PARTIAL REGULARITY FOR A LIOUVILLE SYSTEM

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ABSTRACT. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. We prove that the singular set of any extremal solution of the system

$$-\Delta u = \mu e^v, \quad -\Delta v = \lambda e^u \quad \text{in } \Omega,$$

with $u = v = 0$ on $\partial\Omega$, $\mu, \lambda \geq 0$, has Hausdorff dimension at most $n - 10$.

1. INTRODUCTION

In this article we consider the issue of partial regularity of extremal solutions to the Liouville system

$$\begin{cases} -\Delta u = \mu e^v & \text{in } \Omega, \\ -\Delta v = \lambda e^u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

with Ω a bounded smooth open subset of $\mathbb{R}^n$, and λ, μ nonnegative parameters.

This system is a generalization of the equation

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

where λ denotes a positive parameter. It is well known that there is a maximal parameter $\lambda^* > 0$ for existence of solutions of (2) and for $0 < \lambda < \lambda^*$ there is a minimal solution u_λ. As $\lambda \to \lambda^*, \lambda < \lambda^*$ the solution u_λ converges to the so-called extremal solution, which turns out to be smooth for $n \leq 9$, see [3, 11]. The interested reader may find in the book [7] the developments of the theory for the last six decades, with a particular focus on stable solutions.

Recently it was proved by K. Wang [13] that for $n \geq 10$ the extremal solution of (2) has a singular set of dimension at most $n - 10$. F. Da Lio [5] obtained partial regularity for any weak *stationary* solution in dimension 3 (not necessarily stable). See related results for the Lane-Emden equation in [14, 6].

Here we generalize the results of [13] to the system (1). For this system, M. Montenegro [12] proved the existence of a nonempty open set $\mathcal{U}$ in the quarter plane $\lambda, \mu > 0$ such that for a couple of parameters (μ, λ) in $\mathcal{U}$ there is a smooth *minimal* solution (u, v) and no smooth solution exists if the couple is in the complement of

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1991 *Mathematics Subject Classification.* Primary: 35G30, 35B65; Secondary: 35P30.
Key words and phrases. Partial regularity, Liouville system, extremal solution.

$\overline{\mathcal{U}}$. Minimality means $u \le \tilde{u}$ and $v \le \tilde{v}$ in Ω for any other smooth solution $(\tilde{u}, \tilde{v})$ for the same (μ, λ).

For each slope $m > 0, \mathcal{U}$ intersected with the line $\mu = m\lambda$ is a segment $\{(m\lambda, \lambda) : \lambda \in (0, \lambda^*(m))\}$ and at the extremal point $(m\lambda^*(m), \lambda^*(m)) \in \partial\mathcal{U}$ there is a solution, called the extremal solution. It is defined as the limit as $\lambda \uparrow \lambda^*(m)$ of the minimal solution with parameters $(m\lambda, \lambda)$ and it may be singular. In a recent work [8], L. Dupaigne, A. Farina and B. Sirakov proved that the extremal solutions for the Liouville system (1) are smooth if $n \le 9$. C. Cowan [1] had obtained the same conclusion under the restrictions $3 \le n \le 9$ and $\frac{n-2}{8} \le \frac{\mu}{\lambda} \le \frac{8}{n-2}$. In higher dimensions this fails at least in the radial case and for $\lambda = \mu$, where (1) reduces to (2).

Let us recall that en extremal solution (u, v) satisfies (1) in the sense that $u, v \in L^1(\Omega)$, $e^u\, dist(\cdot, \partial\Omega), e^v dist(\cdot, \partial\Omega) \in L^1(\Omega)$, and

$$\int_\Omega u(-\Delta\varphi) = \int_\Omega \mu e^v \varphi, \quad \int_\Omega v(-\Delta\varphi) = \int_\Omega \lambda e^u \varphi,$$

for all $\varphi \in C^2(\overline{\Omega})$ with $\varphi = 0$ on $\partial\Omega$.

We define the singular set Σ of an extremal solution (u, v) by $x \notin \Sigma$ if there is a neighborhood W of x such that u, v are bounded in W. By elliptic regularity, u, v are then smooth in this neighborhood.

Theorem 1.1. *Assume $n \ge 10$ and let (u, v) be an extremal solution of the Liouville system (1) and Σ be its singular set. Then the Hausdorff dimension of Σ is less or equal than $n - 10$.*

The rest of the article is devoted to the proof of this theorem. We first recall a useful inequality which is valid for stable solutions of the system, obtained in C. Cowan, N. Ghoussoub [2] and L. Dupaigne, A. Farina, B. Sirakov [8]. We then state a comparison result between u and v. Next, we perform a Moser iteration scheme to control the growth of some integrals of e^u and e^v on balls. The final step is an adaptation of an argument of K. Wang [13] using an ε-regularity result. The result in this paper is also closely related to the work of L. Dupaigne, M. Ghergu, O. Goubet and G. Warnault [9] on stable solutions of $\Delta^2 u = e^u$ in a bounded domain or entire space.

2. Proof of Theorem 1.1

From [12] we know that for $(\mu, \lambda) \in \mathcal{U}$, the associated minimal solution (u, v) of (1), which is smooth, is stable in the sense that there exist $\varphi, \psi : \Omega \to \mathbb{R}$, smooth and positive in Ω, satisfying

$$\begin{cases} -\Delta\varphi - \mu e^v \psi = \eta\varphi & \text{in } \Omega, \\ -\Delta\psi - \lambda e^u \varphi = \eta\psi & \text{in } \Omega, \\ \varphi = \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

for some $\eta > 0$. C. Cowan, N. Ghoussoub [2] and independently L. Dupaigne, A. Farina, B. Sirakov [8] have showed that this stability condition implies the following estimate.

Lemma 2.1. *Let (u, v) be a smooth stable solution of the system (1). For any φ in $H_0^1(\Omega)$*

$$\sqrt{\lambda\mu} \int_\Omega \exp(\frac{u+v}{2})\varphi^2 \le \int_\Omega |\nabla\varphi|^2. \tag{3}$$

2.1. **Comparison.** It will be useful later to have the following inequalities between the components of a solution of (1).

Lemma 2.2. *Assume* $\lambda \geq \mu$. *Then for any smooth solution to the Liouville system* (1) *we have:*

$$u \leq v \leq u + \log \lambda - \log \mu. \tag{4}$$

Proof. Introduce $w = v - u - \log \lambda + \log \mu$. Then $w \leq 0$ on $\partial\Omega$. We have $-\Delta w = \lambda e^u - \mu e^v = -\lambda e^u(e^w - 1)$, and then

$$-\Delta w + \lambda e^u \left(\frac{e^w - 1}{w}\right) w = 0.$$

Then due to the maximum principle $w \leq 0$ in Ω. For the first inequality in (4) introduce $\tilde{w} = v - u$. Then $-\Delta \tilde{w} = \lambda e^u - \mu e^v \geq \lambda(e^u - e^v) = -a(x)\tilde{w}$ where $a(x) \geq 0$. Then by the maximum principle $\tilde{w} \geq 0$ in Ω. □

2.2. **Reverse Hölder inequality.** The following estimate is similar to the one obtained in [8] and [9], see also [4] for the scalar case. We assume that (u, v) is a smooth stable solution of (1).

Lemma 2.3. *For any* $0 < \alpha < 4$ *there exists a constant* $C = C(n, \alpha, \lambda, \mu)$ *such that for any* $\varphi \in C_c^\infty(\Omega)$ *we have*

$$\begin{aligned} &\|\nabla(\exp(\frac{\alpha u}{2})\varphi)\|^2_{L^2(\Omega)} + \|\nabla(\exp(\frac{\alpha v}{2})\varphi)\|^2_{L^2(\Omega)} \\ &\qquad \leq C\int_\Omega e^{\alpha u}(|\nabla\varphi|^2 + |\varphi\Delta\varphi|^2) + C\int_\Omega e^{\alpha v}(|\nabla\varphi|^2 + |\varphi\Delta\varphi|^2). \end{aligned} \tag{5}$$

Remark 1. Although the constant C depends on μ, λ it remains bounded as (μ, λ) approaches any extremal couple on $\partial\mathcal{U}$.

Proof. Multiply $-\Delta u = \mu e^v$ by $e^{\alpha u}\varphi^2$ and integrate by parts to obtain

$$\mu\int_\Omega e^{v+\alpha u}\varphi^2 = \int_\Omega \nabla u \nabla(e^{\alpha u}\varphi^2) = \frac{4}{\alpha}\int_\Omega \varphi^2|\nabla(e^{\frac{\alpha u}{2}})|^2 + \frac{1}{\alpha}\int_\Omega \nabla(e^{\frac{\alpha u}{2}})\nabla\varphi^2.$$

This reads also

$$\mu\int_\Omega e^{v+\alpha u}\varphi^2 = \frac{4}{\alpha}\int_\Omega |\nabla(e^{\frac{\alpha u}{2}}\varphi)|^2 - \frac{2}{\alpha}\int_\Omega e^{\alpha u}(|\nabla\varphi|^2 - \varphi\Delta\varphi).$$

A similar equality is valid replacing respectively u by v and μ by λ. Introducing $X = \int_\Omega |\nabla(e^{\frac{\alpha u}{2}}\varphi)|^2$, $Y = \int_\Omega |\nabla(e^{\frac{\alpha v}{2}}\varphi)|^2$, $A = \frac{2}{\alpha}\int_\Omega e^{\alpha u}(|\nabla\varphi|^2 - \varphi\Delta\varphi)$, and $B = \frac{2}{\alpha}\int_\Omega e^{\alpha v}(|\nabla\varphi|^2 - \varphi\Delta\varphi)$, we then have

$$\begin{aligned} \frac{4}{\alpha}X &= \mu\int_\Omega e^{v+\alpha u}\varphi^2 + A, \\ \frac{4}{\alpha}Y &= \lambda\int_\Omega e^{u+\alpha v}\varphi^2 + B. \end{aligned}$$

We combine Hölder's inequality and the stability estimate (3) to obtain

$$\mu\int_\Omega e^{v+\alpha u}\varphi^2 \leq \mu\left(\int_\Omega e^{\frac{u+v}{2}}e^{\alpha u}\varphi^2\right)^{1-\frac{1}{2\alpha}}\left(\int_\Omega e^{\frac{u+v}{2}}e^{\alpha v}\varphi^2\right)^{\frac{1}{2\alpha}} \leq \left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}} X^{1-\frac{1}{2\alpha}} Y^{\frac{1}{2\alpha}}.$$

Analogously, we have the same inequality replacing u by v and μ by λ. Hence we obtain

$$\frac{4}{\alpha}X \le (\frac{\mu}{\lambda})^{\frac{1}{2}} X^{1-\frac{1}{2\alpha}} Y^{\frac{1}{2\alpha}} + A, \tag{6}$$

$$\frac{4}{\alpha}Y \le (\frac{\lambda}{\mu})^{\frac{1}{2}} X^{\frac{1}{2\alpha}} Y^{1-\frac{1}{2\alpha}} + B. \tag{7}$$

Multiplying these inequalities leads to

$$(\frac{16}{\alpha^2} - 1)XY \le A(\frac{\lambda}{\mu})^{\frac{1}{2}} X^{\frac{1}{2\alpha}} Y^{1-\frac{1}{2\alpha}} + B(\frac{\mu}{\lambda})^{\frac{1}{2}} X^{1-\frac{1}{2\alpha}} Y^{\frac{1}{2\alpha}} + AB.$$

Set $\delta = (\frac{16}{\alpha^2} - 1)$. This implies that either

$$(\frac{\mu}{\lambda})^{\frac{1}{2}} X^{1-\frac{1}{2\alpha}} Y^{\frac{1}{2\alpha}} \le \frac{A}{\delta}(1 + \sqrt{1+\delta}), \tag{8}$$

or

$$(\frac{\lambda}{\mu})^{\frac{1}{2}} X^{\frac{1}{2\alpha}} Y^{1-\frac{1}{2\alpha}} \le \frac{B}{\delta}(1 + \sqrt{1+\delta}) \tag{9}$$

hold. Assuming that (8) is true and combining with (6) we get $X \le CA$. Using Young's inequality in (7) we obtain $Y \le C(A+B)$ so that $X+Y \le C(A+B)$ holds, which is (5). Assuming the validity of (9) we obtain the same conclusion. □

A consequence of the previous lemma is the following.

Lemma 2.4. *Set $2^* = \frac{2n}{n-2}$. For any $0 < \alpha < \beta < 2(2^*)$, if $B_{2r}(x) \subset \Omega$ we have*

$$\left(r^{-n} \int_{B_r(x)} (e^{\beta u} + e^{\beta v}) \right)^{\alpha/\beta} \le C r^{-n} \int_{B_{2r}(x)} e^{\alpha u} + e^{\alpha v} \tag{10}$$

Proof. Follows from repeated applications of Lemma 2.3, using Sobolev's embedding and Hölder's inequality. □

Remark 2. Lemmas 2.3 and 2.4 are independent of the boundary conditions of u and v, and do not use the comparison of u to v of Lemma 2.2.

2.3. **Integrability of solutions.**

Lemma 2.5. *Assume (u, v) is a stable smooth solution of (1) with parameter (μ, λ) of the form $\mu = m\lambda$ for some fixed $m > 0$. For $1 \le \alpha < 5$ there is C independent of λ such that*

$$\int_\Omega e^{\alpha u} + e^{\alpha v} \le C.$$

We note that C in general depends on the slope m. In this lemma we need the inequalities between u and v of Lemma 2.2. For the proof, we refer to [8] where the following was proved.

Lemma 2.6. *Assume $\lambda \ge \mu$. If (u, v) is a stable smooth solution of (1) with parameter (μ, λ) of the form $\mu = m\lambda$ for some fixed $m > 0$, then for $1 \le \alpha < 5$ there is C independent of λ such that*

$$\int_\Omega e^{\alpha u} \le C.$$

Lemma 2.5 follows from Lemmas 2.6 and 2.2 in the case $\lambda \ge \mu$. By a symmetric argument we obtain the same conclusion if $\lambda \le \mu$.

2.4. **ε-regularity.** A crucial step is the following ε-regularity result, whose version for stable solutions in the scalar case is due to K. Wang [13], see also [9] for a biharmonic equation with exponential nonlinearity.

Lemma 2.7. *Let (u, v) be an extremal solution of (1). Then there is $\varepsilon_2 > 0$ such that if for some $r_0 > 0$ with $B_{r_0}(x) \subset \Omega$ one has*

$$r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \leq \varepsilon_2$$

then there is a neighborhood of x such that u, v are smooth in this neighborhood.

For the proof we need the following key step, which is adapted from [13] in the scalar case.

Lemma 2.8. *There exists $\varepsilon_0 > 0$ and $\theta > 0$ depending only on n such that for any $0 < \varepsilon \leq \varepsilon_0$, if (u, v) is a stable smooth solution of (1), $B_{r_0}(x) \subset \Omega$ and*

$$r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \leq \varepsilon \tag{11}$$

then

$$(\theta r_0)^{2-n} \int_{B_{\theta r_0}(x)} (e^u + e^v) \leq \varepsilon. \tag{12}$$

Proof. Let us assume that $x = 0$ by shifting coordinates. We rescale the functions by setting

$$\tilde{u}(x) = u(r_0 x) + 2\log(r_0), \quad \tilde{v}(x) = v(r_0 x) + 2\log(r_0), \tag{13}$$

and note that the new functions (where the $\tilde{}$ in the notation will be dropped) satisfy

$$-\Delta u = \mu e^v, \quad -\Delta v = \lambda e^u, \quad \text{in } B_1(0).$$

Let us decompose $u = u_1 + u_2$, $v = v_1 + v_2$ where

$$\begin{aligned} \Delta u_1 &= 0 \quad \text{in } B_{1/2}(0), & u_1 &= u \quad \text{on } \partial B_{1/2}(0), \\ -\Delta u_2 &= \mu e^v \quad \text{in } B_{1/2}(0), & u_2 &= 0 \quad \text{on } \partial B_{1/2}(0), \\ \Delta v_1 &= 0 \quad \text{in } B_{1/2}(0), & v_1 &= v \quad \text{on } \partial B_{1/2}(0), \\ -\Delta v_2 &= \lambda e^u \quad \text{in } B_{1/2}(0), & v_2 &= 0 \quad \text{on } \partial B_{1/2}(0). \end{aligned}$$

Let $\gamma > 0$, $0 < \theta < 1/4$ to be fixed later on and $\varepsilon > 0$. Let us estimate

$$\theta^{2-n} \int_{B_\theta(0)} e^u = \theta^{2-n} \int_{B_\theta(0) \cap [u_2 \leq \varepsilon^\gamma]} e^{u_1 + u_2} + \theta^{2-n} \int_{B_\theta(0) \cap [u_2 > \varepsilon^\gamma]} e^u. \tag{14}$$

For the first term we proceed by noting that e^{u_1} is subharmonic in $B_{1/2}(0)$ and $u_2 \geq 0$, so

$$\begin{aligned} \theta^{2-n} \int_{B_\theta(0) \cap [u_2 \leq \varepsilon^\gamma]} e^{u_1 + u_2} &\leq \theta^{2-n} e^{\varepsilon^\gamma} \int_{B_\theta(0) \cap [u_2 \leq \varepsilon^\gamma]} e^{u_1} \\ &\leq \theta^{2-n} e^{\varepsilon^\gamma} \int_{B_\theta(0)} e^{u_1} \\ &\leq C \theta^2 e^{\varepsilon^\gamma} \int_{B_{1/2}(0)} e^{u_1} \\ &\leq C \theta^2 e^{\varepsilon^\gamma} \int_{B_{1/2}(0)} e^u \leq C \theta^2 e^{\varepsilon^\gamma} \varepsilon, \end{aligned} \tag{15}$$

where we have used (11). For the second term in (14) we have

$$\begin{aligned}\theta^{2-n}\int_{B_\theta(0)\cap[u_2>\varepsilon^\gamma]} e^u &\le \theta^{2-n}\varepsilon^{-\gamma}\int_{B_\theta(0)\cap[u_2>\varepsilon^\gamma]} u_2 e^u\\ &\le \theta^{2-n}\varepsilon^{-\gamma}\int_{B_{1/2}(0)} u_2 e^u\\ &\le \theta^{2-n}\varepsilon^{-\gamma}\|u_2\|_{L^2(B_{1/2}(0))}\|e^u\|_{L^2(B_{1/2}(0))}. \end{aligned} \tag{16}$$

To estimate $\|e^u\|_{L^2(B_{1/2}(0))}$ we apply (10) with $\alpha = 1$, $\beta = 2$ to get

$$\|e^u\|_{L^2(B_{1/2}(0))} \le C\varepsilon^{1/2}. \tag{17}$$

For $\|u_2\|_{L^2(B_{1/2}(0))}$, first note that

$$\|e^v\|_{L^2(B_{1/2}(0))} \le C\varepsilon^{1/2}.$$

Hence by L^2 regularity theory

$$\|u_2\|_{W^{2,2}(B_{1/2}(0))} \le C\varepsilon^{1/2}.$$

By using the Sobolev embedding $W^{2,2} \subset L^{\frac{2n}{n-4}}$ we get

$$\|u_2\|_{L^{\frac{2n}{n-4}}(B_{1/2}(0))} \le C\varepsilon^{1/2}. \tag{18}$$

By interpolation

$$\|u_2\|_{L^2(B_{1/2}(0))} \le \|u_2\|^m_{L^1(B_{1/2}(0))}\|u_2\|^{1-m}_{L^{\frac{2n}{n-4}}(B_{1/2}(0))} \tag{19}$$

where $m = \frac{4}{n+4} \in (0,1)$. But

$$\|u_2\|_{L^1(B_{1/2}(0))} \le C\lambda\|e^v\|_{L^1(B_{1/2}(0))} \le C\varepsilon, \tag{20}$$

so (19) combined with (18) and (20) yields

$$\|u_2\|_{L^2(B_{1/2}(0))} \le C\varepsilon^m\varepsilon^{(1-m)/2} = C\varepsilon^{\frac{1+m}{2}}. \tag{21}$$

Therefore, using (16), (17) and (21) we find

$$\theta^{2-n}\int_{B_\theta(0)\cap[u_2>\varepsilon^\gamma]} e^u \le C\theta^{2-n}\varepsilon^{1+m/2-\gamma}.$$

Combining this and (15) we obtain

$$\theta^{2-n}\int_{B_\theta(0)} e^u \le C\theta^2 e^{\varepsilon^\gamma}\varepsilon + C\theta^{2-n}\varepsilon^{1+m/2-\gamma}.$$

Since $m > 0$ we may choose $0 < \gamma < m/2$. Then fix $\theta > 0$ so that $Ce\theta^2 \le 1/2$ and then choose $\varepsilon_0 > 0$ sufficiently small so that $C\theta^{2-n}\varepsilon_0^{m/2-\gamma} \le 1/2$. It follows that for any $0 < \varepsilon \le \varepsilon_0$

$$\theta^{2-n}\int_{B_\theta(0)} e^u \le \varepsilon.$$

A similar argument yields the corresponding estimate for e^v. Rescaling back we obtain (12). □

Applying the previous lemma we can prove

Lemma 2.9. *There exists $\varepsilon_1 > 0$ and $\theta > 0$ depending only on n such that for any $0 < \varepsilon \le \varepsilon_1$, if (u, v) is a stable smooth solution of (1), $B_{r_0}(x) \subset \Omega$ and*

$$r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \le \varepsilon$$

then

$$r^{2-n} \int_{B_r(y)} (e^u + e^v) \le 2^{n-2}\theta^{2-n}\varepsilon$$

for any $y \in B_{r_0/2}(x)$ and any $0 < r \le r_0/2$.

Proof. By shifting coordinates we can assume that $x = 0$ and by the scaling (13) that $r_0 = 1$. Let ε_0, θ be the constants of Lemma 2.8. We choose ε_1 so that $2^{n-2}\varepsilon_1 = \varepsilon_0$. Then, for any $y \in B_{1/2}(0)$ and $0 < \varepsilon \le \varepsilon_1$ we have

$$(\frac{1}{2})^{2-n} \int_{B_{1/2}(y)} (e^u + e^v) \le 2^{n-2} \int_{B_1(0)} (e^u + e^v) \le 2^{n-2}\varepsilon \le \varepsilon_0.$$

Applying inductively Lemma 2.8, for any integer $k \ge 1$ we have

$$(\theta^k)^{2-n} \int_{B_{\theta^k}(y)} (e^u + e^v) \le 2^{n-2}\varepsilon.$$

If $0 < r \le 1/2$ is arbitrary we select $k \ge 1$ an integer such that $\theta^{k+1} \le r \le \theta^k$. Then

$$r^{2-n} \int_{B_r(y)} (e^u + e^v) \le (\theta^{k+1})^{2-n} \int_{B_{\theta^k}(y)} (e^u + e^v) \le 2^{n-2}\theta^{2-n}\varepsilon.$$

□

Proof of Lemma 2.7. The result of Lemma 2.9 holds also for any extremal solution. This can be proved by approximating an extremal solution (u, v) of parameters $(m\lambda^*(m), \lambda^*(m)) \in \partial\mathcal{U}$ by minimal solutions with parameters $(m\lambda, \lambda)$ and $\lambda \uparrow \lambda^*(m)$. In this process, the constants appearing in the estimates remain bounded, see Remark 1.

Let ε_1, θ be the constants of Lemma 2.9. We take $0 < \varepsilon_2 < \varepsilon_1$ to be fixed later on. By the change of variables (13) we can assume that $x = 0$ and $r_0 = 1$, so now the hypothesis is

$$\int_{B_1(0)} e^u + e^v \le \varepsilon_2.$$

Then by Lemma 2.9 we have

$$r^{2-n} \int_{B_r(y)} (e^u + e^v) \le 2^{n-2}\theta^{2-n}\varepsilon_2$$

for any $y \in B_{1/2}(0)$ and any $0 < r \le 1/2$. This says that e^u, e^v are in the Morrey space $M_{n/2}(B_{1/2}(0))$ and

$$\|e^u\|_{M_{n/2}} + \|e^v\|_{M_{n/2}} \le 2^{n-2}\theta^{2-n}\varepsilon_2. \tag{22}$$

Let $\tilde{u}$, $\tilde{v}$ be the Newtonian potentials of $e^u\chi_{B_{1/2}}(0)$ and $e^v\chi_{B_{1/2}}(0)$ respectively. Then by [10] Lemma 7.20 we have

$$\int_{B_1(0)} e^{\beta|\tilde{u}|} + e^{\beta|\tilde{v}|} \le C_2 \tag{23}$$

for $\beta \leq \min(\frac{c_1}{\|e^u\|_{M_{n/2}}}, \frac{c_1}{\|e^v\|_{M_{n/2}}})$ where $c_1, C_2 > 0$ depend only on dimension. By (22), choosing $\varepsilon_2 > 0$ small, we obtain that (23) holds for some $\beta > n/2$. Then $e^u, e^v \in L^\beta(B_{1/4}(0))$ for some $\beta > n/2$. By standard L^p regularity $u, v \in L^\infty(B_{1/8}(0))$. Scaling back we have the conclusion. □

2.5. **Proof of Theorem 1.1.**

Proof. Let $1 \leq \alpha < 5$. We claim that

$$\Sigma \subset \Big\{ x \in \Omega : \limsup_{r \to 0} r^{2\alpha - n} \int_{B_r(x) \cap \Omega} (e^{\alpha u} + e^{\alpha v}) > 0 \Big\}.$$

Indeed, if $x \in \Omega$ and

$$\lim_{r \to 0} r^{2\alpha - n} \int_{B_r(x) \cap \Omega} (e^{\alpha u} + e^{\alpha v}) = 0$$

then by Hölder's inequality also

$$\lim_{r \to 0} r^{2-n} \int_{B_r(x) \cap \Omega} (e^u + e^v) = 0.$$

Therefore for some $r_0 > 0$ so that $B_{r_0}(x) \subset \Omega$ we have

$$r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \leq \varepsilon_2$$

where $\varepsilon_2 > 0$ is the constant from Lemma 2.7. Then by the same lemma u, v are bounded in a neighborhood of x and hence $x \notin \Sigma$.

Since $e^{\alpha u} + e^{\alpha v} \in L^1(\Omega)$ by Lemma 2.5, we obtain that $\mathcal{H}^{n-2\alpha}(\Sigma) = 0$, see e.g. [7, Theorem 5.3.4]. Letting $\alpha \uparrow 5$ we deduce that the Hausdorff dimension of Σ is less or equal than $n - 10$. □

Acknowledgments

We would like to thank L. Dupaigne for interesting and helpful discussions on the subject.

This article is part of the Ecos Sud C09E06 exchange program "Heterogeneous ecological models and singular nonlinear PDEs". In addition, J.D. was supported by Fondecyt 1090167, CAPDE-Anillo ACT-125 and Fondo Basal CMM.

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Received February 2013; revised March 2013.

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