10). Thus it is natural to assume $\overline{w} = \overline{w_R} = \overline{w_e}$ for all $R > 0$ and $e \in \mathcal{O}(N-1)$. In particular a constant multiple of $\overline{w}$ solves

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^N_+, \\ w = |x|^{-\frac{N-2}{2}} & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

Unfortunately, such a function w does not belong to $H^1(\mathbb{R}^N_+)$. Let $w = w_\alpha$ with $\alpha = \frac{N-2}{2}$ as defined in (12). Observe that $C(N, \frac{N-2}{2}) = H_N$ by (16) and hence w is indeed a solution of(36).

Following an idea of Brezis and Vázquez (equation (4.6) on page 453 of [4]), we restate (9) in terms of the new variable $v = \varphi/w$.

Proof of Theorem 1.4. When $N \geq 3$, $C_0^\infty(\mathbb{R}^N_+ \setminus \{0\})$ is dense in $H^1(\mathbb{R}^N_+)$. So it suffices to prove (9) for $\varphi \in C_0^\infty(\mathbb{R}^N_+ \setminus \{0\})$. Fix such a $\varphi \not\equiv 0$ and let w be the function defined by (12). Notice that, on supp φ, w is smooth and bounded from above and from below by some positive constants. Hence $v := \frac{\varphi}{w} \in C_0^\infty(\overline{\mathbb{R}^N_+})$ is well defined. Now, $\varphi = vw$, $\nabla\varphi = v\nabla w + w\nabla v$ and

$$|\nabla\varphi|^2 = v^2|\nabla w|^2 + w^2|\nabla v|^2 + 2vw\nabla v\nabla w.$$

Integrating

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 = \int_{\mathbb{R}^N_+} v^2|\nabla w|^2 + \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 + 2\int_{\mathbb{R}^N_+} vw\nabla v\nabla w$$

and by Green's formula

$$\begin{aligned} \int_{\mathbb{R}^N_+} v^2|\nabla w|^2 &= \int_{\partial\mathbb{R}^N_+} v^2 w \frac{\partial w}{\partial\nu} - \int_{\mathbb{R}^N_+} w\nabla(v^2\nabla w) \\ &= \int_{\partial\mathbb{R}^N_+} v^2 w \frac{\partial w}{\partial\nu} - 2\int_{\mathbb{R}^N_+} wv\nabla w\nabla v, \end{aligned}$$

since w is harmonic in $\mathbb{R}^N_+$. Thus,

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 = \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 + \int_{\partial\mathbb{R}^N_+} v^2 w\frac{\partial w}{\partial\nu} = \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 + \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{w}\frac{\partial w}{\partial\nu}. \tag{37}$$

But by (16) $\frac{\frac{\partial w}{\partial\nu}(x)}{w(x)} = \frac{H_N}{|x|}$ for $x \in \partial\mathbb{R}^N_+$ and hence,

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 \geq H_N \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|} + \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 \quad \forall\varphi \in H^1(\mathbb{R}^N_+). \tag{38}$$

The second term in the right hand side of the above inequality yields the improvement of Kato's inequality when φ has support in the unit ball.

Now we assume $\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+} \setminus \{0\} \cap B)$ and, as before, set $v = \frac{\varphi}{w}$. Our aim is to prove that given $1 \le q < 2$ there exists $C > 0$ such that

$$I := \int_{\mathbb{R}^N_+} w^2 |\nabla v|^2 \ge \frac{1}{C} \|\varphi\|_{W^{1,q}}. \tag{39}$$

In spherical coordinates

$$I = \int_0^1 r^{N-1} \int_{S_1^+} w^2(r\theta) |\nabla v(r\theta)|^2 d\theta \, dr$$

where $S_1^+ = S_1 \cap \mathbb{R}^N_+$ and $S_1 = \{x \in \mathbb{R}^N \,/\, |x| = 1\}$ is the sphere of radius 1. From (15) we have $w(x) \ge \frac{1}{C}|x|^{-\frac{N-2}{2}}$ for some $C > 0$ and all $x \in B \cap \mathbb{R}^N_+$. Hence

$$I \ge \frac{1}{C} \int_0^1 r \int_{S_1^+} |\nabla v(r\theta)|^2 d\theta \; dr.$$

Let us compute the Sobolev norm of φ :

$$\begin{aligned}
\|\varphi\|^q_{W^{1,q}} &= \int_{\mathbb{R}^N_+ \cap B} |\nabla \varphi|^q dx = \int_0^1 r^{N-1} \int_{S_1^+} |\nabla \varphi(r\theta)|^q d\theta \; dr \\
&= \int_0^1 r^{N-1} \int_{S_1^+} |\nabla v(r\theta) \; w(r\theta) + \nabla w(r\theta) \; v(r\theta)|^q d\theta \; dr \\
&\le C_q \int_0^1 r^{N-1} \int_{S_1^+} |\nabla v(r\theta)|^q |w(r\theta)|^q + |\nabla w(r\theta)|^q |v(r\theta)|^q d\theta \; dr.
\end{aligned}$$

Define

$$\begin{aligned}
I_1 &:= \int_0^1 r^{N-1} \int_{S_1^+} |\nabla v(r\theta)|^q |w(r\theta)|^q d\theta \; dr \\
I_2 &:= \int_0^1 r^{N-1} \int_{S_1^+} |\nabla w(r\theta)|^q |v(r\theta)|^q d\theta \; dr.
\end{aligned}$$

Since $w(x) \le C|x|^{-\frac{N-2}{2}}$ we have by Hölder's inequality

$$\begin{aligned}
I_1 &\le C \int_0^1 r^{N-1-\frac{(N-2)q}{2}} \int_{S_1^+} |\nabla v(r\theta)|^q d\theta \; dr \\
&\le C\Big[\int_0^1 r \int_{S_1^+} |\nabla v(r\theta)|^2 d\theta dr\Big]^{\frac{q}{2}} \Big[\int_0^1 r^{(N-1-\frac{Nq}{2}+\frac{q}{2})\frac{2}{2-q}} dr\Big]^{\frac{2-q}{2}} = C I^{\frac{q}{2}},
\end{aligned} \tag{40}$$

since $q < 2$.

Using $|\nabla w(x)| \le C|x|^{-\frac{N}{2}}$ we estimate I_2 :

$$I_2 \le C \int_{S_1^+} \int_0^1 r^{N-1-\frac{Nq}{2}} |v(r\theta)|^q \; dr \; d\theta.$$

From the classical Hardy inequality

$$\int_0^1 r^\gamma |f(r)|^p \; dr \le \Big(\frac{p}{\gamma+1}\Big)^p \int_0^1 r^{\gamma+p} |f'(r)|^p \; dr$$

($p \ge 1$, $\gamma > -1$, $f \in C_0^\infty(0,1)$) we deduce

$$\int_0^1 r^{N-1-\frac{Nq}{2}} |v(r\theta)|^q \; dr \le C \int_0^1 r^{N-1-\frac{Nq}{2}+q} |\nabla v(r\theta)|^q \; dr$$

and therefore

$$I_2 \le C \int_{S_1^+} \int_0^1 r^{N-1-\frac{Nq}{2}+q} |\nabla v(r\theta)|^q \, dr \, d\theta.$$

Hölder's inequality yields

(41)

$$I_2 \le C\Big[\int_{S_1^+} \int_0^1 r |\nabla v(r\theta)|^2 \, dr \, d\theta \Big]^{\frac{q}{2}} \Big[\int_{S_1^+} \int_0^1 r^{(N-1-\frac{Nq}{2}+\frac{q}{2})\frac{2}{2-q}} \, dr \, d\theta \Big]^{1-\frac{q}{2}} = C I^{\frac{q}{2}},$$

where we have used $q < 2$. Gathering (40) and (41) we conclude that (39) holds. □

Now we pass to the proof of item (d) of Remark 3.1.

Proof of (35)**.** We shall first show the inequality

$$\frac{N-3}{2} \le H_N, \quad \forall N \ge 4.$$

One may assume that $u = u(r,t)$ where $r = |(x_1, \dots, x_{N-1})|$ and $t = x_N$. Then

$$\int_{\partial \mathbb{R}^N_+} \frac{u^2}{|x|} = (N-1)\omega_{N-1} \int_0^\infty u(r,0)^2 r^{N-3} \, dr,$$

where ω_{N-1} is the volume of the unit ball in $\mathbb{R}^{N-1}$. But

$$u(r,0) = -2 \int_0^\infty u(r,t) \frac{\partial u}{\partial t}(r,t) \, dt.$$

So,

$$\begin{aligned} \int_{\partial \mathbb{R}^n_+} \frac{u^2}{|x|} &= -2(N-1)\omega_{N-1} \int_0^\infty \int_0^\infty u(r,t) \frac{\partial u}{\partial t}(r,t) r^{N-3} \, dr \, dt \\ &\le 2(N-1)\omega_{N-1} \int_0^\infty \Big(\int_0^\infty u(r,t)^2 r^{N-4} \, dr \Big)^{1/2} \Big(\int_0^\infty \Big(\frac{\partial u}{\partial t}(r,t) \Big)^2 r^{N-2} \, dr \Big)^{1/2} \, dt. \end{aligned}$$

We use now the inequality

$$\int_0^\infty u(r,t)^2 r^{N-4} \, dr \le \frac{4}{(N-3)^2} \int_0^\infty \Big(\frac{\partial u}{\partial r}(r,t) \Big)^2 r^{N-2} \, dr,$$

which is one of the classical version of Hardy's inequality (in dimension $N-1$). We obtain

$$\begin{aligned} &\int_{\partial \mathbb{R}^N_+} \frac{u^2}{|x|} \\ &\le \frac{4}{N-3}(N-1)\omega_{N-1} \int_0^\infty \Big[\int_0^\infty \Big(\frac{\partial u}{\partial r}(r,t) \Big)^2 r^{N-2} \, dr \Big]^{\frac{1}{2}} \Big[\int_0^\infty \Big(\frac{\partial u}{\partial t}(r,t) \Big)^2 r^{N-2} \, dr \Big]^{\frac{1}{2}} \, dt \\ &\le \frac{2}{N-3}(N-1)\omega_{N-1} \int_0^\infty \int_0^\infty \Big[\Big(\frac{\partial u}{\partial t} \Big)^2 + \Big(\frac{\partial u}{\partial r} \Big)^2 \Big] r^{N-2} \, dr \, dt \\ &= \frac{2}{N-3} \int_{\mathbb{R}^N_+} |\nabla u|^2. \end{aligned}$$

To prove

(42)
$$H_N \le \frac{\sqrt{(N-3)^2+1}}{2},$$

we consider, for fixed $a > 0$ and $\varepsilon \downarrow 0$, the function

$$\tilde{\varphi}(r, x_N) = \begin{cases} r^{\frac{2-N}{2}} e^{-ax_N/r} & \text{if } r > \varepsilon \\ \varepsilon^{\frac{2-N}{2}} e^{-ax_N/\varepsilon} & \text{if } r \leq \varepsilon, \end{cases}$$

where $x = (x', x_N) \in \mathbb{R}_+^{N-1} \times \mathbb{R}_+$, $r = |x'|$. With the test function

$$\varphi = \eta \tilde{\varphi}$$

where $\eta \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B_1(0)$ and $\eta \equiv 0$ outside of $B_2(0)$ and a suitable choice of a one obtains (42). We omit the details.

4. The exponential case

We need the following result that characterizes extremal singular solutions belonging to $H^1(\Omega)$, see [4, Theorem 3.1]. The proof is an adaptation of the one in this reference.

Lemma 4.1. *Assume that $v \in H^1(\Omega)$ is an unbounded solution of* (1) *for some $\lambda > 0$. Assume furthermore the stability condition*

$$\lambda \int_{\Gamma_1} f'(v)\varphi^2 \leq \int_\Omega |\nabla \varphi|^2 \quad \forall \varphi \in C^1(\overline{\Omega}), \varphi = 0 \text{ on } \Gamma_2. \tag{43}$$

Then $\lambda = \lambda^*$ and $v = u^*$.

Remark 4.2. *We have not shown that there is a unique weak solution of* (1) *when $\lambda = \lambda^*$. A result of Martel* [20] *guarantees that this is indeed the case for problem* (5) *and this was used by Brezis and Vázquez in the proof of* [4, Theorem 3.1]*. In our context, we take $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ as the* definition *of the extremal solution. Knowing that $u^* \in H^1(\Omega)$, the proof of* [4, Theorem 3.1] *shows that $\lambda = \lambda^*$ and $v = u^*$.*

To prove Theorems 1.3 and 1.5 it will be convenient to study the function u_0 defined by

$$u_0(x) = \int_{\partial \mathbb{R}_+^N} K(x, y) \log \frac{1}{|y|} \, dy \qquad \text{for } x \in \mathbb{R}_+^N, \tag{44}$$

where as before $K(x, y) = \frac{2x_N}{N\omega_N} |x - y|^{-N}$. Then u_0 is harmonic in $\mathbb{R}_+^N$ and

$$u_0(x) = \log \frac{1}{|x|} \quad \text{for } x \in \partial \mathbb{R}_+^N, x \neq 0.$$

Note that

$$u_0(Rx) = u_0(x) + \log \frac{1}{R}.$$

Let $r = |x'|$. Then

$$u_0(x', x_N) = v(\frac{x_N}{r}) + \log \frac{1}{r}, \tag{45}$$

for some $v : [0, \infty) \to \mathbb{R}$ such that $v(0) = 0$. We see that

$$\frac{\partial u_0}{\partial \nu} = -\frac{\partial u_0}{\partial x_N}\Big|_{x_N = 0} = -\frac{1}{r} v'(0)$$

so

$$\frac{\partial u_0}{\partial \nu} = \lambda_{0,N} e^{u_0} \qquad \text{on } \partial \mathbb{R}_+^N,$$

where we let

$$\lambda_{0,N} = -v'(0).$$

Let

$$\Omega_0 = \{x \in \mathbb{R}^N_+ : u_0(x) > 0\}$$
$$\Gamma_1 = \partial\Omega \cap \partial\mathbb{R}^N_+ \qquad \Gamma_2 = \partial\Omega \setminus \partial\mathbb{R}^N_+.$$

The boundary $\partial\Omega_0$ is not smooth itself but Γ_1, Γ_2 are, and it can be checked that Proposition 1.1 still holds in this case.

It can be verified that Ω_0 can be written as $\Omega_0 = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}_+ : |x'| < e^{v(x_N/|x'|)}\}$.

Lemma 4.3. *We have*

$$\lambda_{0,N} = \begin{cases} (N-3)\frac{\sqrt{\pi}\Gamma(\frac{N}{2}-\frac{3}{2})}{2\Gamma(\frac{N}{2}-1)} & \text{if } N \geq 4, \\ 1 & \text{if } N = 3. \end{cases}$$

Proof. We give details for $N \geq 4$, the case $N = 3$ being similar. We need to compute $v'(0)$. Calculating Δu_0 in terms of v (see (45)) we obtain that v satisfies

$$(1+t^2)v''(t) + (4-N)tv'(t) + 3 - N = 0$$

and thus v' is given by

$$v'(t) = (N-3)(1+t^2)^{\frac{N-4}{2}} \int_0^t (1+s^2)^{\frac{2-N}{2}}\, ds + (1+t^2)^{\frac{N-4}{2}} v'(0).$$

Integrating and using $v(0) = 0$ yields

(46)
$$v(t) = (N-3)\int_0^t (1+\tau^2)^{\frac{N-4}{2}} \int_0^\tau (1+s^2)^{\frac{2-N}{2}}\, ds\, d\tau + v'(0)\int_0^t (1+\tau^2)^{\frac{N-4}{2}}\, d\tau.$$

We look at the asymptotics of the two integrals above, as $t \to \infty$. For the second integral, we have

$$\lim_{t\to\infty} \frac{\int_0^t (1+\tau^2)^{\frac{N-4}{2}}\, d\tau}{t^{N-3}} = \frac{(1+t^2)^{\frac{N-4}{2}}}{(N-3)t^{N-4}} = \frac{1}{N-3}.$$

And for the first integral,

$$\begin{aligned} \lim_{t\to\infty} \frac{\int_0^t (1+\tau^2)^{\frac{N-4}{2}} \int_0^\tau (1+s^2)^{\frac{2-N}{2}}\, ds\, d\tau}{t^{N-3}} &= \lim_{t\to\infty} \frac{(1+t^2)^{\frac{N-4}{2}} \int_0^t (1+s^2)^{\frac{2-N}{2}}\, ds}{(N-3)t^{N-4}} \\ &= \frac{1}{N-3}\int_0^\infty (1+s^2)^{\frac{2-N}{2}}\, ds \\ &= \frac{1}{N-3}\frac{\sqrt{\pi}\Gamma(\frac{N}{2}-\frac{3}{2})}{2\Gamma(\frac{N}{2}-1)}. \end{aligned}$$

Going back to (46), we obtain that

$$v(t) = \left(\frac{\sqrt{\pi}\Gamma(\frac{N}{2}-\frac{3}{2})}{2\Gamma(\frac{N}{2}-1)} + \frac{v'(0)}{N-3}\right) t^{N-3} + o(t^{N-3}).$$

Now, recall that for $x_N > 0$, $\lim_{r\to 0} v(x_N/r) + \log \frac{1}{r} = u_0(0, x_N) \in \mathbb{R}$ exists and is finite. Hence, we must have

$$v'(0) = -(N-3)\frac{\sqrt{\pi}\Gamma(\frac{N}{2}-\frac{3}{2})}{2\Gamma(\frac{N}{2}-1)}.$$

□

Proof of Theorem 1.3. We have shown that u_0 defined in (44) is a solution to (1) with $\Omega = \Omega_0$ and $\lambda = \lambda_{0,N}$. This solution satisfies the stability condition (43) if and only if (by scaling)

$$\lambda_{0,N}\int_{\partial\mathbb{R}^N_+}\frac{\varphi^2}{|x|} \le \int_{\mathbb{R}^N_+}|\nabla\varphi|^2, \qquad \forall\varphi\in C^1_0(\overline{\mathbb{R}^N_+}\setminus\{0\}).$$

In the Appendix we prove that

$$H_N \ge \lambda_{0,N} \text{ if and only if } N \ge 10 \tag{47}$$

and this completes the proof of the theorem. □

Proof of Theorem 1.5.

We prove the theorem by contradiction, assuming that u^* is unbounded. We use an idea of Crandall and Rabinowitz [11], but with different test functions.

Let $\phi(x) = \int_{\partial\mathbb{R}^N_+} K(x,y)|y|^{2-N+\varepsilon}dy$ and $\psi(x) = \int_{\partial\mathbb{R}^N_+} K(x,y)|y|^{\frac{2-N+\varepsilon}{2}}dy$. Then,

$$\frac{\partial\phi}{\partial\nu} = K_\phi|x|^{1-N+\varepsilon} \qquad \frac{\partial\psi}{\partial\nu} = K_\psi|x|^{\frac{-N+\varepsilon}{2}}, \tag{48}$$

where the constants K_ϕ, K_ψ are given by

$$K_\phi = \lambda_{0,N}\varepsilon + O(\varepsilon^2) \quad \text{ and } \quad K_\psi = H_N + O(\varepsilon).$$

Indeed, since u_0 and ϕ are harmonic in Ω,

$$\int_{\partial\Omega} u_0\frac{\partial\phi}{\partial\nu} = \int_{\partial\Omega}\phi\frac{\partial u_0}{\partial\nu}.$$

Clearly, $\int_{\Gamma_2}\left|\phi\frac{\partial u_0}{\partial\nu}\right| \le C$, for some constant C independent of ε. So

$$K_\phi\int_0^1 \ln\left(\frac{1}{r}\right)\frac{1}{r}r^{2-N+\varepsilon}r^{N-2}dr = \lambda_{0,N}\int_0^1\frac{1}{r}r^{2-N+\varepsilon}r^{N-2}dr + O(1) = \frac{\lambda_{0,N}}{\varepsilon} + O(1).$$

Now, $\int_0^1 \ln\frac{1}{r}r^{-1+\varepsilon}dr = \frac{1}{\varepsilon^2}$ so we end up with

$$K_\phi = \lambda_{0,N}\varepsilon + O(\varepsilon^2).$$

Similarly, since ψ and w (defined in (12)) are harmonic in Ω, we have

$$\int_{\partial\Omega} w\frac{\partial\psi}{\partial\nu} = \int_{\partial\Omega}\psi\frac{\partial w}{\partial\nu}.$$

As before the boundary terms on Γ_2 are bounded independently of ε so

$$K_\psi\int_0^1 r^{-1+\varepsilon}dr = H_N\int_0^1 r^{-1+\varepsilon}dr + O(1).$$

Hence,

$$K_\psi = H_N + O(\varepsilon).$$

Multiplying the equation (1) by ϕ and integrating by parts twice yields

$$\int_{\partial\Omega} u^* \frac{\partial \phi}{\partial \nu} = \lambda^* \int_{\partial\Omega} \phi e^{u^*}. \tag{49}$$

Let $\eta \in C^\infty(\mathbb{R}^N)$ be such that $\eta \equiv 1$ in $B_R(0)$ where $R > 0$ is small and fixed, and $\eta = 0$ on Γ_2. Using the stability condition (7) with $\eta\psi$ yields

$$\begin{aligned} \lambda^* \int_{\Gamma_1 \cap B_R(0)} e^{u^*} \psi^2 &\le \int_\Omega |\nabla(\eta\psi)|^2 = \int_{\partial\Omega} \frac{\partial}{\partial\nu}(\eta\psi)(\eta\psi) - \int_\Omega (\eta\psi)\Delta(\eta\psi) \\ &\le \int_{\Gamma_1 \cap B_R(0)} \frac{\partial \psi}{\partial \nu}\psi + C \end{aligned} \tag{50}$$

where the constant C does not depend on ε. Since $\psi^2 = \phi$ on $\partial\mathbb{R}^N_+$ combining (49) and (50) we obtain

$$\int_{\partial\Omega} u^* \frac{\partial \phi}{\partial \nu} \le \int_{\Gamma_1 \cap B_R(0)} \frac{\partial \psi}{\partial \nu}\psi + C.$$

Using (48) we arrive at

$$K_\phi \int_{\Gamma_1 \cap B_R(0)} u^* |x|^{1-N+\varepsilon} \le K_\psi \int_{\Gamma_1 \cap B_R(0)} |x|^{1-N+\varepsilon} + C$$

and thus

$$\int_{\Gamma_1 \cap B_R(0)} u^* |x|^{1-N+\varepsilon} \le \omega_{N-1} \frac{H_N}{\lambda_{0,N}} \frac{1}{\varepsilon^2} + O(\frac{1}{\varepsilon}), \tag{51}$$

where ω_{N-1} is the area of the $N-1$ dimensional sphere.

Next we claim that for any given $0 < \sigma < 1$ there exists $r(\sigma) > 0$ such that

$$u^*(x) \ge (1-\sigma)\log\frac{1}{|x|} \qquad \forall x \in \Gamma_1,\ |x| \le r(\sigma). \tag{52}$$

Observe first that for all $0 < \lambda < \lambda^*$ the minimal solution u_λ is symmetric in the variables $x_1, \dots, x_{N-1}$ by uniqueness of the minimal solution and it achieves its maximum at the origin by the moving plane method (see Proposition 5.2 in [7]).

Assume by contradiction that (52) is false. Then there exists $\sigma > 0$ and a sequence $x_k \in \Gamma_1$ with $x_k \to 0$ such that

$$u^*(x_k) < (1-\sigma)\log\frac{1}{|x_k|}. \tag{53}$$

Let $s_k = |x_k|$ and choose $0 < \lambda_k < \lambda^*$ such that

$$\max_{\overline{\Omega}} u_{\lambda_k} = u_{\lambda_k}(0) = \log\frac{1}{s_k}. \tag{54}$$

Note that $\lambda_k \to \lambda^*$, otherwise u_{λ_k} would remain bounded. Let

$$v_k(x) = \frac{u_{\lambda_k}(s_k x)}{\log\frac{1}{s_k}} \qquad x \in \Omega_k \equiv \frac{1}{s_k}\Omega.$$

Then $0 \le v_k \le 1$, $v_k(0) = 1$, $\Delta v_k = 0$ in Ω_k and

$$\begin{aligned}\frac{\partial v_k}{\partial \nu}(x) &= \frac{1}{\log \frac{1}{s_k}} s_k \lambda_k \exp(u_{\lambda_k}(s_k x)) \\ &\le \frac{\lambda_k}{\log \frac{1}{s_k}} \to 0.\end{aligned}$$

by (54). By elliptic regularity $v_k \to v$ uniformly on compact sets of $\overline{\mathbb{R}^N_+}$ to a function v satisfying $0 \le v \le 1$, $v(0) = 1$, $\Delta v = 0$ in $\mathbb{R}^N_+$, $\frac{\partial v}{\partial \nu} = 0$ on $\partial \mathbb{R}^N_+$. Extending v evenly to $\mathbb{R}^N$ we deduce that $v \equiv 1$. Since $|x_k| = s_k$ we deduce that

$$\frac{u_{\lambda_k}(x_k)}{\log \frac{1}{s_k}} \to 1,$$

which contradicts (53).

Going back to (51) and using (52) we find

$$(1-\sigma) \int_0^{r(\sigma)} \log \frac{1}{r}\, r^{\varepsilon - 1}\, dr \le \frac{K_\psi}{K_\phi} \frac{1}{\varepsilon} + C = \frac{H_N}{\lambda_{0,N}} \frac{1}{\varepsilon^2} + O(\frac{1}{\varepsilon}). \tag{55}$$

Integrating

$$(1-\sigma)\left(\frac{1}{\varepsilon^2} r(\sigma)^\varepsilon + \frac{1}{\varepsilon} r(\sigma)^\varepsilon \log \frac{1}{r(\sigma)}\right) \le \frac{H_N}{\lambda_{0,N}} \frac{1}{\varepsilon^2} + O(\frac{1}{\varepsilon}).$$

Letting $\varepsilon \to 0$ yields

$$(1-\sigma) \le \frac{H_N}{\lambda_{0,N}}.$$

As σ is arbitrarily small we deduce $\frac{H_N}{\lambda_{0,N}} \ge 1$ which by (47) forces $N \ge 10$, a contradiction. □

Proof of Proposition 1.7. Let indeed $u = u_\lambda$ be the minimal solution of (1). Working as in [11] we take $\varphi = e^{ju} - 1$, $j > 0$ in (6) and multiply (1) by $\psi = e^{2ju} - 1$. We obtain

$$\frac{\lambda}{j^2} \int_{\Gamma_1} e^u \left(e^{ju} - 1\right)^2 ds \le \frac{\lambda}{2j} \int_{\Gamma_1} e^u \left(e^{2ju} - 1\right) ds.$$

It follows that

$$\begin{aligned}\left(\frac{1}{j} - \frac{1}{2}\right) \int_{\Gamma_1} e^{(2j+1)u}\, ds &\le \frac{2}{j} \int_{\Gamma_1} e^{(j+1)u}\, ds \\ &\le \frac{2}{j} \int_{\Gamma_1 \cap A} e^{(j+1)u}\, ds + \frac{2}{j} \int_{\Gamma_1 \cap B} e^{(j+1)u}\, ds,\end{aligned}$$

where $A = [(1/j - 1/2)e^{(2j+1)u} < \frac{4}{j} e^{(j+1)u}]$ and $B = [(1/j - 1/2)e^{(2j+1)u} \ge \frac{4}{j} e^{(j+1)u}]$. Given $j \in (0,2)$, we see that u remains uniformly bounded on A, while

$$\frac{2}{j} \int_{\Gamma_1 \cap B} e^{(j+1)u}\, ds \le \frac{1}{2}\left(\frac{1}{j} - \frac{1}{2}\right) \int_{\Gamma_1} e^{(j+1)u}\, ds.$$

We conclude that e^u is bounded in $L^{2j+1}(\partial\Omega)$ independently of λ. If $2j+1 > N-1$ we obtain by elliptic estimates a bound for u in $C^\alpha(\overline{\Omega})$, for some $\alpha \in (0,1)$. Thus if $N < 6$ we can choose $j \in (0,2)$ such that $N - 1 < 2j + 1 < 5$ and obtain a bound for u in $C^\alpha(\overline{\Omega})$ independent of λ. However, for $N = 6, 7, 8$ or 9, this argument does not prove that $u^* \in L^\infty(\Omega)$. □

5. The power case

Proof of Theorem 1.8. We shall give here the proof of the case $p\,C(N,\frac{1}{p-1}) > H_N$. If $p < \frac{N}{N-2}$, the boundedness of u^* follows from standard techniques, using the Sobolev trace embedding theorem $H^1(\Omega) \to L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$.

Let $v = C(N,\frac{1}{p-1})^{\frac{1}{p-1}} w_{\frac{1}{p-1}}$. Then v satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^N_+ \\ \dfrac{\partial v}{\partial \nu} = v^p & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

Observe that $pv^{p-1} = \frac{pC(N,\frac{1}{p-1})}{|x|} > \frac{H_N}{|x|}$ on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and hence

$$\inf \frac{\int_{\partial\mathbb{R}^N_+} |\nabla\varphi|^2 - p\int_{\partial\mathbb{R}^N_+} v^{p-1}\varphi^2}{\int_{\partial\mathbb{R}^N_+} \varphi^2} = -\infty \tag{56}$$

where the infimum is taken over the functions $\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+})$ that do not vanish identically on $\partial\mathbb{R}^N_+$.

Assume that u^* is singular. For $R > 0$ and $0 < \lambda < \lambda^*$ let

$$u_R(x) = \lambda^{\frac{1}{p-1}} R^{\frac{1}{p-1}} u_\lambda(Rx + x_\lambda),$$

where x_λ denotes a point of maximum of u_λ. Observe that since u_λ is positive and harmonic in Ω, $x_\lambda \in \Gamma_1$.

For $0 < \lambda < \lambda^*$, we choose R such that $u_R(0) = 1$ i.e. such that $\lambda^{\frac{1}{1-p}} R^{\frac{1}{p-1}} u_\lambda(x_\lambda) = 1$. Since $u_\lambda(x_\lambda) \to \infty$ as $\lambda \uparrow \lambda^*$ we have $R \to 0$ as $\lambda \uparrow \lambda^*$.

Then u_R verifies

$$\begin{cases} \Delta u_R = 0 & \text{in } \Omega_R \\ \dfrac{\partial u_R}{\partial \nu} = (\lambda^{\frac{1}{p-1}} R^{\frac{1}{p-1}} + u_R)^p & \text{on } \Gamma_1^R \\ u_R = 0 & \text{on } \Gamma_2^R, \end{cases}$$

where

$$\Omega_R = (\Omega - x_\lambda)/R, \quad \Gamma_1^R = (\Gamma_1 - x_\lambda)/R, \quad \Gamma_2^R = (\Gamma_2 - x_\lambda)/R.$$

Furthermore u_R satisfies the stability condition

$$\int_{\Omega_R} |\nabla\varphi|^2 \ge p\int_{\Gamma_1^R} (\lambda^{\frac{1}{p-1}} R^{\frac{1}{p-1}} + u_R)^{p-1}\varphi^2 \qquad \forall\varphi \in C_0^\infty(\Omega_R \cup \Gamma_1^R).$$

Let

$$\phi(R) = \sup\{r > 0 \,/\, B_r \cap \partial\mathbb{R}^N_+ \subset \Gamma_1^R\}.$$

The moving plane method implies that the distance of the point $x_\lambda \in \Gamma_1$ to $\Gamma_1 \cap \Gamma_2$ stays bounded away from zero, see [7] for this method in the context of non-linear Neumann condition. Thus implies that

$$\phi(R) \to +\infty \quad \text{as } R \to 0. \tag{57}$$

Step 1. We have

$$u_R \le v \quad \text{in } \Gamma_1^R \cap B_{\phi(R)}. \tag{58}$$

Proof. Suppose not. Define

$$r_0 = \sup\{\, r > 0 \,|\, r < \phi(R),\, u_R \le v \text{ in } B_r \cap \Gamma_1^R \,\}.$$

Since v is singular at 0, $r_0 > 0$ and we have $u_R \le v$ in $B_{r_0} \cap \Gamma_1^R$. Furthermore, there exists $x_0 \in \partial\mathbb{R}^N_+$ such that $|x_0| = r_0$ and $u_R(x_0, 0) = v(x_0, 0)$.

Let $x \in \Gamma_1^R$ be such that $|x| = r_0$. If $u_R(x, 0) = v(x, 0)$ then $\frac{\partial v}{\partial \nu}(x, 0) = v(x, 0)^p < (\lambda^{\frac{1}{1-p}} R^{\frac{1}{p-1}} + u_R(x, 0))^p = \frac{\partial u_R}{\partial \nu}(x, 0)$ and hence for some $\delta_x > 0$

$$\frac{\partial v}{\partial x_N}(y, t) > \frac{\partial u_R}{\partial x_N}(y, t) \qquad |y - x|^2 + t^2 < \delta_x^2. \tag{59}$$

It follows that for some $m_x > 0$ (and decreasing if necessary δ_x)

$$u_R(y, t) < v(y, t), \quad \text{if } |y - x|^2 + t^2 < \delta_x^2,\ t > m(\,|y| - r_0),\ t > 0. \tag{60}$$

Indeed, because of (59) and $u_R(y, 0) \le v(y, 0)$ for $|y| \le r_0$ we immediately obtain

$$u_R(y, t) < v(y, t) \quad \text{for } |y - x|^2 + t^2 < \delta_x^2, |y| \le r_0,\ t > 0.$$

If (60) is false, then there are sequences $y_k \to x$, $t_k \to 0$ with $|y_k| > r_0$ and $\frac{t_k}{|y_k - x|} \to \infty$ such that $v(y_k, t_k) \le u_R(y_k, t_k)$. Then by the mean value theorem there exists a point ξ_k in the segment from (y_k, t_k) to $(x, 0)$ such that $\nabla(v(\xi_k) - u_R(\xi_k)) \cdot w_k \le 0$ where w_k is the unit vector $w_k = \frac{(y_k - x, t_k)}{\|w_k\|}$. Taking the limit we obtain $\frac{\partial}{\partial x_N}(v(x, 0) - u_R(x, 0)) \le 0$ which contradicts (59) (recall that $\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial x_N}$). This proves (60).

Suppose $x \in \Gamma_1^R$ is such that $|x| = r_0$ and $u_R(x, 0) < v(x, 0)$. The by continuity there is $\delta_x > 0$ such that (60) still holds.

Then by compactness for some $\delta > 0$ and $m > 0$ we have

$$u_R(x, t) < v(x, t), \quad \text{if } (\,|x| - r_0)^2 + t^2 < \delta^2,\ t > m(\,|x| - r_0),\ t > 0. \tag{61}$$

Now consider

$$z(x, t) = \rho^\alpha(\sin(\alpha(\theta - \theta_0)) + b(\theta - \theta_0)^2),$$

where (ρ, θ) are polar coordinates around $(r_0, 0)$ i.e.

$$|x| = r_0 + \rho\cos(\theta), \quad t = \rho\sin(\theta).$$

We choose $\theta_0 \in (0, \frac{\pi}{2})$ is close enough to $\frac{\pi}{2}$ so that $\tan\theta_0 > m$. The parameters α, b are chosen later on.

We shall use the maximum principle to prove that for sufficiently small $\delta > 0$, $\varepsilon > 0$ we have

$$v(x, 0) - u_R(x, 0) \ge \varepsilon z(x, 0), \quad r_0 - \delta < |x| < r_0.$$

We have

$$\begin{aligned}
\Delta z = \frac{\rho^{\alpha-2}}{r_0 + \rho\cos(\theta)} \Bigg[& 2br_0 + \alpha^2 br_0(\theta - \theta_0)^2 + \rho\Big(-(N-2)\alpha\sin(\theta)\cos(\alpha(\theta - \theta_0)) \\
& - 2b(N-2)\sin(\theta)(\theta - \theta_0) + 2b\cos(\theta) + (N-2)\alpha\cos(\theta)\sin(\alpha(\theta - \theta_0)) \\
& + b\alpha(\alpha + N - 2)\cos(\theta)(\theta - \theta_0)^2\Big)\Bigg]
\end{aligned}$$

$$= \rho^{\alpha-2}\Big(2b + \alpha^2 b(\theta - \theta_0)^2 + O(\rho)\Big), \quad \text{as } \rho \to 0 \tag{62}$$

and, observing that for $\theta = \pi$ we are on $\partial\mathbb{R}^N_+$

$$\frac{\partial z}{\partial \nu} = \frac{1}{\rho}\frac{\partial z}{\partial \theta}\Big|_{\theta=\pi} = \rho^{\alpha-1}(\alpha\cos(\alpha(\pi-\theta_0)) + 2b(\pi-\theta_0)).$$

But $\pi - \theta_0 > \frac{\pi}{2}$. We fix $0<\alpha<1$ close enough to 1 such that $\cos(\alpha(\pi-\theta_0)) < 0$, and then take $b>0$ small enough so that $\alpha\cos(\alpha(\pi-\theta_0)) + 2b(\pi-\theta_0) < 0$. Thus, setting

$$a \equiv -(\alpha\cos(\alpha(\pi-\theta_0)) + 2b(\pi-\theta_0)) > 0,$$

we have

$$\frac{\partial z}{\partial \nu} = -a\rho^{\alpha-1} \quad \text{for } \rho > 0 \text{ small.}$$

On the other hand, by (62)

$$\Delta z \ge \rho^{\alpha-2}\,(2b + O(\rho)) \quad \text{as } \rho\to 0 \tag{63}$$

uniformly for $\theta_0 < \theta < \pi$. Now consider the region

$$D = \{\, (x,t) \mid (|x|-r_0)^2 + t^2 < \delta^2,\ t > m(|x|-r_0),\ t>0\,\} = \{\,(\rho,\theta)\mid 0<\rho<\delta,\ \theta_0<\theta<\pi\,\}$$

and write $\partial D = S_1\cup S_2\cup A$ where

$$S_1 = \{\,(\rho,\theta)\mid 0<\rho<\delta,\ \theta=\theta_0\,\},\quad S_2 = \{\,(\rho,\theta)\mid 0<\rho<\delta,\ \theta=\pi\,\}$$

and

$$A = \{\,(\rho,\theta)\mid \rho=\delta,\ \theta_0<\theta<\pi\,\}.$$

By (63) and choosing $\delta>0$ smaller if necessary we achieve $\Delta z>0$ in D. On S_1 we have $z=0$ and $v-u_R>0$. Now we seek $\varepsilon>0$, $\delta>0$ smaller than before such that

$$\inf_{S_2}\Big(\frac{\partial v}{\partial \nu} - \frac{\partial u_R}{\partial \nu}\Big) \ge \varepsilon\sup_{S_2}\frac{\partial z}{\partial \nu} = -a\varepsilon\delta^{\alpha-1}$$

and

$$\inf_A (v-u_R) > \varepsilon\sup_A z = \varepsilon\delta^\alpha C_1$$

where $C_1 = \sin(\alpha(\pi-\theta_0)) + b(\pi-\theta_0)^2$. Writing $K = -\inf_{S_2}(\frac{\partial v}{\partial\nu} - \frac{\partial u_R}{\partial \nu}) < \infty$ and $c_0 = \inf_A(v-u_R) > 0$ we first choose $\delta>0$ small such that

$$\delta\frac{K}{a} < \frac{c_0}{C_1}$$

and then ε such that $\delta^{1-\alpha}\frac{K}{a} \le \varepsilon < \frac{c_0}{C_1}\delta^{-\alpha}$.

The calculations above and the maximum principle then yield $v-u_R\ge\varepsilon z$ in D, which was the desired conclusion. Now this implies that $v-u_R$ is not differentiable at $(x_0,0)$, a contradiction.

Step 2. We let $\lambda\uparrow\lambda^*$ and hence $R\to 0$. Since $0\le u_R\le u_R(0)=1$, $u_R\to u$ uniformly on compact sets of $\bar{\mathbb{R}}^N_+$ and u satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N_+ \\ \dfrac{\partial u}{\partial \nu} = u^p & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{64}$$

and

$$u(0) = 1.$$

Also, u satisfies

$$\int_{\mathbb{R}^N_+} |\nabla \varphi|^2 \geq p \int_{\partial \mathbb{R}^N_+} u^{p-1} \varphi^2 \qquad \forall \varphi \in C_0^\infty(\overline{\mathbb{R}^N_+}). \tag{65}$$

By (57) and the previous step we deduce

$$u \leq v \quad \text{ in } \partial \mathbb{R}^N_+.$$

Let

$$\mu = \sup_{\partial \mathbb{R}^N_+} \frac{u}{v} \leq 1 \tag{66}$$

We claim that $\mu = 1$. Let indeed

$$\tilde{u}(x) = c_1 \int_{\partial \mathbb{R}^N_+} \frac{u(y)^p}{|x-y|^{N-2}}\, dy \qquad x \in \partial \mathbb{R}^N_+.$$

Then $\tilde{u}$ is harmonic in $\mathbb{R}^N_+$ and agrees with u on $\partial \mathbb{R}^N_+$. Since u is bounded by 1 and $\tilde{u}$ is bounded, we see that $\tilde{u} - u$ must be a constant. But then, since $\tilde{u}(x,0) \to 0$ and $u(x,0) \to 0$ as $|x| \to \infty$ we see that $u \equiv \tilde{u}$.

Thus

$$u(x) = \int_{\partial \mathbb{R}^N_+} \frac{u(y)^p}{|x-y|^{N-2}}\, dy \leq c_1 \mu^p \int_{\partial \mathbb{R}^N_+} \frac{v(y)^p}{|x-y|^{N-2}}\, dy = \mu^p v(x) \qquad x \in \partial \mathbb{R}^N_+.$$

This implies $\mu \leq \mu^p$ and since $\mu \neq 0$, we conclude $\mu = 1$.

Step 3. Observe that the supremum in (66) is not attained. Otherwise $v - u$ would achieve a minimum at a point $x \in \partial \mathbb{R}^N_+$, where the normal derivative would be zero. By Hopf's lemma, we would have $u \equiv v$, which is impossible since u is bounded and v is not. Let $x_k \in \partial \mathbb{R}^N_+$ be such that $|x_k| \to \infty$ and $\frac{u(x_k)}{v(x_k)} \to 1$. Let

$$u_k(x) = |x_k|^{\frac{1}{p-1}} u(|x_k| x).$$

Since v is invariant under the above transformation we have $u_k \leq v$ in $\partial \mathbb{R}^N_+$. Thus, for a subsequence we have $u_k \to u_0$ and u_0 solves (64). Since $u_k(\frac{x_k}{|x_k|}) \to v(y)$ where $y = \lim \frac{x_k}{|x_k|}$, again using Hopf's lemma we see that $u_0 \equiv v$. But u_0 satisfies the condition (65), contradicting (56).

Proof of Theorem 1.10. Set $u = w_{\frac{1}{p-1}} - 1$ so that

$$\Delta u = 0 \qquad \text{in } \mathbb{R}^N_+ \tag{67}$$

$$\frac{\partial u}{\partial \nu} = C(N, \tfrac{1}{p-1})(1+u)^p \qquad \text{on } \partial \mathbb{R}^N_+ \tag{68}$$

Let $\Omega = \{x \in \mathbb{R}^N_+ \mid u(x) > 0\}$, $\Gamma_1 = \partial\Omega \cap \partial \mathbb{R}^N_+$, $\Gamma_2 = \partial \Omega \setminus \partial \mathbb{R}^N_+$. Then u is a singular solution to (67), (68) with

$$u = 0 \quad \text{on } \Gamma_2.$$

To apply Lemma 4.1 we need to verify that $u \in H^1(\Omega)$. We are assuming that $pC(N, \frac{1}{p-1}) \le H_N$ and $p \ge \frac{N}{N-2}$. Actually we must have $p > \frac{N}{N-2}$. For this it is convenient to observe that :

$$C(N,\alpha) = C(N, N-2-\alpha) \quad \forall 0 < \alpha < N-2 \tag{69}$$

$$\alpha \mapsto C(N,\alpha) \quad \text{is increasing for } 0 < \alpha < \tfrac{N-2}{2}. \tag{70}$$

Property (69) is direct from (16) and we leave (70) to the appendix. From these properties we deduce that $p > \frac{N}{N-2}$ and therefore $u \in H^1(\Omega)$. This solution satisfies the stability condition (43) if and only if (by scaling)

$$pC(N, \tfrac{1}{p-1}) \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|} \le \int_{\mathbb{R}^N_+} |\nabla\varphi|^2, \qquad \forall \varphi \in C^1_0(\overline{\mathbb{R}^N_+} \setminus \{0\})$$

which is guaranteed by Kato's inequality (9). Thus we may apply Lemma 4.1 and conclude that u is the extremal solution.

□

Appendix

Proof of (47)**.** We have $H_N = 2\left[\frac{\Gamma(\frac{N}{4})}{\Gamma(\frac{N}{4}-\frac{1}{2})}\right]^2 = 2f(N/4)^2$ where $f(z) = \frac{\Gamma(z)}{\Gamma(z-\frac{1}{2})}$ and similarly we have $\lambda_{0,N} = \sqrt{\pi}\frac{N-3}{2}\frac{\Gamma(\frac{N}{2}-\frac{3}{2})}{\Gamma(\frac{N}{2}-1)} = \sqrt{\pi}\frac{\Gamma(\frac{N}{2}-\frac{1}{2})}{\Gamma(\frac{N}{2}-1)} = \sqrt{\pi} f\left(\frac{N}{2}-\frac{1}{2}\right)$. Then

$$\frac{H_N}{\lambda_{0,N}} = \frac{2f(N/4)^2}{\sqrt{\pi} f\left(\frac{N}{2}-\frac{1}{2}\right)}.$$

Since $H_{10} = \frac{9\pi}{8} > \lambda_{0,10} = \frac{35\pi}{32}$ it follows that

$$\frac{H_{10}}{\lambda_{0,10}} > 1.$$

On the other hand

$$\frac{H_9}{\lambda_{0,9}} = \frac{\left(\frac{5\Gamma(\frac{1}{4})^2}{12\pi}\right)^2}{\frac{16}{5}} \approx \frac{3.039}{3.333} < 1.$$

Let us compute

$$\begin{aligned} \frac{d}{dx}\frac{f(\frac{x}{4})^2}{f\left(\frac{x}{2}-\frac{1}{2}\right)} &= \frac{\frac{1}{2}f(\frac{x}{4})f'(\frac{x}{4})f\left(\frac{x}{2}-\frac{1}{2}\right) - \frac{1}{2}f(\frac{x}{4})^2 f'\left(\frac{x}{2}-\frac{1}{2}\right)}{f\left(\frac{x}{2}-\frac{1}{2}\right)^2} \\ &= \frac{f(\frac{x}{4})^2}{2f\left(\frac{x}{2}-\frac{1}{2}\right)}\left[\frac{f'(\frac{x}{4})}{f(\frac{x}{4})} - \frac{f'\left(\frac{x}{2}-\frac{1}{2}\right)}{f\left(\frac{x}{2}-\frac{1}{2}\right)}\right] \end{aligned}$$

Recall that

$$\Gamma'(z) = \psi_0(z)\Gamma(z), \qquad \text{where } \psi_0(z) = -\left(\frac{1}{z} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right)\right).$$

In particular, ψ_0 is a positive increasing function on $]\Xi,\infty[$ where $\Xi \approx 1.4$ is the unique zero of ψ_0 in $\mathbb{R}_+$, and

$$f'(z) = \left[\psi_0(z) - \psi_0\left(z-\frac{1}{2}\right)\right]\frac{\Gamma(z)}{\Gamma(z-\frac{1}{2})} > 0 \qquad \text{for } z \in]1/2,\infty[.$$

Calculating

$$\frac{d}{dx}\frac{f'(x)}{f(x)} = \psi_0'(x) - \psi_0'(x-\frac{1}{2}) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2} - \frac{1}{(n+x-\frac{1}{2})^2} < 0 \quad \text{for } x > \frac{1}{2}.$$

So $\frac{f'(x)}{f(x)}$ is decreasing for $x > \frac{1}{2}$ and it follows that $\frac{f'(\frac{x}{4})}{f(\frac{x}{4})} - \frac{f'\left(\frac{x}{2}-\frac{1}{2}\right)}{f\left(\frac{x}{2}-\frac{1}{2}\right)} > 0$ for $x > 2$. Hence

$$H_N > \lambda_{0,N} \quad \text{only for } N \geq 10.$$

□

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